Convergence of Brownian motions on RCD spaces

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1 Introduction & Result

In this talk, we consider the following problem:

(Q) Does the weak convergence of Brownian motions follow only from geometrical convergence of the underlying spaces (or, vice versa)?

As a main result in this talk, we show that the weak convergence of the laws of Brownian motions is equivalent to the measured Gromov–Hausdorff (mGH) convergence of the underlying metric measure spaces under the following assumption:

Assumption 1.1 Let $N, K$ and $D$ be constants with $1 < N < \infty$, $K \in \mathbb{R}$ and $0 < D < \infty$. For $n \in \mathbb{N} := \mathbb{N} \cup \{\infty\}$, let $\mathcal{X}_n = (X_n, d_n, m_n)$ be a metric measure space satisfying the RCD$^*(K, N)$ condition with $\text{Diam}(X_n) \leq D$ and $m_n(X_n) = 1$.

Under Assumption 1.1, it is known that there exists a conservative Hunt process on $\mathcal{X}_n$ associated with the Cheeger energy and unique in all starting points in $\mathcal{X}_n$. We denote it by $(\{P_n^x\}_{x \in X_n}, \{B_n^x\}_{t \geq 0})$, called the Brownian motion on $\mathcal{X}_n$. We state our main theorem precisely:

Theorem 1.2 Suppose that Assumption 1.1 holds. Then the following statements (i) and (ii) are equivalent:

(i) (mGH-convergence of the underlying spaces)

$\mathcal{X}_n$ converges to $\mathcal{X}_\infty$ in the measured Gromov–Hausdorff sense.

(ii) (Weak convergence of the laws of Brownian motions)

There exist

\[
\begin{aligned}
&\text{a compact metric space } (X, d) \\
&\text{isometric embeddings } \iota_n : X_n \to X \ (n \in \mathbb{N}) \\
x_n \in X_n \ (n \in \mathbb{N})
\end{aligned}
\]

such that

$\iota_n(B_n^x)^\# \# P_n^x \to \iota_\infty(B_\infty^x)^\# \# P_\infty^x$ weakly in $\mathcal{P}(C([0, \infty); X))$.

The subscript $\#$ means the operation of the push-forward of measures.
RCD*(K, N) (Riemannian Curvature-Dimension) spaces, introduced by Erbar–Kuwada–Sturm [2], are metric measure spaces satisfying a generalized notion of “Ricci ≥ K, dim ≤ N”, which include several important classes of non-smooth spaces. For example, measured Gromov–Hausdorff (mGH) limit spaces of complete Riemannian manifolds with Ricci ≥ K, dim = N, or Alexandrov spaces with Curv ≥ K/(N – 1), dim = N are included in RCD*(K, N) spaces.

Remark 1.3 We give comments to several related works.

(i) In [4], Ogura studied the weak convergence of the laws of the Brownian motions on Riemannian manifolds by a different approach from this talk. He push-forwarded all Brownian motions not to the ambient space X, but to the limit space M∞ with respect to approximation maps fn : Mn → M∞ of the Kasue–Kumura convergence with certain time-discretization of Brownian motions.

More precisely, he assumed uniform upper bounds for heat kernels, and the Kasue–Kumura spectral convergence (3) of the underlying manifolds Mn. He push-forward each Brownian motions on Mn to the Kasue–Kumura spectral limit space M∞ with respect to εn-isometry fn : Mn → M∞, and show the convergence in law on the càdlàg space of the push-forwarded and time-discretized Brownian motions on M∞.

(ii) In [1], Albeverio and Kusuoka studied diffusion processes associated with SDEs on thin tubes in \( \mathbb{R}^d \) shrinking to one-dimensional spider graphs. They studied the weak convergence of these diffusions to one-dimensional diffusions on the limit graphs. Their setting does not satisfy the RCD*(K, N) condition because Ricci curvatures are not bounded below at points of conjunctions in spider graphs.

References


