Stochastic heat equation arising from a certain branching systems in random environment

Makoto Nakashima
University of Tsukuba, Graduate School of Pure and Applied Sciences

In this talk, we will consider the stochastic heat equations on the line which have been studied for four decades. Especially, we will construct a non-negative solution to a certain stochastic heat equation by using a branching systems in random environment.

1 Stochastic heat equation

In this talk, we consider the stochastic heat equations as follows:

\[ \frac{\partial}{\partial t} X_t = \frac{1}{2} \Delta X_t(x) + a(X_t(x)) \dot{W}(t, x), \]  

where \( W \) is a time-space white noise and \( a \) is a continuous function with \( a(0) = 0 \).

The study of stochastic heat equation was started around 1970’s. In particular, the existence and the uniqueness of the strong solution to (1.1) are known if \( a \) is Lipschitz continuous [8] et.al.

Also, the existence of the solution to (1.1) are verified for more general \( a \) under some initial conditions [7]. On the other hand, the uniqueness of solutions to (1.1) are very difficult problem attacked by many mathematicians [4, 3] et.al.

The stochastic heat equations (1.1) appear as some limit process. One of the most famous examples is a one-dimensional super-Brownian motion which is a measure-valued process arising as a scaling limit of some critical branching Brownian motion or branching random walks.

2 Super-Brownian motion

Before giving a definition of super-Brownian motion, we recall the branching random walks.*

**Definition 1.** Branching random walks are defined as follows:

1. There are particles at \( x_1, \cdots, x_{M_N} \in \mathbb{Z}^d \) at time 0.
2. The particles at time \( n \) choose a nearest neighbor site independently and uniformly, and move there.
3. Then, each of them independently splits into two particles with probability \( \frac{1}{2} \) or vanishes with probability \( \frac{1}{2} \).

**Remark:** The total number at time \( n \), \( B_n \), is a critical Galton-Watson process.

We set a measure-valued process \( \{X_t^{(N)}\} \) as follows: For every Borel set \( A \)

\[ X_0^{(N)}(dx) = \frac{1}{N} \sum_{i=1}^{M_N} \delta_{x_i/N^{1/2}}(dx), \]

\[ X_t^{(N)}(A) = \frac{1}{N} \sharp \{ \text{particles locates in } N^{1/2}A \text{ at time } \lfloor Nt \rfloor \}. \]

Then, we have the following theorem:

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*nakamako@math.tsukuba.ac.jp

*In this talk, we consider the most simple case.
Theorem 2. ([9, 1]) If $X_0^{(N)} \Rightarrow X_0$ in $\mathcal{M}_F(\mathbb{R}^d)$, then $\{X_t^{(N)}\}$ weakly converges to a measure valued process $X_t$ as $N \to \infty$.

Moreover, [2, 6] if $d = 1$, then $X_t$ is absolutely continuous with respect to the Lebesgue measure for any $t > a.s.$ and its density $X_t(x)$ is the unique non-negative weak solution to the stochastic heat equation
\[
\frac{\partial}{\partial t} X_t(x) = \frac{1}{2} \Delta X_t(x) + \sqrt{X_t(x)} W(t,x), \quad \lim_{t \to 0} X_t(x) dx = X_0(dx).
\]

3 Main result

We construct a solution to (1.1) with $a(u) = \sqrt{u}$ from a certain branching system in random environment.

Theorem 3. ([5]) For any $X_0 \in \mathcal{M}_F(\mathbb{R})$, there exists the unique, weak, and non-negative solution to the stochastic heat equation
\[
\frac{\partial}{\partial t} X_t(x) = \frac{1}{2} \Delta X_t(x) + \sqrt{X_t(x) + X_t(x)^2} W(t,x), \quad \lim_{t \to \infty} X_t(x) dx = X_0(dx).
\]

Remark: Mytnik gave a remark on the above construction in his paper.

References


