A Brief Introduction to Brownian Motion on a Riemannian Manifold

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Contents

Introductory Remarks

These lecture notes constitute a brief introduction to stochastic analysis on manifolds in general, and Brownian motion on Riemannian manifolds in particular. Instead of going into detailed proofs and not accomplishing much, I will outline main ideas and refer the interested reader to the literature for more thorough discussion. This is especially true for the last lecture, in which I only discuss the flat space case. Therefore it should be only served as a guide to what one should expect for the path and loop spaces over a Riemannian manifold.

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Lecture 1. Brownian Motion on a Riemannian Manifold

1.1. Brownian motion on euclidean space

Brownian motion on euclidean space is the most basic continuous time Markov process with continuous sample paths. By general theory of Markov processes, its probabilistic behavior is uniquely determined by its initial distribution and its transition mechanism. The latter can be specified by either its transition density function or its infinitesimal generator. For Brownian motion on ^{n}, its transition density function is the Gaussian heat kernel

$$p\,(t,x,y)=\left(\frac{1}{2\pi t}\right)^{n/2}e^{-|x-y|^2/2t}, \tag{1.1.1}$$

and its infinitesimal generator is half of the Laplace operator:

$$\frac{1}{2}\Delta=\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}.$$

The law $\mathbb{P}_x$ of Brownian motion starting from x is therefore a probability measure on the euclidean path space $C(\mathbb{R}_+,\mathbb{R}^n)$. If we use $\mathbb{E}_x$ to denote the integration with respect to $\mathbb{P}_x$, then we have Dynkin's formula

$$\mathbb{E}_x f(X_t)=f(x)+\frac{1}{2}\mathbb{E}_x\int_0^t\Delta f(X_s)\,ds.$$

Here X stands for the so-called coordinate process on $C(\mathbb{R}_+,\mathbb{R}^n)$:

$$X(\omega)_t=X_t(\omega)=\omega_t,\quad \omega\in C(\mathbb{R}_+,\mathbb{R}^n).$$

Dynkin's formula is the starting point of applications of Brownian motion to analysis. If we want to do *stochastic analysis* (as opposed to analysis), then we need Itô's formula:

$$f(X_t)=f(X_0)+\int_0^t\langle\nabla f(X_s),dX_s\rangle+\frac{1}{2}\int_0^t\Delta f(X_s)\,ds.$$

This formula can be regarded as a microscopic formulation (a refinement) of Dynkin's formula. Lying between the two is the martingale characterization of Brownian motion:

$$f(X_t)=f(X_0)+\text{martingale}+\frac{1}{2}\int_0^t\Delta f(X_s)\,ds.$$

All these formulations (Dynkin's formula, Itô's formula, martingale characterization) are equivalent in the sense that each one of them defines uniquely the probability measure $\mathbb{P}_x$, under which the coordinate process is a Markov process with the Gaussian transition density function (1.1.1).

All of the above formulations of Brownian motion will find its counterpart when $\mathbb{R}^n$ is replaced by a Riemannian manifold.

As a way of thinking, it is useful to regard paths of Brownian motion as the characteristic lines of the Laplace operator Δ. Since Δ is an elliptic operator, we know that it has no real characteristic lines in the classical sense. It is a basic fact in the theory of partial differential equations that the solution of the initial value problem is a weighted average of the data on the characteristic lines emanating from the point at which we are seeking the solution. Take, for example, the initial value problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,x) = f(x), \quad \frac{\partial u}{\partial x}(0,x) = 0.$$

The characteristic lines of the wave operator are $x \pm t = \text{const}$. The solution of the problem is

$$u(t,x) = \frac{f(x+t) + f(x-t)}{2},$$

which is the expected value of the initial value $f(x+t)$ and $f(x-t)$ on the characteristic lines from (x,t) as if we assign each characteristic line the probability $1/2$. Now consider the Dirichlet problem on a smooth domain D in $\mathbb{R}^n$:

$$\begin{cases} \Delta u = 0 & \text{on } D, \\ \quad u = f & \text{on } \partial D. \end{cases}$$

Each curve X starting from x meets the boundary at X_{τ_D}, where τ_D is the first exit time of D:

$$\tau_D = \inf \{t \geq 0 : X_t \notin D\}.$$

The value of the data for this "characteristic line" is $f(X_{\tau_D})$. The probability is assigned to these "characteristic lines" according to the law of Brownian motion $\mathbb{P}_x$. In analogy with the case of the wave equation, we arrive heuristically the formula

$$u_f(x) = \mathbb{E}_x f(X_{\tau_D}), \quad x \in D.$$

This is Doob's representation of the solution of the Dirichlet problem.

1.2. Laplace-Beltrami operator and the heat kernel

As we have seen in Section 1.1, the Laplace operator and the Gaussian transition density function (heat kernel) are the basic analytic objects associated with Brownian motion on $\mathbb{R}^n$. Of the two, the Laplace operator is the more fundamental one and the heat kernel can be obtained as the (minimal) fundamental solution of the heat equation associated with the Laplace operator, namely, the function $p\,(t,x,y)$ is the smallest positive solution of the following initial value problem:

$$\frac{\partial p}{\partial t}=\frac{1}{2}\Delta p,\quad \lim_{t\downarrow 0}p\,(t,x,y)=\delta_x(y).$$

The counterpart of the Laplace operator on a Riemannian manifold M is the Laplace-Beltrami operator Δ_M, which will serves as the infinitesimal generator for Brownian motion on M. This operator can be briefly described as follows. We denote the Riemannian metric on M by $\langle\cdot,\cdot\rangle$. The gradient $\mathrm{grad} f$ of a function f on M is a vector field on M defined uniquely by

$$\langle \mathrm{grad} f, X\rangle = X(f),\quad X\in\Gamma(M).$$

[Here $\Gamma(M)$ is the space of smooth vector field on M.] The divergence $\mathrm{div}X$ of a vector field X is characterized by

$$\int_M X(f)d\mu=-\int_M f\mathrm{div}X\,d\mu.$$

Here μ is the Riemannian volume measure. The Laplace-Beltrami operator is

$$\Delta_M f=\mathrm{div}(\mathrm{grad} f).$$

In local coordinates $x^1,\cdots,x^n$, the Riemannian metric is written in the form

$$ds^2=g_{ij}dx^i dx^j.$$

The Laplace-Beltrami operator is given by

$$\Delta_M f=\frac{1}{\sqrt{G}}\frac{\partial}{\partial x^j}\left(\sqrt{G}g^{ij}\frac{\partial f}{\partial x^i}\right).$$

Here G is the determinant of the matrix $\{g_{ij}\}$ and $\{g^{ij}\}$ is its inverse. We have

$$\Delta_M f=g^{ij}\frac{\partial^2 f}{\partial x^i\partial x^j}+b^i\frac{\partial f}{\partial x^i},$$

where

$$b^i=\frac{1}{\sqrt{G}}\frac{\partial(\sqrt{G}g^{ij})}{\partial x^j}=g^{jk}\Gamma^i_{jk},$$

where Γ^i_{jk} are the Christoffel symbols of the metric ds^2 in the local coordinates. Therefore Δ_M is a second order, strictly elliptic operator.

The construction of the heat kernel (minimal fundamental solution) $p\,(t,x,y)$ associated with the Laplace-Beltrami operator is not a trivial task and belongs to the fields of partial differential equations and differential geometry. It is carried out in great detail in Chavel[**1**]. We will see later that the theory of stochastic differential equations allows us in some sense to avoid this construction.

A general stochastic differential equation in Stratonovich formulation has the form

$$dX_t=V_\alpha(X)_t\circ dW^\alpha_t+V_0(X_t)\,dt.$$

It generates a diffusion process (i.e., Markov process with continuous sample paths) whose infinitesimal generator is

$$L = \frac{1}{2}\sum_{i=1}^{l} V_\alpha^2 + V_0.$$

Thus it has the form of a "sum of squares." Unfortunately, on a general Riemannian manifold, there is no natural way of writing the Laplace-Beltrami operator as a sum of squares. Such a representation of Δ_M is possible if M is embedded isometrically in some euclidean space $\mathbb{R}^l$. For a point $x \in M$, let $P_\alpha(x)$ denote the projection of the unit coordinate vector ξ_α on the tangent space T_xM. We obtain in this way l vector fields P_α on M. It can be shown that ([**11**], 77–78)

$$\Delta_M = \sum_{\alpha=1}^{l} P_\alpha^2.$$

According to Nash's embedding theorem, one can always embed a Riemannian manifold isometrically in some euclidean space. This fact and the above expression for Δ_M can be used to give an extrinsic construction of Brownian motion on M.

1.3. Brownian motion on a Riemannian manifold

We define Brownian motion on M to be a Markov process whose transition density function is $p(t, x, y)$, the heat kernel associated with the Laplace-Beltrami operator. General theory of Markov processes shows how such a process can be constructed, see Chung[**4**]. It turns out to be a diffusion process, i.e., a strong Markov process with continuous sample paths.

On a general Riemannian manifold it may happen that

$$\int_M p(t, x, y)dy < 1.$$

Probabilistically this means that Brownian motion may not run for all time. More precisely, there is a finite stopping time e, called the lifetime of Brownian motion such that

$$\lim_{t\uparrow\uparrow e} X_t = \infty_M.$$

where $\widehat{M} = M \cup \{\infty_M\}$ is the one-point compactification of M and the above limit is understood in the topology of $\widehat{M}$. Intuitively speaking, Brownian motion may go off the manifold in a finite amount of time. Naturally this case does not happen if M is compact. We have

$$\mathbb{P}_x\{e \geq t\} = \int_M p(t, x, y)\, dy.$$

When $\mathbb{P}_x\{e < \infty\} = 1$ for some x and t (hence for all x and t), we say that the manifold M is stochastically complete. We will address later the question when a Riemannian manifold is stochastically complete.

Once Brownian motion is constructed as a diffusion processes with transition density function $p(t, x, y)$, the fact that $p(t, x, y)$ is the minimal fundamental solution of the heat equation for the Laplace-Beltrami operator give immediately Dynkin's formula

$$\mathbb{E}_x f(X_t) = f(x) + \frac{1}{2}\mathbb{E}_x \int_0^t \Delta_M f(X_s)\, ds$$

for all reasonable functions on M, say smooth with compact support. With a little extra effort, this can be expanded to read

$$f(X_t) = f(X_0) + M_t^f + \frac{1}{2}\int_0^t \Delta f(X_s)\, ds, \qquad 0 \le t < e. \tag{1.3.1}$$

Here M^f is a (local) martingale depending on f. The quadratic variation process of M^f can be identified.

PROPOSITION 1.3.1. *We have*

$$\langle M^f \rangle_t = \int_0^t |\nabla f(X_s)|^2 ds.$$

PROOF. We decompose $f(X_t)^2$ in two ways. First, we use Itô's formula and (1.3.1) to obtain

$$\begin{aligned} f(X_t)^2 &= f(x)^2 + 2\int_0^t f(X_s)\, df(X_s) + \langle f(X)\rangle_t \\ &= f(x)^2 + 2\int_0^t f(X_s) dM_s^f + \int_0^t f(X_s)\Delta_M f(X_s)\, ds + \langle M^f\rangle_t. \end{aligned}$$

Second, we use (1.3.1) with f replaced by f^2 and obtain

$$f(X_t)^2 = f(x)^2 + M_t^{f^2} + \frac{1}{2}\int_0^t \Delta_M(f^2)(X_s)\, ds.$$

Equating the bounded variation parts of the two expressions, we have

$$\langle M^f\rangle_t = \frac{1}{2}\int_0^t \left\{\Delta_M(f^2)(X_s) - 2f(X_s)\Delta_M f(X_s)\right\} ds.$$

Finally,

$$\Delta_M(f^2) - 2f\Delta_M f = 2|\nabla f|^2.$$

□

We thus have established the following fact about Brownian motion on M:

$$f(X_t) = f(X_0) + M_t^f + \frac{1}{2}\int_0^t \Delta_M f(X_s)\, ds, \qquad 0 \le t < e,$$

with where M^f is a local martingale with

$$\langle M^f \rangle_t = \int_0^t |\nabla f(X_s)|^2 ds.$$

This property of Brownian motion is sufficient for most applications in analysis and geometry, for in these applications we rarely need to know the joint distribution of the martingale M^f and the stochastic integral $\int_0^t \Delta_M f(X_s)\, ds$. For more delicate stochastic analysis, we need to have an explicit description of the martingale M^f. If $M = \mathbb{R}^n$, we have

$$M_t^f = \int_0^t \langle \nabla f(X_s), dX_t \rangle.$$

In the present formulation of Brownian motion, there is no direct ways of writing out M^f.

1.4. Brownian motion by embedding

If M is a submanifold of a euclidean space $\mathbb{R}^l$, Brownian motion on M can be obtained by solving a stochastic differential equation on M. In SECTION 1.2 we mentioned that the Laplace-Beltrami operator Δ_M can be written in the form of a sum of squares:

$$\Delta_M = \sum_{\alpha=1}^{l} P_\alpha^2,$$

where P_α is the projection of the αth coordinate unit vector ξ_α on the tangent space $T_x M$. Each P_α is a vector field on M. Consider the following Stratonovich stochastic differential equation on M driven by a l-dimensional euclidean Brownian motion W:

$$dX_t = \sum_{\alpha=1}^{l} P_\alpha(X_t) \circ dW_t^\alpha, \qquad X_0 \in M. \tag{1.4.1}$$

This is a stochastic differential equation on M because P_α are vector fields on M. Extending P_α arbitrarily to the whole ambient space we can solve this equation as if it is an equation on $\mathbb{R}^l$ by the usual Picard's iteration. It can be verified that if the initial value X_0 lies on the manifold M, then the solution lies on M for all time:

$$\mathbb{P}_x \{X_t \in M \quad \text{for all } t < e | X_0 \in M\} = 1,$$

see [**11**], 22–23. Furthermore, the solution is a diffusion process generated by $2^{-1} \sum_{\alpha=1}^{l} P_\alpha^2 = 2^{-1}\Delta_M$. Using Itô's formula we have

$$f(X_t) = f(X_0) + M_t^f + \frac{1}{2}\int_0^t \Delta_M f(X_s)\, ds, \qquad 0 \le t < e,$$

where

$$M_t^f = \int_0^t \langle P_\alpha f(X_s), dW_s^\alpha \rangle.$$

[The summation convention is used.]

EXAMPLE 1.4.1. (Brownian motion on $\mathbb{S}^n$) Consider the unit sphere $\mathbb{S}^n$ canonically embedded in $\mathbb{R}^{n+1}$. The projection to the tangent sphere at x is given by

$$P(x)\,\xi = \xi - \langle \xi, x\rangle x, \qquad x \in \mathbb{S}^n, \quad \xi \in \mathbb{R}^{n+1}.$$

Hence the matrix $P = \{P_1, \cdots, P_{n+1}\}$ is

$$P(x)_{ij} = \delta_{ij} - x_i x_j.$$

Brownian motion on $\mathbb{S}^n$ is the solution of the stochastic differential equation

$$X_t^i = X_0^i + \int_0^t (\delta_{ij} - X_s^i X_s^j) \circ dW_s^j, \quad X_0 \in \mathbb{S}^n.$$

This is Stroock's representation of spherical Brownian motion.

The representation (1.4.1) is a good way to visualize Brownian motion on M as a pathwise object (rather than a measure on the path space $C(\mathbb{R}_+, M)$). It is an extrinsic representation because it depends on the embedding of M into some euclidean space $\mathbb{R}^l$. It has the drawback that the equation is driven by a Brownian motion W whose dimension l is in general larger than the dimension n of the manifold M, whereas in some sense Brownian motion on M should still be an n-dimensional object. Full strength of Brownian motion on M can only be revealed after we write it faithfully as an n-dimensional object, i.e., as the solution of a stochastic differential equation driven by an n-dimensional euclidean Brownian motion.

1.5. Brownian motion in local coordinates

As we have mentioned in SECTION 1.2, the Laplace-Beltrami operator can be written as

$$\Delta_M f = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^j} \left(\sqrt{G} g^{ij} \frac{\partial f}{\partial x^i} \right) = g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + b^i \frac{\partial f}{\partial x^i}.$$

This gives a way of constructing Brownian motion valid up to the first time it exits from the local coordinate chart. Let $\sigma = \left\{ \sigma_i^j \right\}$ be the unique symmetric square root of $g^{-1} = \left\{ g^{ij} \right\}$. Consider the solution of the stochastic differential equation for a process $X_t = \left\{ X_t^1, \ldots, X_t^n \right\}$:

$$dX_t^i = \sigma_j^i(X_t)\, dB_t^j + \frac{1}{2} b^i(X_t)\, dt.$$

Then it is easy to verify by Itô's formula that X is a diffusion process generated by $2^{-1}\Delta_M$, i.e., X is a Brownian motion on M. Brownian motion can be studied this way by choosing an appropriate local coordinate system in which the Laplace-Beltrami operator Δ_M takes special a special form.

Lecture 2. Brownian Motion and Geometry

We study the effect of curvature on the behavior of Brownian motion and hope that it will lead to interesting results about the manifold itself. We will concentrate some problems which can be studied through the distance function $r(x) = d(x, o)$, where o is a fixed point on the manifold. In this respect, the radial process $r_t = r(X_t)$ is a natural object of investigation. We often compare this process with the same process on a radially symmetric manifold satisfying certain curvature conditions. On such a manifold, problems often becomes one-dimensional and can be solved explicitly.

2.1. Radially symmetric manifolds

A radially symmetric manifold M has a distinguished point o, call the pole of M. In the polar coordinates (r, θ) induced by the exponential map $\exp_o : \mathbb{R}^n \to M$ based at o, the metric has the following form

$$ds^2 = dr^2 + G(r)^2 d\theta^2.$$

Here $d\theta^2$ denotes the standard Riemannian metric on the $(n-1)$-sphere $\mathbb{S}^{n-1}$, and G is a smooth function on an interval $[0, D)$ satisfying

$$G(0) = 0, \ G'(0) = 1, \quad 0 \le r < D.$$

In terms of these coordinates, the Laplace-Beltrami operator has the form

$$\Delta_M = L_r + \frac{1}{G(r)^2} \Delta_{\mathbb{S}^{n-1}}, \tag{2.1.1}$$

where L_r is the radial Laplacian

$$L_r = \left(\frac{\partial}{\partial r} \right)^2 + (n-1) \frac{G'(r)}{G(r)} \frac{\partial}{\partial r},$$

and $\Delta_{\mathbb{S}^{n-1}}$ is the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$. The main feature of this case is that the radial component is completely decoupled from the angular component and the angular component is a scaling of $\Delta_{\mathbb{S}^{n-1}}$ by the factor $1/G(r)^2$ depending on the radial component. In the terminology of differential geometry, the metric ds^2 has the form of a warped product.

Let $X_t = (r_t, \theta_t)$ be a Brownian motion on a radially symmetric manifold M written in polar coordinates. Using polar coordinates, we find that the radial component is the solution of the stochastic differential equation:

$$r_t = r_0 + W_t + \frac{n-1}{2} \int_0^t \frac{G'(r_s)}{G(r_s)} ds, \tag{2.1.2}$$

where W is a 1-dimensional Brownian motion. The angular component can also be described easily. Let $Y = \{Y_t\}$ be a Brownian motion on $\mathbb{S}^{n-1}$ independent of W (hence also independent of $\{r_t\}$). Define a new time scale

$$l_t = \int_0^t \frac{ds}{G(r_s)^2}, \tag{2.1.3}$$

and let $\theta_t = Y_{l_t}$ be the time change of the spherical Brownian motion Y. Then $X_t = (r_t, \theta_t)$ constructed this way is a Brownian motion on M (see [**11**], 84–85). From this description of Brownian motion it is clear that the behavior of Brownian motion on a radially symmetric manifold is controlled by and large by its radial process. The radially process is a one-dimensional diffusion process, which has been well studied in probability theory.

Let's study a special case more closely. A complete, simply connected manifolds of negative sectional curvature is called a Cartan-Hadamard manifold. Suppose that M is such a manifold with constant negative curvature $-K^2$ (space form). Then it is a radially symmetric manifold with the metric $ds^2 = dr^2 + G(r)^2 d\theta^2$ is given by

$$G(r) = \frac{\sinh Kr}{K}.$$

The behavior of Brownian motion on this manifold is typical for Brownian motion on Cartan-Hadamard manfiold whose curvature is pinned between two negative constant.

The radial process is the solution of

$$dr_t = dW_t + \frac{n-1}{2} K \coth Kr_t \, dt.$$

We write the equation in the form

$$r_t = W_t + \frac{(n-1)}{2} \int_0^t K \coth Kr_s \, ds.$$

We have $\coth Kr_t \geq 1$ for all t because $r_t \geq 0$. Hence

$$r_t \geq r_0 + W_t + \frac{n-1}{2} K\, t. \tag{2.1.4}$$

According to the law of the iterated logarithm,

$$\limsup_{t\to\infty} \frac{W_t}{\sqrt{2t \ln\ln t}} = 1,$$

Thus the term $r_0 + W_t$ on the right side of (2.1.4) is of lower order than the last term and we have (with probability one) $r_t \to \infty$. Now that $\coth Kr_t \to 1$ as $t \to \infty$, from

$$\frac{r_t}{t} = \frac{r_0 + W_t}{t} + \frac{n-1}{2} \frac{K}{t} \int_0^t \coth Kr_s \, ds$$

we find the asymptotic behavior of the radial process

$$\lim_{t\to\infty} \frac{r_t}{t} = \frac{(n-1)K}{2}. \tag{2.1.5}$$

We now examine the angular process. From the above discussion, we know that it is a time-change of a Brownian motion on the sphere $\mathbb{S}^{n-1}$. In the present case, the time change is

$$l_t = \int_0^t \left(\frac{K}{\sinh K r_s} \right)^2 ds.$$

From (2.1.5) the integral converges as $t \uparrow \infty$. It follows that

$$\lim_{t \to \infty} \theta_t = Y_{l_\infty}. \tag{2.1.6}$$

(2.1.5) and (2.1.6) give a fairly good picture of the asymptotic behavior of Brownian motion on a complete, simply connected manifold of constant negative curvature. See [**12**] for more recent work on angular convergence of Brownian motion and its relation with the Dirichlet problem at infinity for Cartan-Hadamard manifolds.

2.2. Radial process

The concept of a radial process can be introduced for Brownian motion on a general Riemannian manifold M. Fix a reference point $o \in M$, and let $r(x) = d(x, o)$ be the Riemannian distance between x and o. We define the radial process $r_t = r(X_t)$. It is natural to try to use Itô's formula to decompose this into a martingale part and a bounded variation part. The function $r : M \to \mathbb{R}_+$ has a well behaved singularity at the origin. In particular,

$$\Delta_M r \sim \frac{n-1}{r} \qquad \text{near } r = 0.$$

This singularity will not cause any problem for us because, except for the trivial one-dimensional case, Brownian motion X never hits o for $t > 0$. However, $x \mapsto r(x)$ is not a smooth function on $M \backslash \{o\}$. Differential geometry ([**8**]) tells us exactly where it is smooth.

For simplicity we assume that M is geodesically complete. Every geodesic segment can be extended in both directions indefinitely and every pair of points can be connected by a distance-minimizing geodesic. For each unit vector $e \in T_oM$, there is a unique geodesic $C_e : [0, \infty) \to M$ such that $\dot{C}_e(0) = e$. The exponential map $\exp : T_oM \to M$ is

$$\exp te = C_e(t).$$

If we identify T_oM with $\mathbb{R}^n$ by an orthonormal frame, the exponential map becomes a map from $\mathbb{R}^n$ onto M. For small t, the geodesic $C_e[0, t]$ is the unique distance-minimizing geodesic between its endpoints. Let $t(e)$ be the largest t such that the geodesic $C_e[0, t]$ is distance-minimizing from $C_e(0)$ to $C_e(t)$. Define

$$\widetilde{C}_o = \{t(e)e : e \in T_oM, \ |e| = 1\}.$$

Then the cutlocus of o is the set $C_o = \exp \widetilde{C}_o$. Sometimes we also call $\widetilde{C}_o$ the cutlocus of o. The set within the cutlocus is the star-shaped domain

$$\widehat{E}_o = \{te \in T_oM : e \in T_oM, \, 0 \le t < t(e), \, |e| = 1\}.$$

On M the set within cutlocus is $E_o = \exp \widetilde{E}_o$. We have the following basic results from differential geometry (see [**8**] [**2**] [**3**]).

THEOREM 2.2.1. *(i) The map* $\exp : \widetilde{E}_o \to E_o$ *is a diffeomorphism.*
(ii) The cutlocus C_o *is a closet subset of measure zero.*
(iii) If $x \in C_y$*, then* $y \in C_x$.
(iv) E_o *and* C_o *are disjoint and* $M = E_0 \cup C_o$. □

According to the above theorem the polar coordinates (r, θ) are well behaved on the region $M \backslash C_o$ within the cutlocus. The set they do not cover is the cutlocus C_o, a set of measure zero. The radial function $r(x) = d(x, o)$ is smooth on $M \backslash C_o$ and Lipschitz on all of M. Furthermore, $|\nabla r| = 1$ everywhere on $M \backslash C_o$.

If X is a Brownian motion on M starting within E_o, then, before it hits the cutlocus C_o,

$$r(X_t) = r(X_0) + W_t + \frac{1}{2} \int_0^t \Delta_M r(X_s) \, ds, \quad t < T_{C_o}, \tag{2.2.1}$$

where T_{C_o} is the first hitting time X of the cutlocus C_o and W is a martingale. Its quadratic variation is

$$\langle W \rangle_t = \int_0^t |\nabla r(X_s)|^2 \, ds = t.$$

Hence by Lévy's criterion, W is a Brownian motion. (2.2.1) shows that the behavior of the radial process is largely controlled by the Laplacian of the distance function $\Delta_M r$. In practice we try to bound $\Delta_M r$ by a known function of r and then control $r(X_t)$ by comparing it with a one-dimensional diffusion process.

(2.2.1) is good enough if the cutlocus C_o is empty, e.g., if M is a Cartan-Hadamard manifold. What happens to the radial process when Brownian motion crosses the cutlocus? Very complicated. A very detailed study of this problem can be found in [**6**]. In most cases, the following result due to W. Kendall is sufficient.

THEOREM 2.2.2. *Suppose that* X *is a Brownian motion on Riemannian manifold* M*. Let* $r(x) = d(x, o)$ *be the distance function from a fixed point* $o \in M$*. Then there exist a one-dimensional euclidean Brownian motion* W *and a nondecreasing process* L *which increases only when* $X_t \in C_o$ *(the cutlocus of* o*) such that*

$$r(X_t) = r_0 + W_t + \frac{1}{2} \int_0^t \Delta_M r(X_s) \, ds - L_t, \quad t < e.$$

According this theorem, we always have an upper bound

$$r_t \le r_0 + W_t + \frac{1}{2}\int_0^t \Delta_M r(X_s)\,ds, \quad t < e.$$

If we need to bound the radial process from above, we have to assume that the cutlocus is empty. In this case,

$$r(X_t) = r_0 + W_t + \frac{1}{2}\int_0^t \Delta_M r(X_s)\,ds, \quad t < e.$$

2.3. Comparison theorems

The next step is to study the Laplacian $\Delta_M r$ of the distance function. The goal is to bound $\Delta_M r$ by a simple function of r. There is a host of Laplacian (more generally, Hessian) comparison theorems of this type one can draw from differential geometry (see [**3**] [**15**]). We cite two simple ones which compare an arbitrary Riemannian manifold with manifolds of constant curvature (see [**15**]).

THEOREM 2.3.1. *Let $K_M(x)$ denote any sectional curvature at $x \in M$ and assume that*

$$-K_1^2 \le K_M(x) \le K_2^2.$$

Then we have at any smooth point of the distance function $r(x)$,

$$(n-1)K_2 \cot K_2 r(x) \le \Delta_M r(x) \le (n-1)K_1 \coth K_1 r(x).$$

This result can be used to control the radial process, or more precisely, to compare the radial process on M with those on manifolds of constant curvatures K_1^2 and $-K_2^2$. Let's first bound the radial process from above. We have

$$r_t \le r_0 + W_t + \frac{1}{2}\int_0^t \Delta_M r(X_s)\,ds.$$

Next, consider the equation

$$r_t^1 = r_0 + W_t + \frac{n-1}{2}\int_0^t K_1 \coth K_1 r_s^1\,ds.$$

r_t^1 is the radial process of a Brownian motion on the space form of constant curvature $-K_1^2$. Note that it is driven by the same Brownian motion W. Since we have $\Delta_M r(x) \le (n-1)K_1 \coth K_1 r(x)$, the drift of r_t is smaller that the drift of r_t^1. By the standard comparison theorem for one-dimensional processes (see [**13**]), we have $r_t \le r_t^1$ for all $t \ge 0$.

Next, if M does not have cutlocus, or if $t \le T_{C_o}$, we have

$$r_t = r_0 + W_t + \frac{1}{2}\int_0^t \Delta_M r(X_s)\,ds.$$

This time we compare with the process r_t^2 determined by

$$r_t^2 = r_0 + W_t + \frac{n-1}{2}\int_0^t K_2 \cot K_2 r_s^2\,ds.$$

r_t^2 is the radial process on the $(n-1)$-dimensional sphere of radius $1/K_2$. By the same argument as before, we have $r_t \geq r_t^2$.

Let's see two applications. The next result is due to S. T. Yau.

THEOREM 2.3.2. *Let M be a complete manifold whose sectional curvature is bounded from below by a constant. Then it is stochastically complete, i.e.,*

$$\int_M p\,(t,x,y)\,dy = 1$$

for all $x \in M, t > 0$.

PROOF. If M is not stochastically complete, that Brownian motion blows up with positive probability, i.e., it goes to infinity in finite amount of time. Suppose that $K_M(x) \geq -K_1^2$. Then we have shown that $r_t \leq r_t^2$. If r_t goes to infinity in finite amount of time, certain r_t^2 will do the same. But we have shown that $r_t^2 \sim (n-1)K_1 t/2$ as $t \to \infty$, which means that r_t^2 does not blow up. Nor will r_t. □

We say that Brownian motion on M is transient if for some $x \in M$ (hence for all $x \in M$),

$$\mathbb{P}_x \left\{ \lim_{t \uparrow\uparrow e} X_t = \infty_M \right\} = 1.$$

Otherwise, we say Brownian motion is recurrent on M. There is a simple analytic criterion for recurrence and transience. Let

$$G(x,y) = \int_0^\infty p(t,x,y)\,dt$$

be Green's function of M. Then Brownian motion on M is transient if and only if $G(x,y) < \infty$ for some pair of points $x \neq y$ (hence for all such pairs of points). It is well known that euclidean Brownian motion of dimension 1 and 2 is recurrent, and of dimension 3 or higher is transient.

THEOREM 2.3.3. *Suppose that M is a Cartan-Hadamard manifold of dimension greater than 2. Then Brownian motion on M is transient.*

PROOF. M does not have cutlocus and $K_M(x) \leq 0$. Therefore $r_t \geq r_t^2$, where r_t^2 is the radial process of euclidean Brownian motion of dimension n. Since $r_t^2 \to \infty$ as $t \uparrow \infty$, we must have $r_t \to \infty$ as $t \uparrow\uparrow e$, which means Brownian motion on M must be transient. □

More refined results along these lines can be found in [**11**].

Lecture 3. Stochastic Calculus on manifolds

From a theoretical point of view, the most satisfactory construction of Brownian motion on a manifold is that of Eells-Elworthy-Malliavin.

3.1. Orthonormal frame bundle

Let $\mathscr{O}_x(M)$ be the set of orthonormal frames of the tangent space T_xM. The orthonormal frame bundle

$$\mathscr{O}(M) = \bigcup_{x \in M} \mathscr{O}_x(M)$$

has a natural structure of a smooth manifold of dimension $n(n+1)/2$. Let $\pi : \mathscr{O}(M) \to M$ be the canonical projection. Each element $u \in \mathscr{O}(M)$ is therefore an isometry

$$u : \mathbb{R}^n \to T_{\pi u}M.$$

Let $u \in \mathscr{O}(M)$ and $\pi u = x$. The fibre $\pi^{-1}x = \mathscr{O}_x M$ is naturally a smooth submanifold of dimension $n(n-1)/2$. Its tangent space $V_u\mathscr{O}(M)$ is a subspace of the same dimension of the full tangent space $T_u\mathscr{O}(M)$. A curve $\{u_t\}$ in $\mathscr{O}(M)$ is horizontal if u_t is the parallel transport of u_0 along the projection curve $\{\pi u_t\}$. The set of tangent vectors of horizontal curves passing through a fixed point $u \in \mathscr{O}(M)$ is the horizontal subspace $H_u\mathscr{O}(M)$ of dimension n of $T_u\mathscr{O}(M)$ and we have the relation

$$T_u\mathscr{O}(M) = H_u\mathscr{O}(M) \oplus V_u\mathscr{O}(M),$$

and the projection $\pi : \mathscr{O}(M) \to M$ induces an isomorphism $\pi_* : H_u\mathscr{O}(M) \to T_xM$. On the orthonormal frame bundle, we have n well defined horizontal vector field H_i. At each $u \in \mathscr{O}(M)$, $H_i(u)$ is the unique horizontal vector in $H_u\mathscr{O}(M)$ whose projection is the ith unit vector ue_i of the orthonormal frame; i.e.,

$$\pi_* H_i(u) = ue_i, \qquad H_i(u) \in H_u\mathscr{O}(M).$$

The operator

$$\Delta_{\mathscr{O}(M)} = \sum_{i=1}^n H_i^2$$

is called Bochner's horizontal Laplacian on $\mathscr{O}(M)$. The Eells-Elworthy-Malliavin construction is based on the following relation.

PROPOSITION 3.1.1. *For any smooth function f on M, we have*

$$\Delta_M f(x) = \Delta_{\mathscr{O}(M)}(f \circ \pi)(u)$$

for any $u \in \mathscr{O}(M)$ such that $\pi u = x$.

3.2. Eells-Elworthy-Malliavin construction of Brownian motion

We note that $\Delta_{\mathscr{O}(M)}$ is in the form of the sum of n squares, where n is the dimension of the manifold M. Of course the price to pay is that it is an operator on a much larger space $\mathscr{O}(M)$ instead of on the manifold M itself. Consider the following stochastic differential equation on $\mathscr{O}(M)$:

$$dU_t = \sum_{i=1}^n H_i(U_t) \circ dW_t^i.$$

It is driven by an n-dimensional Brownian motion W. A solution of this equation is called a horizontal Brownian motion (on $\mathscr{O}(M)$). It is a diffusion process generated by $\Delta_{\mathscr{O}(M)}$. Itô's formula takes the following form:

$$F(U_t) = F(U_0) + \sum_{i=1}^n \int_0^t H_i F(U_s)\, dW_s^i + \frac{1}{2}\int_0^t \Delta_{\mathscr{O}(M)} F(U_s)\, ds,$$

where F is a smooth function on $\mathscr{O}(M)$. Now, if we apply this to a function of the form $F = f \circ \pi$, the lift of a smooth function f on M, then by PROPOSITION 3.1.1,

$$f(X_t) = f(X_0) + \sum_{i=1}^n \int_0^t H_i(f\circ\pi)(U_s)\, dW_s^i + \frac{1}{2}\int_0^t \Delta_M f(X_s)\, ds.$$

Here $X_t = \pi U_t$ is the projection of the horizontal Brownian motion U_t on the manifold M. It follows that X_t is a Brownian motion on M starting from $X_0 = \pi U_0$.

Suppose that we want to construct a Brownian motion starting from x. We fix an orthonormal frame $u \in \mathscr{O}(M)$ over x, i.e., $\pi u = x$. There is a unique horizontal Brownian motion U_t starting from the frame u. The projection $X_t = \pi U_t$ is a Brownian motion from x. This Brownian motion is not uniquely determined by the driving Brownian motion W because the initial frame u can be chosen arbitrarily. Of course, the law of X is completely determined by the initial point x and does not depend on the choice of either u or W. Once a frame u is fixed, $W \mapsto X$ establishes a measure preserving map

$$J : (P_o(\mathbb{R}^n), \mu) \to (P_x(M), \nu),$$

where μ is the Wiener measure on the euclidean path space $P_o(\mathbb{R}^n) = C_o(\mathbb{R}_+, \mathbb{R}^n)$ (the law of euclidean Brownian motion) and ν is the law of Brownian motion on M starting from x. The map J is usually called the Itô map for the reason that it is obtained through solving an Itô type stochastic differential equations. We will see later that this map is invertible. The image $X = JW$ is called a stochastic development of W.

3.3. Stochastic horizontal lift

In differential geometry, a smooth curve on M can be lifted to a horizontal curve with the help of the Riemannian connection (or equivalently, the concept of parallel transport). If $c : I \to M$ is such a curve, we choose a frame $u(0)$ over $c(0$ and simply let $u(t)$ be the parallel transport of $u(0)$ along $c[0,t]$, which is accomplished by solving an ordinary differential equation in local charts. The lift $\{u(t),\, t \in I\}$ in turn defines a smooth curve w in $\mathbb{R}^n$ by

$$w(t) = u(t)^{-1}\dot{c}(t).$$

The curve w is called an anti-development of c. The standard reference for this part of differential geometry is [**14**].

A similar procedure can be carried out if the smooth curve c is replaced by a Brownian motion, or more generally, a semimartingale X on M. We expect that the horizontal lift U of X is obtained by solving a stochastic differential equation driven by X. Unlike a smooth curve on M, a semimartingale on M is not a local object. A construction of U using local charts is possible but technically unwieldy. If we assume that M is embedded in some euclidean space, then a relatively clean construction is possible.

Before we proceed further, let us give the definition of a semimartingale on M. Let $(\Omega, \mathscr{F}_*, \mathbb{P})$ be a probability space with filtration $\mathscr{F}_* = \{\mathscr{F}_t, t \geq 0\}$. A semimartingale $X = \{X_t,\, t \geq 0\}$ on M is an M-valued, $\mathscr{F}_*$-adapted process such that $\{f(X_t),\, t \geq 0\}$ is a real-valued semimartingale for all smooth functions f on M.

Let M be a submanifold of $\mathbb{R}^l$ and recall that $P_\alpha(x)$ is the projection of the αth coordinate unit vector ξ_α to the tangent space T_xM at $x \in M$. Suppose that X is a semimartingale on M. Since M is a submanifold of $\mathbb{R}^l$, X can be regarded as a semimartingale on $\mathbb{R}^l$, i.e., $X = \left\{X^1, \ldots, X^l\right\}$. Let $P^*_\alpha(u)$ be the horizontal lift of $P_\alpha(\pi u)$ to $u \in \mathscr{O}(M)$. Then we obtain l horizontal vector field. Consider the following stochastic differential equation on $\mathscr{O}(M)$ driven by X:

$$dU_t = \sum_{\alpha=1}^{l} P^*_\alpha(U_t) \circ dX^\alpha_t. \tag{3.3.1}$$

It has a unique solution once an initial frame U_0 is given.

THEOREM 3.3.1. *The solution of (3.3.1) is a horizontal lift of X to $\mathscr{O}(M)$.*

PROOF. We sketch a proof, see [**11**] for details.

The proof is based on the following identity which holds for any semimartingale X on M:

$$X_t = X_0 + \sum_{\alpha=1}^{l} \int_0^t P_\alpha(X_s) \circ dX_s^\alpha. \tag{3.3.2}$$

This can be regarded as a stochastic differential equation for X on M. Laurent Schwartz observed once that every semimartingale on a manifold M is a solution of a stochastic differential equation on M.

Let $f : \mathscr{O}(M) \to M \subseteq \mathbb{R}^l$ be the projection $\pi : \mathscr{O}(M) \to M$ regarded as an $\mathbb{R}^l$-valued function on $\mathscr{O}(M)$. Let $Y_t = f(U_t) = \pi U_t$. We have to show that $Y_t = X_t$. Apply Itô's formula to $f(U_t)$, we obtain

$$Y_t = Y_0 + \sum_{\alpha=1}^{l} \int_0^t P_\alpha^* f(U_s) \circ dX_s^\alpha.$$

A not so difficult calculation shows that

$$P_\alpha^* f(u) = P_\alpha(\pi u).$$

Hence

$$Y_t = Y_0 + \sum_{\alpha=1}^{l} \int_0^t P_\alpha(Y_s) \circ dX_s^\alpha.$$

Therefore Y satisfies the same stochastic differential equation as X (see (3.3.2)). By the uniqueness of solutions we must have $X = Y = \pi U$; namely, U is a horizontal lift of X. □

Once we have found the horizontal lift U of X, it is not hard to write down the anti-development of X:

$$W_t = \int_0^t U_s^{-1} P_\alpha(U_s) \circ dX_s^\alpha.$$

It can be verified easily that this $\mathbb{R}^l$-valued semimartingale drives an equation for U, namely,

$$dU_t = \sum_{i=1}^{n} H_i(U_t) \circ dW_t^i.$$

Furthermore, if X is a Brownian motion on M, then W is a Brownian motion on $\mathbb{R}^n$. This can be verified using Lévy's criterion.

The correspondences

$$W \longleftrightarrow U \longleftrightarrow X$$

are very useful because it converts a manifold-valued process X into a euclidean space valued process W, which is much easier to handle. We emphasize two points: (1) these correspondences are valid for any M-valued semimartingale X; (2) they depend on the connection we have used to define horizontal lift for vectors. Here we used the Riemannian connection, but the whole construction can be carried out for any affine connection ∇. In this case the orthonormal frame bundle $\mathscr{O}(M)$ should be replaced by the general linear frame bundle $\mathscr{F}(M)$, otherwise everything remains the same.

3.4. Stochastic integrals on a manifold

As applications of the concepts of stochastic horizontal lift and anti-development, we define some stochastic integrals on a manifold.

Let X be a semimartingale and U and X its horizontal lift and anti-development, respectively. Let θ a 1-form on M. The stochastic line integral of θ along $X[0,t]$ is defined by

$$\int_{X[0,t]} \theta = \sum_{i=1}^{n} \int_0^t \theta(U_s e_i) \circ dW_s^i.$$

It can be verified that this definition is independent of the choice of the connection. It is possible to write some other definitions in which the connection does not show up. For example, if M is a submanifold of $\mathbb{R}^l$, then we have

$$\int_{X[0,t]} \theta = \sum_{\alpha=1}^{l} \int_0^t \theta(P_\alpha)(X) \circ dX_t^\alpha.$$

If $\theta = df$ is an exact 1-form, then

$$\int_{X[0,t]} \theta = f(X_t) - f(X_0).$$

Another interesting fact is that the stochastic anti-development W itself is a stochastic line integral. Let Θ be the $\mathbb{R}^n$-valued solder form on $\mathscr{O}(M)$, i.e.,

$$\Theta(Z)(u) = u^{-1}\pi_* Z, \qquad Z \in \Gamma(\mathscr{O}(M)).$$

We have

$$W_t = \int_{U[0,t]} \Theta.$$

Let h be a (0,2)-tensor on M. The h-quadratic variation of X is defined by

$$\int_0^t h(dX_s, dX_s) = \int_0^t h(U_s e_i, U_s e_j) d\langle W^i, W^j \rangle_s.$$

Again we have

$$\int_0^t h(dX_s, dX_s) = \int_0^t h(P_\alpha, P_\beta)(X_s)\, d\langle X^i, X^j \rangle_s,$$

so the definition is independent of the choice of the connection.

If we take $h = df_1 \otimes df_2$ for some smooth functions f_1, f_2 on M, then

$$\int_0^t (df_1 \otimes df_2)(dX_s, dX_s) = \langle f_1(X), f_2(X) \rangle_t.$$

These concepts are useful in the study of manifold-valued martingales.

Lecture 4. Analysis on Path and Loop Spaces

Let $P_o(M) = C_o([0,1], M)$ be the space of continuous functions from $[0,1$ to M starting from a fixed point $o \in M$. The loop space is $L_o(M) = \{\gamma \in P_o(M) : \gamma(1) = o\}$. These are typical examples of infinite-dimensional spaces. We want to do analysis on these spaces. The measure we use for $P_o(M)$ is the Wiener measure ν, the law of Brownian motion on M starting from o. For the loop space $L_o(M)$, we use the law ν_o of Brownian bridge based at o. To do analysis, we need the concept of a gradient operator. Due to time limit, we will only discuss the case of the flat path space $P_o(\mathbb{R}^n)$. For generalization of the results discuss here to a general Riemannian manifold see [**11**].

4.1. Quasi-invariance of the Wiener measure

If an $h \in P_o(\mathbb{R}^n)$ is absolutely continuous and $\dot{h} \in L^2(I; \mathbb{R}^n)$ we define

$$|h|_{\mathscr{H}} = \sqrt{\int_0^1 |\dot{h}_s|^2 ds}\,;$$

otherwise we set $|h|_{\mathscr{H}} = \infty$. The ($\mathbb{R}^n$-valued) Cameron-Martin space is

$$\mathscr{H} = \{h \in P_o(\mathbb{R}^n) : |h|_{\mathscr{H}} < \infty\}\,.$$

Theorem 4.1.1. *(Cameron-Martin-Maruyama) Let $h \in \mathscr{H}$ and*

$$\xi_h \omega = \omega + h, \qquad \omega \in P_o(\mathbb{R}^n)$$

a Cameron-Martin shift on the path space. Then the shifted Wiener measure $\mu^h = \mu \circ (\xi_h)^{-1}$ is absolutely continuous with respect to μ and

$$\frac{d\mu^h}{d\mu}(\omega) = \exp\left[\langle h_s, \omega\rangle_{\mathscr{H}} - \frac{1}{2}|h|^2_{\mathscr{H}}\right]. \tag{4.1.1}$$

Here

$$\langle h, \omega\rangle_{\mathscr{H}} = \int_0^1 \langle \dot{h}_s, d\omega_s\rangle.$$

Cameron-Martin shifts are the only shift which preserves the measure class of the Wiener measure. More precisely we have the following converse of the above theorem.

THEOREM 4.1.2. *Let $h \in P_o(\mathbb{R}^n)$, and let*

$$\xi_h \omega = \omega + h, \qquad \omega \in P_o(\mathbb{R}^n)$$

be the shift on the path space by h. Let μ be the Wiener measure on $P_o(\mathbb{R}^d)$. If the shifted Wiener measure $\mu^h = \mu \circ (\xi_h)^{-1}$ is absolutely continuous with respect to μ, then $h \in \mathscr{H}$.

PROOF. We show that if $h \notin \mathscr{H}$, then the measures μ and μ^h are mutually singluar, i.e., there is a set A such that $\mu A = 1$ and $\mu^h A = 0$. Let

$$\langle f, g \rangle_{\mathscr{H}} = \int_0^1 \dot{f}_s dg_s,$$

whenever the integral is well defined. If $f \in \mathscr{H}$ such that $\dot{f}$ is a step function on $[0,1]$:

$$\dot{f} = \sum_{i=0}^{l-1} f_i I_{[s_i, s_{i+1})},$$

where $f_i \in \mathbb{R}^n$ and $0 = s_0 < s_1 < \cdots < s_l = 1$, then

$$\langle f, h \rangle_{\mathscr{H}} = \sum_{i=0}^{l-1} f_i \left(h_{s_{i+1}} - h_{s_i} \right)$$

is well defined. It is an easy exercise to show that if there is a constant C such that

$$\langle f, h \rangle_{\mathscr{H}} \le C |f|_{\mathscr{H}}$$

for all step functions $\dot{f}$, then h is absolutely continuous and $\dot{h}$ is square-integrable, namely, $h \in \mathscr{H}$.

Suppose that $h \notin \mathscr{H}$. Then there is a sequence $\{f_l\}$ such that

$$|f_l|_{\mathscr{H}} = 1 \quad \text{and} \quad \langle h, f_l \rangle_{\mathscr{H}} \ge 2l.$$

Let W be the coordinate process on $P_o(\mathbb{R}^d)$. Then it is a Brownian motion under μ and the stochastic integral

$$\langle f_l, W \rangle_{\mathscr{H}} = \int_0^1 \left\langle \dot{f}_{l,s}, dW_s \right\rangle$$

is well defined. Let

$$A_l = \{ \langle f_l, W \rangle_{\mathscr{H}} \le l \}$$

and $A = \limsup_{l \to \infty} A_l$. Since $|f_l|_{\mathscr{H}} = 1$, the random variable $\langle f_l, W \rangle_{\mathscr{H}}$ is standard Gaussian under μ; hence

$$\mu A_l \ge 1 - e^{-l^2/2}.$$

This shows that $\mu A = 1$. On the other hand,

$$\mu^h A_l = \mu \{ \langle f_l, W + h \rangle_{\mathscr{H}} \le l \} \le \mu \{ \langle f_l, W \rangle_{\mathscr{H}} \le -l \}.$$

Hence $\mu^h A_l \le e^{-l^2/2}$ and $\mu^h A = 0$. Therefore μ and μ^h are mutually singular. □

The above quasi-invariance result can be carried over to the flat loop space. Let

$$\mathscr{H}_o = \{h \in \mathscr{H} : h(1) = 0\}.$$

We will show that the Wiener measure μ_o on $L_o(\mathbb{R}^n)$ is quasi-invariant under the Cameron-Martin shift $\xi^h : L_o(\mathbb{R}^n) \to L_o(\mathbb{R}^n)$ for $h \in \mathscr{H}_o$.

Let $\{\omega_s\}$ be the coordinate process on the space $P_o(\mathbb{R}^n)$ and consider the stochastic differential equation for Brownian bridge

$$d\gamma_s = d\omega_s - \frac{\gamma_s ds}{1-s}, \quad \gamma_0 = o.$$

The assignment $J\omega = \gamma$ defines a measurable map $J : P_o(\mathbb{R}^n) \to L_o(\mathbb{R}^n)$. The map J can also be viewed as an $L_o(\mathbb{R}^n)$-valued random variable. Suppose that $h \in \mathscr{H}_o$. A simple computation shows that

$$d\{\gamma_s + h_s\} = d\{\omega_s + k_s\} - \frac{\gamma_s + h_s}{1-s} ds,$$

where

$$k_s = h_s + \int_0^s \frac{h_\tau}{1-\tau} d\tau.$$

This shows that through the map J, the shift $\gamma \mapsto \gamma + h$ in the loop space $L_o(\mathbb{R}^n)$ is equivalent to a shift $\omega \mapsto \omega + k$ in the path space $P_o(\mathbb{R}^n)$. The following lemma shows that the latter is a Cameron-Martin shift.

LEMMA 4.1.3. *(Hardy's inequality)*

$$\int_0^1 \left|\frac{h_s}{1-s}\right|^2 ds \le 4\int_0^1 |\dot{h}_s|^2 ds.$$

PROOF. We have for any $t \in (0,1)$,

$$\begin{aligned}
\int_0^t \left|\frac{h_s}{1-s}\right|^2 ds = & \int_0^t |h_s|^2 d\left[\frac{1}{1-s}\right] \\
=& 2\int_0^t \frac{h_s \cdot \dot{h}_s}{1-s} ds + \frac{|h_t|^2}{1-t} \\
\le& \frac{1}{2}\int_0^t \left|\frac{h_s}{1-s}\right|^2 ds + 2\int_0^t |\dot{h}_s|^2 ds + \frac{|h_t|^2}{1-t}.
\end{aligned}$$

In the last step we have used inequality

$$2ab \le \frac{1}{2}a^2 + 2b^2.$$

Therefore

$$\int_0^t \left|\frac{h_s}{1-s}\right|^2 ds \le 4\int_0^t |\dot{h}_s|^2 ds + \frac{2|h_t|^2}{1-t}.$$

The desired inequality follows by letting $t \to 1$ in the above inequality because

$$\frac{|h_t|^2}{1-t} = \frac{1}{1-t}\left|\int_t^1 \dot{h}_s ds\right|^2 \le \int_t^1 |\dot{h}_s|^2 ds \to 0.$$

□

The lemma implies that $k \in \mathscr{H}$. Define the exponential martingale

$$e_s = \exp\left[\int_0^s \langle \dot{k}_\tau, d\omega_\tau\rangle - \frac{1}{2}\int_0^s |\dot{k}_\tau|^2 d\tau\right].$$

Let μ^k be the probability measure on $P_o(\mathbb{R}^n)$ defined by

$$\frac{d\mu^k}{d\mu} = e_1. \tag{4.1.2}$$

By the Cameron-Martin-Maruyama theorem, μ^k is the law of $\omega + k$. Since it is absolutely continuous with respect to μ, the random variable $\omega \mapsto J(\omega + k)$ is well-defined and

$$J(\omega + k) = \gamma + h. \tag{4.1.3}$$

Let μ_o^h be the law of the shifted Brownian bridge $\gamma + h$. Then

$$\begin{aligned}\mu_o^h(C) =& \mu_o(C - h) = \mu(J^{-1}C - k)\\ =& \mu^k(J^{-1}C) = \mu(e_1; J^{-1}C)\\ =& \mu_o(e_1 \circ J; C),\end{aligned}$$

where we have used (4.1.3) and (4.1.2) in the second and the fourth steps, respectively. Now it is clear that μ_o^h and μ_o are mutually equivalent on $L_o(\mathbb{R}^n)$ and

$$\frac{d\mu_o^h}{d\mu_o} = e_1 \circ J. \tag{4.1.4}$$

Finally it is easy to verify that

$$\int_0^1 \langle \dot{k}_s, d\omega_s\rangle = \int_0^1 \langle \dot{h}_s, d\gamma_s\rangle, \qquad \int_0^1 |\dot{k}_s|^2 ds = \int_0^1 |\dot{h}_s|^2 ds. \tag{4.1.5}$$

This means that

$$e_1(J\gamma) = e_1(\omega) = \exp\left[\int_0^1 \langle \dot{h}_s, d\gamma_s\rangle - \frac{1}{2}\int_0^1 |\dot{h}_s|^2 ds\right].$$

We have proved the following result.

THEOREM 4.1.4. *Let $h \in \mathscr{H}_o$ and $\xi^h\gamma = \gamma + h$ for $\gamma \in L_o(\mathbb{R}^n)$. Then the shifted Wiener measure $\mu_o^h = \mu_o \circ (\xi^h)^{-1}$ on the loop space $L_o(\mathbb{R}^n)$ is equivalent to μ_o and*

$$\frac{d\mu_o^h}{d\mu_o} = \exp\left[\int_0^1 \langle \dot{h}_s, d\gamma_s\rangle - \frac{1}{2}\int_0^1 |\dot{h}_s|^2 ds\right].$$

4.2. Gradient operator

We now define the gradient operator in the path space $P_o(\mathbb{R}^n)$. By analogy with finite dimensional space, each element $h \in P_o(\mathbb{R}^n)$ represents a direction along which one can differentiate a nice functions F on $P_o(\mathbb{R}^n)$. Naturally the directional derivative of F along h should be defined by the formula

$$D_h F(\omega) = \lim_{t\to 0} \frac{F(\omega + th) - F(\omega)}{t} \tag{4.2.1}$$

if the limit exists in some sense. The preliminary class of functions on $P_o(M)$ for which the above definition of $D_h F$ makes immediate sense is that of cylinder functions.

DEFINITION 4.2.1. *Let $\mathbb{E}$ be a Banach space. A function $F : P_o(\mathbb{R}^n) \to \mathbb{E}$ is called an $\mathbb{E}$-valued cylinder function if it has the form*

$$F(\omega) = f(\omega_{s_1}, \cdots, \omega_{s_l}), \tag{4.2.2}$$

where $0 < s_1 < \cdots < s_l \le 1$ and f is an $\mathbb{E}$-valued smooth function on $(\mathbb{R}^n)^l$ such that all its derivatives have at most polynomial growth. The set of $\mathbb{E}$-valued cylinder functions is denoted by $\mathcal{C}(\mathbb{E})$. Typically $\mathbb{E} = \mathbb{R}^1, \mathbb{R}^n$, or $\mathscr{H}$. We denote $\mathcal{C}(\mathbb{R}^1)$ simply by $\mathcal{C}$.

If $F \in \mathcal{C}$ is given by (4.2.2), then it is clear that the limit (4.2.1) exists everywhere and we have

$$D_h F(\omega) = \sum_{i=1}^{l} \langle \nabla^i F(\omega), h_{s_i} \rangle_{\mathbb{R}^n}, \tag{4.2.3}$$

where

$$\nabla^i F(\omega) = \nabla^i f(\omega_{s_1}, \cdots, \omega_{s_l}).$$

Here $\nabla^i f$ denotes the gradient of f with respect to the ith variable.

It is natural to define the gradient DF of a function $F \in \mathcal{C}$ to be an $\mathscr{H}$-valued functions on $P_o(\mathbb{R}^n)$ such that

$$\langle DF(\omega), h \rangle_{\mathscr{H}} = D_h F(\omega).$$

A simple calculation shows that

$$DF(\omega)_s = \sum_{i=1}^{l} \min(s, s_i) \nabla^i F(\omega) \tag{4.2.4}$$

and

$$|DF(\omega)|^2_{\mathscr{H}} = \sum_{i=1}^{l} (s_i - s_{i-1}) \left| \sum_{j=i}^{l} \nabla^j F(\omega) \right|^2. \tag{4.2.5}$$

4.3. Integration by parts

We will use the notation

$$(F, G) = \int_{P_o(\mathbb{R}^n)} F(\omega)G(\omega)\mu(d\omega).$$

THEOREM 4.3.1. *Let $F, G \in \mathcal{C}$ and $h \in \mathscr{H}$. Then*

$$(D_h F, G) = (F, D_h^* G), \tag{4.3.1}$$

where

$$D_h^* = -D_h + \int_0^1 \langle \dot{h}_s, d\omega_s \rangle.$$

PROOF. Let $\xi^{th}\omega = \omega + th$ and $\mu^{th} = \mu \circ \left(\xi^{th}\right)^{-1}$. Then μ^{th} and μ are mutually absolutely continuous. We have

$$\int (F \circ \xi^{th}) G d\mu = \int F(G \circ \xi^{-th}) d\mu^{th} = \int F(G \circ \xi^{-th}) \frac{d\mu^{th}}{d\mu} d\mu.$$

We differentiate with respect to t and set $t = 0$. Using the formula for the Radon-Nikodym derivative (4.1.1) in the Cameron-Martin-Maruyama theorem we have at $t = 0$

$$\frac{d}{dt}\left\{\frac{d\mu^{th}}{d\mu}\right\} = \int_0^1 \langle \dot{h}_s, d\omega_s \rangle = \langle h, \omega \rangle_{\mathscr{H}}.$$

The formula follows immediately. □

To understand the integration by parts formula better, let's look at its finite dimensional analog and find out the proper replacement for the stochastic integral in D_h^*. Let $h \in \mathbb{R}^N$ and consider the differential operator

$$D_h = \sum_{i=1}^N h^i \frac{d}{dx^i}.$$

Let μ be the Gaussian measure on $\mathbb{R}^N$, i.e.,

$$\frac{d\mu}{dx} = \left(\frac{1}{2\pi}\right)^{N/2} e^{-|x|^2/2}.$$

[dx is the Lebesgue measure.] For smooth functions F, G on $\mathbb{R}^N$ with compact support we have by the usual integration by parts for the Lebesgue measure

$$(D_h F, G) = (F, D_h^* G), \tag{4.3.2}$$

where $D_h^* = -D_h + \langle h, x \rangle$ at $x \in \mathbb{R}^N$.

REMARK 4.3.2. Since D_h is a derivation we have

$$D_h^* G = -D_h G + (D_h^* 1) G.$$

Therefore to find the formal adjoint of D_h it is enough compute $D_h^* 1$, which is denoted by $\operatorname{div}(D_h)$ or $\delta(h)$ by various authors. We have

$$D_h^* 1(\omega) = \int_0^1 \langle \dot{h}_s, d\omega_s \rangle.$$

An integration by parts formula for the gradient operator D can be obtained as follows. Fix an orthonormal basis $\{h^j\}$ for $\mathscr{H}$. Denote by $\mathcal{C}_0(\mathscr{H})$ the set of $\mathscr{H}$-valued functions G of the form $G = \sum_j G_j h^j$, where each $G_j \in \mathcal{C}$ and almost all of them are equal to zero. It is easy to check that $\mathcal{C}_0(\mathscr{H})$ is dense in $L^p(\mu; \mathscr{H})$ for all $p \in [1, \infty)$.

Since $DF = \sum_j (D_{h^j} F) h^j$ in $L^2(\mu; \mathscr{H})$, we have

$$(DF, G) = \sum_j (D_{h^j} F, G_j) = \sum_j (F, D_{h^j}^* G_j).$$

The assumption that $G \in \mathcal{C}_o(\mathscr{H})$ means that the sums are finite. Let

$$D^* G = \sum_{j=0}^{\infty} D_{h^j}^* G_j = -\sum_{j=0}^{\infty} D_{h^j} G_j + \sum_{j=0}^{\infty} G_j \int_0^1 \langle \dot{h}_s^j, d\omega_s \rangle.$$

We rewrite this formula in a more compact form. If

$$J = \sum_{j,k} J_{jk} h^j \otimes h^k$$

is an $\mathscr{H} \otimes_{\mathbb{R}} \mathscr{H}$-valued function, we write

$$\operatorname{Trace} J = \sum_{j=0}^{\infty} J_{jj}.$$

For $G = \sum_k G_k h^j$ we define its gradient to be

$$DG = \sum_{j,k} (D_{h^j} G_k) h^j \otimes h^k.$$

Then it is clear that

$$\operatorname{Trace} DG = \sum_j D_{h^j} G_j.$$

For $G = \sum_j G_j h^j$ we define

$$\int_0^1 \langle \dot{G}_s, d\omega_s \rangle = \sum_j G_j \int_0^1 \langle \dot{h}_s^j, d\omega_s \rangle.$$

[This is a term-by-term integration with respect to a specific basis for $\mathscr{H}$, not anticipative stochastic integral!] Then we can write

$$D^* G = -\operatorname{Trace} DG + \int_0^1 \langle \dot{G}_s, d\omega_s \rangle. \tag{4.3.3}$$

THEOREM 4.3.3. *Let $F \in \mathcal{C}$ and $G \in \mathcal{C}_0(\mathscr{H})$. Then*

$$(DF, G) = (F, D^*G),$$

where D^*G is given by (4.3.3).

The Gradient operator can also be defined on the loop space $L_o(\mathbb{R}^n)$. Its integration by parts formula takes the same form as in the path space.

4.4. Ornstein Uhlenbeck operator

Let $\mathrm{Dom}(\mathcal{E}) = \mathrm{Dom}(D)$ and define the positive symmetric quadratic form $\mathcal{E} : \mathrm{Dom}(\mathcal{E}) \times \mathrm{Dom}(\mathcal{E}) \to \mathbb{R}$ by

$$\mathcal{E}(F, F) = (DF, DF)_{L^2(\mu;\mathscr{H})} = E|DF|^2_{\mathscr{H}}.$$

Then $(\mathcal{E}, D(\mathcal{E}))$ is a closed quadratic form, i.e., $\mathrm{Dom}(\mathcal{E})$ is complete with respect to the inner product

$$\mathcal{E}_1(F, F) = \mathcal{E}(F, F) + (F, F).$$

Furthermore the pair $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is a Dirichlet form; see [**9**].

By general theory of closed symmetric forms (Fukushima[**9**], 17-19), there exists a non-positive self-adjoint operator L such that $\mathrm{Dom}(\mathcal{E}) = \mathrm{Dom}(\sqrt{-L})$ and

$$\mathcal{E}(F, F) = (\sqrt{-L}F, \sqrt{-L}F). \tag{4.4.1}$$

L is called the Ornstein-Uhlenbeck operator on the path space $P_o(\mathbb{R}^n)$. In fact we have $L = -D^*D$, where D is the gradient operator and D^* its adjoint.

The Ornstein-Uhlenbeck operator is an infinite-dimensional generalization of the usual Ornstein-Uhlenbeck operator on $\mathbb{R}^1$:

$$L = -D^*D = \frac{d^2}{dx^2} - x\frac{d}{dx},$$

where

$$D = \frac{d}{dx}, \qquad D^* = -\frac{d}{dx}.$$

D^* is the adjoint of D with respect to the standard Gaussian measure μ on $\mathbb{R}^1$:

$$\frac{d\mu}{dx} = \frac{e^{-x^2/2}}{\sqrt{2\pi}}.$$

It is a classical result (see [**5**]) that the Hermite polynomials

$$H_N(x) = \frac{(-1)^N}{\sqrt{N!}}\, e^{x^2/2} \frac{d^N}{dx^N}\, e^{-x^2/2}, \qquad N \in \mathbb{Z}_+$$

form a complete set of L^2-eigenfunctions for the self-adjoint operator L on $L^2(\mathbb{R}^1, \mu)$:

$$LH_N = -NH_N, \qquad N \in \mathbb{Z}_+.$$

Returning to the path space, for simplicity we will assume in the following that the base space has dimension $n = 1$. Let $\{h^i\}$ be an orthonormal basis for $\mathscr{H}$. Then the $\{\langle h^i, \omega\rangle_{\mathbb{H}}\}$ is i.i.d. sequence with standard Gaussian distribution. Hence the map $T : \omega \mapsto \{\langle h^i, \omega\rangle_{\mathbb{H}}\}$ is an isometry (measure-preserving map) between the two measure spaces ($\tilde{\mu}$ is the Gaussian measure on $\mathbb{R}^{\mathbb{Z}_+}$). With this isometry in mind, the following construction is in order.

Let $\mathcal{I}$ denote the set of indices $I = \{n_i\}$ such that $n_i \in \mathbb{Z}_+$ and almost all of them are equal to zero. Denote $|I| = n_1 + n_2 + \cdots$. For $I \in \mathcal{I}$ define

$$H_I(\omega) = \prod_i H_{n_i}(\langle h^i, \omega\rangle_{\mathscr{H}}).$$

Then the fact that the Hermite polynomials $\{H_N, N \in \mathbb{Z}_+\}$ form an orthonormal basis for $L^2(\mathbb{R}, \mu)$ implies immediately that $\{H_I, I \in \mathcal{I}\}$ is an orthonormal basis for $L^2(P_o(\mathbb{R}), \mu)$. Moreover, the eigenspace of L for the eigenvalue N is

$$C_N = \text{the linear span of } \{H_I : |I| = N\}$$

and

$$L^2(P_o(\mathbb{R}), \mu) = C_0 \oplus C_1 \oplus C_2 \oplus \cdots .$$

This is the Wiener chaos decomposition. The following theorem completely describes the spectrum $\mathrm{Spec}(-L)$ of $-L$.

THEOREM 4.4.1. $\mathrm{Spec}(-L) = \mathbb{Z}_+$ *and* C_N *is the eigenspace for the eigenvalue* N. *Let* $P_N : L^2(P_o(\mathbb{R}), \mu) \to C_N$ *be the orthogonal projection to* C_N. *Then*

$$LF = -\sum_{N=0}^{\infty} N P_N F.$$

Note that all eigenspaces are infinite dimensional except for $C_0 = \mathbb{R}$.

Let $\mathscr{P}_t = e^{tL/2}$ in the sense of spectral theory. The Ornstein-Uhlenbeck semigroup $\{\mathscr{P}_t\}$ is the strongly continuous L^2-semigroup generated by the Ornstein-Uhlenbeck operator. Clearly,

$$\mathscr{P}_t F = e^{-Nt/2} F, \qquad F \in C_N.$$

Each $\mathscr{P}_t$ is a conservative, L^2-contraction, i.e., $P_t 1 = 1$ and $\|P_t F\|_2 \le \|F\|_2$.

PROPOSITION 4.4.2. *The semigroup* $\{\mathscr{P}_t\}$ *is positive, namely* $\mathscr{P}_t F \ge 0$ *if* $F \ge 0$. *For each* $p \in [1, \infty]$, *the Ornstein-Uhlenbeck semigroup* $\{\mathscr{P}_t\}$ *is a positive, conservative, and contractive* L^p*-semigroup.*

PROOF. The positivity follows from the fact that the semigroup comes from a Dirichlet form, see [**9**], 22-24. The L^p-contraction follows from the positivity and L^2-contraction. □

4.5. Logarithmic Sobolev inequality

Infinite dimensional analysis, Sobolev inequalities in general do not hold. In their stead, under certain conditions, we can prove a weaker inequality called logarithmic Sobolev inequality. In its form presented here the inequality is due to E. Nelson.

THEOREM 4.5.1. *Let μ be the standard Gaussian measure on $\mathbb{R}^N$ and ∇ the usual gradient operator. Suppose that f is a smooth function on $\mathbb{R}^N$ such that both f and ∇f have at most polynomial growth. Then we have*

$$\int_{\mathbb{R}^N} |f|^2 \log |f| d\mu \le \int_{\mathbb{R}^N} |\nabla f|^2 d\mu + \|f\|_2^2 \log \|f\|_2. \tag{4.5.1}$$

Here $\|f\|_2$ is the norm of f in $L^2(\mathbb{R}^N, \mu)$.

PROOF. For a positive s let μ_s be the Gaussian measure

$$\mu_s(dx) = \left(\frac{1}{2\pi s}\right)^{l/2} e^{-|x|^2/2s} dx,$$

where dx denotes the Lebesgue measure. Then $\mu = \mu_1$. Let $g = f^2$ and

$$P_s g(x) = \int_{\mathbb{R}^l} g(x-y)\mu_s(dy).$$

Consider the function $H_s = P_s \phi(P_{1-s} g)$, where $\phi(t) = 2^{-1} t \log t$. Differentiating with respect to s and noting that Δ commutes with P_s we have

$$\begin{aligned}
\frac{dH_s}{ds} &= \frac{1}{2} P_s \Delta \phi(P_{1-s} g) - \frac{1}{2} P_s \left\{\phi'(P_{1-s} g) \Delta P_{1-s} g\right\} \\
&= \frac{1}{2} P_s \left\{\phi'(P_{1-s} g)\Delta P_{1-s} g + \phi''(P_{1-s} g) \left|\nabla P_{1-s} g\right|^2\right\} \\
&\quad - \frac{1}{2} P_s \left\{\phi'(P_{1-s} g) \Delta P_{1-s} g\right\} \\
&= \frac{1}{2} P_s \left\{\phi''(P_{1-s} g) \left|\nabla P_{1-s} g\right|^2\right\} \\
&\le \frac{1}{4} P_s \left\{\frac{(P_{1-s}|\nabla g|)^2}{P_{1-s} g}\right\} \\
&\le P_s \left\{P_{1-s} |\nabla f|^2\right\} \\
&= P_1 |\nabla f|^2.
\end{aligned}$$

Here we have used the fact that $|\nabla P_{1-s} g| \le P_{1-s} |\nabla g|$ in the fourth step and the inequality

$$(P_{s-r}|\nabla g|)^2 \le 4 P_{s-r} g P_{s-r} |\nabla f|^2$$

in the fifth step, the latter being a consequence of the Cauchy-Schwarz inequality. Now integrating from 0 to 1 we obtain the desired result immediately. □

Translating the finite dimensional logarithmic Sobolev inequality in Theorem 4.5.1 to the path space $P_o(\mathbb{R}^n)$ we obtain the following result.

THEOREM 4.5.2. *If $F \in \mathrm{Dom}(D)$, then we have*

$$\int_{P_o(\mathbb{R}^n)} |F|^2 \log |F| d\mu \le \int_{P_o(\mathbb{R}^n)} |DF|^2_{\mathscr{H}} d\mu + \|F\|_2^2 \log \|F\|_2. \tag{4.5.2}$$

PROOF. We assume $n = 1$ for simplicity. The set of functions of the form

$$F(\omega) = f(\langle h_1, \omega\rangle_{\mathscr{H}}, \cdots, \langle h_0, \omega\rangle_{\mathscr{H}})$$

is dense in Dom(D). For such an F, we have

$$DF(\omega) = \sum_{i=0}^{l} f_{x_i}(\omega) h^i.$$

It is then clear that

$$|DF(\omega)|_{\mathscr{H}} = |\nabla f(\langle h^0, \omega\rangle_{\mathscr{H}}, \cdots, \langle h^l, \omega\rangle_{\mathscr{H}})|.$$

On the other hand, the distribution of $\left\{\langle h^0, \omega\rangle_{\mathscr{H}}, \cdots, \langle h^l, \omega\rangle_{\mathscr{H}}\right\}$ is the standard Gaussian measure on $\mathbb{R}^{l+1}$. Therefore (4.5.2) reduces to (4.5.1). □

There is a general result due to L. Gross which says that a logarithmic Sobolev inequality is equivalent to an hypercontractivity property of the corresponding semigroup.

THEOREM 4.5.3. *Let $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ be a Dirichlet form on a probability space $(X, \mathscr{B}, \mu)$ and $\{\mathscr{P}_t\}$ be the associated semigroup. The following statements are equivalent for a positive constant C .*

(I) Hypercontractivity for $\{\mathscr{P}_t\}$: $\|\mathscr{P}_t\|_{q,p} = 1$ for all (t, p, q) such that $t > 0, 1 < p < q$ and

$$e^{t/C} \geq \frac{q-1}{p-1}.$$

(II) The logarithmic Sobolev inequality for $\mathcal{E}$:

$$\mathbb{E}\left(F^2 \ln F^2\right) \leq 2C\mathcal{E}(F, F) + \mathbb{E}\, F^2 \,\ln \mathbb{E}\, F^2.$$

PROOF. See [**7**] or [**11**]. □

THEOREM 4.5.4. *The Ornstein-Uhlenbeck semigroup $\{\mathscr{P}_t\}$ is hypercontractive. More precisely $\|\mathscr{P}_t\|_{q,p} = 1$ for all (t, p, q) such that $t > 0, 1 < p < q$ and*

$$e^t \geq \frac{q-1}{p-1}.$$

4.6. Concluding remarks

For a compact Riemannian manifold, a logarithmic Sobolev inequality for the gradient operator on the path space is known [**11**]. In the proof of the logarithmic Sobolev inequality presented here, we have taken advantage of the Gaussian structure of the underlying linear Gaussian structure. For a general manifold a completely different approach is needed. In general a logarithmic inequality implies the existence of a positive spectrum gap. In the flat case we simply verify this fact by direct computation. A parallel (or maybe not so parallel) theory for a manifold with boundary is a current area of active research.

For the flat loop space $L_o(\mathbb{R}^n)$ the theory is the same as the flat path space because from the Gaussian point of view, the path and loop spaces are isometric. For a general loop space $L_o(M)$, where M is a compact Riemannian manifold, S. Aida proved that the 0-eigenspace of the Ornstein-Uhlenbeck operator L is simple for each homotopy class. Even for simply connected M, A. Erbele proved that there is in general no positive spectral gap. It is believed that this anomaly is due to the presence of negative curvature on M. Therefore the current effort is aiming at proving the existence of a logarithmic Sobolev inequality for a compact, simply connected manifold of positive curvature.

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