Optimal Markovian Couplings

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Contents

• Couplings and Markovian couplings.
• Optimal Markovian coupling with respect to distances.
• Optimal Markovian coupling with respect to nonnegative, lower semi-continuous functions.
1. Couplings and Markovian Couplings.

Given prob. meas. $\mu_1, \mu_2$, a prob. meas. $\tilde{\mu}$ on $(E_1 \times E_2, \mathcal{E}_1 \times \mathcal{E}_2)$ is called a coupling of $\mu_1$ and $\mu_2$ if following marginality holds:

\[
\tilde{\mu}(A_1 \times E_2) = \mu_1(A_1)
\]
\[
\tilde{\mu}(E_1 \times A_2) = \mu_2(A_2),
\]

\[
A_k \in \mathcal{E}_k, \quad k = 1, 2.
\]

\[\text{(M)}\]
Example 1. Independent coupling $\tilde{\mu}_0$:
$\tilde{\mu}_0 = \mu_1 \times \mu_2$.

Application. $\mu_k = \mu$ on $\mathbb{R}$. FKG-inequality:

$$\int f \, g \, d\mu \geq \int f \, d\mu \int g \, d\mu, \quad f, g \in \mathcal{M}.$$
Example 1. Independent coupling $\tilde{\mu}_0$:

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$$\int f g d\mu \geq \int f d\mu \int g d\mu, \quad f, g \in \mathcal{M}.$$  

$$\iint \tilde{\mu}_0(dx, dy)[f(x) - f(y)][g(x) - g(y)] \geq 0.$$  

Example 2. $E_k = E$. Basic coupling $\tilde{\mu}_b$:

$$\tilde{\mu}_b(dx_1, dx_2) = (\mu_1 \wedge \mu_2)(dx_1)I_\Delta + \frac{(\mu_1 - \mu_2)^+(dx_1)(\mu_2 - \mu_1)^+(dx_2)}{(\mu_1 - \mu_2)^+(E)}I_{\Delta c},$$

$(\nu_1 - \nu_2)^\pm$: Jordan-Hahn decomposition.

$\nu_1 \wedge \nu_2 = \nu_1 - (\nu_1 - \nu_2)^+$. $\Delta = $ diagonals.

$$\tilde{\mu}_b(\rho) = \frac{1}{2}\|\mu_1 - \mu_2\|_{\text{Var}} = \inf_{\tilde{\mu}} \tilde{\mu}(\rho).$$

$\rho$: discrete distance. $\tilde{\mu}_b$: $\rho$-optimal coupling.

Markov processes: $P_k(t, x_k, dy_k)$.

$P_k(t)$ on $(E_k, \mathcal{E}_k)$, $k = 1, 2$. $\tilde{P}^{t;x_1,x_2}(A_1 \times A_2)$. 
Markovian couplings

A Markov process \( \tilde{P}(t) \) on \((E_1 \times E_2, \mathcal{E}_1 \times \mathcal{E}_2)\) having marginality:

\[
\tilde{P}(t; x_1, x_2; A_1 \times E_2) = P_1(t, x_1, A_1), \\
\tilde{P}(t; x_1, x_2; E_1 \times A_2) = P_2(t, x_2, A_2), \\
t \geq 0, \ x_k \in E_k, \ A_k \in \mathcal{E}_k, \ k = 1, 2.
\]

Equivalently,

\[
\tilde{P}(t) f(x_1, x_2) = P_1(t) f(x_1), \\
\tilde{P}(t) f(x_1, x_2) = P_2(t) f(x_2), \\
t \geq 0, \ x_k \in E_k, \ f \in b\mathcal{E}_k, \ k = 1, 2,
\]

\(b\mathcal{E}\): bounded \(\mathcal{E}\)-measurable functions.

On LHS, \(f\) regarded as a bivariate function.
Jump processes

Jump condition:
\[ \lim_{t \to 0} P(t, x, \{x\}) = 1, \quad x \in E. \]

\(q\)-pair \((q(x), q(x, dy))\):
\[ 0 \leq q(x) = \lim_{t \to 0} \frac{1 - P(t, x, \{x\})}{t} \leq \infty, \quad x \in E \]
\[ q(x, A) = \lim_{t \to 0} \frac{P(t, x, A \setminus \{x\})}{t} \leq q(x), \quad x \in E, \ A \in \mathcal{R} \]
\[ \mathcal{R} = \left\{ A \in \mathcal{E} : \lim_{t \to 0} \sup_{x \in A} \left[ 1 - P(t, x, \{x\}) \right] = 0 \right\}. \]

Totally stable: \(q(x) < \infty\) for all \(x\).
Conservative: \(q(x, E) = q(x)\) for all \(x\).
Operator: \(\Omega f(x) = \int q(x, dy)[f(y) - f(x)]\]
\[ -[q(x) - q(x, E)]f(x), \quad f \in b \mathcal{E}. \]
Coupling \(q\)-pair:

\[
\tilde{q}(x_1, x_2) = \lim_{t \to 0} \frac{1 - \tilde{P}(t; x_1, x_2; \{x_1\} \times \{x_2\})}{t},
\]

\((x_1, x_2) \in E_1 \times E_2\)

\[
\tilde{q}(x_1, x_2; \tilde{A}) = \lim_{t \to 0} \frac{1 - \tilde{P}(t; x_1, x_2; \tilde{A})}{t},
\]

\((x_1, x_2) \notin \tilde{A} \in \tilde{R}\).

**Theorem**

- (C. 1994). Coupling \(q\)-pair totally stable \(\iff\) so are marginals.

- (Y. H. Zhang, 1994). Coupling \(q\)-pair conservative \(\iff\) so are marginals.
Marginality for operators:

Given $\Omega_1$ and $\Omega_2$, from (MP), it follows that any

$$\tilde{\Omega}f(x_1, x_2) = \int_{E \times E} \tilde{q}(x_1, x_2; dy_1, dy_2)[f(y_1, y_2) - f(x_1, x_2)].$$

must satisfies

$$\tilde{\Omega}f(x_1, x_2) = \Omega_1 f(x_1), \quad f \in \mathcal{B}_{\mathcal{E}_1}$$
$$\tilde{\Omega}f(x_1, x_2) = \Omega_2 f(x_2), \quad f \in \mathcal{B}_{\mathcal{E}_2}, \quad (\text{MO})$$

$$x_k \in E_k, \; k = 1, 2.$$

—-coupling operator.
Markov chain: \( P(t) = (P_{ij}(t) : i, j \in E) \).

\( Q \)-matrix: \( Q = (q_{ij}) := \frac{d}{dt} P(t) \bigg|_{t=0} \).

\( Qf(i) := \sum_{j \in E} q_{ij} f_j = \sum_{j \neq i} q_{ij} [f_j - f_i] \).

\( \tilde{P}(t) \longrightarrow \tilde{Q} \). Marginality for operators:

\[
\tilde{Q}f(i_1, i_2) = Q_1 f(i_1) \tag{MO}
\]

\[
\tilde{Q}f(i_1, i_2) = Q_2 f(i_2) \quad i_1, i_2 \in E.
\]

\( \tilde{Q} \): coupling \( Q \)-matrix of \( Q_1 \) and \( Q_2 \).

**Question:** Does there exist a coupling operator?
Examples of coupling for jump processes

Example 1. Independent coupling $\tilde{\Omega}_0$:

\[
\tilde{\Omega}_0 f(x_1, x_2) = [\Omega_1 f(\cdot, x_2)](x_1) + [\Omega_2 f(x_1, \cdot)](x_2)
\]

\[
x_k \in E_k, \ k = 1, 2.
\]

Example 2. Classical coupling (Doeblin (1938)) $\tilde{\Omega}_c$: $\Omega_1 = \Omega_2 = \Omega$.

\[
x_1 \neq x_2, \ (x_1, x_2) \rightarrow (y_1, x_2) \text{ at rate } q(x_1, dy_1)
\]

\[
\rightarrow (x_1, y_2) \text{ at rate } q(x_2, dy_2).
\]

Otherwise, $(x, x) \rightarrow (y, y) \text{ at rate } q(x, dy)$.

Character. Chinese idiom: fall in love at first sight.

Omit the last property from now on.
Example 3. Basic coupling (Wasserstein (1969))

\( \tilde{\Omega}_b \): For \( x_1, x_2 \in E \),

\[(x_1, x_2) \to (y, y) \text{ at rate } [q_1(x_1, \cdot) \wedge q_2(x_2, \cdot)](dy) \]

\[\to (y_1, x_2) \text{ at rate } [q_1(x_1, \cdot) - q_2(x_2, \cdot)]^+(dy_1)\]

\[\to (x_1, y_2) \text{ at rate } [q_2(x_2, \cdot) - q_1(x_1, \cdot)]^+(dy_2)\]

\((\nu_1 - \nu_2)^\pm\): Jordan-Hahn decomposition.

\(\nu_1 \wedge \nu_2 = \nu_1 - (\nu_1 - \nu_2)^+\).

Example 4. Coupling of marching soldiers (C. (1986)) \( \tilde{\Omega}_m \). \( E \): Addition group.

\((x_1, x_2) \to (x_1 + y, x_2 + y)\)

at rate \([q_1(x_1, x_1 + y) \wedge q_2(x_2, x_2 + y)](dy)\).

Marching: A Chinese command to soldiers to start marching.
Birth-death $Q$-matrix:

\[ i \rightarrow i + 1 \text{ at rate } b_i = q_{i,i+1} \]
\[ \rightarrow i - 1 \text{ at rate } a_i = q_{i,i-1} \cdot \]

Example 5. Coupling by inner reflection (C. (1990)) $	ilde{\Omega}_{ir}$:

Take $\tilde{\Omega}_{ir} = \tilde{\Omega}_c$ if $|i_1 - i_2| \leq 1$. For $i_2 \geq i_1 + 2$, take

\[
(i_1, i_2) \rightarrow (i_1 + 1, i_2 - 1) \text{ at rate } b_{i_1} \wedge a_{i_2}
\]
\[ \rightarrow (i_1 - 1, i_2) \text{ at rate } a_{i_1} \]
\[ \rightarrow (i_1, i_2 + 1) \text{ at rate } b_{i_2}. \]

Exchange $i_1$ and $i_2$ to get $\tilde{\Omega}_{ir}$ for $i_1 \geq i_2 + 2$. 


Infinite many choices of coupling operator $\tilde{\Omega}$!

Non-explosive?

**Fundamental Theorem for Couplings of Jump Processes** (C. (1986)).

- If a coupling operator is non-explosive, then so are their marginals.
- If the marginals are both non-explosive, then so is every coupling operator.
- If so, then $(\text{MP}) \iff (\text{MO})$. 
Markovian Couplings for diffusions

Couplings for elliptic operators in \( \mathbb{R}^d \).

\[
L = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i}.
\]

An elliptic (may be degenerated) operator \( \tilde{L} \) on the product space \( \mathbb{R}^d \times \mathbb{R}^d \) is called a **Coupling of** \( L \) if it satisfies the following **marginality**:

\[
\tilde{L} f(x, y) = L f(x) \quad \text{(resp. } \tilde{L} f(x, y) = L f(y)),
\]

\[
f \in C^2_b(\mathbb{R}^d), \quad x \neq y.
\]

LHS, \( f \) bivariate function.
Coefficients of $\tilde{L}$:

$$a(x, y) = \begin{pmatrix} a(x) & c(x, y) \\ c(x, y)^* & a(y) \end{pmatrix}, \quad b(x, y) = \begin{pmatrix} b(x) \\ b(y) \end{pmatrix}.$$ 

Condition: $a(x, y)$ non-negative definite.
Only freedom: $c(x, y)$.

Three examples:
(1) Classical coupling $c(x, y) \equiv 0, x \neq y$.
(2) Coupling of marching soldiers (C. and S. F. Li(1989)).
Let $a(x) = \sigma(x)^2$. Take $c(x, y) = \sigma(x)\sigma(y)$.
The couplings given below are due to Lindvall & Rogers (1986) and C. & Li (1989) respectively.

(3) Coupling by reflection. Take

\[ c(x, y) = \sigma(x) \left[ \sigma(y)^* - 2 \frac{\sigma(y)^{-1} \bar{u} \bar{u}^*}{|\sigma(y)^{-1} \bar{u}|^2} \right], \]

\[ \text{det} \sigma(y) \neq 0, \quad x \neq y \]

\[ c(x, y) = \sigma(x) [I - 2\bar{u} \bar{u}^*] \sigma(y)^*, \quad x \neq y \]

where \( \bar{u} = (x - y)/|x - y| \). Extended to manifold by W.S.Kendall[1986]. Also M. Cranston[1991].

In the case that \( x = y \), the first and the third ones are defined to be the same as the second one. Reduce higher dim. to dim. one.
Conjecture: The fundamental theorem holds for diffusions. Facts:

- Sufficient condition for well-posed. There exists \( \varphi_k \) such that \( \lim_{|x| \to \infty} \varphi_k(x) = \infty \) and \( L\varphi \leq c\varphi \) for some constant \( c \). Then, take \( \tilde{\varphi}(x_1, x_2) = \varphi_1(x_1) + \varphi_2(x_2) \).

- Let \( \tau_{n,k} \) be the first time leaving from the cube with length \( n \) of the \( k \)-th process and let \( \tilde{\tau}_n \) be the first time leaving the product cube of coupled process, then we have \( \tau_{n,1} \lor \tau_{n,2} \leq \tilde{\tau}_n \leq \tau_{n,1} + \tau_{n,2} \). Well-posed iff \( \lim_{n \to \infty} \mathbb{P}_k[\tau_{n,k} < t] = 0 \).

Problem: Markovian couplings for Levý processes.
2. Optimal Markovian Coupling w.r.t. distances, $\rho$-optimal Markovian couplings

There $\infty$ many Markovian couplings. Does there exist an optimal one?
For birth-death processes, we have an order as follows:

$$\tilde{\Omega}_{ir} \succ \tilde{\Omega}_{b} \succ \tilde{\Omega}_{c} \succ \tilde{\Omega}_{m},$$

Probability distances: $\geq 16$, the total variation, Lévy-Prohorov distance for the weak convergence. Another probability distance.
Typical convergences in probability theory

\[
\text{convergence in } L^p \\
\text{a.s. conv.} \quad \rightarrow \quad \text{convergence in } \mathbb{P} \quad \rightarrow \quad \text{vague conv.}
\]

weak convergence
$L^p$-convergence, a.s. convergence and convergence in $\mathbb{P}$ depend on the reference frame: $(\Omega, \mathcal{F}, \mathbb{P})$. Vague (weak) convergence does not.

Skorohod Thm. (cf. Ikeda and Watanabe (1981), p.9 Thm.2.7): if $P_n \Rightarrow P$, choose reference frame $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\xi_n \sim P_n, \xi \sim P, \xi_n \to \xi$ a.s.

Except the $L^p$-convergence.

Let $\xi_1, \xi_2: (\Omega, \mathcal{F}, \mathbb{P}) \to (E, \rho, \mathcal{E})$. $L^p$-distance:

$$\|\xi_1 - \xi_2\|_p = \left\{ \mathbb{E}[\rho(\xi_1, \xi_2)^p] \right\}^{1/p}, \quad p \geq 1.$$
Let $\xi_i \sim P_i$, $i = 1, 2$ and $(\xi_1, \xi_2) \sim \tilde{P}$. Then

$$\|\xi_1 - \xi_2\|_p = \left\{ \int \rho(x_1, x_2)^p \tilde{P}(dx_1, dx_2) \right\}^{1/p}.$$ 

$\tilde{P}$: a coupling of $P_1$ and $P_2$. Ignore our reference frame $(\Omega, \mathcal{F}, \mathbb{P})$, then there are a lot of choices of $\tilde{P}$ for given $P_1$ and $P_2$. Thus, the intrinsic distance should be defined as follows:

$$W_p(P_1, P_2) = \inf_{\tilde{P}} \left\{ \int \rho(x_1, x_2)^p \tilde{P}(dx_1, dx_2) \right\}^{1/p}$$

—-$p$-th Wasserstein distance.
Intrinsic: $\xi = \eta + x$, $W_\rho(P_\xi, P_\eta) = |x|$.

Set $W = W_1$. $W$ in dim. one. $W_2$ for Gaussian.

**Dobrushin Theorem**: For discrete $\rho$, $W = \| \cdot \| \text{var/2}$.

**Definition**: Coupling $\overline{P}$ of $P_1$ and $P_2$ is called $\rho$-optimal if

$$
\int \rho(x_1, x_2) \overline{P}(dx_1, dx_2) = W(P_1, P_2).
$$

Definition: \((E, \rho, \mathcal{E})\) metric space. Coupling operator \(\Omega\) is called \(\rho\)-optimal if
\[
\Omega \rho(x_1, x_2) = \inf \Omega \rho(x_1, x_2) \quad \text{for all } x_1 \neq x_2.
\]

Example 6 (Coupling by reflection) (C. (1994))
If \(i_2 = i_1 + 1\), then
\[
(i_1, i_2) \rightarrow (i_1 - 1, i_2 + 1) \quad \text{at rate } a_i \wedge b_i
\]
\[
\rightarrow (i_1 + 1, i_2) \quad \text{at rate } b_i
\]
\[
\rightarrow (i_1, i_2 - 1) \quad \text{at rate } a_i.
\]

If \(i_2 \geq i_1 + 2\), then
\[
(i_1, i_2) \rightarrow (i_1 - 1, i_2 + 1) \quad \text{at rate } a_i \wedge b_i
\]
\[
\rightarrow (i_1 + 1, i_2 - 1) \quad \text{at rate } b_i \wedge a_i.
\]

\(i_1 > i_2\): by symmetry. Reflect. outside strange.
There exist infinitely many choices of $\tilde{Q}$!

**Theorem** (C. (1994)). Birth-death processes. Given positive $(u_i)$.

- Take distance $\rho(i, j) = \sum_{k<|i-j|} u_k$. If $u_k$ is non-increasing in $k$, then the coupling by reflection is $\rho$-optimal.

- Take $\rho$ as above. If $u_k$ is non-decreasing in $k$, then the coupling of marching soldiers is $\rho$-optimal.

- Take $\rho(i, j) = |\sum_{k<i} u_k - \sum_{k<j} u_k|$. Then the above couplings, except the independent one, are all $\rho$-optimal.

Far away from probabilistic intuition.
**Definition.** Given $\rho \in C^2(\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, x) : x \in \mathbb{R}^d\})$, a coupling operator $\bar{L}$ is called $\rho$-optimal if

$$\bar{L}\rho(x, y) = \inf_{\tilde{L}} \tilde{L}\rho(x, y), \quad x \neq y,$$

where $\tilde{L}$ varies over all coupling operator.

**Theorem (C. (1994)).** Let $f \in C^2(\mathbb{R}_+; \mathbb{R}_+)$: $f(0) = 0$, $f' > 0$, $f'' \leq 0$. Set $\rho(x, y) = f(|x - y|)$. Then, the $\rho$-optimal solution $c(x, y)$ is given as follows.

- If $d = 1$, then $c(x, y) = -\sqrt{a_1(x)a_2(y)}$ and moreover,
\[ \bar{L}f(|x - y|) = \frac{1}{2} \left( \sqrt{a_1(x)} + \sqrt{a_2(y)} \right)^2 f''(|x - y|) \]
\[ + \frac{(x - y)(b_1(x) - b_2(y))}{|x - y|} f'(|x - y|). \]

Next, suppose that \( a_k = \sigma_k^2 \ (k = 1, 2) \) is non-degenarated and write
\[ c(x, y) = \sigma_1(x) H^*(x, y) \sigma_2(y). \]

• If \( f'''(r) < 0 \) for all \( r > 0 \), then
\[ H(x, y) = U(\gamma)^{-1} \left[ U(\gamma)U(\gamma)^* \right]^{1/2}, \]
where
\[ \gamma = 1 - \frac{|x - y|f''(|x - y|)}{f'(|x - y|)}, \quad U(\gamma) = \sigma_1(x)(I - \gamma \bar{u} \bar{u}^*) \sigma_2(y). \]
If \( f(r) = r \), then \( H(x, y) \) is a solution to the equation:

\[
U(1)H = (U(1)U(1)^*)^{1/2}.
\]

In particular, if \( a_k(x) = \varphi_k(x)\sigma^2 \) for some positive function \( \varphi_k \) \((k = 1, 2)\), where \( \sigma \) is independent of \( x \) and \( \det \sigma > 0 \). Then

- when \( \rho(x, y) = \|x - y\| \),
  \[
  H(x, y) = I - 2\sigma^{-1}\bar{u}\bar{u}^*\sigma^{-1}/\|\sigma^{-1}\bar{u}\|^2.
  \]
- In the last term, one can replace \( \rho(x, y) = \|x - y\| \) by \( \rho(x, y) = f(\|\sigma^{-1}(x - y)\|) \).

3. Optimal Markovian Coupling w.r.t. nonnegative, lower semi-continuous functions. \( \varphi \)-optimal Markovian couplings

Given metric space \((E, \rho, \mathcal{E})\). \( \varphi \): a nonnegative, lower semi-continuous function.

**Definition:** \( \varphi \)-optimal (Markovian) coupling, replace \( \rho \) with \( \varphi \).
Examples:

• \( \varphi = f \circ \rho, \ f: f(0) = 0, f' > 0, f'' \leq 0. \) Cont.

• \( \varphi(x, y) = 1 \iff x \neq y, \) otherwise, \( \varphi(x, y) = 0. \)

• Let \( E \) have a measurable semi-order “\( \leq \)” and \( F := \{(x, y) : x \leq y\} \) is a closed set. Take \( \varphi = I_{F^c}. \)
**Definition.** Let $\mathcal{M}$ be set of bdd monotone functions $f: x \leq y \implies f(x) \leq f(y)$.

- $\mu_1 \leq \mu_2$: $\mu_1(f) \leq \mu_2(f), \forall f \in \mathcal{M}$.
- $P_1 \leq P_2$: $P_1(f)(x_1) \leq P_2(f)(x_2)$, $\forall x_1 \leq x_2, f \in \mathcal{M}$.
- $P_1(t) \leq P_2(t)$: $P_1(t)(f)(x_1) \leq P_2(t)(f)(x_2)$, $\forall t \geq 0, x_1 \leq x_2, f \in \mathcal{M}$.

**Theorem** (V. Strassen, 1965). For Polish space, $\mu_1 \leq \mu_2 \iff \exists$ a coupling measure $\bar{\mu}$ such that $\bar{\mu}(F^c) = 0$. 
Existence theorem of optimal couplings (S. Y. Zhang, 2000).

Let \((E, \rho, \mathcal{E})\) be Polish and \(\varphi \geq 0\) be l.s.c.

- Given \(P_k(x_k, dy_k), \ k = 1, 2,\) there exists \(\bar{P}(x_1, x_2; dy_1, dy_2)\) such that \(\bar{P}\varphi = \inf \tilde{P}\varphi.\)

- Given jump operators \(\Omega_k, \ k = 1, 2,\) there exists a coupling operator \(\bar{\Omega}\) such that \(\bar{\Omega}\varphi = \inf \tilde{\Omega}\varphi.\)

Strassen’s theorem: \(I_{F^c}\)-optimal Markovian coupling satisfies \(\bar{\mu}(F^c) = 0.\)

Polish space. Jump processes.

\[ P_1(t) \leq P_2(t) \text{ iff} \]

\[ \Omega_1 I_B(x_1) \leq \Omega_2 I_B(x_2), \quad \forall x_1 \leq x_2, A \in \mathcal{M}. \]

Theorem (T. Lindvall, 1999). \( \Delta \): diagonals.

\bullet \ \mu_1 \leq \mu_2 \implies \inf \tilde{\mu}(F^c) = 0 \quad \tilde{\mu}(\Delta^c) = \frac{1}{2} \| \mu_1 - \mu_2 \|_{\Var}.

\bullet \text{Let } P_1 \leq P_2. \text{ Then }

\[ \inf \tilde{P}(x_1, x_2; \Delta^c) = \frac{1}{2} \| P_1(x_1, \cdot) - P_2(x_2, \cdot) \|_{\Var} \]

\[ \tilde{P}(x_1, x_2; F^c) = 0 \]

for all \( x_1 \leq x_2. \)
For diffusions, “$P_1(t) \leq P_2(t)$”, solved.

**Problem**: Existence of $\varphi$-optimal Markovian couplings for diffusions. $\varphi \in C^2(\mathbb{R}^{2d} \setminus \Delta)$.

**Problem**: Construction of optimal Markovian couplings.
http://www.bnu.edu.cn/~chenmf

The end!

Thank you, everybody!

谢谢大家！