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Recalling (3.13) and (5.2), we have $e^{-i\langle x,E_\varepsilon^*\rangle} = \nu_\alpha(p_{E_\varepsilon^*}(n(x,u)t))$. On the other hand, since $\langle \mathfrak{h}_{(\varepsilon)}, -E_\varepsilon^*\rangle = 0$, we get $\nu_{-E_\varepsilon^*+\zeta}(t) = \nu_\zeta(t) = \chi_{i\zeta}(t)$ by (5.7). Thus, introducing a surjective group homomorphism

$$p : G_{(\varepsilon)} \ni n(x,u)t \mapsto (p_{E_\varepsilon^*}(n(x,u)t),\, t) \in \mathrm{Heis}(E_\varepsilon^*) \times H_{(\varepsilon)},$$

we summarize these observations as $\nu_{-E_\varepsilon^*+\zeta} = (\nu_\alpha \otimes \chi_{i\zeta}) \circ p$. On the other hand, it is easy to check that $p^{-1}(\exp \tilde{\mathfrak{r}}_{(\varepsilon)} \times H_{(\varepsilon)}) = \exp \mathfrak{r}_{(\varepsilon)}$ (recall (5.3)). Therefore Lemma 5.2 (i) leads us to

$$\begin{aligned} \mathrm{Ind}_{\exp \mathfrak{r}_{(\varepsilon)}}^{G_{(\varepsilon)}} \nu_{-E_\varepsilon^*+\zeta} &= \mathrm{Ind}_{p^{-1}(\exp \tilde{\mathfrak{r}}_{(\varepsilon)} \times H_{(\varepsilon)})}^{G_{(\varepsilon)}} (\nu_\alpha \otimes \chi_{i\zeta}) \circ p \\ &\simeq \left(\mathrm{Ind}_{\exp \tilde{\mathfrak{r}}_{(\varepsilon)} \times H_{(\varepsilon)}}^{\mathrm{Heis}(E_\varepsilon^*) \times H_{(\varepsilon)}} \nu_\alpha \otimes \chi_{i\zeta} \right) \circ p. \end{aligned} \tag{5.11}$$

Moreover we see also from Lemma 5.2 (ii) that

$$\mathrm{Ind}_{\exp \tilde{\mathfrak{r}}_{(\varepsilon)} \times H_{(\varepsilon)}}^{\mathrm{Heis}(E_\varepsilon^*) \times H_{(\varepsilon)}} \nu_\alpha \otimes \chi_{i\zeta} \simeq \left(\mathrm{Ind}_{\exp \tilde{\mathfrak{r}}_{(\varepsilon)}}^{\mathrm{Heis}(E_\varepsilon^*)} \nu_\alpha \right) \otimes \chi_{i\zeta}. \tag{5.12}$$

Recalling $\tau_{(\varepsilon)} = \tilde{\tau}_{E_\varepsilon^*} \circ p_{E_\varepsilon^*}$, we obtain by (5.11), (5.12) and (5.5) that

$$\mathrm{Ind}_{\exp \mathfrak{r}_{(\varepsilon)}}^{G_{(\varepsilon)}} \nu_{-E_\varepsilon^*+\zeta} \simeq (\tilde{\tau}_{E_\varepsilon^*} \otimes \chi_{i\zeta}) \circ p = \tau_{(\varepsilon)} \otimes \chi_{i\zeta}.$$

Therefore

$$\rho_{-E_\varepsilon^*+\zeta} \simeq \mathrm{Ind}_{G_{(\varepsilon)}}^{G} \mathrm{Ind}_{\exp \mathfrak{r}_{(\varepsilon)}}^{G_{(\varepsilon)}} \nu_{-E_\varepsilon^*+\zeta} \simeq \mathrm{Ind}_{G_{(\varepsilon)}}^{G} \tau_{(\varepsilon)} \otimes \chi_{i\zeta}. \tag{5.13}$$

Now comparing (5.10) with (4.6) and recalling the fact that the restriction of $\chi_{-i\Im s/2}$ to $H_{(\varepsilon)}$ depends only on the s_k's such that $\varepsilon_k = 0$, we see that the condition $s \in \Theta(\varepsilon,\zeta)$ is equivalent to that $s \in \mathfrak{X}(\varepsilon)$ and $\chi_{i\zeta}|_{H_{(\varepsilon)}} = \chi_{-i\Im s/2}|_{H_{(\varepsilon)}}$. Hence, thanks to (5.13) and Theorem 4.12, we conclude that

$$\rho_{-E_\varepsilon^*+\zeta} \simeq \mathrm{Ind}_{G_{(\varepsilon)}}^{G} \tau_{(\varepsilon)} \otimes \chi_{-i\Im s/2} \simeq \pi_s.$$

(ii) Suppose that $s^i \in Z(\varepsilon^i,\zeta^i)$ ($\varepsilon^i \in \{0,1\}^r$, $\zeta^i \in Z(\varepsilon^i)$, $i = 1,2$) and let $\mathcal{O}^i$ be the coadjoint orbit through $-E_{\varepsilon^i}^* + \zeta^i$. Then the theory of orbit method together with (i) states that $\pi_{s^1} \simeq \pi_{s^2}$ if and only if $\mathcal{O}^1 = \mathcal{O}^2$. When $\varepsilon^1 \neq \varepsilon^2$, it is clear from Proposition 5.1 (iv) that $\mathcal{O}^1 \neq \mathcal{O}^2$, so that $\pi_{s^1} \not\simeq \pi_{s^2}$. Consider the case $\varepsilon^1 = \varepsilon^2 (=: \varepsilon)$ and $\zeta^1 \neq \zeta^2$. Let P be the subset $\{\, \eta \in \mathfrak{g}^* \,;\, \eta|_V = -E_\varepsilon^* \,\}$ of $\mathfrak{g}^*$. Then Proposition 5.1 (iv) tells us that $\mathcal{O}^i \cap P = -E_\varepsilon^* + \zeta^i + (V \oplus N_{(\varepsilon)} \oplus \mathfrak{h}_{(\varepsilon)})^\perp$. On the other hand, we see from (5.9) and Theorem 2.3 (ii) that $Z(\varepsilon) \cap \mathfrak{h}_{(\varepsilon)}^\perp = \{0\}$. Thus $\zeta^1 + \mathfrak{h}_{(\varepsilon)}^\perp \neq \zeta^2 + \mathfrak{h}_{(\varepsilon)}^\perp$, so that

$$-E_\varepsilon^* + \zeta^1 + (V \oplus N_{(\varepsilon)} \oplus \mathfrak{h}_{(\varepsilon)})^\perp \neq -E_\varepsilon^* + \zeta^2 + (V \oplus N_{(\varepsilon)} \oplus \mathfrak{h}_{(\varepsilon)})^\perp.$$

Hence $\pi_{s^1} \not\simeq \pi_{s^2}$ and the proof is completed. ■

so that

$$\mathrm{Ad}^*(t)(V \oplus N_{(\varepsilon)} \oplus \mathfrak{h}_{(\varepsilon)})^\perp = (V \oplus N_\xi \oplus \mathfrak{h}_\xi)^\perp.$$

This observation together with (iii) proves the assertion (iv) because $G = H \cdot N(Q)$. ■

Here we need the following lemma. Routine proof is left to the reader.

LEMMA 5.2. *Let G_0 be a Lie group, H_0 a subgroup of G_0, (σ, U) a unitary representation of H_0 and π the induced representation $\mathrm{Ind}_{H_0}^{G_0}\sigma$.*

(i) *Let G_1 be a Lie group and $p : G_1 \to G_0$ a surjective group homomorphism. Then the unitary representation $\pi \circ p$ of G_1 is equivalent to the induced representation $\mathrm{Ind}_{p^{-1}(H_0)}^{G_1}(\sigma \circ p)$.*

(ii) *Let G_2 be a Lie group and $\chi : G_2 \to \mathbb{C}$ a unitary character of G_2. Then the unitary representation $\pi \otimes \chi$ of $G_0 \times G_2$ is equivalent to the induced representation $\mathrm{Ind}_{H_0 \times G_2}^{G_0 \times G_2}(\sigma \otimes \chi)$.*

Put

$$Z(\varepsilon) := \left\{ \zeta = \sum_{k=1}^{r} \zeta_k \hat{A}_k \in \mathfrak{g}^* \,;\, \zeta_k = 0 \text{ for all } k \text{ such that } \varepsilon_k = 1 \right\}, \tag{5.9}$$

and define the set $\Theta(\varepsilon, \zeta)$ of parameters s for $\zeta \in Z(\varepsilon)$ by

$$\Theta(\varepsilon, \zeta) := \left\{ s \in \mathbb{C}^r; \begin{array}{ll} \Re s_k > q_k(\varepsilon)/2 & (\text{if } \varepsilon_k = 1) \\ s_k = q_k(\varepsilon)/2 - 2i\zeta_k & (\text{if } \varepsilon_k = 0) \end{array} \right\}, \tag{5.10}$$

where $q_k(\varepsilon) = \sum_{m>k} \varepsilon_m \dim \mathfrak{g}_{(\alpha_m - \alpha_k)/2}$ $(k = 1, \ldots, r)$ (see (2.14)). We see from (4.6), (5.9) and (5.10) that $\mathfrak{X}(\varepsilon) = \bigsqcup_{\zeta \in Z(\varepsilon)} \Theta(\varepsilon, \zeta)$. Thus Theorem 4.8 states that $\mathcal{H}_s(D) \neq \{0\}$ if and only if s belongs to some $\Theta(\varepsilon, \zeta)$.

Now we arrive at the classification of $(\pi_s, \mathcal{H}_s(D))$.

THEOREM 5.3. (i) *If $s \in \Theta(\varepsilon, \zeta)$, then π_s corresponds to the coadjoint orbit through $-E_\varepsilon^* + \zeta \in \mathfrak{g}^*$ by the Kirillov-Bernat correspondence.*

(ii) *Two IUR's π_{s^1} and π_{s^2} are equivalent if and only if s^1 and s^2 belong to the same $\Theta(\varepsilon, \zeta)$.*

Proof. Since $\mathfrak{r}_{(\varepsilon)}$ is a real polarization at $-E_\varepsilon^* + \zeta \in \mathfrak{g}^*$ satisfying the Pukanszky condition by Proposition 5.1, we have $\rho_{-E_\varepsilon^*+\zeta} = \mathrm{Ind}_{\exp \mathfrak{r}_{(\varepsilon)}}^{G} \nu_{-E_\varepsilon^*+\zeta}$. Let $n(x,u)t$ be an element of $\exp \mathfrak{r}_{(\varepsilon)}$ with $x \in V, u \in \sum_{1 \le l \le d}^{\oplus} \mathbb{R} U_l^\varepsilon \oplus N_{(\varepsilon)}$ and $t \in H_{(\varepsilon)}$. Since $\nu_{-E_\varepsilon^*+\zeta}$ is a unitary character on $\exp \mathfrak{r}_{(\varepsilon)}$, we have

$$\nu_{-E_\varepsilon^*+\zeta}(n(x,u)t) = e^{-i\langle x, E_\varepsilon^* \rangle} \nu_{-E_\varepsilon^*+\zeta}(t).$$

annihilator of $\mathfrak{h}_{(\varepsilon)}$ in $\mathfrak{h}^*$. Then

$$-E_\varepsilon^* + (V \oplus N_{(\varepsilon)} \oplus \mathfrak{h}_{(\varepsilon)})^\perp = \left\{ \eta \in \mathfrak{g}^* \,;\, \eta|_V = -E_\varepsilon^*,\ \eta|_{W_\mathbb{R}} \in N_{(\varepsilon)}^\perp \text{ and } \eta|_\mathfrak{h} \in \mathfrak{h}_{(\varepsilon)}^\perp \right\}.$$

Put $\eta_0 := \mathrm{Ad}^*(n(x_0,u_0))(-E_\varepsilon^*)$ $(n(x_0,u_0) \in N(Q))$. Observing that

$$\begin{aligned}\mathrm{Ad}(n(x_0,u_0)^{-1})(x+u+T) &= \sum_{n\geq 0} \frac{1}{n!}\mathrm{ad}(-x_0-u_0)^n(x+u+T)\\ &= x+u+T-[x_0,T]-[u_0,u]-[u_0,T]+[u_0,[u_0,T]]/2\end{aligned}$$

for $x \in V$, $u \in W$ and $T \in \mathfrak{h}$, we see from (3.1) that

$$\begin{aligned}\langle x+u+T, \eta_0\rangle &= -\langle x, E_\varepsilon^*\rangle + \langle [x_0,T], E_\varepsilon^*\rangle + \langle [u_0,u], E_\varepsilon^*\rangle - \langle [u_0,[u_0,T]], E_\varepsilon^*\rangle/2\\ &= -\langle x, E_\varepsilon^*\rangle + 2\Im Q_{(\varepsilon)}(u_0,u) + \langle x_0, T\cdot E_\varepsilon^*\rangle + \Im Q_{(\varepsilon)}(u_0,[T,u_0]).\end{aligned}$$

Thus $\eta_0|_V = -E_\varepsilon^*$ and $\eta_0|_{W_\mathbb{R}} = 2\Im Q_{(\varepsilon)}(u_0,\,\cdot\,) \in N_{(\varepsilon)}^\perp$. When $T \in \mathfrak{h}_{(\varepsilon)}$, we have $[T,u_0] \in N_{(\varepsilon)}$ by (3.5) and $T\cdot E_\varepsilon^* = 0$, so that the above formula tells us $\langle T, \eta_0\rangle = 0$. It follows that

$$\mathrm{Ad}^*(N(Q))(-E_\varepsilon^*) \subset -E_\varepsilon^* + (V \oplus N_{(\varepsilon)} \oplus \mathfrak{h}_{(\varepsilon)})^\perp.$$

We shall show the converse inclusion. Let ϕ and ψ be elements of $N_{(\varepsilon)}^\perp$ and $\mathfrak{h}_{(\varepsilon)}^\perp$ respectively. Since $N_{(\varepsilon)}^\perp$ is canonically identified with the space of linear forms on $M_{(\varepsilon)} = W/N_{(\varepsilon)}$, and since $2\Im Q_{(\varepsilon)}$ induces a non-degenerate bilinear form on $M_{(\varepsilon)}$, we can take $u_1 \in W$ for which $2\Im Q_{(\varepsilon)}(u_1,\,\cdot\,) = \phi$. Put $\psi' := \psi - \Im Q_{(\varepsilon)}(u_1,[\,\cdot\,,u_1]) \in \mathfrak{h}^*$. Then we can check $\psi' \in \mathfrak{h}_{(\varepsilon)}^\perp$ by (3.5). We consider the map $\Psi : V \ni x \mapsto \langle [x,\,\cdot\,], E_\varepsilon^*\rangle \in \mathfrak{h}^*$. Then

$$\begin{aligned}(\mathrm{Image}\,\Psi)^\perp &= \{\, T \in \mathfrak{h} \,;\, \langle T, \Psi(x)\rangle = 0 \text{ for all } x \in V \,\}\\ &= \{\, T \in \mathfrak{h} \,;\, T\cdot E_\varepsilon^* = 0 \,\} = \mathfrak{h}_{(\varepsilon)}.\end{aligned}$$

Therefore $\mathrm{Image}\,\Psi = \mathfrak{h}_{(\varepsilon)}^\perp$, so that we can take $x_1 \in V$ for which $\psi' = \Psi(x_1) = \langle [x_1,\,\cdot\,], E_\varepsilon^*\rangle$. Put $\eta_1 := \mathrm{Ad}^*(n(x_1,u_1))(-E_\varepsilon^*)$. Then the same calculation as for η_0 yields that $\eta_1|_{W_\mathbb{R}} = 2\Im Q(\,\cdot\,,u_1) = \phi$ and that

$$\eta_1|_\mathfrak{h} = \Im Q_{(\varepsilon)}(u_1,[\,\cdot\,,u_1]) + \langle [x_1,\,\cdot\,], E_\varepsilon^*\rangle = \Im Q_{(\varepsilon)}(u_1,[\,\cdot\,,u_1]) + \psi' = \psi.$$

Since we can take such x_1 and u_1 for any $\phi \in N_{(\varepsilon)}^\perp$ and any $\psi \in \mathfrak{h}_{(\varepsilon)}^\perp$, the assertion (iii) holds.

(iv) Take $t \in H$ and put $\xi := t\cdot E_\varepsilon^* \in \mathcal{O}_\varepsilon^*$. It is easy to see that $tH_{(\varepsilon)}t^{-1} = H_\xi$, so that $\mathrm{Ad}(t)\mathfrak{h}_{(\varepsilon)} = \mathfrak{h}_\xi$. Then we have by (3.3)

$$\mathrm{Ad}(t)(V \oplus N_{(\varepsilon)} \oplus \mathfrak{h}_{(\varepsilon)}) = V \oplus N_\xi \oplus \mathfrak{h}_\xi,$$

real polarization at $-E_\varepsilon^* + \zeta$.

(iii) *One has*

$$\mathrm{Ad}^*(N(Q))(-E_\varepsilon^* + \zeta) = -E_\varepsilon^* + \zeta + (V \oplus N_{(\varepsilon)} \oplus \mathfrak{h}_{(\varepsilon)})^\perp.$$

In particular, $\mathfrak{r}_{(\varepsilon)}$ *satisfies the Pukanszky condition.*

(iv) *The coadjoint orbit in* $\mathfrak{g}^*$ *through* $-E_\varepsilon^* + \zeta$ *is described as*

$$\bigsqcup_{\xi \in \mathcal{O}_\varepsilon^*} \left\{ -\xi + \zeta + (V \oplus N_\xi \oplus \mathfrak{h}_\xi)^\perp \right\}.$$

Proof. (i) For $x, x_0 \in V$, $u, u_0 \in W$, and $T, T_0 \in \mathfrak{h}$, we observe that

$$\begin{aligned} \Lambda_{-E_\varepsilon^*}(x + u + T, x_0 + u_0 + T_0) &= -\langle [x, T_0], E_\varepsilon^* \rangle - \langle [u, u_0], E_\varepsilon^* \rangle - \langle [T, x_0], E_\varepsilon^* \rangle \\ &= -\langle x, T_0 \cdot E_\varepsilon^* \rangle - 2\Im Q_{(\varepsilon)}(u, u_0) + \langle x_0, T \cdot E_\varepsilon^* \rangle. \end{aligned} \tag{5.8}$$

Then the last term vanishes for all x, u, T if and only if $x_0 \in I_{(\varepsilon)}^\perp$, $u_0 \in N_{(\varepsilon)}$ and $T_0 \in \mathfrak{h}_{(\varepsilon)}$.

(ii) Let $Y_i = x_i + u_i + v_i + T_i$ $(i = 1, 2)$ be elements of $\mathfrak{r}_{(\varepsilon)}$ with $x_i \in V$, $u_i \in \sum_{1 \le l \le d}^{\oplus} \mathbb{R} U_l^\varepsilon$, $v_i \in N_{(\varepsilon)}$ and $T_i \in \mathfrak{h}_{(\varepsilon)}$. Then $[Y_1, Y_2] \in V \oplus N_{(\varepsilon)} \oplus \mathfrak{h}_{(\varepsilon)} \subset \mathfrak{r}_{(\varepsilon)}$ because (3.5) tells us that $[T_i, W] \subset N_{(\varepsilon)}$. Thus $\mathfrak{r}_{(\varepsilon)}$ is a subalgebra. Since $\Lambda_{-E_\varepsilon^* + \zeta} = \Lambda_{-E_\varepsilon^*}$ owing to (5.6), the same calculation as in (5.8) yields that

$$\Lambda_{-E_\varepsilon^* + \zeta}(Y_1, Y_2) = -\langle x_1, T_2 \cdot E_\varepsilon^* \rangle - 2\Im Q_{(\varepsilon)}(u_1 + v_1, u_2 + v_2) + \langle x_2, T_1 \cdot E_\varepsilon^* \rangle.$$

Since $T_i \in \mathfrak{h}_{(\varepsilon)}$, $v_i \in N_{(\varepsilon)}$ and $u_i \in \sum_{1 \le l \le d}^{\oplus} \mathbb{R} U_l^\varepsilon$, the right-hand side vanishes. Now we consider the dimensions of the subspaces. Since the kernel of the linear map $\mathfrak{h} \ni T \mapsto T \cdot E_\varepsilon^* \in V^*$ coincides with $\mathfrak{h}_{(\varepsilon)}$, we have $\dim I_{(\varepsilon)} = \dim \mathfrak{h} - \dim \mathfrak{h}_{(\varepsilon)}$. Theorem 1.2 (iv) implies $\dim V = \dim \mathfrak{h}$, so that we obtain by (i)

$$\begin{aligned} \dim \mathrm{Ker}\, \Lambda_{-E_\varepsilon^* + \zeta} &= (\dim V - \dim I_{(\varepsilon)}) + \dim_{\mathbb{R}} N_{(\varepsilon)} + \dim \mathfrak{h}_{(\varepsilon)} \\ &= \dim_{\mathbb{R}} N_{(\varepsilon)} + 2 \dim \mathfrak{h}_{(\varepsilon)}. \end{aligned}$$

Since $d = \dim_{\mathbb{C}} M_{(\varepsilon)} = (\dim W_{\mathbb{R}} - \dim_{\mathbb{R}} N_{(\varepsilon)})/2$, we then have

$$\begin{aligned} \dim \mathfrak{r}_{(\varepsilon)} &= \dim V + (\dim W_{\mathbb{R}} - \dim_{\mathbb{R}} N_{(\varepsilon)})/2 + \dim_{\mathbb{R}} N_{(\varepsilon)} + \dim \mathfrak{h}_{(\varepsilon)} \\ &= (2 \dim V + \dim W_{\mathbb{R}})/2 + (\dim_{\mathbb{R}} N_{(\varepsilon)} + 2 \dim \mathfrak{h}_{(\varepsilon)})/2 \\ &= (\dim \mathfrak{g} + \dim \mathrm{Ker}\, \Lambda_{-E_\varepsilon^* + \zeta})/2. \end{aligned}$$

Hence $\mathfrak{r}_{(\varepsilon)}$ is a real polarization at $-E_\varepsilon^* + \zeta$.

(iii) Owing to (5.6), we have $\mathrm{Ad}^*(G)\zeta = \zeta$. Thus it is sufficient to consider the case $\zeta = 0$. We denote by $N_{(\varepsilon)}^\perp$ the annihilator of $N_{(\varepsilon)}$ in $W_{\mathbb{R}}^*$, and by $\mathfrak{h}_{(\varepsilon)}^\perp$ the

and take an orthonormal basis $\{[U_1^\varepsilon]_{(\varepsilon)}, [U_2^\varepsilon]_{(\varepsilon)}, \dots, [U_d^\varepsilon]_{(\varepsilon)}\}$ $(d := \dim_{\mathbb{C}} M_{(\varepsilon)})$ of $M_{(\varepsilon)}$ with respect to the Hermitian form $Q_{(\varepsilon)}$. Put

$$\tilde{\mathfrak{r}}_{(\varepsilon)} := \mathbb{R} \oplus \sum_{l=1}^{d}{}^{\oplus} \mathbb{R}\,[U_l^\varepsilon]_{(\varepsilon)}. \tag{5.3}$$

Then $\tilde{\mathfrak{r}}_{(\varepsilon)}$ is a subalgebra of $\mathfrak{n}_{(\varepsilon)}$. By (5.1) and (5.2) we have

$$\Lambda_\alpha(c + [u]_{(\varepsilon)}, c' + [u']_{(\varepsilon)}) = -2\Im Q_{(\varepsilon)}(u, u') \quad (c, c' \in \mathbb{R},\ [u]_{(\varepsilon)}, [u']_{(\varepsilon)} \in M_{(\varepsilon)}), \tag{5.4}$$

so that $\Lambda_\alpha(\tilde{\mathfrak{r}}_{(\varepsilon)}, \tilde{\mathfrak{r}}_{(\varepsilon)}) = 0$. On the other hand, since $\operatorname{Ker}\Lambda_\alpha = \mathbb{R}$ by (5.4), we get $(\dim \mathfrak{n}_{(\varepsilon)} + \dim \operatorname{Ker}\Lambda_\alpha)/2 = ((1+2d)+1)/2 = \dim \tilde{\mathfrak{r}}_{(\varepsilon)}$. Hence $\tilde{\mathfrak{r}}_{(\varepsilon)}$ is a real polarization at $\alpha \in (\mathfrak{n}_{(\varepsilon)})^*$. Then $\rho_\alpha = \operatorname{Ind}_{\exp \tilde{\mathfrak{r}}_{(\varepsilon)}}^{\operatorname{Heis}(E_\varepsilon^*)} \nu_\alpha$ is an IUR of $\operatorname{Heis}(E_\varepsilon^*)$. Note that $\rho_\alpha(a, 0)$ is the scalar multiplication by $\nu_\alpha(a, 0) = e^{-a}$ for elements $(a, 0) \in \operatorname{Heis}(E_\varepsilon^*)$. Comparing this fact with the definition (3.12) of the Fock representation $\tilde{\tau}_{E_\varepsilon^*}$, we obtain

$$\operatorname{Ind}_{\exp \tilde{\mathfrak{r}}_{(\varepsilon)}}^{\operatorname{Heis}(E_\varepsilon^*)} \nu_\alpha \simeq \tilde{\tau}_{E_\varepsilon^*} \tag{5.5}$$

by Stone-von Neumann's theorem.

Now we apply the orbit method to our split solvable group G. As in Section 2, we denote by $\hat{Y}$ the linear form $B(\,\cdot\,, Y)$ on $\mathfrak{g}$. We first consider elements $\zeta = \sum_{k=1}^r \zeta_k \hat{A}_k$ of $\hat{\mathfrak{a}}$. Since $\mathfrak{a}$ is orthogonal to $[\mathfrak{g}, \mathfrak{g}]$ with respect to B, we see that

$$\Lambda_\zeta(Y, Y') = \langle [Y, Y'], \zeta \rangle = 0 \quad (Y, Y' \in \mathfrak{g}). \tag{5.6}$$

Thus the whole space $\mathfrak{g}$ is a real polarization at ζ and the corresponding representation ρ_ζ is equivalent to the unitary character ν_ζ. Since $\langle \sum_{k=1}^r c_k A_k, \zeta \rangle = \sum_{k=1}^r \zeta_k c_k$, we have $\nu_\zeta(\exp \sum c_k A_k) = e^{i(\zeta_1 c_1 + \cdots + \zeta_r c_r)}$. Hence, identifying $\zeta \in \mathfrak{g}^*$ with $(\zeta_1, \dots, \zeta_r) \in \mathbb{R}^r$, we get by (1.12)

$$\nu_\zeta(g) = \chi_{i\zeta}(g) \quad (g \in G). \tag{5.7}$$

Next we study real polarizations at elements $-E_\varepsilon^* + \zeta \in \mathfrak{g}^*$ $(\varepsilon \in \{0,1\}^r)$. For the sake of convenience, we denote by $T \cdot \xi$ $(T \in \mathfrak{h}, \xi \in V^*)$ the element of V^* given by $\langle x, T \cdot \xi \rangle := -\langle [T, x], \xi \rangle$ $(x \in V)$. We put $I_{(\varepsilon)} := \{\, T \cdot E_\varepsilon^* \in V^* \,;\, T \in \mathfrak{h} \,\}$.

Proposition 5.1. (i) *One has*

$$\operatorname{Ker}\Lambda_{-E_\varepsilon^*} = I_{(\varepsilon)}^\perp \oplus N_{(\varepsilon)} \oplus \mathfrak{h}_{(\varepsilon)},$$

where $I_{(\varepsilon)}^\perp := \{\, x \in V \,;\, \langle x, \xi \rangle = 0$ *for all* $\xi \in I_{(\varepsilon)} \,\}$.

(ii) *Let* $\mathfrak{r}_{(\varepsilon)}$ *be the subspace* $V \oplus \sum_{1 \le l \le d}^{\oplus} \mathbb{R} U_l^\varepsilon \oplus N_{(\varepsilon)} \oplus \mathfrak{h}_{(\varepsilon)}$ *of* $\mathfrak{g}$. *Then* $\mathfrak{r}_{(\varepsilon)}$ *is a*

Let $s, s' \in \mathfrak{X}(\varepsilon)$ and suppose that $\chi_{-i\Im s/2}|_{H_{(\varepsilon)}} = \chi_{-i\Im s'/2}|_{H_{(\varepsilon)}}$. Then Theorem 4.12 tells us that π_s and $\pi_{s'}$ are equivalent. Note that, in view of Theorem 2.3 (ii), the restriction of $\chi_{-i\Im s/2}$ to $H_{(\varepsilon)}$ depends only on the s_k's such that $\varepsilon_k = 0$.

5. THE ORBIT METHOD AND THE CLASSIFICATION OF π_s

In this section, we describe the equivalence classes of the IUR's π_s of G through the theory of orbit method. Here we review the orbit method for exponential solvable Lie groups briefly (see [**2**] for details). Let G_1 be an exponential solvable Lie group, $\mathfrak{g}_1$ its Lie algebra, and f a linear form on $\mathfrak{g}_1$. We define an alternative form Λ_f on $\mathfrak{g}_1$ by $\Lambda_f(X, X') := \langle [X, X'], f\rangle$ $(X, X' \in \mathfrak{g}_1)$. A subalgebra $\mathfrak{r}$ is said to be *a real polarization at* f if $\mathfrak{r}$ is a Lagrangian subspace of Λ_f, that is, $\mathfrak{r}$ satisfies the following two conditions:

(i) $\dim \mathfrak{r} = (\dim \mathfrak{g}_1 + \dim \mathrm{Ker}\, \Lambda_f)/2$,

(ii) $\Lambda_f(\mathfrak{r}, \mathfrak{r}) = 0$.

Thanks to (ii), the function ν_f on $\exp \mathfrak{r}$ given by $\nu_f(\exp X) := e^{i\langle X, f\rangle}$ $(X \in \mathfrak{r})$ is a unitary character of $\exp \mathfrak{r}$. Let $\rho_{f,\mathfrak{r}} := \mathrm{Ind}_{\exp \mathfrak{r}}^{G_1} \nu_f$. Then $\rho_{f,\mathfrak{r}}$ is irreducible if and only if $f + \mathfrak{r}^\perp \subset \mathrm{Ad}^*(G_1)f$. This condition is called *the Pukanszky condition*. For any $f \in \mathfrak{g}_1^*$, there exists a real polarization $\mathfrak{r}$ at f satisfying the Pukanszky condition. The equivalence class of $\rho_{f,\mathfrak{r}}$ is independent of the choice of such $\mathfrak{r}$, so that we denote this IUR of G_1 by ρ_f. Then any IUR π of G_1 is equivalent to ρ_f for some $f \in \mathfrak{g}_1^*$. For $f, f' \in \mathfrak{g}_1^*$, the IUR's $\rho_f, \rho_{f'}$ are equivalent if and only if f and f' are contained in the same coadjoint orbit. Hence we obtain a bijection from the orbit space $\mathrm{Ad}^*(G_1)\backslash \mathfrak{g}_1^*$ to the unitary dual of G_1. We call this correspondence *the Kirillov-Bernat correspondence*.

We first apply the orbit method to the nilpotent Lie group $\mathrm{Heis}(E_\varepsilon^*)$ with $\varepsilon \in \{0, 1\}^r$ (see (3.11)). The Lie algebra of $\mathrm{Heis}(E_\varepsilon^*)$ is identified with $\mathfrak{n}_{(\varepsilon)} := \mathbb{R} \oplus M_{(\varepsilon)}$ with the bracket product

$$[c + [u]_{(\varepsilon)}, c' + [u']_{(\varepsilon)}] := 2\Im Q_{(\varepsilon)}(u, u') + [0]_{(\varepsilon)} \ (c, c' \in \mathbb{R},\ [u]_{(\varepsilon)}, [u']_{(\varepsilon)} \in M_{(\varepsilon)}). \quad (5.1)$$

Let α be a linear form on $\mathfrak{n}_{(\varepsilon)}$ defined by

$$\alpha(c + [u]_{(\varepsilon)}) := -c \quad (c \in \mathbb{R},\ [u]_{(\varepsilon)} \in M_{(\varepsilon)}) \quad (5.2)$$

where $t \in H$ so that $\xi = t \cdot E_\varepsilon^*$. Then the condition (a), (4.19) and Lemma 3.4 tell us that $f'(\xi, \cdot) \in \mathcal{F}_\xi$ (a.a. $\xi \in \mathcal{O}_\varepsilon^*$) and the calculation similar to (4.18) yields

$$\int_{\mathcal{O}_\varepsilon^*} \|f'(\xi, \cdot)\|_\xi^2 \, d\mathcal{R}^*_{\Re s^*}(\xi) = \int_{H/H_{(\varepsilon)}} \|\varphi(t, \cdot)\|^2_{(\varepsilon)} \, d\rho_s(tG_{(\varepsilon)}) = \|\varphi\|^2 < \infty.$$

Therefore $f' \in \mathcal{L}_s(\mathcal{O}_\varepsilon^* \times W)$. Moreover we see from (4.17), (4.19) and the condition (b) that

$$\begin{aligned}
\Phi f'(tn, \cdot) &= \chi_{-i\Im s/2}(t^{-1})\tau_{(\varepsilon)}(n^{-1})S_{t^{-1}} f'(t \cdot E_\varepsilon^*, \cdot) \\
&= \chi_{-i\Im s/2}(t^{-1})\tau_{(\varepsilon)}(n^{-1})S_{t^{-1}}[\chi_{-i\Im s/2}(t)S_t\varphi(t, \cdot)] \\
&= \tau_{(\varepsilon)}(n^{-1})\varphi(t, \cdot) = \varphi(tn, \cdot).
\end{aligned}$$

Hence $\Phi : \mathcal{L}_s(\mathcal{O}_\varepsilon^* \times W) \to \mathcal{L}$ is surjective. It remains only to show that Φ is an intertwining operator between $(\check{\pi}_s, \mathcal{L}_s(\mathcal{O}_\varepsilon^* \times W))$ and $(\pi, \mathcal{L})$, that is, to show that

$$\Phi^{-1}\pi(g_0)\Phi f = \check{\pi}_s(g_0)f \quad (g_0 \in G,\ f \in \mathcal{L}_s(\mathcal{O}_\varepsilon^* \times W)).$$

Put $\varphi := \Phi f$, $\varphi_0 := \pi(g_0)\varphi$ and $f_0 := \Phi^{-1}\varphi_0$. Let ξ be an element of $\mathcal{O}_\varepsilon^*$ and take $t \in H$ for which $\xi = t \cdot E_\varepsilon^*$. We first consider the case $g_0 = t_0 \in H$. Then by (4.19) and (4.16),

$$\begin{aligned}
f_0(\xi, \cdot) &= \chi_{-i\Im s/2}(t)S_t\varphi_0(t, \cdot) \\
&= \chi_{-i\Im s/2}(t)\chi_{\Re s/2}(t_0)S_t\varphi(t_0^{-1}t, \cdot).
\end{aligned}$$

By (4.17), we have

$$\begin{aligned}
f_0(\xi, \cdot) &= \chi_{-i\Im s/2}(t)\chi_{\Re s/2}(t_0)S_t[\chi_{-i\Im s/2}(t^{-1}t_0)S_{t^{-1}t_0} f(t_0^{-1}t \cdot E_\varepsilon^*, \cdot)] \\
&= \chi_{\bar{s}/2}(t_0)S_{t_0} f(t_0^{-1} \cdot \xi, \cdot),
\end{aligned}$$

which equals $\check{\pi}_s(t_0)f(\xi, \cdot)$ by (4.12). Next we observe the case $g_0 = n_0 \in N(Q)$. Similarly to the above, (4.19) and (4.16) yields

$$\begin{aligned}
f_0(\xi, \cdot) &= \chi_{-i\Im s/2}(t)S_t\varphi_0(t, \cdot) = \chi_{-i\Im s/2}(t)S_t\varphi(n_0^{-1}t, \cdot) \\
&= \chi_{-i\Im s/2}(t)S_t\varphi(t \cdot t^{-1}n_0^{-1}t, \cdot).
\end{aligned}$$

Using (4.17) and Lemma 3.5, we obtain

$$\begin{aligned}
f_0(\xi, \cdot) &= \chi_{-i\Im s/2}(t)S_t[\chi_{-i\Im s/2}(t^{-1})\tau_{(\varepsilon)}(t^{-1}n_0t)S_{t^{-1}} f(t \cdot E_\varepsilon^*, \cdot)] \\
&= \tau_{t \cdot E_\varepsilon^*}(n_0) f(t \cdot E^*, \cdot) = \tau_\xi(n_0) f(\xi, \cdot).
\end{aligned}$$

Then the last term equals $\check{\pi}_s(n_0)f(\xi, \cdot)$ by (4.13). Hence Proposition 4.11 completes the proof. ■

where $n_1 := t_1^{-1}nt_1 \in N(Q)$. Since Lemma 3.4 says that $S_{t_1^{-1}}$ is the identity on $\mathcal{F}_{(\varepsilon)}$ and since

$$\tau_{(\varepsilon)}(n_1^{-1}) = \tau_{(\varepsilon)}(t_1^{-1})\tau_{(\varepsilon)}(n^{-1})\tau_{(\varepsilon)}(t_1) = \tau_{(\varepsilon)}(n^{-1}),$$

we obtain by (4.17)

$$\begin{aligned}\varphi'(tnt_1,\,\cdot) &= \chi_{-i\Im s/2}(t_1^{-1})\cdot\chi_{-i\Im s/2}(t^{-1})\tau_{(\varepsilon)}(n^{-1})S_{t^{-1}}f(t\cdot E_\varepsilon^*,\,\cdot)\\ &= \chi_{-i\Im s/2}(t_1^{-1})\varphi'(tn,\,\cdot).\end{aligned}$$

Next, for the case $g_1 = n_2 \in N(Q)$ we have by (4.17)

$$\begin{aligned}\varphi'(tnn_2,\,\cdot) &= \chi_{-i\Im s/2}(t^{-1})\tau_{(\varepsilon)}(n_2^{-1})\tau_{(\varepsilon)}(n^{-1})S_{t^{-1}}f(t\cdot E_\varepsilon^*,\,\cdot)\\ &= \tau_{(\varepsilon)}(n_2^{-1})\varphi'(tn,\,\cdot),\end{aligned}$$

whence (b) follows. Since $\chi_{-i\Im s/2}(t^{-1})\tau_{(\varepsilon)}(n^{-1})$ is a unitary operator on $\mathcal{F}_{(\varepsilon)}$, we see from (4.14) and (4.17) that the integral in (c) equals

$$\|\varphi'\|^2 = \int_{H/H_{(\varepsilon)}} \|S_{t^{-1}}f(t\cdot E_\varepsilon^*,\,\cdot)\|_{(\varepsilon)}^2\, d\mathcal{R}^*_{\Re s^*}(t\cdot E_\varepsilon^*).$$

Thanks to Lemma 3.4, we get

$$\begin{aligned}\|\varphi'\|^2 &= \int_{H/H_{(\varepsilon)}} \|f(t\cdot E_\varepsilon^*,\,\cdot)\|^2_{t\cdot E_\varepsilon^*}\, d\mathcal{R}^*_{\Re s^*}(t\cdot E_\varepsilon^*)\\ &= \int_{\mathcal{O}_\varepsilon^*} \|f(\xi,\,\cdot)\|_\xi^2\, d\mathcal{R}^*_{\Re s^*}(\xi) = \|f\|^2_{\mathcal{L}_s} < \infty.\end{aligned} \tag{4.18}$$

Hence $\varphi' \in \mathcal{L}$ and the map $\Phi : \mathcal{L}_s(\mathcal{O}_\varepsilon^* \times W) \ni f \mapsto \varphi' \in \mathcal{L}$ is an isometry. We shall prove that Φ is surjective. First of all, we note that (4.17) implies

$$f(\xi,\,\cdot) = \chi_{-i\Im s/2}(t)S_t\varphi'(t,\,\cdot) \quad (\xi \in \mathcal{O}_\varepsilon^*),$$

where $t \in H$ is taken in such a way that $\xi = t\cdot E_\varepsilon^*$. Let φ be an element of $\mathcal{L}$, t_1 an element of H and t_0 an element of $H_{(\varepsilon)}$. Then we have by the condition (b) and Lemma 3.4,

$$\begin{aligned}\chi_{-i\Im s/2}(t_1t_0)S_{t_1t_0}\varphi(t_1t_0,\,\cdot) &= \chi_{-i\Im s/2}(t_1t_0)\chi_{-i\Im s/2}(t_0^{-1})S_{t_1}S_{t_0}\varphi(t_1,\,\cdot)\\ &= \chi_{-i\Im s/2}(t_1)S_{t_1}\varphi(t_1,\,\cdot),\end{aligned}$$

which means that we can define a function f' on $\mathcal{O}_\varepsilon^* \times W$ by

$$f'(\xi,\,\cdot) := \chi_{-i\Im s/2}(t)S_t\varphi(t,\,\cdot) \quad (\xi \in \mathcal{O}_\varepsilon^*), \tag{4.19}$$

from which it follows that $\pi_s(n(x_0,u_0))F = \Phi_s\check{\pi}_s(n(x_0,u_0))f$. ■

Proposition 4.11 together with (4.13) states that Φ_s induces the direct integral decomposition of $\pi_s|_{N(Q)}$:

$$\pi_s|_{N(Q)} \simeq \int_{\mathcal{O}_\varepsilon^*}^{\oplus} \tau_\xi|_{N(Q)}\, d\mathcal{R}^*_{\Re s^*}(\xi).$$

Let $G_{(\varepsilon)} := N(Q) \rtimes H_{(\varepsilon)}$.

THEOREM 4.12. *When $s \in \mathfrak{X}(\varepsilon)$, one has $\pi_s \simeq \mathrm{Ind}_{G_{(\varepsilon)}}^G \tau_{(\varepsilon)} \otimes \chi_{-i\Im s/2}$.*

Proof. Noting that $G/G_{(\varepsilon)} \simeq H/H_{(\varepsilon)} \simeq \mathcal{O}_\varepsilon^*$, we define a measure ρ_s on $G/G_{(\varepsilon)}$ by

$$d\rho_s(gG_{(\varepsilon)}) := d\mathcal{R}^*_{\Re s^*}(t\cdot E_\varepsilon^*) \quad (g = tn \in G,\ t \in H,\ n \in N(Q)). \tag{4.14}$$

The measure ρ_s is relatively invariant under G. In fact, if $g_0 = t_0n_0 \in G$ ($t_0 \in H$, $n_0 \in N(Q)$), then $g_0g = t_0t(t^{-1}n_0t)n$, so that we get by Proposition 2.4 (i)

$$\begin{aligned} d\rho_s(g_0gG_{(\varepsilon)}) &= d\rho_s(t_0tG_{(\varepsilon)}) = d\mathcal{R}^*_{\Re s^*}(t_0t\cdot E_\varepsilon^*) \\ &= \chi_{-\Re s}(t_0)d\mathcal{R}^*_{\Re s^*}(t\cdot E_\varepsilon^*) = \chi_{-\Re s}(g_0)d\rho_s(gG_{(\varepsilon)}). \end{aligned} \tag{4.15}$$

The induced representation $\pi := \mathrm{Ind}_{G_{(\varepsilon)}}^G \tau_{(\varepsilon)} \otimes \chi_{-i\Im s/2}$ is realized on the space $\mathcal{L}$ of equivalence classes of measurable functions φ on $G\times W$ such that

(a) $\varphi(g,\cdot) \in \mathcal{F}_{(\varepsilon)}$ (a.a. $g \in G$),

(b) $\varphi(gg_1,\cdot) = \chi_{-i\Im s/2}(g_1^{-1})\tau_{(\varepsilon)}(g_1^{-1})\varphi(g,\cdot)$ $(g_1 \in G_{(\varepsilon)})$,

(c) $\|\varphi\|^2 := \int_{G/G_{(\varepsilon)}} \|\varphi(g,\cdot)\|^2_{(\varepsilon)} d\rho_s(gG_{(\varepsilon)}) < \infty$.

Then $\mathcal{L}$ is a Hilbert space (cf. [**18**, p. 374]). The representation operators for π are given by

$$\pi(g_0)\varphi(g,\cdot) := \chi_{\Re s/2}(g_0)\varphi(g_0^{-1}g,\cdot) \quad (g_0 \in G). \tag{4.16}$$

For $f \in \mathcal{L}_s(\mathcal{O}_\varepsilon^*\times W)$, let φ' be a function on $G\times W$ defined by

$$\varphi'(tn(x,u),u') := \chi_{i\Im s/2}(t)e^{i\langle x,E_\varepsilon^*\rangle - Q_{(\varepsilon)}(u',u) - Q_{(\varepsilon)}(u,u)/2} f(t\cdot E_\varepsilon^*, t\cdot u'),$$

that is, with $n = n(x,u)$,

$$\varphi'(tn,\cdot) := \chi_{-i\Im s/2}(t^{-1})\tau_{(\varepsilon)}(n^{-1})S_{t^{-1}}f(t\cdot E_\varepsilon^*,\cdot) \in \mathcal{F}_{(\varepsilon)} \tag{4.17}$$

(note that Lemma 3.4 tells us $S_{t^{-1}}f(t\cdot E_\varepsilon^*,\cdot) \in \mathcal{F}_{(\varepsilon)}$). Then $\varphi' \in \mathcal{L}$. In fact, the condition (a) is satisfied. To show (b), we first observe the case $g_1 = t_1 \in H_{(\varepsilon)}$. Noting that $t_1\cdot E_\varepsilon^* = E_\varepsilon^*$, we have by (4.17)

$$\begin{aligned} \varphi'(tnt_1,\cdot) &= \varphi'(tt_1n_1,\cdot) \\ &= \chi_{-i\Im s/2}(t_1^{-1}t^{-1})\tau_{(\varepsilon)}(n_1^{-1})S_{t_1^{-1}}S_{t^{-1}}f(t\cdot E_\varepsilon^*,\cdot), \end{aligned}$$

Let $\tilde{\mathcal{L}}$ be the linear span of $\{k_{(z,u)}\}_{(z,u)\in D}$. By (4.1), (4.8) and (4.11) we see immediately that Φ_s is an isometric isomorphism from $\tilde{\mathcal{L}}$ onto $\tilde{\mathcal{H}}_s(D)$. If $f_0 \in \mathcal{L}_s(\mathcal{O}_\varepsilon^* \times W)$ is orthogonal to $\tilde{\mathcal{L}}$, then for any $(z,u) \in D$ we have by (4.10)

$$0 = (f_0|k_{(z,u)})_{\mathcal{L}_s} = \Phi_s f_0(z,u) = \int_{\mathcal{O}_\varepsilon^*} e^{i\langle z,\xi\rangle} f_0(\xi,u)\, d\mathcal{R}^*_{\Re s^*}(\xi),$$

so that $f_0 = 0$ by the injectivity of the Fourier-Laplace transform. Therefore $\tilde{\mathcal{L}}$ is dense in $\mathcal{L}_s(\mathcal{O}_\varepsilon^* \times W)$. This together with the fact that $\tilde{\mathcal{H}}_s(D)$ is dense in $\mathcal{H}_s(D)$ completes the proof. ■

Define a representation $\check{\pi}_s$ of $G = N(Q) \rtimes H$ on $\mathcal{L}_s(\mathcal{O}_\varepsilon^* \times W)$ by

$$\begin{aligned}
\check{\pi}_s(t_0)f(\xi,u) &:= \chi_{\bar{s}/2}(t_0) f(t_0^{-1}\cdot\xi, t_0^{-1}\cdot u) && (t_0 \in H),\\
\check{\pi}_s(n(x_0,u_0))f(\xi,u) &:= e^{-i\langle x_0,\xi\rangle + Q_\xi(u,u_0) - Q_\xi(u_0,u_0)/2} f(\xi, u-u_0) && (n(x_0,u_0) \in N(Q)),
\end{aligned}$$

where $\bar{s} := (\bar{s}_1, \dots, \bar{s}_r)$. Bearing (3.9) and (3.14) in mind, we rewrite these formulas as follows:

$$\check{\pi}_s(t_0)f(\xi,\cdot) := \chi_{\bar{s}/2}(t_0) S_{t_0} f(t_0^{-1}\cdot\xi,\cdot) \qquad (t_0 \in H), \tag{4.12}$$

$$\check{\pi}_s(n(x_0,u_0))f(\xi,\cdot) := \tau_\xi(n(x_0,u_0))f(\xi,\cdot) \qquad (n(x_0,u_0) \in N(Q)). \tag{4.13}$$

PROPOSITION 4.11. *Suppose that $s \in \mathfrak{X}(\varepsilon)$. One has*

$$\check{\pi}_s(g_0) = \Phi_s^{-1}\pi_s(g_0)\Phi_s \quad (g_0 \in G).$$

In other words, $(\check{\pi}_s, \mathcal{L}_s(\mathcal{O}_\varepsilon^* \times W))$ is an IUR of G equivalent to $(\pi_s, \mathcal{H}_s(D))$, and Φ_s is a unitary intertwining operator between $\check{\pi}_s$ and π_s.

Proof. Take $f \in \mathcal{L}_s(\mathcal{O}_\varepsilon^* \times W)$ and set $F := \Phi_s f \in \mathcal{H}_s(D)$. Then it suffices to show that $\pi_s(g_0)F = \Phi_s \check{\pi}_s(g_0) f$. First we observe for $t_0 \in H$

$$\begin{aligned}
\pi_s(t_0)F(z,u) &= \chi_{-s/2}(t_0) \int_{\mathcal{O}_\varepsilon^*} e^{i\langle t_0^{-1}\cdot z,\xi\rangle} f(\xi, t_0^{-1}\cdot u)\, d\mathcal{R}^*_{\Re s^*}(\xi)\\
&= \chi_{-s/2}(t_0) \int_{\mathcal{O}_\varepsilon^*} e^{i\langle z,\xi'\rangle} f(t_0^{-1}\cdot\xi', t_0^{-1}\cdot u)\, d\mathcal{R}^*_{\Re s^*}(t_0^{-1}\cdot\xi').
\end{aligned}$$

Since Proposition 2.4 tells us that $d\mathcal{R}^*_{\Re s^*}(t_0^{-1}\cdot\xi') = \chi_{\Re s}(t_0)\, d\mathcal{R}^*_{\Re s^*}(\xi')$, we obtain $\pi_s(t_0)F = \Phi_s\check{\pi}_s(t_0)f$. Next, for $n(x_0,u_0) \in N(Q)$, we have

$$\begin{aligned}
\pi_s(n(x_0,u_0))F(z,u) &= \int_{\mathcal{O}_\varepsilon^*} e^{i\langle z - x_0 - 2iQ(u,u_0) + iQ(u_0,u_0),\xi\rangle} f(\xi, u-u_0)\, d\mathcal{R}^*_{\Re s^*}(\xi)\\
&= \int_{\mathcal{O}_\varepsilon^*} e^{i\langle z,\xi\rangle} e^{-i\langle x_0,\xi\rangle + Q_\xi(u,u_0) - Q_\xi(u_0,u_0)/2} f(\xi, u-u_0)\, d\mathcal{R}^*_{\Re s^*}(\xi),
\end{aligned}$$

which completes the proof. ■

The following theorem is a direct consequence of Proposition 4.4.

THEOREM 4.9. *The unitary representation $(\pi_s, \mathcal{H}_s(D))$ of G is irreducible provided $\mathcal{H}_s(D) \neq \{0\}$.*

By Remark 4.2, the Hilbert space $\mathcal{H}_s(D)$ is the completion of the space $\tilde{\mathcal{H}}_s(D)$ spanned by $\{K^s_{(z,u)}\}_{(z,u)\in D}$. Now we give a more concrete description of $\mathcal{H}_s(D)$ by making use of the Fourier-Laplace transform. For $s \in \mathfrak{X}(\varepsilon)$, let $\mathcal{L}_s(\mathcal{O}^*_\varepsilon \times W)$ denote the Hilbert space $\mathcal{L}(\mathcal{O}^*_\varepsilon \times W; \mathcal{R}^*_{\Re s^*})$ (see Proposition 3.6). For $(z_0, u_0) \in D$, we set

$$k_{(z_0,u_0)}(\xi, u) := e^{-i\langle \bar{z}_0, \xi\rangle} e^{Q_\xi(u,u_0)} \qquad (u \in W, \xi \in \overline{\Omega^*}).$$

Then $k_{(z_0,u_0)}(\xi, \cdot) \in \mathcal{F}_\xi$ for all $\xi \in \overline{\Omega^*}$. We see from (4.7) that

$$\int_{\mathcal{O}^*_\varepsilon} \|k_{(z_0,u_0)}(\xi, \cdot)\|^2_\xi \, d\mathcal{R}^*_{\Re s^*}(\xi) = K^s((z_0, u_0), (z_0, u_0)) < \infty.$$

Thus $k_{(z_0,u_0)} \in \mathcal{L}_s(\mathcal{O}^*_\varepsilon \times W)$. Similarly we have

$$(k_{(z_1,u_1)} | k_{(z_0,u_0)})_{\mathcal{L}_s} = K^s((z_0, u_0), (z_1, u_1)) \quad ((z_1, u_1) \in D). \tag{4.8}$$

For $f \in \mathcal{L}_s(\mathcal{O}^*_\varepsilon \times W)$, let $\Phi_s f$ be the function on D given by

$$\Phi_s f(z, u) := (f | k_{(z,u)})_{\mathcal{L}_s} \quad ((z, u) \in D).$$

THEOREM 4.10. *Assume that $s \in \mathfrak{X}(\varepsilon)$. Then $\Phi_s f$ is expressed as the absolutely converge integral*

$$\Phi_s f(z, u) = \int_{\mathcal{O}^*_\varepsilon} e^{i\langle z,\xi\rangle} f(\xi, u) \, d\mathcal{R}^*_{\Re s^*}(\xi), \tag{4.9}$$

and Φ_s induces a unitary isomorphism from $\mathcal{L}_s(\mathcal{O}^*_\varepsilon \times W)$ onto $\mathcal{H}_s(D)$.

Proof. Since $f(\xi, \cdot) \in \mathcal{F}_\xi$ for almost all $\xi \in \mathcal{O}^*_\varepsilon$, Fubini's theorem and Lemma 3.3 tell us that

$$\begin{aligned}
\Phi_s f(z, u) &= (f | k_{(z,u)})_{\mathcal{L}_s} \\
&= \int_{\mathcal{O}^*_\varepsilon} e^{i\langle z,\xi\rangle} \left\{ \int_{M_\xi} f(\xi, u') \overline{e^{Q_\xi(u',u)}} e^{-Q_\xi(u',u')} dm_\xi([u']_\xi) \right\} d\mathcal{R}^*_{\Re s^*}(\xi) \qquad (4.10) \\
&= \int_{\mathcal{O}^*_\varepsilon} e^{i\langle z,\xi\rangle} f(\xi, u) \, d\mathcal{R}^*_{\Re s^*}(\xi).
\end{aligned}$$

Next, we see from (4.8) that

$$\Phi_s k_{(z_1,u_1)} = K^s_{(z_1,u_1)}. \tag{4.11}$$

Now we state the necessary and sufficient condition for the non-vanishing of $\mathcal{H}_s(D)$. For $\varepsilon \in \{0,1\}^r$, let $\mathfrak{X}(\varepsilon)$ (resp. $\mathfrak{X}$) be the set of $s \in \mathbb{C}^r$ such that $\Re s^* = (\Re s_r, \dots, \Re s_1)$ belongs to $\Xi^*(\varepsilon^*)$ (resp. Ξ^*). Then we have by Theorem 2.3 (iii)

$$\mathfrak{X}(\varepsilon) := \{\, s \in \mathbb{C}^r \,;\, \Re s_k > q_k(\varepsilon)/2 \ \ (\text{if } \varepsilon_k = 1), \quad \Re s_k = q_k(\varepsilon)/2 \ \ (\text{if } \varepsilon_k = 0) \,\} \tag{4.6}$$

and $\mathfrak{X} = \bigsqcup_{\varepsilon \in \{0,1\}^r} \mathfrak{X}(\varepsilon)$.

THEOREM 4.8. *Non-zero $\mathcal{H}_s(D)$ exists if and only if s belongs to $\mathfrak{X}$.*

Proof. We first show the "only if" part. By Proposition 4.6, we have

$$\Delta_{-\Re s}(y + y') = K^s((iy, 0), (iy', 0)) \quad (y, y' \in \Omega),$$

so that $\Delta_{-\Re s}$ is of positive type. Moreover (1.14) tells us that $\Delta_{-\Re s}$ satisfies the condition (4.5) for $\kappa = -|\Re s| \in \mathbb{R}$. On the other hand, Proposition 2.4 says that

$$\Delta_{-\Re s}(y) = \langle \mathcal{R}^*_{\Re s^*}, e^{-\langle y, \cdot \rangle} \rangle.$$

Hence we see from Lemma 4.7 and the uniqueness of the Laplace transform that $\mathcal{R}^*_{\Re s^*}$ is a positive measure and thanks to Theorem 2.3 (iii), we conclude that $s \in \mathfrak{X}$.

Next we show the "if" part. Suppose that $s \in \mathfrak{X}(\varepsilon)$. We denote the right-hand side of (4.4) by $K((z,u),(z',u'))$. In view of Propositions 4.1 and 4.3, it suffices to show that K is a kernel function of positive type. Corollary 2.5 says that

$$K((z,u),(z',u')) = \langle \mathcal{R}^*_{\Re s^*}, e^{-\langle (z-\bar{z}')/i - 2Q(u,u'), \xi \rangle} \rangle.$$

Since Theorem 2.3 (iii) says that $\mathcal{R}^*_{\Re s^*}$ is a positive measure on the H-orbit $\mathcal{O}^*_\varepsilon$, the right-hand side is rewritten as

$$\int_{\mathcal{O}^*_\varepsilon} e^{i\langle z, \xi \rangle - i\langle \bar{z}', \xi \rangle + Q_\xi(u,u')} d\mathcal{R}^*_{\Re s^*}(\xi).$$

Using Lemma 3.3, we see that the integrand is equal to

$$e^{i\langle z, \xi \rangle - i\langle \bar{z}', \xi \rangle} (e^{Q_\xi(\cdot, u')} | e^{Q_\xi(\cdot, u)})_\xi = (e^{-i\langle \bar{z}', \xi \rangle} e^{Q_\xi(\cdot, u')} | e^{-i\langle \bar{z}, \xi \rangle} e^{Q_\xi(\cdot, u)})_\xi.$$

Therefore, for $N \in \mathbb{N}$, $a_1, \dots, a_N \in \mathbb{C}$ and $(z_1, u_1), \dots, (z_N, u_N) \in D$ we have

$$\begin{aligned}
&\sum_{k,l=1}^N a_k \bar{a}_l K((z_l, u_l), (z_k, u_k)) \\
&\quad = \sum_{k,l=1}^N a_k \bar{a}_l \int_{\mathcal{O}^*_\varepsilon} (e^{-i\langle \bar{z}_k, \xi \rangle} e^{Q_\xi(\cdot, u_k)} | e^{-i\langle \bar{z}_l, \xi \rangle} e^{Q_\xi(\cdot, u_l)})_\xi \, d\mathcal{R}^*_{\Re s^*}(\xi) \qquad (4.7) \\
&\quad = \int_{\mathcal{O}^*_\varepsilon} \| \textstyle\sum_{k=1}^N a_k e^{-i\langle \bar{z}_k, \xi \rangle} e^{Q_\xi(\cdot, u_k)} \|^2_\xi \, d\mathcal{R}^*_{\Re s^*}(\xi) \geq 0,
\end{aligned}$$

Let $\mathcal{H}(D)$ be the space of all holomorphic functions on D and π_s ($s \in \mathbb{C}^r$) be the representation of G given by

$$\pi_s(g)F(z,u) := \chi_{-s/2}(g)F(g^{-1}\cdot(z,u)) \quad (g \in G,\ F \in \mathcal{H}(D),\ (z,u) \in D).$$

DEFINITION 4.5. For $s \in \mathbb{C}^r$, let $\mathcal{H}_s(D)$ be a subspace of $\mathcal{H}(D)$ such that

(i) $\mathcal{H}_s(D)$ has a Hilbert space structure with reproducing kernel K^s,

(ii) $(\pi_s, \mathcal{H}_s(D))$ is a unitary representation of G.

We note that non-zero $\mathcal{H}_s(D)$ is unique, though it might not exist for some s.

PROPOSITION 4.6. *Suppose that $\mathcal{H}_s(D) \neq \{0\}$ and that the norm is normalized in such a way that $\|K^s_{(iE,0)}\|^2 = 2^{-|\Re s|}$. Then the reproducing kernel K^s is given by*

$$K^s((z,u),(z',u')) = \Delta_{-\Re s}((z-\bar{z}')/i - 2Q(u,u')) \quad ((z,u),(z',u') \in D), \tag{4.4}$$

where $\Delta_{-\Re s}$ is a holomorphic function on $\Omega + iV$ defined as in Corollary 2.5.

Proof. Take $t \in H$ for which $t\cdot E = \Im z - Q(u,u) \in \Omega$. Then we have $(z,u) = n(\Re z, u)t\cdot(iE,0)$, so that Proposition 4.3 tells us that

$$\begin{aligned} K^s((z,u),(z,u)) &= |\chi_{-s/2}(n(\Re z,u)t)|^2 K^s((iE,0),(iE,0)) = 2^{-|\Re s|}\chi_{-\Re s}(t) \\ &= 2^{-|\Re s|}\Delta_{-\Re s}(\Im z - Q(u,u)) \\ &= \Delta_{-\Re s}((z-\bar{z})/i - 2Q(u,u)), \end{aligned}$$

where the last equality follows from (1.14). Hence (4.4) is verified by uniqueness theorem. ∎

For a function ψ on Ω, set

$$K_\psi(y,y') = \psi(y+y') \quad (y,y' \in \Omega).$$

We call ψ *a function of positive type* if K_ψ is a kernel function of positive type on $\Omega\times\Omega$. Note that ψ is necessarily non-negative in order to be of positive type. The following lemma is known ([**5**, Satz 5.1]).

LEMMA 4.7. *Suppose that a continuous function ψ of positive type satisfies the condition*

$$\psi(\lambda x) = \lambda^\kappa \psi(x) \quad (\lambda > 0,\ x \in \Omega) \tag{4.5}$$

for some $\kappa \in \mathbb{R}$. Then there exists a positive measure ν on $\overline{\Omega^*}$ such that

$$\psi(y) = \int_{\overline{\Omega^*}} e^{-\langle y,\xi\rangle}\, d\nu(\xi) \quad (y \in \Omega).$$

Assume that a group G_0 acts on X. Let χ be a one-dimensional representation of G_0 and π_χ the action of G_0 on functions f on X given by

$$\pi_\chi(g)f(x) := \chi(g)f(g^{-1}\cdot x) \quad (g \in G_0, x \in X). \tag{4.2}$$

PROPOSITION 4.3. *Let $\mathcal{H}$ be a Hilbert space with reproducing kernel K. Then π_χ preserves $\mathcal{H}$ and defines a unitary representation of G_0 if and only if K has the following relative G_0-invariance:*

$$K(g\cdot x, g\cdot y) = |\chi(g)|^2 K(x,y) \quad (x, y \in X, g \in G_0). \tag{4.3}$$

Proof. First we prove the "if" part. For a linear combination $f = \sum_{k=1}^N a_k K_{x_k} \in \mathcal{H}$, we have by (4.2) and (4.3)

$$\pi_\chi(g)f(x) = \sum_{k=1}^N a_k \chi(g) K_{x_k}(g^{-1}\cdot x) = \sum_{k=1}^N a_k \overline{\chi(g)}^{-1} K_{g\cdot x_k}(x),$$

so that we get by (4.1) and (4.3)

$$\|\pi_\chi(g)f\|^2 = \sum_{k,l=1}^N a_k \bar{a}_l |\chi(g)|^{-2} K(g\cdot x_l, g\cdot x_k) = \sum_{k,l=1}^N a_k \bar{a}_l K(x_l, x_k) = \|f\|^2.$$

Since the space of such f is dense in $\mathcal{H}$, the operator $\pi_\chi(g)$ preserves $\mathcal{H}$ and is unitary on $\mathcal{H}$.

Next we show the "only if" part. For $f \in \mathcal{H}$, we have by the unitarity of π_χ

$$\pi_\chi(g)f(g\cdot y) = (\pi_\chi(g)f | K_{g\cdot y}) = (f | \pi_\chi(g^{-1})K_{g\cdot y}),$$

and by (4.2)

$$\pi_\chi(g)f(g\cdot y) = \chi(g)f(y) = (f | \overline{\chi(g)}K_y).$$

Thus we get $\pi_\chi(g^{-1})K_{g\cdot y}(x) = \overline{\chi(g)}K_y(x)$, that is,

$$\chi(g)^{-1}K(g\cdot x, g\cdot y) = \overline{\chi(g)}K(x,y),$$

whence (4.3) follows. ∎

The following proposition follows from Kunze's theorem [**10**].

PROPOSITION 4.4. *Let $(\pi_\chi, \mathcal{H})$ be the unitary representation of G_0 as in Proposition* 4.3. *Assume that X is a complex domain and that $\mathcal{H}$ consists of holomorphic functions on X. If G_0 acts on X transitively, then $(\pi_\chi, \mathcal{H})$ is irreducible.*

We apply the preceding propositions to our split solvable group G and function spaces on D. Recall the one-dimensional representation χ_s of G defined by (1.12).

that is, $f(t\cdot E_\varepsilon^*,\,\cdot) := S_t\varphi(t\cdot E_\varepsilon^*,\,\cdot) \in \mathcal{F}_{t\cdot E_\varepsilon^*}$. This definition is valid because Lemma 3.4 tells us that $S_t\varphi(t\cdot E_\varepsilon^*,\,\cdot) = S_{t'}\varphi(t'\cdot E_\varepsilon^*,\,\cdot)$ when $t\cdot E_\varepsilon^* = t'\cdot E_\varepsilon^*$. Lemma 3.4 also implies that f belongs to $\mathcal{L}(\mathcal{O}_\varepsilon^*\times W;\nu)$ and that the map $\varphi\mapsto f$ is an isometry. This map is surjective, because the inverse map $f\mapsto\varphi$ is given by

$$\varphi(t\cdot E_\varepsilon^*,\cdot) := S_{t^{-1}}f(t\cdot E_\varepsilon^*,\cdot) \in \mathcal{F}_{(\varepsilon)}.$$

Hence $\mathcal{L}(\mathcal{O}_\varepsilon^*\times W;\nu)$ is a Hilbert space. ■

4. REPRESENTATIONS OF G REALIZED ON SPACES OF HOLOMORPHIC FUNCTIONS ON D

We begin this section with general propositions about reproducing kernels of Hilbert spaces. Let X be a set and K a function on $X\times X$. We say that K is a *kernel function of positive type* if the matrix $(K(x_k,x_l))_{k,l=1}^N$ is a positive semi-definite Hermitian matrix for any $N\in\mathbb{N}$ and $x_1,\dots,x_N\in X$. We put $K_x := K(\cdot,x)$. It is easily verified that reproducing kernels are kernel functions of positive type. Conversely, the following proposition is known.

PROPOSITION 4.1. (i) *If K is a kernel function of positive type, then K is the reproducing kernel of a Hilbert space $\mathcal{H}$ of functions on X.*

(ii) *If, in addition, X is a complex domain and K is holomorphic in the first variable and anti-holomorphic in the second, then $\mathcal{H}$ consists of holomorphic functions on X.*

The point of the proof is that the space $\tilde{\mathcal{H}}$ of linear combinations $\sum_{k=1}^M a_kK_{x_k}$ $(M\in\mathbb{N}, a_k\in\mathbb{C}, x_k\in X)$ endowed with inner product

$$\Big(\sum_{k=1}^M a_kK_{x_k}\Big|\sum_{l=1}^N b_lK_{y_l}\Big) := \sum_{k=1}^M\sum_{l=1}^N a_k\bar{b}_lK(y_l,x_k) \quad (N\in\mathbb{N}, b_l\in\mathbb{C}, y_l\in X) \tag{4.1}$$

is a pre-Hilbert space with the reproducing property

$$(f|K_x) = f(x) \qquad (f\in\tilde{\mathcal{H}},\, x\in X).$$

The space $\mathcal{H}$ is obtained as the completion of $\tilde{\mathcal{H}}$. See [**6**, Theorem IX.2.7] for details.

Remark 4.2. If a Hilbert space $\mathcal{H}$ has the reproducing kernel K, then (4.1) holds and $\mathcal{H}$ is "reproduced" from K as described above. In particular, the linear span $\tilde{\mathcal{H}}$ of $\{K_x\}_{x\in X}$ is dense in $\mathcal{H}$.

(b) $\|f\|^2 := \int_{\mathcal{O}_\varepsilon^*} \|f(\xi,\cdot)\|_\xi^2 \, d\nu(\xi) < \infty$.

Then $\mathcal{L}(\mathcal{O}_\varepsilon^* \times W; \nu)$ is a Hilbert space unitarily isomorphic to L.

The assertions might be intuitively clear. However we include a proof here because of a certain delicacy on the arguments concerning exceptions for null sets.

Proof. (i) To show that L is a Hilbert space, we have only to prove that a Cauchy sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ in L has a limit in L. Throughout the proof of (i), we regard functions in $\mathcal{F}_{(\varepsilon)}$ as functions on the quotient space $M_{(\varepsilon)}$. Similarly, L is regarded as a function space on $\mathcal{O}_\varepsilon^* \times M_{(\varepsilon)}$. Then the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ has a limit $\tilde{\varphi}$ in $L^2(\mathcal{O}_\varepsilon^* \times M_{(\varepsilon)}, d\nu \otimes d\sigma)$, where σ is the Gaussian measure on $M_{(\varepsilon)}$ given by $d\sigma([u]_{(\varepsilon)}) := e^{-Q_{(\varepsilon)}(u,u)} dm_{E_\varepsilon^*}([u]_{(\varepsilon)})$ (we write $[u]_{(\varepsilon)}$ for $[u]_{E_\varepsilon^*}$). We take a subsequence $\{\varphi_{n(i)}\}_{i\in\mathbb{N}}$ such that $\varphi_{n(i)}(\xi,\cdot) \to \tilde{\varphi}(\xi,\cdot)$ in $L^2(M_{(\varepsilon)}, d\sigma)$ as $i \to \infty$ for almost all $\xi \in \mathcal{O}_\varepsilon^*$. Then $\{\varphi_{n(i)}(\xi,\cdot)\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{F}_\xi$, so that the limit $\varphi_\xi \in \mathcal{F}_\xi$ exists. Clearly φ_ξ is equal to $\tilde{\varphi}(\xi,\cdot)$ as an element of $L^2(M_{(\varepsilon)}, d\sigma)$. Therefore, for $[u]_{(\varepsilon)} \in M_{(\varepsilon)}$, Lemma 3.3 tells us that

$$\begin{aligned}\varphi_\xi([u]_{(\varepsilon)}) &= \int_{M_{(\varepsilon)}} \varphi_\xi([u']_{(\varepsilon)}) e^{Q_{(\varepsilon)}(u,u')} d\sigma([u']_{(\varepsilon)}) \\ &= \int_{M_{(\varepsilon)}} \tilde{\varphi}(\xi,[u']_{(\varepsilon)}) e^{Q_{(\varepsilon)}(u,u')} d\sigma([u']_{(\varepsilon)}).\end{aligned}$$

Let $\varphi(\xi,[u]_{(\varepsilon)})$ denote the last term. Then φ is a measurable function on $\mathcal{O}_\varepsilon^* \times M_{(\varepsilon)}$ satisfying the condition (a) in (i) and equals $\tilde{\varphi}$ for almost everywhere in $\mathcal{O}_\varepsilon^* \times M_{(\varepsilon)}$. Thus φ is nothing but the limit of $\{\varphi_n\}_{n\in\mathbb{N}}$ in L, so that L is a Hilbert space. Since Φ is an isometry by definition, it remains to show that Φ is surjective. Let φ_0 be an element of L such that $(\varphi_0|\psi)_L = 0$ for all $\psi \in \mathrm{Image}\,\Phi$. Take $\psi(\xi,u) := \phi(\xi) e^{Q_{(\varepsilon)}(u,u_0)}$ with $\phi \in C_c(\mathcal{O}_\varepsilon^*)$ and $[u_0]_{(\varepsilon)} \in M_{(\varepsilon)}$. Then using Fubini's theorem and Lemma 3.3, we have

$$\begin{aligned}0 &= \int_{\mathcal{O}_\varepsilon^*} \overline{\phi(\xi)} \int_{M_{(\varepsilon)}} \varphi_0(\xi,[u]_{(\varepsilon)}) \overline{e^{Q_{(\varepsilon)}(u,u_0)}} \, d\sigma([u]_{(\varepsilon)}) \, d\nu(\xi) \\ &= \int_{\mathcal{O}_\varepsilon^*} \overline{\phi(\xi)} \varphi_0(\xi,[u_0]_{(\varepsilon)}) \, d\nu(\xi).\end{aligned}$$

Since $\phi \in C_c(\mathcal{O}_{\varepsilon *}^*)$ is arbitrary, we obtain $\varphi_0(\cdot,[u_0]_{(\varepsilon)}) = 0$ (a.e. $\mathcal{O}_\varepsilon^*$) for all $[u_0]_{(\varepsilon)} \in M_{(\varepsilon)}$. Hence $\varphi_0 = 0$ (a.e. $\mathcal{O}_\varepsilon^* \times M_{(\varepsilon)}$) and (i) is verified.

(ii) Let φ be an element of L (here we consider φ as a function on $\mathcal{O}_\varepsilon^* \times W$) and put

$$f(t \cdot E_\varepsilon^*, u) := \varphi(t \cdot E_\varepsilon^*, t^{-1} \cdot u) \quad (t \in H, u \in W),$$

Thus p_ξ is a group homomorphism. Therefore we obtain an IUR $\tau_\xi := \tilde{\tau}_\xi \circ p_\xi$ of G_ξ realized on $\mathcal{F}_\xi$:

$$\tau_\xi(n(x_0,u_0)t_0)f(u) := e^{-i\langle x_0,\xi\rangle + Q_\xi(u,u_0) - Q_\xi(u_0,u_0)/2} f(u-u_0) \qquad (3.14)$$
$$(f \in \mathcal{F}_\xi,\ n(x_0,u_0) \in N(Q),\ t_0 \in H).$$

For the elements $n(x,0)$ in the center of $N(Q)$, the operator $\tau_\xi(n(x,0))$ is the scalar multiplication by $e^{-i\langle x,\xi\rangle}$, so that τ_ξ and $\tau_{\xi'}$ are not equivalent when $\xi \neq \xi'$. Furthermore the following relation holds.

LEMMA 3.5. *As operators on $\mathcal{F}_{t\cdot\xi}$,*

$$S_t \circ \tau_\xi(n(x,u)) \circ S_{t^{-1}} = \tau_{t\cdot\xi}(n(t\cdot x, t\cdot u)) \quad (n(x,u) \in N(Q)).$$

Proof. Let f be an element of $\mathcal{F}_\xi$. Using (3.2) and (3.14), we have for $u' \in W$

$$\begin{aligned} S_t\tau_\xi(n(x,u))f(u') &= \tau_\xi(n(x,u))f(t^{-1}\cdot u') \\ &= e^{-i\langle x,\xi\rangle + Q_\xi(t^{-1}\cdot u',u) - Q_\xi(u,u)/2} f(t^{-1}\cdot u' - u) \\ &= e^{-i\langle t\cdot x, t\cdot\xi\rangle + Q_{t\cdot\xi}(u', t\cdot u) - Q_{t\cdot\xi}(t\cdot u, t\cdot u)/2} f(t^{-1}\cdot(u' - t\cdot u)) \\ &= \tau_{t\cdot\xi}(n(t\cdot x, t\cdot u))S_t f(u'), \end{aligned}$$

whence the lemma follows. ∎

Let $\varepsilon \in \{0,1\}^r$ and ν be a measure on the H-orbit $\mathcal{O}_\varepsilon^*$. We denote by $\mathcal{F}_{(\varepsilon)}$ the Fock space $\mathcal{F}_{E_\varepsilon^*}$ on $M_{(\varepsilon)} := M_{E_\varepsilon^*}$. In the following proposition, we shall give realizations of the Hilbert space tensor product $L^2(\mathcal{O}_\varepsilon^*,\, d\nu) \otimes \mathcal{F}_{(\varepsilon)}$ as well as the direct integral

$$\int_{\mathcal{O}_\varepsilon^*}^{\oplus} \mathcal{F}_\xi \, d\nu(\xi)$$

as concrete function spaces.

PROPOSITION 3.6. (i) *Let L be the space of equivalence classes of measurable functions φ on $\mathcal{O}_\varepsilon^* \times W$ such that*

(a) *$\varphi(\xi,\,\cdot) \in \mathcal{F}_{(\varepsilon)}$ for almost all $\xi \in \mathcal{O}_\varepsilon^*$,*

(b) *$\|\varphi\|^2 := \int_{\mathcal{O}_\varepsilon^*} \|\varphi(\xi,\,\cdot)\|_{(\varepsilon)}^2 \, d\nu(\xi) < \infty$.*

Then L is a Hilbert space and the map $\Phi : L^2(\mathcal{O}_\varepsilon^*,\, d\nu) \otimes \mathcal{F}_{(\varepsilon)} \ni \phi\otimes\psi \mapsto \phi\psi \in L$ is a unitary isomorphism.

(ii) *Let $\mathcal{L}(\mathcal{O}_\varepsilon^* \times W; \nu)$ be the space of equivalence classes of measurable functions f on $\mathcal{O}_\varepsilon^* \times W$ such that*

(a) *$f(\xi,\,\cdot) \in \mathcal{F}_\xi$ for almost all $\xi \in \mathcal{O}_\varepsilon^*$,*

Similarly elements $[u']_{t\cdot\xi}$ of $M_{t\cdot\xi}$ are written as $[u']_{t\cdot\xi} = t\cdot[u]_\xi$ ($[u]_\xi \in M_\xi$). Then

$$dm_{t\cdot\xi}(t\cdot[u]_\xi) = dm_\xi([u]_\xi) \tag{3.10}$$

as measures on M_ξ. In fact, we have by (3.7) and (3.2)

$$1 = \int_{M_{t\cdot\xi}} e^{-Q_{t\cdot\xi}(u',u')} dm_{t\cdot\xi}([u']_{t\cdot\xi}) = \int_{M_\xi} e^{-Q_\xi(u,u)} dm_{t\cdot\xi}(t\cdot[u]_\xi),$$

so that the normalization rule (3.7) leads us to (3.10). Hence

$$\begin{aligned}\|S_t f\|^2_{t\cdot\xi} &= \int_{M_{t\cdot\xi}} |S_t f(u')|^2 e^{-Q_{t\cdot\xi}(u',u')} dm_{t\cdot\xi}([u']_{t\cdot\xi}) \\ &= \int_{M_\xi} |f(u)|^2 e^{-Q_\xi(u,u)} dm_{t\cdot\xi}(t\cdot[u]_\xi) = \|f\|^2_\xi.\end{aligned}$$

The rest of the claims follows from Lemma 3.1. ■

We define the Heisenberg groups to be $\mathrm{Heis}(\xi) := \mathbb{R}\rtimes M_\xi$ with multiplication rule

$$\begin{aligned}(a,[u]_\xi)\cdot(a',[u']_\xi) := (a+a'+\Im Q_\xi(u,u'),\ [u]_\xi+[u']_\xi) \\ (a,a'\in\mathbb{R},\ u,u'\in W).\end{aligned} \tag{3.11}$$

Then an IUR $\tilde\tau_\xi$ of $\mathrm{Heis}(\xi)$, called *the Fock representation*, is realized on the Fock space $\mathcal{F}_\xi$ by the formula

$$\tilde\tau_\xi(a_0,[u_0]_\xi)f(u) := e^{-ia_0+Q_\xi(u,u_0)-Q_\xi(u_0,u_0)/2} f(u-u_0) \quad (f\in\mathcal{F}_\xi). \tag{3.12}$$

Set $G_\xi := N(Q)\rtimes H_\xi$ and define a map $p_\xi : G_\xi \to \mathrm{Heis}(\xi)$ by

$$p_\xi(n(x,u)t) := (\langle x,\xi\rangle, [u]_\xi). \tag{3.13}$$

Then we have for $t\in H_\xi$

$$\begin{aligned}p_\xi(t\,n(x,u)t^{-1}) &= p_\xi(n(t\cdot x, t\cdot u)) = (\langle x, t^{-1}\cdot\xi\rangle,\ t\cdot[u]_\xi) \\ &= (\langle x,\xi\rangle,[u]_\xi) = p_\xi(n(x,u))\end{aligned}$$

(we have used Lemma 3.1 for the third equality) and we get by (1.11), (3.1) and (3.11),

$$\begin{aligned}p_\xi(n(x,u)\,n(x',u')) &= p_\xi(n(x+x'+2\Im Q(u,u'),u+u')) \\ &= (\langle x,\xi\rangle+\langle x',\xi\rangle+\Im Q_\xi(u,u'),\ [u]_\xi+[u']_\xi) \\ &= (\langle x,\xi\rangle,[u]_\xi)\cdot(\langle x',\xi\rangle,[u']_\xi) \\ &= p_\xi(n(x,u))\cdot p_\xi(n(x',u')).\end{aligned}$$

Therefore for $t = \exp T$ $(T \in \mathfrak{h}_{(\varepsilon)})$ and $u \in W$, we have

$$t \cdot u - u = \sum_{n \geq 1} \frac{1}{n!} \mathrm{ad}(T)^n u \in N_{(\varepsilon)}. \tag{3.6}$$

LEMMA 3.1. *Let $\xi \in \overline{\Omega^*}$, and $t, t' \in H$ such that $t \cdot \xi = t' \cdot \xi$. Then*

$$t \cdot [u]_\xi = t' \cdot [u]_\xi \in M_{t \cdot \xi}$$

for all $[u]_\xi \in M_\xi$. In particular, if $t \cdot \xi = \xi$, then $t \cdot [u]_\xi = [u]_\xi$.

Proof. Since Theorem 2.3 (i) tells us that there exist $\varepsilon \in \{0,1\}^r$ and $t_0 \in H$ for which $\xi = t_0 \cdot E_\varepsilon^*$, it suffices to prove the lemma for $\xi = E_\varepsilon^*$. Then $t_1 := (t')^{-1} t$ belongs to $H_{(\varepsilon)}$ and (3.6) says that $t_1 \cdot [u]_{E_\varepsilon^*} = [u]_{E_\varepsilon^*}$. Therefore $t \cdot [u]_{E_\varepsilon^*} = t' t_1 \cdot [u]_{E_\varepsilon^*} = t' \cdot [u]_{E_\varepsilon^*}$. ∎

DEFINITION 3.2. Let $\mathcal{F}_\xi$ be the space of holomorphic functions f on W such that

(i) $f(u+v) = f(u)$ for all $u \in W$ and $v \in N_\xi$,

(ii) $\|f\|_\xi^2 := \int_{M_\xi} |f(u)|^2 e^{-Q_\xi(u,u)}\, dm_\xi([u]_\xi) < \infty$,

where dm_ξ is the Lebesgue measure on M_ξ normalized in such a way that

$$\int_{M_\xi} e^{-Q_\xi(u,u)} dm_\xi([u]_\xi) = 1. \tag{3.7}$$

The condition (i) says that each $f \in \mathcal{F}_\xi$ can be regarded as a holomorphic function on M_ξ, so that $\mathcal{F}_\xi$ is the Fock space on M_ξ. The following lemma follows from the general theory of Fock spaces ([**1**]).

LEMMA 3.3. *The function $e^{Q_\xi(\cdot,\cdot)}$ is the reproducing kernel of $\mathcal{F}_\xi$. In particular,*

$$\int_{M_\xi} e^{Q_\xi(u,u_2)} \overline{e^{Q_\xi(u,u_1)}} e^{-Q_\xi(u,u)} dm_\xi([u]_\xi) = e^{Q_\xi(u_1,u_2)} \quad (u_1, u_2 \in W). \tag{3.8}$$

For $t \in H$ let S_t denote the translation by t given by

$$S_t \varphi(u) := \varphi(t^{-1} \cdot u) \quad (u \in W) \tag{3.9}$$

for functions φ on W.

LEMMA 3.4. *Let $\xi \in \overline{\Omega^*}$. For $t \in H$, the translation S_t $(t \in H)$ induces a unitary isomorphism from $\mathcal{F}_\xi$ to $\mathcal{F}_{t \cdot \xi}$. Moreover, if $t, t' \in H$ satisfies $t' \cdot \xi = t \cdot \xi$, then $S_{t'} = S_t$ on $\mathcal{F}_\xi$. In particular, S_t is the identity operator on $\mathcal{F}_\xi$ when $t \in H_\xi$.*

Proof. By (3.3), elements v' of $N_{t \cdot \xi}$ are of the form $v' = t \cdot v$ with $v \in N_\xi$. Then, if $f \in \mathcal{F}_\xi$,

$$S_t f(u + v') = f(t^{-1} \cdot u + v) = f(t^{-1} \cdot u) = S_t f(u).$$

COROLLARY 2.5. *The function Δ_s on Ω is analytically continued to a holomorphic function on $\Omega + iV$ by*

$$\Delta_s(y+ix) = \langle \mathcal{R}^*_{-s^*}, e^{-\langle y+ix,\,\cdot\,\rangle} \rangle \quad (y \in \Omega, x \in V).$$

3. FOCK SPACES ON W

From now on, we denote by $t \cdot \xi$ the element $\iota(t) \cdot \xi$ ($t \in H$, $\xi \in V^*$) for simplicity. For $\xi \in \overline{\Omega^*}$, let Q_ξ be the Hermitian form on W defined by

$$Q_\xi(u,u') := \langle 2Q(u,u'), \xi \rangle = \langle [ju,u'] + i[u,u'], \xi \rangle /2 \quad (u,u' \in W). \tag{3.1}$$

Since Q is Ω-positive, we see that $Q_\xi(u,u) \geq 0$ for all $u \in W$. Let N_ξ be the kernel of Q_ξ, that is, $N_\xi := \{\, u \in W \,;\, Q_\xi(u,u) = 0 \,\}$. Then, regarded as a function on the quotient space $M_\xi := W/N_\xi$, the form Q_ξ is positive definite. For $u \in W$, we denote by $[u]_\xi$ the element $u + N_\xi$ of M_ξ. Since (1.10) and (2.6) lead us to

$$Q_{t\cdot\xi}(t\cdot u, t\cdot u') = 2\langle t \cdot Q(u,u'), t\cdot\xi\rangle = 2\langle Q(u,u'),\xi\rangle = Q_\xi(u,u') \tag{3.2}$$

for $u, u' \in W$ and $t \in H$, we see that

$$N_{t\cdot\xi} = t \cdot N_\xi. \tag{3.3}$$

Thus the action of t induces an isomorphism

$$M_\xi \ni [u]_\xi \mapsto t \cdot [u]_\xi := t\cdot u + N_{t\cdot\xi} \in M_{t\cdot\xi}.$$

Now we set $Q_{(\varepsilon)} := Q_{E^*_\varepsilon}$ and $N_{(\varepsilon)} := N_{E^*_\varepsilon}$. For $u = U_1 + U_2 + \cdots + U_r$ ($U_k \in \mathfrak{g}_{\alpha_k/2}$), noting that $[jU_m, U_k] \in \mathfrak{g}_{(\alpha_m+\alpha_k)/2}$, we have

$$Q_{(\varepsilon)}(u,u) = \sum_{k=1}^r \varepsilon_k \langle [U_k, jU_k], -E^*_k\rangle/2 = \sum_{k=1}^r \varepsilon_k B(U_k,U_k)/2,$$

where B is the inner product on $\mathfrak{g}$ given by (1.7). Thus $Q_{(\varepsilon)}(u,u) = 0$ if and only if $U_k = 0$ for all k such that $\varepsilon_k = 1$. In other words, we obtain

$$N_{(\varepsilon)} = \sum_{\varepsilon_k=0}^{\oplus} \mathfrak{g}_{\alpha_k/2}. \tag{3.4}$$

Since Theorem 1.2 tells us $[A_k, W] = \mathfrak{g}_{\alpha_k/2}$ ($1 \leq k \leq r$) and $[\mathfrak{g}_{(\alpha_k-\alpha_i)/2}, W] \subset \mathfrak{g}_{\alpha_k/2}$ ($1 \leq i < k \leq r$), we see from Theorem 2.3 (ii) that

$$[\mathfrak{h}_{(\varepsilon)}, W] \subset \sum_{\varepsilon_k=0}^{\oplus} \mathfrak{g}_{\alpha_k/2} = N_{(\varepsilon)}. \tag{3.5}$$

Proof. (i) It suffices to note

$$E^*_\varepsilon = \sum_{k=1}^r \varepsilon_k \hat{E}_k = \sum_{k=1}^r \varepsilon_{r+1-k}\mathfrak{E}_k = \mathfrak{E}_{\varepsilon^*}. \tag{2.15}$$

(ii) Using (2.4) and (2.5), we rewrite (2.8) as

$$\begin{aligned}\mathfrak{h}^*_{\mathfrak{E}_{\varepsilon^*}} &= \left(\sum_{\varepsilon^*_k=0} \mathbb{R}A_{r+1-k} \oplus \sum_{\varepsilon^*_k=0, m>k}^{\oplus} \mathfrak{g}_{(\alpha_{r+1-k}-\alpha_{r+1-m})/2}\right)\widehat{\ }. \\ &= \left(\sum_{\varepsilon_k=0} \mathbb{R}A_k \oplus \sum_{\varepsilon_k=0, i<k}^{\oplus} \mathfrak{g}_{(\alpha_k-\alpha_i)/2}\right)\widehat{\ }.\end{aligned}$$

On the other hand, we see from (2.15) that $\mathfrak{h}_{(\varepsilon)} = \{\, T \in \mathfrak{h}\,;\, \hat{T} \in \mathfrak{h}^*_{\mathfrak{E}_{\varepsilon^*}}\}$. Hence the assertion (ii) holds.

(iii) The first and second assertions are clear from (2.15). It is also clear from (2.5) and (2.12) that $p^*_{r+1-k}(\varepsilon^*) = q_k(\varepsilon)$. Thus we see from (2.13) that

$$\begin{aligned}\Xi^*(\varepsilon^*) &= \left\{ s \in \mathbb{R}^r;\ \begin{array}{l} s_{r+1-k} > p^*_{r+1-k}(\varepsilon^*)/2 \ \ (\text{if } \varepsilon^*_{r+1-k} = 1) \\ s_{r+1-k} = p^*_{r+1-k}(\varepsilon^*)/2 \ \ (\text{if } \varepsilon^*_{r+1-k} = 0) \end{array} \right\} \\ &= \{\, s \in \mathbb{R}^r\,;\, s_{r+1-k} > q_k(\varepsilon)/2 \ (\text{if } \varepsilon_k = 1), \quad s_{r+1-k} = q_k(\varepsilon)/2 \ (\text{if } \varepsilon_k = 0)\,\},\end{aligned}$$

which completes the proof. ■

Let C be the element $\sum_{k=1}^r c_k A_k$ of $\mathfrak{a}$. By (2.4) we have $\hat{C} = -\sum_{k=1}^r c_{r+1-k}\mathfrak{A}_k$, so that (2.9) and (1.12) imply

$$\chi^*_s(\iota(\exp C)) = \chi^*_s(\exp \hat{C}) = e^{-s_1 c_r - s_2 c_{r-1} - \cdots - s_r c_1} = \chi_{-s^*}(\exp C),$$

where $s^* = (s_r, s_{r-1}, \ldots, s_1)$. Hence we obtain

$$\Delta^*_s(\iota(t)\cdot E^*) = \chi^*_s(\iota(t)) = \chi_{-s^*}(t) \quad (t \in H), \tag{2.16}$$

$$\Delta^*_s(\iota(t)\cdot \xi) = \chi_{-s^*}(t)\Delta^*_s(\xi) \quad (\xi \in \Omega^*, t \in H). \tag{2.17}$$

Proposition 2.4. (i) *The Riesz distribution $\mathcal{R}^*_s$ is relatively invariant under the action of H^*:*

$$\langle \mathcal{R}^*_s, \varphi \circ \iota(t)^{-1}\rangle = \chi_{-s^*}(t)\,\langle \mathcal{R}^*_s, \varphi\rangle \quad (t \in H, \varphi \in \mathcal{S}(V^*)).$$

(ii) *One has $\langle \mathcal{R}^*_{s^*}, e^{-\langle y,\,\cdot\rangle}\rangle = \Delta_{-s}(y)$ for all $y \in \Omega$ and $s \in \mathbb{C}^r$.*

Proof. (i) Confer [**9**, Theorem 4.2].

(ii) Take $t \in H$ for which $y = t\cdot E$. Then, thanks to (i), we get $\langle \mathcal{R}^*_{s^*}, e^{-\langle y,\,\cdot\rangle}\rangle = \chi_{-s}(t)\langle \mathcal{R}^*_{s^*}, e^{-\langle E,\,\cdot\rangle}\rangle$. The right-hand side equals $\Delta_{-s}(y)$ by (1.13) and (2.11). ■

setting

$$\Gamma_{\Omega^*}(s) := \int_{\Omega^*} e^{-\langle E,\xi\rangle}\Delta_s^*(\xi)\,d\mu^*(\xi),$$

we define the tempered distribution $\mathcal{R}_s^*$ on V^* by

$$\langle \mathcal{R}_s^*, \psi\rangle := \frac{1}{\Gamma_{\Omega^*}(s)}\int_{\Omega^*}\psi(\xi)\Delta_s^*(\xi)\,d\mu^*(\xi) \qquad (\psi\in\mathcal{S}(V^*)), \tag{2.10}$$

and for general $s\in\mathbb{C}^r$, $\langle \mathcal{R}_s^*,\psi\rangle$ is defined by analytic continuation. Clearly we have

$$\langle \mathcal{R}_s^*, e^{-\langle E,\,\cdot\rangle}\rangle = 1. \tag{2.11}$$

For $\varepsilon\in\{0,1\}^r$, we set

$$p_k^*(\varepsilon) := \sum_{i<k}\varepsilon_i \dim \mathfrak{s}^*_{(\beta_k-\beta_i)/2} \quad (k=1,\dots,r) \tag{2.12}$$

and

$$\Xi^*(\varepsilon) := \{\, s\in\mathbb{R}^r\,;\ s_k > p_k^*(\varepsilon)/2 \ \ (\text{if } \varepsilon_k=1),\quad s_k = p_k^*(\varepsilon)/2 \ \ (\text{if } \varepsilon_k=0)\,\}. \tag{2.13}$$

Theorem 6.2 in [**9**] states that, if $s\in\Xi^*(\varepsilon)$, then $\mathcal{R}_s^*$ is an H^*-relatively invariant measure on $\mathcal{O}_\varepsilon$. Moreover every positive $\mathcal{R}_s^*$ is a measure on some orbit $\mathcal{O}_\varepsilon$. We now summarize the above in a form convenient to us. Set $\varepsilon^* := (\varepsilon_r,\varepsilon_{r-1},\dots,\varepsilon_1)$ for $\varepsilon=(\varepsilon_1,\dots,\varepsilon_{r-1},\varepsilon_r)\in\{0,1\}^r$.

THEOREM 2.3. (i) *Let E_ε^* be the element of V^* given by*

$$\left\langle \sum_{k=1}^r x_{kk}E_k + \sum_{m>k} X_{mk}, E_\varepsilon^*\right\rangle = \sum_{k=1}^r \varepsilon_k x_{kk} \quad (x_{kk}\in\mathbb{R}, X_{mk}\in\mathfrak{g}_{(\alpha_m+\alpha_k)/2})$$

and $\mathcal{O}_\varepsilon^*$ the H^*-orbit in V^* through E_ε^*. Then $\overline{\Omega^*} = \bigsqcup_{\varepsilon\in\{0,1\}^r}\mathcal{O}_\varepsilon^*$.

(ii) *For $\xi\in V^*$, let H_ξ be the inverse image $\iota^{-1}(H_\xi^*)$, that is, $H_\xi := \{\, t\in H\,;\ \iota(t)\cdot\xi=\xi\,\}$. Then the Lie algebra $\mathfrak{h}_{(\varepsilon)}$ of $H_{(\varepsilon)} := H_{E_\varepsilon^*}$ is equal to*

$$\sum_{\varepsilon_k=0}^{\oplus}\mathbb{R}A_k \oplus \sum_{\varepsilon_k=0,i<k}^{\oplus}\mathfrak{g}_{(\alpha_k-\alpha_i)/2}.$$

(iii) *The distribution $\mathcal{R}_s^*$ is positive if and only if s belongs to $\Xi^* := \bigsqcup_{\varepsilon\in\{0,1\}^r}\Xi^*(\varepsilon)$. Moreover $\mathcal{R}_s^*$ is a positive measure on $\mathcal{O}_\varepsilon^*$ if and only if $s\in\Xi^*(\varepsilon^*)$, and this condition is equivalent to*

$$s_{r+1-k} > q_k(\varepsilon)/2 \quad (\text{if } \varepsilon_k=1),\qquad s_{r+1-k} = q_k(\varepsilon)/2 \quad (\text{if } \varepsilon_k=0),$$

where

$$q_k(\varepsilon) := \sum_{m>k}\varepsilon_m \dim \mathfrak{g}_{(\alpha_m-\alpha_k)/2} \quad (k=1,\dots,r). \tag{2.14}$$

Hence the inclusions "$\subset$" can be replaced by equalities "$=$". ■

If one puts

$$\beta_k := -\alpha^*_{r+1-k} \in (\hat{\mathfrak{a}})^*, \quad \mathfrak{A}_k := -\hat{A}_{r+1-k} \in \hat{\mathfrak{a}}, \quad \mathfrak{E}_k := \hat{E}_{r+1-k} \in V^* \tag{2.4}$$

for $k = 1, \ldots, r$, then Proposition 2.2 is written in the form of Theorem 1.2 with

$$\mathfrak{s}^*_{(\beta_m \pm \beta_k)/2} = (\mathfrak{g}_{(\alpha_{r+1-k} \pm \alpha_{r+1-m})/2})\widehat{\ } \quad (1 \leq k < m \leq r). \tag{2.5}$$

These data are convenient and will be used when we make a direct translation of the results in [**9**] into the present situation.

Let H^* be the Lie group corresponding to the Lie algebra $\mathfrak{h}^*$. The group H^* acts on V^* by the adjoint action of $\mathfrak{s}^*$. Using the Lie group isomorphism $\iota : H \ni \exp T \mapsto \exp \hat{T} \in H^*$, we see that the action of H^* on V^* is the transfer of the coadjoint action of H on V^* by means of ι. Hence we have

$$\langle x, \iota(t) \cdot \xi \rangle = \langle t^{-1} \cdot x, \xi \rangle \qquad (t \in H). \tag{2.6}$$

Since the dual cone Ω^* is the H-orbit through E^* under the the coadjoint action ([**15**, Theorem 4.15]), and since $\sum_{k=1}^r \mathfrak{E}_k = \sum_{k=1}^r \hat{E}_{r+1-k} = \hat{E} = E^*$, (2.6) convinces us that Ω^* is the cone corresponding to the normal j-algebra $(\mathfrak{s}^*, j^*, -E)$. Thus if $\mathfrak{E}^*_\varepsilon$ is the element $\sum_{k=1}^r \varepsilon_k \mathfrak{E}_k$ for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \{0,1\}^r$, then the H^*-orbit decomposition of the closure $\overline{\Omega^*}$ is described as

$$\overline{\Omega^*} = \bigsqcup_{\varepsilon \in \{0,1\}^r} \mathfrak{O}_\varepsilon \quad (\mathfrak{O}_\varepsilon := H^* \cdot \mathfrak{E}^*_\varepsilon). \tag{2.7}$$

Moreover for $\xi \in V^*$, let H^*_ξ be the stabilizer at ξ in H^* and $\mathfrak{h}^*_\xi$ the Lie algebra of H^*_ξ. Then we have

$$\mathfrak{h}^*_{\mathfrak{E}^*_\varepsilon} = \sum_{\varepsilon_k = 0}^{\oplus} \mathbb{R}\mathfrak{A}_k \oplus \sum_{\varepsilon_k = 0,\, m > k}^{\oplus} \mathfrak{s}^*_{(\beta_m - \beta_k)/2}. \tag{2.8}$$

Now we define the Riesz distributions $\mathcal{R}^*_s$ ($s \in \mathbb{C}^r$) on Ω^*. Let χ^*_s be the one-dimensional representation of H^* given by

$$\chi^*_s\big(\exp\big(\textstyle\sum c_k \mathfrak{A}_k\big)\big) := e^{s_1 c_1 + \cdots + s_r c_r} \quad (c_1, \ldots, c_r \in \mathbb{R}) \tag{2.9}$$

and Δ^*_s be the function on Ω^* defined by

$$\Delta^*_s(h \cdot E^*) = \chi^*_s(h) \quad (h \in H^*).$$

We denote by μ^* the H^*-invariant measure on Ω^*. Recalling (2.5), we put $p^*_k := \sum_{i<k} \dim \mathfrak{s}^*_{(\beta_k - \beta_i)/2}$ for $k = 1, \ldots, r$. For $s \in \mathbb{C}^r$ such that $\Re s_k > p^*_k/2$ $(k = 1, \ldots, r)$,

we get

$$[\mathfrak{s}^*, \mathfrak{s}^*] = [\mathfrak{h}^*, \mathfrak{h}^*] \oplus [\mathfrak{h}^*, V^*] = [\mathfrak{h}, \mathfrak{h}]\hat{} \oplus \hat{V} = [\mathfrak{s}, \mathfrak{s}]\hat{}.$$

Therefore, thanks to Proposition 2.1 (iii), $\hat{\mathfrak{a}}$ is the orthogonal complement of $[\mathfrak{s}^*, \mathfrak{s}^*]$ with respect to B^*. For a linear form β on $\hat{\mathfrak{a}}$, put

$$\mathfrak{s}^*_\beta := \{\, \eta \in \mathfrak{s}^* \,;\, [\mathfrak{C}, \eta] = \langle \mathfrak{C}, \beta \rangle \eta \text{ for all } \mathfrak{C} \in \hat{\mathfrak{a}} \,\}.$$

Let $\alpha_1^*, \ldots, \alpha_r^*$ be the basis of $(\hat{\mathfrak{a}})^*$ dual to the basis $\hat{A}_1, \ldots, \hat{A}_r$ of $\hat{\mathfrak{a}}$.

PROPOSITION 2.2. *One has the root space decomposition*

$$\mathfrak{h}^* = \hat{\mathfrak{a}} \oplus \sum_{1 \le k < m \le r}^{\oplus} \mathfrak{s}^*_{(\alpha_m^* - \alpha_k^*)/2},$$

$$V^* = \sum_{k=1}^{r}{}^{\oplus} \mathfrak{s}^*_{-\alpha_k^*} \oplus \sum_{1 \le k < m \le r}^{\oplus} \mathfrak{s}^*_{-(\alpha_m^* + \alpha_k^*)/2}.$$

Moreover

$$\mathfrak{s}^*_{(\alpha_m^* - \alpha_k^*)/2} = (\mathfrak{g}_{(\alpha_m - \alpha_k)/2})\hat{}, \quad \mathfrak{s}^*_{-(\alpha_m^* + \alpha_k^*)/2} = (\mathfrak{g}_{(\alpha_m + \alpha_k)/2})\hat{} \quad (1 \le k < m \le r),$$

and $\mathfrak{s}^*_{-\alpha_k^*} = \mathbb{R}\hat{E}_k$ $(k = 1, \ldots, r)$.

Proof. For $C = \sum_{k=1}^r c_k A_k \in \mathfrak{a}$, we have $\langle \hat{C}, \alpha_k^* \rangle = c_k = \langle C, \alpha_k \rangle$ $(k = 1, \ldots, r)$ by definition. Take $T \in \mathfrak{g}_{(\alpha_m - \alpha_k)/2}$. Then

$$[\hat{C}, \hat{T}] = [C, T]\hat{} = \{(1/2)(c_m - c_k)T\}\hat{} = \langle \hat{C}, (\alpha_m^* - \alpha_k^*)/2 \rangle \hat{T},$$

which tells us that $(\mathfrak{g}_{(\alpha_m - \alpha_k)/2})\hat{} \subset \mathfrak{s}^*_{(\alpha_m^* - \alpha_k^*)/2}$. Next, for $X_{mk} \in \mathfrak{g}_{(\alpha_m + \alpha_k)/2}$ and $x' = \sum_{i=1}^r x'_{ii} E_i + \sum_{l>i} X'_{li} \in V$ $(x'_{ii} \in \mathbb{R}, X'_{li} \in \mathfrak{g}_{(\alpha_l + \alpha_i)/2})$, we have

$$\begin{aligned} \langle x', [\hat{C}, \hat{X}_{mk}] \rangle &= -\langle [C, x'], \hat{X}_{mk} \rangle \\ &= -B\left(\sum_{i=1}^r c_i x'_{ii} E_i + \sum_{l>i} (1/2)(c_l + c_i) X'_{li},\, X_{mk} \right). \end{aligned}$$

Since all the root spaces $\mathfrak{g}_\alpha$ are mutually orthogonal with respect to B, we obtain

$$\langle x', [\hat{C}, \hat{X}_{mk}] \rangle = -(1/2)(c_m + c_k) B(X'_{mk}, X_{mk}) = \langle x', -(1/2)(c_m + c_k) \hat{X}_{mk} \rangle.$$

Thus $(\mathfrak{g}_{(\alpha_m + \alpha_k)/2})\hat{} \subset \mathfrak{s}^*_{-(\alpha_m^* + \alpha_k^*)/2}$. Similarly we get $(\mathfrak{g}_{\alpha_k})\hat{} = \mathbb{R}\hat{E}_k \subset \mathfrak{s}^*_{-\alpha_k^*}$. Therefore

$$\begin{aligned} \mathfrak{s}^* = \hat{\mathfrak{s}} &= \hat{\mathfrak{a}} \oplus \sum_{k=1}^{r}{}^{\oplus} \mathbb{R}\hat{E}_k \oplus \sum_{1 \le k < m \le r}^{\oplus} (\mathfrak{g}_{(\alpha_m + \alpha_k)/2})\hat{} \oplus \sum_{1 \le k < m \le r}^{\oplus} (\mathfrak{g}_{(\alpha_m - \alpha_k)/2})\hat{} \\ &\subset \hat{\mathfrak{a}} \oplus \sum_{k=1}^{r}{}^{\oplus} \mathfrak{s}^*_{-\alpha_k^*} \oplus \sum_{1 \le k < m \le r}^{\oplus} \mathfrak{s}^*_{-(\alpha_m^* + \alpha_k^*)/2} \oplus \sum_{1 \le k < m \le r}^{\oplus} \mathfrak{s}^*_{(\alpha_m^* - \alpha_k^*)/2} \\ &\subset \mathfrak{s}^*. \end{aligned}$$

It is clear that $\mathfrak{s}^*$ is split solvable. Let $j^* : \mathfrak{s}^* \to \mathfrak{s}^*$ be the adjoint operator of $j : \mathfrak{s} \to \mathfrak{s}$. Clearly $(j^*)^2 = -\mathrm{id}_{\mathfrak{s}^*}$. Let B^* be the bilinear form on $\mathfrak{s}^*$ defined by

$$B^*(\eta_1, \eta_2) := -\langle E, [\eta_1, j^*\eta_2]\rangle \quad (\eta_1, \eta_2 \in \mathfrak{s}^*). \tag{2.3}$$

PROPOSITION 2.1. (i) $j^*\hat{Y} = -(jY)\hat{}\ \ (Y \in \mathfrak{s})$.

(ii) $j^*\hat{T} = -[\hat{T}, E^*] \in V^* \quad (T \in \mathfrak{h})$.

(iii) $B^*(\hat{Y}, \hat{Y}') = B(Y, Y') \quad (Y, Y' \in \mathfrak{s})$.

(iv) *The triple* $(\mathfrak{s}^*, j^*, -E)$ *is a normal* j*-algebra, where the element* $-E$ *of* V *is regarded as a linear form on* $\mathfrak{s}^*$.

Proof. (i) Clear from the fact that B is j-invariant and $j^2 = -\mathrm{id}_{\mathfrak{s}}$.

(ii) By (i) and (2.1), we have for $x \in V$

$$\langle x, j^*\hat{T}\rangle = -\langle [x, j(-jT)], E^*\rangle = -\langle x, [\hat{T}, E^*]\rangle.$$

(iii) We express $Y = x + T$ and $Y' = x' + T'$ with $x, x' \in V$ and $T, T' \in \mathfrak{h}$. Then we have by (2.3) and (i)

$$B^*(\hat{Y}, \hat{Y}') = -\langle E, [\hat{x} + \hat{T}, -(jT')\hat{}\,- (jx')\hat{}\,]\rangle.$$

Using (2.2), (1.9) and the j-invariance of B, we see that the right-hand side equals

$$\begin{aligned} -\langle [T, E], (jT')\hat{}\,\rangle + \langle [jx', E], \hat{x}\rangle &= -B(-jT, jT') + B(-j(jx'), x) \\ &= B(T, T') + B(x, x'), \end{aligned}$$

whence (iii) follows.

(iv) Owing to (i) and (iii), the bilinear form B^* defines a j^*-invariant inner product on $\mathfrak{s}^*$. Thus it remains to show the equality

$$[\eta_1, \eta_2] + j^*[j^*\eta_1, \eta_2] + j^*[\eta_1, j^*\eta_2] - [j^*\eta_1, j^*\eta_2] = 0 \quad (\eta_1, \eta_2 \in \mathfrak{s}^*).$$

Let $N(\eta_1, \eta_2)$ denote the left-hand side. We can check easily that $N(j^*\eta_1, \eta_2) = -j^*N(\eta_1, \eta_2)$ and $N(\eta_2, \eta_1) = -N(\eta_1, \eta_2)$. Therefore it is sufficient to consider the case $\eta_i = \hat{T}_i \in \mathfrak{h}^*$ ($T_i \in \mathfrak{h}$, $i = 1, 2$). Using (ii) and Jacobi's identity, we have

$$\begin{aligned} j^*N(\hat{T}_1, \hat{T}_2) &= j^*[\hat{T}_1, \hat{T}_2] - [j^*\hat{T}_1, \hat{T}_2] - [\hat{T}_1, j^*\hat{T}_2] \\ &= -[[\hat{T}_1, \hat{T}_2], E^*] + [[\hat{T}_1, E^*], \hat{T}_2] + [\hat{T}_1, [\hat{T}_2, E^*]] = 0, \end{aligned}$$

which completes the proof. ∎

Now we study the root space decomposition of $\mathfrak{s}^*$. We have $[\mathfrak{h}^*, V^*] = V^*$ because Proposition 2.1 (ii) tells us that $[\mathfrak{h}^*, V^*] \subset V^* = j^*\mathfrak{h}^* = -[\mathfrak{h}^*, E^*] \subset [\mathfrak{h}^*, V^*]$. Thus

where $|s| := \sum_{k=1}^{r} s_k$. If a function ψ on D is relatively invariant under G, that is,

$$\psi(g \cdot (z,u)) = \chi_s(g)\psi(z,u) \quad (g \in G, (z,u) \in D)$$

for some $s \in \mathbb{C}^r$, then ψ is written as

$$\psi(z,u) = c\,\Delta_s(\Im z - Q(u,u))$$

with some constant $c \in \mathbb{C}$. In fact, taking a unique element $t \in H$ for which $\Im z - Q(u,u) = t \cdot E$, we have $(z,u) = n(\Re z, u)t \cdot (iE, 0)$, so that we get by (1.13)

$$\psi(z,u) = \chi_s(n(\Re z,u)t)\psi(iE,0) = \chi_s(t)\psi(iE,0) = \psi(iE,0)\Delta_s(\Im z - Q(u,u)).$$

2. DUAL CONES AS HOMOGENEOUS CONES

As is well known, the dual cones of homogeneous cones are also homogeneous cones by the contragredient action. In this section, we apply results in [**9**] to the Riesz distributions on dual cones.

The construction of Siegel domains in Section 1 tells us that normal j-algebras $\mathfrak{g}$ such that $\mathfrak{g}(1/2) = 0$ correspond to tube domains $V + i\Omega \subset V_{\mathbb{C}}$. These normal j-algebras are called *normal j-algebras of tube type*. The argument in [**9**] is based on this one-to-one correspondence between normal j-algebras of tube type and homogeneous cones. Let $\mathfrak{s}$ be the normal j-subalgebra $V \oplus \mathfrak{h}$ of $\mathfrak{g}$, which corresponds to the homogeneous cone $\Omega \subset V$. We shall define a normal j-algebra structure on the dual space $\mathfrak{s}^* = V^* \oplus \mathfrak{h}^*$ in such a way that the corresponding homogeneous cone is the dual cone Ω^* of Ω. By Remark 1.3, $(\mathfrak{s}, j, -E^*)$ is a normal j-algebra. Let B be the corresponding j-invariant inner product on $\mathfrak{s}$:

$$B(Y', Y) := -\langle [Y', jY], E^* \rangle \quad (Y, Y' \in \mathfrak{s}). \tag{2.1}$$

For every $Y \in \mathfrak{s}$, let $\hat{Y}$ be the element of $\mathfrak{s}^*$ given by $\hat{Y} := B(\cdot\,, Y)$. Since $V \perp \mathfrak{h}$ with respect to B, it is clear that $\hat{\mathfrak{h}} = \mathfrak{h}^*$ and $\hat{V} = V^*$. By transferring the Lie algebra structure of $\mathfrak{h}$ by $\widehat{\ }$, $\mathfrak{h}^*$ becomes a Lie algebra. Then $\mathfrak{h}^*$ acts on V^* through the coadjoint action of $\mathfrak{h}$ on V^* under the above identification $\mathfrak{h}^* \equiv \mathfrak{h}$. This enables us to introduce a Lie algebra structure in $\mathfrak{s}^*$ as the semidirect product $V^* \rtimes \mathfrak{h}^*$. The bracket product is described as

$$\begin{aligned} \langle x + T, [\xi_1 + \hat{T}_1, \xi_2 + \hat{T}_2] \rangle = -\langle [T_1, x], \xi_2 \rangle + \langle [T_2, x], \xi_1 \rangle + B(T, [T_1, T_2]) \\ (x \in V,\ \xi_1, \xi_2 \in V^*,\ T, T_1, T_2 \in \mathfrak{h}). \end{aligned} \tag{2.2}$$

$u \in \mathfrak{g}(1/2)$ let $n(x,u)$ denote the element $\exp(x+u)$ of $N(Q)$. Noting that $[u,u'] = 4\Im Q(u,u')$ $(u,u' \in \mathfrak{g}(1/2))$, we see that the Campbell-Hausdorff formula gives the multiplication law

$$\begin{aligned} n(x,u)n(x',u') = n(x+x'+2\Im Q(u,u'), u+u') \\ (x,x' \in \mathfrak{g}(1),\ u,u' \in \mathfrak{g}(1/2)). \end{aligned} \tag{1.11}$$

The Lie group $G := \exp \mathfrak{g}$ is the semidirect product $N(Q) \rtimes H$ with the formula

$$tn(x,u)t^{-1} = n(t \cdot x, t \cdot u) \quad (t \in H, x \in \mathfrak{g}(1), u \in \mathfrak{g}(1/2)).$$

Now we define the homogeneous Siegel domain $D = D(\Omega, Q)$ corresponding to the normal j-algebra $(\mathfrak{g}, j, \omega)$ by

$$D(\Omega, Q) := \{\, (z,u) \in V_{\mathbb{C}} \times W \,;\, \Im z - Q(u,u) \in \Omega \,\}.$$

We realize G as an affine transformation group on $V_{\mathbb{C}} \times W$ by

$$\begin{aligned} n(x_0,u_0)t_0 \cdot (z,u) := (t_0 \cdot z + x_0 + 2iQ(t_0 \cdot u, u_0) + iQ(u_0,u_0), t_0 \cdot u + u_0) \\ (t_0 \in H, n(x_0,u_0) \in N(Q), (z,u) \in V_{\mathbb{C}} \times W). \end{aligned}$$

Putting $(z_1,u_1) := n(x_0,u_0)t_0 \cdot (z,u)$, we see from a simple calculation with (1.10) that

$$\Im z_1 - Q(u_1,u_1) = t_0 \cdot (\Im z - Q(u,u)),$$

which implies that the action of G preserves the Siegel domain $D(\Omega, Q)$. As is already mentioned, H acts on Ω simply transitively, so that G acts on D simply transitively.

For parameters $s = (s_1, \cdots, s_r) \in \mathbb{C}^r$, let χ_s be the one-dimensional representations of G such that

$$\chi_s\Big(\exp \sum_{k=1}^{r} c_k A_k\Big) := e^{s_1 c_1 + \cdots + s_r c_r} \quad (c_1, \ldots, c_r \in \mathbb{R}). \tag{1.12}$$

Note that, since $\mathfrak{n}(Q) \subset [\mathfrak{g}, \mathfrak{g}]$, any χ_s equals 1 on $N(Q)$. Let Δ_s be the function on Ω given by

$$\Delta_s(t \cdot E) := \chi_s(t) \quad (t \in H). \tag{1.13}$$

Then $\Delta_s(t \cdot x) = \chi_s(t)\Delta_s(x)$ for all $x \in \Omega$ and $t \in H$. For $\lambda > 0$, put $t(\lambda) := \exp(\log \lambda\, A) \in H$. Thanks to (1.5), we see that $t(\lambda) \cdot x = \lambda x$ for $x \in \Omega \subset V$. Thus we get by (1.12)

$$\Delta_s(\lambda x) = \chi_s(t(\lambda))\Delta_s(x) = \lambda^{|s|}\Delta_s(x), \tag{1.14}$$

where $\mathfrak{g}(\mu) := \{0\}$ for $\mu > 1$. Putting $A := \sum_{k=1}^{r} A_k \in \mathfrak{a}$, we have

$$\mathrm{ad}(A)\mathfrak{g}(\mu) = \mu\mathfrak{g}(\mu) \quad (\mu = 0, 1/2, 1). \tag{1.5}$$

Remark 1.3. Let E^* be the linear form on $\mathfrak{g}(1)$ defined by

$$\left\langle \sum_{k=1}^{r} x_{kk}E_k + \sum_{m>k} X_{mk}, E^* \right\rangle = \sum_{k=1}^{r} x_{kk} \quad (x_{kk} \in \mathbb{R}, X_{mk} \in \mathfrak{g}_{(\alpha_m+\alpha_k)/2}). \tag{1.6}$$

We extend E^* to $\mathfrak{g}$ by zero-extension. Then $(\mathfrak{g}, j, -E^*)$ is also a normal j-algebra and induces the same root decomposition of $\mathfrak{g}$ as in Theorem 1.2. In particular, the root spaces are mutually orthogonal with respect to the corresponding inner product

$$B(Y_1, Y_2) := -\langle [Y_1, jY_2], E^* \rangle \quad (Y_1, Y_2 \in \mathfrak{g}). \tag{1.7}$$

We see from Theorem 1.2 (v) that $j\mathfrak{g}(1/2) = \mathfrak{g}(1/2)$, so that $j|_{\mathfrak{g}(1/2)}$ defines a complex structure on $\mathfrak{g}(1/2)$. We denote by W the complex vector space $(\mathfrak{g}(1/2), j)$ and by $W_{\mathbb{R}}$ the real vector space $\mathfrak{g}(1/2)$ itself. By (1.4), $\mathfrak{g}(1)$ is a commutative ideal and $\mathfrak{g}(0)$ is a subalgebra of $\mathfrak{g}$. We write V and $\mathfrak{h}$ for $\mathfrak{g}(1)$ and $\mathfrak{g}(0)$ respectively. Putting

$$E := E_1 + \cdots + E_r \in V, \tag{1.8}$$

we have by Theorem 1.2 (iv)

$$jT = -[T, E] \in V \qquad (T \in \mathfrak{h}). \tag{1.9}$$

We denote by H the solvable Lie group $\exp \mathfrak{h}$. Then the relation (1.4) implies that the group H acts on V and W by the adjoint action. Let Ω be the H-orbit in V through E. Then Ω is a regular cone and H acts simply transitively on Ω. Since (1.1) together with (1.4) leads us to the relation $[T, ju] = j[T, u]$ $(T \in \mathfrak{h}, u \in W)$, the action of H on W is complex linear. Let $Q : W_{\mathbb{R}} \times W_{\mathbb{R}} \to V_{\mathbb{C}}$ be a real bilinear map defined by

$$Q(u, u') := \frac{1}{4}([ju, u'] + i[u, u']) \quad (u, u' \in W_{\mathbb{R}}).$$

Since (1.1) and (1.4) also yield the equality $[ju, ju'] = [u, u']$ $(u, u' \in W_{\mathbb{R}})$, we see that Q is Hermitian on W. It is known that Q is Ω-positive, that is, $Q(u, u)$ belongs to the closure $\overline{\Omega}$ for all $u \in W$ and $Q(u, u) = 0$ implies $u = 0$. Moreover the map Q is H-equivariant:

$$Q(t \cdot u, t \cdot u') = t \cdot Q(u, u') \quad (t \in H,\ u, u' \in W). \tag{1.10}$$

Set $\mathfrak{n}(Q) := \mathfrak{g}(1) \oplus \mathfrak{g}(1/2)$. Then (1.4) tells us that $\mathfrak{n}(Q)$ is an (at most 2-step) nilpotent ideal of $\mathfrak{g}$. Let $N(Q)$ be the Lie group $\exp \mathfrak{n}(Q)$, and for $x \in \mathfrak{g}(1)$ and

DEFINITION 1.1. Let $\mathfrak{g}$ be a real split solvable Lie algebra, j a linear automorphism on $\mathfrak{g}$ such that $j^2 = -\mathrm{id}_{\mathfrak{g}}$, and ω a linear form on $\mathfrak{g}$. The triple $(\mathfrak{g}, j, \omega)$ is called *a normal j-algebra* if the following two conditions are satisfied:

(i) For all $Y_1, Y_2 \in \mathfrak{g}$,

$$[Y_1, Y_2] + j[Y_1, jY_2] + j[jY_1, Y_2] - [jY_1, jY_2] = 0. \tag{1.1}$$

(ii) The bilinear form $B_\omega(Y_1, Y_2) := \langle [Y_1, jY_2], \omega \rangle$ $(Y_1, Y_2 \in \mathfrak{g})$ defines a j-invariant inner product on $\mathfrak{g}$.

Let $\mathfrak{a}$ be the orthogonal complement of the subspace $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$ with respect to B_ω. It is known that $\mathfrak{a}$ is a commutative subalgebra of $\mathfrak{g}$ and $\mathrm{ad}(\mathfrak{a})$ is a commutative family of self-adjoint operators on $\mathfrak{g}$. For a linear form α on $\mathfrak{a}$, we set

$$\mathfrak{g}_\alpha := \{\, Y \in \mathfrak{g} \,;\, [C, Y] = \langle C, \alpha \rangle Y \text{ for all } C \in \mathfrak{a} \,\}.$$

Then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, and if $\alpha \neq \beta$, we have $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ with respect to B_ω. If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$, we call α *a root* and $\mathfrak{g}_\alpha$ *the root space* corresponding to α. We have the following root space decomposition of $\mathfrak{g}$.

THEOREM 1.2 (Piatetskii-Shapiro [**13**]). (i) *There is a linear basis $\{A_1, \ldots, A_r\}$ of $\mathfrak{a}$ such that if one puts $E_l := -jA_l$, then $[A_k, E_l] = \delta_{kl} E_l$ $(1 \leq k, l \leq r)$.*

(ii) *Let $\alpha_1, \ldots, \alpha_r$ be the basis of $\mathfrak{a}^*$ dual to $A_1, \ldots, A_r$. Then the possible roots are of the following forms:*

$$\alpha_k, \qquad \alpha_k/2 \qquad (1 \leq k \leq r),$$
$$(\alpha_m - \alpha_k)/2, \quad (\alpha_m + \alpha_k)/2 \quad (1 \leq k < m \leq r).$$

(iii) *The root space $\mathfrak{g}_{\alpha_k}$ $(1 \leq k \leq r)$ is equal to $\mathbb{R}E_k$.*

(iv) *If $m > k$, then $j\mathfrak{g}_{(\alpha_m - \alpha_k)/2} = \mathfrak{g}_{(\alpha_m + \alpha_k)/2}$, and the action of j is given by*

$$jY = -[Y, E_k] \quad (Y \in \mathfrak{g}_{(\alpha_m - \alpha_k)/2}). \tag{1.2}$$

(v) *One has $j\mathfrak{g}_{\alpha_k/2} = \mathfrak{g}_{\alpha_k/2}$ for all $k = 1, \ldots, r$.*

We set

$$\begin{aligned} \mathfrak{g}(1) &:= \sum_{k=1}^{r}{}^{\oplus} \mathbb{R}E_k \oplus \sum_{1 \leq k < m \leq r}^{\oplus} \mathfrak{g}_{(\alpha_m + \alpha_k)/2}, \quad \mathfrak{g}(1/2) := \sum_{k=1}^{r}{}^{\oplus} \mathfrak{g}_{\alpha_k/2}, \\ \mathfrak{g}(0) &:= \mathfrak{a} \oplus \sum_{1 \leq k < m \leq r}^{\oplus} \mathfrak{g}_{(\alpha_m - \alpha_k)/2}. \end{aligned} \tag{1.3}$$

Then we have the grading $\mathfrak{g} = \mathfrak{g}(1) \oplus \mathfrak{g}(1/2) \oplus \mathfrak{g}(0)$ with the bracket relation

$$[\mathfrak{g}(\mu), \mathfrak{g}(\nu)] \subset \mathfrak{g}(\mu + \nu) \quad (\mu, \nu = 0, 1/2, 1), \tag{1.4}$$

Theorems A and B above are established in Section 4. We first show in Proposition 4.6 that, if $\mathcal{H}_s(D) \neq \{0\}$ and if the inner product is normalized appropriately, then the reproducing kernel K^s of $\mathcal{H}_s(D)$ is given by

$$K^s((z,u),(z',u')) = \Delta_{-\Re s}((z-\bar{z}')/i - 2Q(u,u')) \quad ((z,u),(z',u') \in D),$$

where $\Delta_{-\Re s}$ is analytically continued to $\Omega + iV$. Since K^s is a kernel function of positive type, the function $\Delta_{-\Re s}$ on Ω is of positive type, that is,

$$\sum_{k,l=1}^{N} a_k \bar{a}_l \Delta_{-\Re s}(y_k + y_l) \geq 0 \quad (N \in \mathbb{N},\ a_k \in \mathbb{C},\ y_k \in \Omega,\ k = 1, \ldots, N).$$

Then we see from [**5**, Satz 5.1] (Lemma 4.7) that $\Delta_{-\Re s}$ is expressed as the Laplace transform of a positive measure on $\overline{\Omega^*}$. Comparing this with (0.1), we see that the necessary condition for the non-vanishing of $\mathcal{H}_s(D)$ is reduced to the positivity of the Riesz distributions on Ω^*. This is the point of the proof of Theorem A (Theorem 4.8). After proving Theorem B (Theorem 4.10), we consider the IUR $\check{\pi}_s$ ($s \in \mathfrak{X}(\varepsilon)$) of G realized on the space $\int_{\mathcal{O}_\varepsilon^*}^{\oplus} \mathcal{F}_\xi \, d\mathcal{R}^*_{\Re s^*}(\xi)$ as the transfer of $(\pi_s, \mathcal{H}_s(D))$ by means of the unitary map Φ_s in Theorem B. Through the direct integral of the Fock spaces $\mathcal{F}_\xi$, the restriction of $\check{\pi}_s$ to $N(Q)$ is expressed as $\int_{\mathcal{O}_\varepsilon^*}^{\oplus} \tau_\xi|_{N(Q)} \, d\mathcal{R}^*_{\Re s^*}(\xi)$, and the operators $\check{\pi}_s(t)$ ($t \in H$) map every fiber $\mathcal{F}_\xi$ onto $\mathcal{F}_{t\cdot\xi}$. Furthermore we show in Theorem 4.12 that $\check{\pi}_s$ is equivalent to the induced representation $\mathrm{Ind}_{G_{(\varepsilon)}}^{G} \tau_{(\varepsilon)} \otimes \chi_{-i\Im s/2}$, where $G_{(\varepsilon)} := G_{E_\varepsilon^*}$ and $\tau_{(\varepsilon)} := \tau_{E_\varepsilon^*}$.

In the last section, Section 5, we utilize the theory of orbit method in order to determine the equivalence classes of $(\pi_s, \mathcal{H}_s(D))$. We construct a real polarization at $-E_\varepsilon^* + \zeta \in \mathfrak{g}^*$ and give a description of the coadjoint orbit in $\mathfrak{g}^*$ through $-E_\varepsilon^* + \zeta$ in Proposition 5.1. Making use of Theorem 4.12 and some general facts concerning induced representations, we prove Theorem C (Theorem 5.3), which completes the classification.

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1. NORMAL j-ALGEBRAS AND SIEGEL DOMAINS

In this section, we describe the fundamental structure of normal j-algebras $\mathfrak{g}$ and the Siegel domains on which the Lie groups $\exp \mathfrak{g}$ act simply transitively as affine transformation groups. First of all, we state the definition of normal j-algebras.

characterized by the Laplace transform:

$$\langle \mathcal{R}_s^*, e^{-\langle y,\xi\rangle}\rangle_\xi = \Delta_{-s^*}(y) \quad (y \in \Omega), \tag{0.1}$$

where $s^* := (s_r, s_{r-1}, \dots, s_1) \in \mathbb{C}^r$ (see Proposition 2.4 (ii)). Applying our previous results about positive Riesz distributions in [**9**], we see that, with an appropriate choice of the elements E_ε^* ($\varepsilon \in \{0,1\}^r$) of V^*, the closure $\overline{\Omega^*}$ is decomposed into 2^r H-orbits $\mathcal{O}_\varepsilon^* := H \cdot E_\varepsilon^*$, and that the distribution $\mathcal{R}_{\Re s^*}^*$ ($\Re s^* := (\Re s_r, \dots, \Re s_1)$) induces an H-relatively invariant measure on $\mathcal{O}_\varepsilon^*$ if $s \in \mathfrak{X}(\varepsilon)$ (cf. Theorem 2.3). For $\xi \in \overline{\Omega^*}$ let $\mathcal{F}_\xi$ be the Fock space on $\mathfrak{g}(1/2)$ whose reproducing kernel is $e^{2\langle Q(\cdot,\cdot),\xi\rangle}$.

THEOREM B. *If $s \in \mathfrak{X}(\varepsilon)$, then one has the following unitary isomorphism*

$$\Phi_s : \int_{\mathcal{O}_\varepsilon^*}^{\oplus} \mathcal{F}_\xi \, d\mathcal{R}_{\Re s^*}^*(\xi) \ni f \mapsto F \in \mathcal{H}_s(D),$$

where

$$F(z,u) := \int_{\mathcal{O}_\varepsilon^*} e^{i\langle z,\xi\rangle} f(\xi)(u) \, d\mathcal{R}_{\Re s^*}^*(\xi) \quad ((z,u) \in D).$$

If $\mathcal{H}_s(D) \neq \{0\}$, then the unitary representation $(\pi_s, \mathcal{H}_s(D))$ of G is irreducible by Kunze's theorem [**10**]. Now we can state the classification of the equivalence classes of IUR's (irreducible unitary representations) $(\pi_s, \mathcal{H}_s(D))$ of G. For $\varepsilon \in \{0,1\}^r$, set

$$Z(\varepsilon) := \{\, \zeta = (\zeta_1, \dots, \zeta_r) \in \mathbb{R}^r \,;\, \zeta_k = 0 \text{ for all } k \text{ such that } \varepsilon_k = 1 \,\},$$

$$\Theta(\varepsilon,\zeta) := \{\, s \in \mathfrak{X}(\varepsilon) \,;\, -\Im s_k/2 = \zeta_k \text{ for all } k \text{ such that } \varepsilon_k = 0 \,\} \quad (\zeta \in Z(\varepsilon)).$$

Clearly we have $\mathfrak{X}(\varepsilon) = \bigsqcup_{\zeta \in Z(\varepsilon)} \Theta(\varepsilon,\zeta)$. Each element ζ of $Z(\varepsilon)$ is canonically identified with an element of $\mathfrak{g}^*$ vanishing on the subspace $[\mathfrak{g},\mathfrak{g}]$ (see (5.9) for this).

THEOREM C. (i) *If $s \in \Theta(\varepsilon,\zeta)$, then the IUR $(\pi_s, \mathcal{H}_s(D))$ of G corresponds to the coadjoint orbit through $-E_\varepsilon^* + \zeta \in \mathfrak{g}^*$ by the Kirillov-Bernat correspondence.*

(ii) *Two IUR's $(\pi_s, \mathcal{H}_s(D))$ and $(\pi_{s'}, \mathcal{H}_{s'}(D))$ are equivalent if and only if s and s' belong to the same $\Theta(\varepsilon,\zeta)$.*

Let us explain the organization of this paper. In the first section, we describe the basic structure of normal j-algebras and the corresponding Siegel domains. In Section 2, we study the Riesz distributions on the dual cone Ω^* by introducing the normal j-algebra structure on the dual vector space $\mathfrak{g}^*(1) \oplus \mathfrak{g}^*(0)$. Theorem 2.3 translates our previous results in [**9**] into the context of dual cones.

Section 3 is devoted to the study of the Fock spaces $\mathcal{F}_\xi$ ($\xi \in \overline{\Omega^*}$) and of the IUR's τ_ξ of the groups $G_\xi := N(Q) \rtimes H_\xi$ realized on $\mathcal{F}_\xi$ as in (3.14), where $N(Q)$ is the nilpotent Lie subgroup $\exp(\mathfrak{g}(1) \oplus \mathfrak{g}(1/2))$ of G and H_ξ is the stabilizer at ξ in H.

Let us state our results in more detail. Based on the one-to-one correspondence between homogeneous Siegel domains and normal j-algebras established by Piatetskii-Shapiro [**13**], we start the argument with a normal j-algebra $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is graded as $\mathfrak{g} = \mathfrak{g}(1) \oplus \mathfrak{g}(1/2) \oplus \mathfrak{g}(0)$. We have a regular cone $\Omega \subset \mathfrak{g}(1)$ on which the group $H := \exp \mathfrak{g}(0)$ acts simply transitively by the adjoint action. The subspace $\mathfrak{g}(1/2)$ is naturally regarded as a complex vector space, and we have an Ω-positive Hermitian map $Q : \mathfrak{g}(1/2) \times \mathfrak{g}(1/2) \to \mathfrak{g}(1)_{\mathbb{C}}$. Then the Siegel domain $D = D(\Omega, Q)$ corresponding to $\mathfrak{g}$ is defined to be

$$D(\Omega, Q) := \{ (z, u) \in \mathfrak{g}(1)_{\mathbb{C}} \times \mathfrak{g}(1/2) ; \Im z - Q(u, u) \in \Omega \}.$$

The group $G := \exp \mathfrak{g}$ is realized as an affine transformation group on $\mathfrak{g}(1)_{\mathbb{C}} \times \mathfrak{g}(1/2)$ and acts on D simply transitively. Let r be the codimension of $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$. Then one-dimensional representations of G are parameterized as χ_s by $s = (s_1, \ldots, s_r) \in \mathbb{C}^r$ (see (1.12)). When $\chi = \chi_{-s/2}$, we write π_s and $\mathcal{H}_s(D)$ for π_χ and $\mathcal{H}_\chi(D)$ respectively. In order to describe our results, we need the root space decomposition of $\mathfrak{g}$: there is a basis $\alpha_1, \alpha_2, \ldots, \alpha_r$ for the roots of $\mathfrak{g}$, so that

$$\mathfrak{g}(1) = \sum_{k=1}^{r}{}^{\oplus} \mathfrak{g}_{\alpha_k} \oplus \sum_{1 \le k < m \le r}{}^{\oplus} \mathfrak{g}_{(\alpha_m + \alpha_k)/2}, \quad \mathfrak{g}(1/2) = \sum_{k=1}^{r}{}^{\oplus} \mathfrak{g}_{\alpha_k/2},$$
$$\mathfrak{g}(0) = \mathfrak{a} \oplus \sum_{1 \le k < m \le r}{}^{\oplus} \mathfrak{g}_{(\alpha_m - \alpha_k)/2}$$

(see Theorem 1.2 for details). For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \{0, 1\}^r$ we put $q_k(\varepsilon) := \sum_{m>k} \varepsilon_m \dim \mathfrak{g}_{(\alpha_m - \alpha_k)/2}$ $(k = 1, \ldots, r)$,

$$\mathfrak{X}(\varepsilon) := \{ s \in \mathbb{C}^r ; \Re s_k > q_k(\varepsilon)/2 \ (\text{if } \varepsilon_k = 1), \quad \Re s_k = q_k(\varepsilon)/2 \ (\text{if } \varepsilon_k = 0) \},$$

and $\mathfrak{X} := \bigsqcup_{\varepsilon \in \{0,1\}^r} \mathfrak{X}(\varepsilon)$.

THEOREM A. *Non-zero $\mathcal{H}_s(D)$ exists if and only if $s \in \mathfrak{X}$.*

Therefore one can regard the set $\mathfrak{X}$ as a non-symmetric analogue of the Wallach set studied by Vergne and Rossi [**16**] in the case of symmetric Siegel domains. Next we give a concrete description of non-zero $\mathcal{H}_s(D)$ for $s \in \mathfrak{X}$. For this purpose, we shall make use of some facts about Riesz distributions $\mathcal{R}_s^*$ $(s \in \mathbb{C}^r)$ on the dual cone Ω^* in $\mathfrak{g}(1)^*$ (see (2.10) for the definition of $\mathcal{R}_s^*$). The Riesz distributions are supported by the closure $\overline{\Omega^*}$ and are relatively invariant under the coadjoint action of H. Let Δ_s $(s \in \mathbb{C}^r)$ be the function on Ω given by $\Delta_s(t \cdot E) := \chi_s(t)$ $(t \in H)$, where E is a certain fixed element of Ω (see (1.8)). Then the distribution $\mathcal{R}_s^*$ is

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Representations of the affine transformation groups acting simply transitively on Siegel domains

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Abstract. Let G be the split solvable Lie group acting simply transitively on a Siegel domain D. We consider irreducible unitary representations of G realized on Hilbert spaces of holomorphic functions on D. We determine all such Hilbert spaces by connecting them with positive Riesz distributions on the dual cone and describe them through the Fourier-Laplace transform. Moreover we classify the representations of G by making use of the orbit method.

INTRODUCTION

Let D be a homogeneous Siegel domain and G a split solvable Lie group acting simply transitively on D as affine transformations. In the present paper, we study unitary representations of G realized on spaces of holomorphic functions on D by relating them to positive Riesz distributions and classify them by using the theory of orbit method.

Denote by $\mathcal{H}(D)$ the space of all holomorphic functions on D. For a one-dimensional representation $\chi : G \to \mathbb{C}$ of G, we define a representation π_χ of G by

$$\pi_\chi(g)F(p) := \chi(g)F(g^{-1} \cdot p) \quad (g \in G,\ F \in \mathcal{H}(D),\ p \in D).$$

We consider the subspace $\mathcal{H}_\chi(D)$ of $\mathcal{H}(D)$ satisfying the following two conditions:

(i) $\mathcal{H}_\chi(D)$ has a Hilbert space structure with reproducing kernel,

(ii) $(\pi_\chi, \mathcal{H}_\chi(D))$ is a unitary representation of G.

We note that non-zero $\mathcal{H}_\chi(D)$ is unique, though it might not exist for some χ. We shall establish a necessary and sufficient condition for the non-vanishing of such $\mathcal{H}_\chi(D)$ (see Theorem A below) and construct all of the spaces $\mathcal{H}_\chi(D)$ from positive Riesz distributions on the dual cone through the Fourier-Laplace transform (Theorem B). Moreover we describe the equivalence classes of the representations $(\pi_\chi, \mathcal{H}_\chi(D))$ of G by means of the orbit method (Theorem C).