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(ii):

$$H_0(\mathfrak{g}(0,1,\infty),\ L_{\lambda_0,k}(0)\otimes L_{\lambda_1,k}(1)\otimes M_{\lambda_\infty,k}(\infty)) \\ = H^0(\mathfrak{c}, L_{\lambda_0,k}(0)\otimes L_{\lambda_1,k}(1))^{(\lambda_\infty)}.$$

Proof. Similarly to the proof of 4.5, Shapiro's Lemma implies the statement (ii) since we have

$$\mathrm{Ind}_{U(\mathfrak{c})}^{U(\mathfrak{g}(0,1,\infty))}\{L_{\lambda_0,k}(0)\otimes L_{\lambda_1,k}(1)\otimes \mathbb{C}_{\lambda_\infty,\infty,k}\} \simeq L_{\lambda_0,k}(0)\otimes L_{\lambda_1,k}(1)\otimes M_{\lambda_\infty,k}(\infty).$$

Combining Theorem 4.6 and the statement (ii), we obtain the statement (i). Q.E.D.

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for $u \in M_{\lambda_0,k}(0)$ (resp. $v \in M_{\lambda_1,k}(1)$). For $w = 0, 1$, we denote the restriction of $\Psi^{\epsilon,w}$ to the weight space $M_{\lambda_w,k}(w)_{\Lambda - i\alpha - j\delta}$ ($\Lambda = \Lambda_{\lambda_w,k}$) by $\Psi^{\epsilon,w}_{i,j}$ and set

$$\psi^{\epsilon,w}_{i,j} = \begin{cases} \Psi^{\epsilon,w}_{i,j} & \text{if } i \equiv \epsilon \mod 2, \\ f.\ \Psi^{\epsilon,w}_{i-1,j+1} & \text{if } i \not\equiv \epsilon \mod 2. \end{cases}$$

Then we have

Lemma 4.6. *For each $w = 0, 1$, the subalgebra $\tilde{\mathfrak{n}}_-(w)$ acts on the space $\bigoplus_{j,i+2j \in \mathbb{Z}_{\geq 0}} \mathbb{C}\psi^{\epsilon,w}_{i,j}$ and, as a $\tilde{\mathfrak{n}}_-(w)$-module, it is isomorphic to the quotient module $\tilde{\mathcal{G}}_{\lambda,\mu}(w)$, where λ and μ are give by $\lambda = -\lambda_1 - 2$ (if $w = 0$), $\lambda = -\lambda_0 - 2$ (if $w = 1$) and $\mu = \frac{1}{2}(\lambda_\infty - \lambda_0 - \lambda_1) - \epsilon$.*

Proof. If we set $\xi = \xi_{i,\epsilon} = \frac{1}{2}(i + (-1)^\epsilon \delta^{(2)}_{i,1})$, then the following maps

$$\psi^{\epsilon,0}_{i,j} \mapsto \begin{cases} (-1)^\xi F_{(i+\epsilon)/2}\theta \otimes z^j & \text{if } i \equiv \epsilon \mod 2, \\ (-1)^\xi F_{(i-1+\epsilon)/2} \otimes z^j & \text{if } i \not\equiv \epsilon \mod 2, \end{cases}$$

and

$$\psi^{\epsilon,1}_{i,j} \mapsto \begin{cases} (-1)^j F_{(i+\epsilon)/2}\theta \otimes (z-1)^j & \text{if } i \equiv \epsilon \mod 2, \\ (-1)^j F_{(i-1+\epsilon)/2} \otimes (z-1)^j & \text{if } i \not\equiv \epsilon \mod 2, \end{cases}$$

give us isomorphisms of $\tilde{\mathfrak{n}}_-(w)$-modules for $w = 0, 1$. Q.E.D.

We return to the proof of Theorem 4.6. Let $S^{(0)}_{m_0,s_0}|\lambda_0, k\rangle_0$ and $S^{(1)}_{m_1,s_1}|\lambda_1, k\rangle_1$ be the singular vectors of $M_{\lambda_0,k}(0)$ and $M_{\lambda_1,k}(1)$ respectively. For a functional $\Phi \in (M_{\lambda_0,k}(0) \otimes M_{\lambda_1,k}(1))^*$, the condition $d^*\Phi = 0$ is equivalent to

$$\Phi(S^{(0)}_{m_0,s_0}|\lambda_0, k\rangle_0 \otimes |\lambda_1, k\rangle_1) = \Phi(|\lambda_0, k\rangle_0 \otimes S^{(1)}_{m_1,s_1}|\lambda_1, k\rangle_1) = 0.$$

By an argument similar to the proof of Theorem 4.4, we can show this theorem. Q.E.D.

By Theorem 4.5 and 4.6, Theorem 4.1 is proved directly. Finally we obtain Theorem 4.5 from the following lemma, and thus complete the proof of Theorem 4.1.

Lemma 4.7. *Let k be a generic level and λ_0, λ_1, $\lambda_\infty \in \Lambda^k$. Then we have*

(i):

$$\begin{aligned} &H_0(\mathfrak{g}(0,1,\infty),\ L_{\lambda_0,k}(0) \otimes L_{\lambda_1,k}(1) \otimes M_{\lambda_\infty,k}(\infty)) \\ &\quad = H_0(\mathfrak{g}(0,1,\infty),\ L_{\lambda_0,k}(0) \otimes L_{\lambda_1,k}(1) \otimes L_{\lambda_\infty,k}(\infty)), \end{aligned}$$

Theorem 4.5. *Let k be a generic level. Suppose that $\lambda_0,\ \lambda_1,\ \lambda_\infty \in \mathcal{S}_k = \Lambda^k$, then we have*

$$\begin{aligned}&\dim H_0(\mathfrak{g}(0,1,\infty),\ L_{\lambda_0,k}(0)\otimes L_{\lambda_1,k}(1)\otimes L_{\lambda_\infty,k}(\infty))\\ &\quad=\dim H_0(\mathfrak{c}, L_{\lambda_0,k}(0)\otimes L_{\lambda_1,k}(1))^{(\lambda_\infty)}.\end{aligned}$$

where $H_0(\mathfrak{c}, L_{\lambda_0,k}(0)\otimes L_{\lambda_1,k}(1))^{(\lambda_\infty)}$ denotes an eigenspace of $h_\infty\otimes(1-z^{-1})$.

Similarly to the admissible case, we need an explicit description of the coinvariant $H_0(\mathfrak{c}, L_{\lambda_0,k}(0)\otimes L_{\lambda_1,k}(1))^{(\lambda_\infty)}$ to prove the theorem. For $\lambda_0,\ \lambda_1,\ \lambda_\infty \in \mathcal{S}_k = \Lambda^k$, let $(V_{i_0}^{\epsilon_0}, V_{j_0})$, $(V_{i_1}^{\epsilon_1}, V_{j_1})$ and $(V_{i_\infty}^{\epsilon_\infty}, V_{j_\infty})$ be the pairs of irreducible representations for $osp(1|2)$ and sl_2, corresponding to the elements $l_{\lambda_0},\ l_{\lambda_1},\ l_{\lambda_\infty} \in \mathcal{A}_k$ respectively. A triple $(\lambda_0,\lambda_1,\lambda_\infty)$ is called proper if $(V_{i_\infty}^{\epsilon_\infty}, V_{j_\infty})$ is one component of $(V_{i_0}^{\epsilon_0}\otimes V_{i_1}^{\epsilon_1}, V_{j_0}\otimes V_{j_1})$.

Theorem 4.6. *Let k be a generic level. Suppose that $\lambda_0,\ \lambda_1 \in \Lambda^k$, then we have*

$$\dim H_0(\mathfrak{c}, L_{\lambda_0,k}(0)\otimes L_{\lambda_1,k}(1))^{(\lambda_\infty)} = \begin{cases} 1 & \text{if } (\lambda_0,\lambda_1,\lambda_\infty)\text{:proper,}\\ 0 & \text{otherwise.}\end{cases}$$

Proof. From Poincaré duality, it suffices to calculate

$$\dim H^0(\mathfrak{c}, (L_{\lambda_0,k}(0)\otimes L_{\lambda_1,k}(1))^*)^{(-\lambda_\infty)}.$$

For $w = 0, 1 \in \mathbb{C}P^1$, let

$$0 \to M(w) \to M_{\lambda_w,k}(w) \to L_{\lambda_w,k}(w) \to 0,$$

be the resolution for the generalized Weyl module $L_{\lambda_w,k}(w)$. Notice that we have an exact sequence

$$W \xrightarrow{d} M_{\lambda_0,k}(0)\otimes M_{\lambda_1,k}(1) \to L_{\lambda_0,k}(0)\otimes L_{\lambda_1,k}(1) \to 0,$$

where $W = M_{\lambda_0,k}(0)\otimes M(1) + M(0)\otimes M_{\lambda_1,k}(1)$ is a subset of $M_{\lambda_0,k}(0)\otimes M_{\lambda_1,k}(1)$. Hence by definition, we obtain

$$\begin{aligned}&H^0(\mathfrak{c}, (L_{\lambda_0,k}(0)\otimes L_{\lambda_1,k}(1))^*)^{(-\lambda_\infty)}\\ &\quad=(((M_{\lambda_0,k}(0)\otimes M_{\lambda_1,k}(1))^*)^{\mathfrak{c}})^{(-\lambda_\infty)}\cap\ker d^*.\end{aligned}$$

It follows from Lemma 4.3 and the proof of Lemma 4.7 that

$$\dim(((M_{\lambda_0,k}(0)\otimes M_{\lambda_1,k}(1))^*)^{\mathfrak{c}})_\epsilon^{(-\lambda_\infty)} = 1,$$

for any $\epsilon = 0, 1$. Now we determine when $H^0(\mathfrak{c}, (L_{\lambda_0,k}(0)\otimes L_{\lambda_1,k}(1))^*)_\epsilon^{(-\lambda_\infty)}$ has a non-zero element. For $\Psi^\epsilon \in (((M_{\lambda_0,k}(0)\otimes M_{\lambda_1,k}(1))^*)^{\mathfrak{a}})_\epsilon^{(-\lambda_\infty)}$ such that $\Psi^\epsilon \neq 0$, we define a functional $\Psi^{\epsilon,0}$ (resp. $\Psi^{\epsilon,1}$) on $M_{\lambda_0,k}(0)$ (resp. $M_{\lambda_1,k}(1)$) by

$$\Psi^{\epsilon,0}(u) = \Psi^\epsilon(u\otimes|\lambda_1,k\rangle_1)\ (\text{resp. } \Psi^{\epsilon,1}(v) = \Psi^\epsilon(|\lambda_0,k\rangle_0\otimes v)\ )$$

as a $\mathfrak{g}(0,1,\infty)$-module. Therefore Shapiro's lemma implies that

$$H_0(\mathfrak{g}(0,1,\infty),\ M_{\lambda_0,k}(0)\otimes M_{\lambda_1,k}(1)\otimes L_{\lambda_\infty,k}(\infty)) = H_0(\bar{\mathfrak{a}},\mathbb{C}_{0,\lambda_0,k}\otimes\mathbb{C}_{1,\lambda_1,k}\otimes L_{\lambda_\infty,k}(\infty)).$$

It follows from the definition of $\mathbb{C}_{w,\lambda_w,k}$ that the right hand side coincides with

$$H_0(\mathfrak{a}, L_{\lambda_\infty,k}(\infty))^{(\lambda_0,-\lambda_1)}.$$

Thus we obtain the statement (ii).

At last we prove the statement (i). Let

$$\cdots \to M_1 \oplus M_2 \to M_{\lambda_0,k}(0) \to L_{\lambda_0,k}(0) \to 0,$$

be the BGG resolution for the admissible representation $L_{\lambda_0,k}(0)$. From this resolution we obtain the exact sequence

$$\begin{aligned} H_0(\mathfrak{g}(0,1,\infty),\ (M_1\oplus M_2)(0)\otimes M_{\lambda_1,k}(1)\otimes L_{\lambda_\infty,k}(\infty)) &\to \\ H_0(\mathfrak{g}(0,1,\infty),\ M_{\lambda_0,k}(0)\otimes M_{\lambda_1,k}(1)\otimes L_{\lambda_\infty,k}(\infty)) &\to \\ H_0(\mathfrak{g}(0,1,\infty),\ L_{\lambda_0,k}(0)\otimes M_{\lambda_1,k}(1)\otimes L_{\lambda_\infty,k}(\infty)) &\to\ 0. \end{aligned}$$

Notice that the Verma module M_1 and M_2 have non-admissible highest weights. Hence it follows from the statement (ii) and Theorem 4.3 that

$$H_0(\mathfrak{g}(0,1,\infty),\ (M_1\oplus M_2)(0)\otimes M_{\lambda_1,k}(1)\otimes L_{\lambda_\infty,k}(\infty)) = 0,$$

and thus

$$\begin{aligned} &H_0(\mathfrak{g}(0,1,\infty),\ M_{\lambda_0,k}(0)\otimes M_{\lambda_1,k}(1)\otimes L_{\lambda_\infty,k}(\infty)) \\ &\quad = H_0(\mathfrak{g}(0,1,\infty),\ L_{\lambda_0,k}(0)\otimes M_{\lambda_1,k}(1)\otimes L_{\lambda_\infty,k}(\infty)). \end{aligned} \tag{26}$$

Applying the same argument to the second factor in (26), we obtain

$$\begin{aligned} &H_0(\mathfrak{g}(0,1,\infty),\ L_{\lambda_0,k}(0)\otimes M_{\lambda_1,k}(1)\otimes L_{\lambda_\infty,k}(\infty)) \\ &\quad = H_0(\mathfrak{g}(0,1,\infty),\ L_{\lambda_0,k}(0)\otimes L_{\lambda_1,k}(1)\otimes L_{\lambda_\infty,k}(\infty)). \end{aligned} \tag{27}$$

Combining (26) and (27), we have proved the statement (i). Q.E.D.

Second we prove our main theorem in the generic level case (**Case 1**) i.e. Theorem 4.1. We introduce a nilpotent subalgebra $\mathfrak{c}$, which plays the same role as $\mathfrak{a}$ in the above proof. Let $\mathfrak{c}$ be the subalgebra of $\mathfrak{g}(0,1,\infty)$ which consists of the meromorphic functions $f(z)$ such that $f(z)\in[\mathfrak{b}_w,\mathfrak{b}_w]$ at $w=\infty$ i.e.

$$\mathfrak{c} = \mathbb{C}f \oplus \mathbb{C}F \oplus \mathfrak{g}\otimes\mathbb{C}[z^{-1},(1-z)^{-1}]_0,$$

where $\mathbb{C}[x,y]_0$ denotes the maximal ideal (x,y) of $\mathbb{C}[x,y]$. We set $\bar{\mathfrak{c}} = \mathfrak{c}\oplus\mathbb{C}h_\infty\otimes(1-z^{-1})$.

gives us an isomorphism of a $\tilde{\mathfrak{n}}_-(\infty)$-module. Q.E.D.

Here we complete the proof of Theorem 4.4. Since the Verma module $M_{\lambda_\infty,k}(\infty)$ ($\lambda_\infty = \lambda_{m,s} \in \Lambda^k$) has two singular vectors

$$S^{(\infty)}_{m,s}|\lambda_\infty, k\rangle_\infty,\ S^{(\infty)}_{m-p,s-q}|\lambda_\infty, k\rangle_\infty,$$

in the weight spaces $M_{m,\frac{ms}{2}}$ and $M_{m-p,\frac{(m-p)(s-q)}{2}}$. Therefore, the condition $\Phi^\epsilon \in \ker d^*$ is equivalent to

$$\begin{aligned}&\Phi^\epsilon(S^{(\infty)}_{m,s}|\lambda_\infty, k\rangle_\infty) = 0,\\ &\Phi^\epsilon(S^{(\infty)}_{m-p,s-q}|\lambda_\infty, k\rangle_\infty) = 0.\end{aligned}$$

Hence, it follows from Lemma 4.4 (and its proof) that, for each $\epsilon = 0, 1$, the solution of the system

$$\begin{aligned}\pi(a(S^{(\infty)}_{m,s}))\varphi^\epsilon_{m,\frac{ms}{2}} &= 0,\\ \pi(a(S^{(\infty)}_{m-p,s-q}))\varphi^\epsilon_{m-p,\frac{(m-p)(s-q)}{2}} &= 0,\end{aligned}$$

gives us a non-trivial element of $H^0(\mathfrak{a}, L_{\lambda_\infty,k}(\infty))$. Here a denotes the antipode of the Lie superalgebra $\mathfrak{g}(0,1,\infty)$ i.e. $a : U(\mathfrak{g}(0,1,\infty)) \to U(\mathfrak{g}(0,1,\infty))$ is given by

$$\begin{aligned}&a(x) = -(-1)^{\deg x}x && \text{if } x \in \mathfrak{g}(0,1,\infty),\\ &a(xy) = (-1)^{\deg x \deg y}a(y)a(x) && \text{for } x, y \in U(\mathfrak{g}(0,1,\infty)).\end{aligned} \tag{25}$$

Q.E.D.

Remark that Theorem 4.2 is an immediate consequence of Theorem 4.3 and 4.4. Now we return to the proof of Theorem 4.3. To show the theorem, it suffices to prove the following lemma.

Lemma 4.5. *Let k be an admissible level and λ_0, λ_1, $\lambda_\infty \in \Lambda^k$. Then we have*

(i):

$$\begin{aligned}&H_0(\mathfrak{g}(0,1,\infty),\ M_{\lambda_0,k}(0) \otimes M_{\lambda_1,k}(1) \otimes L_{\lambda_\infty,k}(\infty))\\ &\quad = H_0(\mathfrak{g}(0,1,\infty),\ L_{\lambda_0,k}(0) \otimes L_{\lambda_1,k}(1) \otimes L_{\lambda_\infty,k}(\infty)),\end{aligned}$$

(ii):

$$\begin{aligned}&H_0(\mathfrak{g}(0,1,\infty),\ M_{\lambda_0,k}(0) \otimes M_{\lambda_1,k}(1) \otimes L_{\lambda_\infty,k}(\infty))\\ &\quad = H_0(\mathfrak{a}, L_{\lambda_\infty,k}(\infty))^{(\lambda_0,-\lambda_1)}.\end{aligned}$$

Proof. We first prove the statement (ii), since the statement (i) is proved as a consequence of the statement (ii). For $w = 0, 1$, let $\mathbb{C}_{w,\lambda_w,k}$ be the one dimensional representation defined in (15). By direct calculation, we see that

$$\mathrm{Ind}_{U(\tilde{\mathfrak{a}})}^{U(\mathfrak{g}(0,1,\infty))}\{\mathbb{C}_{0,\lambda_0,k} \otimes \mathbb{C}_{1,\lambda_1,k} \otimes L_{\lambda_\infty,k}(\infty))\} \simeq M_{\lambda_0,k}(0) \otimes M_{\lambda_1,k}(1) \otimes L_{\lambda_\infty,k}(\infty),$$

whose cohomology gives us

$$H^0(\mathfrak{a}, L_{\lambda_\infty,k}(\infty)^*)^{(-\lambda_0,\lambda_1)} = ((M_{\lambda_\infty,k}(\infty)^*)^{\mathfrak{a}})^{(-\lambda_0,\lambda_1)} \cap \ker d^*.$$

It follows from Lemma 4.3 that

$$\dim((M_{\lambda_\infty,k}(\infty)^*)^{\mathfrak{a}})^{(-\lambda_0,\lambda_1)}_\epsilon = 1,$$

for any $\epsilon = 0, 1$ (see the proof of Lemma 4.7 (ii)). In particular we have

$$\dim H^0(\mathfrak{a}, L_{\lambda_\infty,k}(\infty)^*)^{(-\lambda_0,\lambda_1)}_\epsilon \leq 1.$$

Now we determine when the eigenspace $H^0(\mathfrak{a}, L_{\lambda_\infty,k}(\infty)^*)^{(\lambda_0,-\lambda_1)}_\epsilon$ has a non-zero element. Let Φ^ϵ be a non-zero element of $((M_{\lambda_\infty,k}(\infty)^*)^{\mathfrak{a}})^{(\lambda_0,-\lambda_1)}_\epsilon$ and $\Phi^\epsilon_{i,j}$ the restriction of Φ^ϵ to the weight space $M_{i,j} = M_{\lambda_\infty,k}(\infty)_{-\Lambda+i\alpha+j\delta}$ ($\Lambda = \Lambda_{\lambda_\infty,k}$). We set

$$\varphi^\epsilon_{i,j} = \begin{cases} \Phi^\epsilon_{i,j} & \text{if } i \equiv \epsilon \mod 2, \\ f \otimes z.\ \Phi^\epsilon_{i-1,j+1} & \text{if } i \not\equiv \epsilon \mod 2. \end{cases}$$

Lemma 4.4. *The subalgebra $\tilde{\mathfrak{n}}_-(\infty)$ acts on the space $\bigoplus_{j,i+2j\in\mathbb{Z}_{\geq 0}} \mathbb{C}\varphi^\epsilon_{i,j}$ and, as a $\tilde{\mathfrak{n}}_-(\infty)$-module, it is isomorphic to the quotient module $\tilde{\mathcal{G}}_{\lambda,\mu}(\infty)$ for $\lambda = -\lambda_1 - 2$ and $\mu = \frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_\infty) - \epsilon$.*

Proof. Set $\lambda_s = \lambda_0 + \lambda_1 - \lambda_\infty$, $\lambda_d = \lambda_0 - \lambda_1 - \lambda_\infty$. It is convenient to use the notation

$$\delta^{(2)}_{i.j} = \begin{cases} 0 & \text{if } i \equiv j \mod 2, \\ 1 & \text{if } i \not\equiv j \mod 2. \end{cases}$$

If $\Phi^\epsilon \in H^0(\mathfrak{a}, M_{\lambda_\infty,k}(\infty)^*)^{(-\lambda_0,\lambda_1)}_\epsilon$, then we have

$$\mathfrak{a}.\Phi^\epsilon = 0,\ h_0 \otimes (z-1).\Phi^\epsilon = -\lambda_0\Phi^\epsilon,\ h_1 \otimes z.\Phi^\epsilon = \lambda_1\Phi^\epsilon. \tag{24}$$

From (24) and the definition of $\varphi^\epsilon_{i,j}$, we obtain

$$\begin{aligned}
E \otimes 1.\varphi^\epsilon_{i,j} &= \tfrac{1}{2}(\lambda_d + 2i - 2 - 2\delta^{(2)}_{i.j})\varphi^\epsilon_{i-2,j}, \\
h \otimes z.\varphi^\epsilon_{i,j} &= (-\lambda_0 + \lambda_\infty - 2i)\varphi^\epsilon_{i,j-1}, \\
F \otimes z.\varphi^\epsilon_{i,j} &= \tfrac{1}{2}(\lambda_s + 2i + 2 + 2\delta^{(2)}_{i.j})\varphi^\epsilon_{i+2,j-1}, \\
e \otimes 1.\varphi^\epsilon_{i,j} &= \begin{cases} -\varphi^\epsilon_{i-1,j} & \text{if } i \equiv \epsilon \mod 2, \\ -\frac{1}{2}(\lambda_d + 2i - 2)\varphi^\epsilon_{i-1,j} & \text{if } i \not\equiv \epsilon \mod 2, \end{cases} \\
f \otimes z.\varphi^\epsilon_{i,j} &= \begin{cases} \varphi^\epsilon_{i+1,j-1} & \text{if } i \equiv \epsilon \mod 2, \\ \frac{1}{2}(\lambda_s + 2i + 2)\varphi^\epsilon_{i+1,j-1} & \text{if } i \not\equiv \epsilon \mod 2, \end{cases}
\end{aligned}$$

Therefore the map

$$\varphi^\epsilon_{i,j} \mapsto \begin{cases} F_{(i+\epsilon)/2}\theta \otimes z^j & \text{if } i \equiv \epsilon \mod 2, \\ F_{(i-1+\epsilon)/2} \otimes z^j & \text{if } i \not\equiv \epsilon \mod 2, \end{cases}$$

Theorem 4.3. *Let k be an admissible level. Suppose that $\lambda_0,\ \lambda_1,\ \lambda_\infty \in \mathcal{S}_k = \Lambda^k$ then we have*

$$\begin{aligned}&\dim H_0(\mathfrak{g}(0,1,\infty),\ L_{\lambda_0,k}(0)\otimes L_{\lambda_1,k}(1)\otimes L_{\lambda_\infty,k}(\infty))\\ &\quad = \dim H_0(\mathfrak{a}, L_{\lambda_\infty,k}(\infty))^{(\lambda_0,-\lambda_1)}.\end{aligned}$$

where $H_0(\mathfrak{a}, L_{\lambda_\infty,k}(\infty))^{(\lambda_0,-\lambda_1)}$ denotes a simultaneous eigenspace of $h_0\otimes(z-1)$ and $h_1\otimes z$.

It follows from this theorem that, to determine the fusion rule at an admissible level, it is enough to see the space of the coinvariant $H_0(\mathfrak{a}, L_{\lambda_\infty,k}(\infty))$. We first give an explicit description of this coinvariant, and then we use it to show Theorem 4.3. For $\lambda_0,\ \lambda_1,\ \lambda_\infty \in \mathcal{S}_k = \Lambda^k$, let $[(V^0_{i_0}, V_{j_0})]$, $[(V^0_{i_1}, V_{j_1})]$ and $[(V^0_{i_\infty}, V_{j_\infty})]$ be the equivalence classes corresponding to l_{λ_0}, l_{λ_0}, $l_{\lambda_\infty} \in \mathcal{A}_k$ respectively.

Theorem 4.4. *Suppose that k is an admissible level.*

(i): $\dim H_0(\mathfrak{a}, L_{\lambda_\infty,k}(\infty)) < \infty$, *if and only if* $\lambda_\infty \in \Lambda_k$.

(ii): *If $\lambda_\infty \in \Lambda^k$ then we have*

$$\begin{aligned}&\dim H_0(\mathfrak{a}, L_{\lambda_\infty,k}(\infty))^{(\lambda_0,-\lambda_1)}\\ &\quad = \begin{cases} 2 & \text{if } \lambda_0,\lambda_1 \in \Lambda_k \text{ and both } C^0 \text{ and } C^1 \text{ hold,}\\ 1 & \text{if } \lambda_0,\lambda_1 \in \Lambda_k \text{ and only one of } C^0 \text{ and } C^1 \text{ holds,}\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

where the conditions C^0 and C^1 are defined as follows:
For $\epsilon = 0$ or 1, set

$$P^\epsilon = \left\{ (i,j)\in(\mathbb{Z})^2 \;\middle|\; \begin{array}{l} |i_0-i_1|+\epsilon \le i \le \min(i_0+i_1, 2q-i_0-i_1-2)-\epsilon,\\ |j_0-j_1| \le j \le \min(j_0+j_1, 2p-j_0-j_1-4),\\ i_0+i_1+i \equiv \epsilon,\ j_0+j_1+j \equiv 0 \pmod 2,\end{array} \right\},$$

and

$$C^0 \Leftrightarrow [(i_\infty, j_\infty)\in P^0], \quad C^1 \Leftrightarrow [(-i_\infty+q-1, -j_\infty+p-2)\in P^1].$$

Proof. Poincaré duality implies that the left hand side of the theorem (ii) is equal to

$$\dim H^0(\mathfrak{a}, L_{\lambda_\infty,k}(\infty)^*)^{(-\lambda_0,\lambda_1)}. \tag{22}$$

Thus we calculate (22) by using the BGG resolution:

$$\cdots \to M_1\oplus M_2 \xrightarrow{d} M_{\lambda_\infty,k}(\infty) \to L_{\lambda_\infty,k}(\infty) \to 0, \tag{23}$$

for $L_{\lambda_\infty,k}(\infty)$. Taking the restricted dual of (23), we have an injective resolution

$$\cdots \leftarrow M_1^*\oplus M_2^* \xleftarrow{d^*} M_{\lambda_\infty,k}(\infty)^* \leftarrow L_{\lambda_\infty,k}(\infty)^* \leftarrow 0,$$

where the subscript ϵ of the coinvariant stands for the even ($\epsilon = 0$) or odd ($\epsilon = 1$) part of the space.

Proof. For $\lambda \in \mathfrak{h}^*$, let $\phi_{w,\lambda}$ be the character of the Borel subalgebra $\mathfrak{b}_w$ in (14). We denote by $\mathbb{C}_{w,\lambda}$ the 1-dimensional representation of $\mathfrak{b}_w$ defined by the character $\phi_{w,\lambda}$. Let $M_{w,\lambda}$ be the Verma module of $\mathfrak{g}$ induced from the representation $\mathbb{C}_{w,\lambda}$. It follows from Shapiro's lemma and Poincaré duality that

$$\begin{aligned} & H_0(\mathfrak{g}(0,1,\infty), M_{\lambda_0,k}(0) \otimes M_{\lambda_1,k}(1) \otimes M_{\lambda_\infty,k}(\infty)) \\ & \quad = H_0(\mathfrak{g}, M_{0,\lambda_0} \otimes M_{1,\lambda_1} \otimes M_{\infty,\lambda_\infty}) \\ & \quad = H^0(\mathfrak{g}, (M_{0,\lambda_0} \otimes M_{1,\lambda_1} \otimes M_{\infty,\lambda_\infty})^*)^* \\ & \quad = \mathrm{Hom}_{\mathfrak{g}}(M_{0,\lambda_0} \otimes M_{1,\lambda_1} \otimes M_{\infty,\lambda_\infty}, \mathbb{C})^*. \end{aligned}$$

Using the fact that $\mathfrak{b}_0 \cap \mathfrak{b}_\infty = \mathfrak{h}$, we obtain

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{g}}(M_{0,\lambda_0} \otimes M_{1,\lambda_1} \otimes M_{\infty,\lambda_\infty}, \mathbb{C}) &= \mathrm{Hom}_{\mathfrak{g}}(M_{0,\lambda_0} \otimes M_{\infty,\lambda_\infty}, M'_{1,\lambda_1}) \\ &= \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}_{0,\lambda_0} \otimes \mathbb{C}_{\infty,\lambda_\infty}, M'_{1,\lambda_1}) \\ &= (M'_{1,\lambda_1})_{\lambda_0-\lambda_\infty}, \end{aligned}$$

where M'_{1,λ_1} stands for the full dual of M_{1,λ_1}. Here we note that the above $\mathrm{Hom}_{\mathfrak{g}}(\cdot,\cdot)$ are the sets of all $\mathfrak{g}$-module (not necessarily $\mathbb{Z}_2$-graded) homomorphisms. To prove the lemma, it is enough to show that

$$\dim((M'_{1,\lambda_1})_{\lambda_0-\lambda_\infty})_\epsilon = 1,$$

for $\epsilon = 0, 1$. Notice that the even (resp. odd) part of the Verma module M_{1,λ_1} is given by $(M_{1,\lambda_1})_0 = \mathbb{C}[F]v_{\lambda_1}$ (resp. $(M_{1,\lambda_1})_1 = \mathbb{C}[F]fv_{\lambda_1}$), where v_{λ_1} is a highest weight vector of M_{1,λ_1}. From this, for any $\phi_\epsilon \in (M^*_{1,\lambda_1})_\epsilon$, $h.\phi_\epsilon = (\lambda_0 - \lambda_\infty)\phi_\epsilon$ is equivalent to

$$\begin{aligned} &\phi_0(F^{j+1}v_{\lambda_1}) = \tfrac{1}{2}(\lambda_0 + \lambda_1 - \lambda_\infty - 4j)\phi_0(F^j v_{\lambda_1}), && \text{if } \epsilon = 0, \\ &\phi_1(F^{j+1}fv_{\lambda_1}) = \tfrac{1}{2}(\lambda_0 + \lambda_1 - \lambda_\infty - 4j - 2)\phi_1(F^j fv_{\lambda_1}), && \text{if } \epsilon = 1. \end{aligned}$$

Hence we have $\dim((M^*_{1,\lambda_1})_{\lambda_0-\lambda_\infty})_\epsilon = 1$ for each $\epsilon = 0, 1$. Q.E.D.

First we consider the fusion algebra at an admissible level (**Case 2**). Let us prove Theorem 4.2. We can reduce our calculation of the dimensions of coinvariants (21) to computation of cohomology of a certain nilpotent subalgebra $\mathfrak{a}$. Let $\mathfrak{a}$ be the subalgebra of $\mathfrak{g}(0,1,\infty)$ which consists of the meromorphic functions $f(z) \in \mathfrak{g}(0,1,\infty)$ such that $f(z) \in [\mathfrak{b}_w, \mathfrak{b}_w]$ at $w = 0, 1$ i.e.

$$\begin{aligned} \mathfrak{a} = ((\mathbb{C}e_0 \oplus \mathbb{C}E_0) \otimes (z-1)) \oplus ((\mathbb{C}e_1 \oplus \mathbb{C}E_1) \otimes z) \\ \oplus \mathfrak{g} \otimes z(z-1)\mathbb{C}[z], \end{aligned}$$

where e_w and E_w ($w = 0, 1$) are defined as in (12). We set $\bar{\mathfrak{a}} = \mathfrak{a} \oplus (\mathbb{C}h_0 \otimes (z-1)) \oplus (\mathbb{C}h_1 \otimes z)$. Till the end of the proof of Theorem 4.2, let $k = p/q - 3$ be an admissible level.

we have

$$\pi(S_{m,s}) = Y^+ X e^m Y^-$$
$$= \left\{ \prod_{i=1}^{\frac{m+1}{2}} P(i - \frac{1}{2} + \frac{1}{4}st) \prod_{i=1}^{\frac{m-1}{2}} Q(-i + \frac{1}{2} + \frac{1}{4}st) \right\} Y^+ e^m Y^-$$
if $\frac{s}{2} \in 2\mathbb{Z} + 1$,

$$= \left\{ \prod_{i=1}^{\frac{m-1}{2}} P(i + \frac{1}{4}st) \prod_{i=1}^{\frac{m+1}{2}} Q(-i + 1 - \frac{1}{4}st) \right\} Y^+ e^m Y^-$$
if $\frac{s}{2} \in 2\mathbb{Z}$.

Applying a similar method to the part $Y^+ e^m Y^-$, we obtain the desired result by induction. Q.E.D.

Let us begin the proof. To do this, we calculate the dimension of coinvariants:

$$H_0(\mathfrak{g}(w_1, w_2, w_3),\ L_{\lambda_1,k}(w_1) \otimes L_{\lambda_2,k}(w_2) \otimes L_{\lambda_3,k}(w_3)). \tag{21}$$

From now on, we fix $w_1 = 0$, $w_2 = 1$ and $w_3 = \infty$ for simplicity, since the dimensions of the coinvariants do not depend on the choice of the points of $\mathbb{C}P^1$.

First we list up the images of singular vectors under the evaluation map π, which we need for our calculation. For each $w \in \mathbb{C}P^1$, we denote a highest weight vector of the Verma module $M_{\lambda_w,k}(w)$ by $|\lambda_w, k\rangle_w$. Let $S^{(w)}_{m,s}|\lambda_w, k\rangle_w$ be the singular vector of $M_{\lambda_w,k}(w)$ for $\lambda_w = \lambda_{m,s}$. Then the image $\pi(S^{(w)}_{m,s})$ $(w = 1, \infty)$ is given by

$$\pi(S^{(w)}_{m,s}) = 2^{-\frac{ms}{2}} \prod_{j=1}^{s} \prod_{i=1}^{m} \left\{ \prod_{i+j \in 2\mathbb{Z}} P^{(w)}(\frac{i}{2} + \frac{j}{2}t) \prod_{i+j \in 2\mathbb{Z}+1} Q^{(w)}(-\frac{i-1}{2} - \frac{j}{2}t) \right\} f^m,$$

if $m > 0$, $s \geq 0$, and

$$\pi(S^{(w)}_{m,s}) = 2^{-\frac{ms}{2}} \prod_{j=1}^{-s-1} \prod_{i=1}^{-m} \left\{ \prod_{i+j \in 2\mathbb{Z}} Q^{(w)}(\frac{i}{2} + \frac{j}{2}t) \prod_{i+j \in 2\mathbb{Z}+1} P^{(w)}(-\frac{i-1}{2} - \frac{j}{2}t) \right\} e^{-m},$$

if $m < 0$, $s < 0$,

where $P^{(w)}(s) = e_w f_w + 2s$ and $Q^{(w)}(s) = f_w e_w - 2s$. Remark that, in the case of $w = 0$, the formula in Lemma 4.2 is nothing but the image $\pi(S^{(0)}_{m,s})$.

The following lemma is a key in our proof.

Lemma 4.3. *For any λ_0, λ_1, $\lambda_\infty \in \mathfrak{h}^*$ and $\epsilon \in \mathbb{Z}/2\mathbb{Z}$, we have*

$$\dim H_0(\mathfrak{g}(0, 1, \infty), M_{\lambda_0,k}(0) \otimes M_{\lambda_1,k}(1) \otimes M_{\lambda_\infty,k}(\infty))_\epsilon = 1,$$

$$\pi(S_{m,s}) = 2^{-\frac{ms}{2}} \prod_{j=1}^{-s-1} \prod_{i=1}^{-m} \left\{ \prod_{i+j\in 2\mathbb{Z}} Q(\frac{i}{2}+\frac{j}{2}t) \prod_{i+j\in 2\mathbb{Z}+1} P(-\frac{i-1}{2}-\frac{j}{2}t) \right\} e^{-m},$$

if $m<0,\ s<0$.

Proof. From $\pi(f_1)=f$ and $\pi(f_0)=\frac{1}{2}E=\frac{1}{4}[e,e]$, we see that

$$\pi(S_{m,s}) = \begin{cases} 2^{-\frac{ms}{2}} fF^{\frac{(m-1)+st}{2}} E^{\frac{m+(s-1)t}{2}} \cdots fF^{\frac{(m-1)-st}{2}} & \text{if } m>0,\ s\geq 0, \\ 2^{-\frac{ms}{2}} E^{-\frac{m+(s+1)t}{2}} fF^{-\frac{(m+1)+(s+2)t}{2}} \cdots E^{-\frac{(m+1)-(s+1)t}{2}} & \text{if } m<0,\ s<0. \end{cases}$$

Here we show the formula of $\pi(S_{m,s})$ in the case of $m>0$, $s\geq 0$ and $m\equiv 1 \mod 2$. The other cases can be calculated similarly. In this case, $\pi(S_{m,s})$ has the form

$$\pi(S_{m,s}) = 2^{-\frac{ms}{2}} fF^{\frac{(m-1)+st}{2}} E^{\frac{m+(s-1)t}{2}} \cdots E^{\frac{m+t}{2}} fF^{\frac{m-1}{2}} E^{\frac{m-t}{2}} \cdots \\ \cdots E^{\frac{m-(s-1)t}{2}} fF^{\frac{(m-1)+st}{2}}.$$

We concentrate on the part $E^{\frac{m+t}{2}} fF^{\frac{m-1}{2}} E^{\frac{m-t}{2}} = E^{\frac{m+t}{2}} f^m E^{\frac{m-t}{2}}$. Since $E^\gamma f = Q(\gamma) e E^{\gamma-1}$ and $eE^\gamma f = P(-\gamma)E^\gamma$ for any $\gamma\in\mathbb{C}$, we obtain

$$E^\gamma f^m = \{ \prod_{i=1}^{\frac{m+1}{2}} Q(\gamma-i+1) \prod_{i=1}^{\frac{m-1}{2}} P(-\gamma+i) \} e E^{\gamma-\frac{m+1}{2}},$$

and then we have

$$E^{\frac{m+t}{2}} f^m E^{\frac{m-t}{2}} = \{ \prod_{i=1}^{\frac{m+1}{2}} Q(\frac{m+t}{2}-i+1) \prod_{i=1}^{\frac{m-1}{2}} P(-\frac{m+t}{2}+i) \} e^m.$$

Here we set

$$X = \prod_{i=1}^{\frac{m+1}{2}} Q(\frac{m+t}{2}-i+1) \prod_{i=1}^{\frac{m-1}{2}} P(-\frac{m+t}{2}+i),$$
$$Y^+ = 2^{-\frac{ms}{2}} fF^{\frac{(m-1)+st}{2}} E^{\frac{m+(s-1)t}{2}} \cdots fF^{\frac{(m-1)+2t}{2}},$$
$$Y^- = fF^{\frac{(m-1)-2t}{2}} \cdots E^{\frac{m-(s-1)t}{2}} fF^{\frac{(m-1)-st}{2}}.$$

Using the commutation relations

$$E^\gamma P(s) = P(s-\gamma)E^\gamma,\ E^\gamma Q(s) = Q(s+\gamma)E^\gamma,$$
$$fF^\gamma P(s) = Q(-s-\gamma)fF^\gamma,\ fF^\gamma Q(s) = P(-s+\gamma+1)fF^\gamma,$$

4.4. Proof of Theorem 4.1 and 4.2. In this subsection, we give the proof of our main theorems. For this purpose we need explicit forms of images of singular vectors under the evaluation map from $\tilde{\mathfrak{g}}$ to $\mathfrak{g}$. In the case of $\tilde{\mathfrak{g}}$, the singular vector formula in Subsection 2.3 is explicitly written as follows: Let k be an admissible or generic level. For $m,\ s \in \mathbb{Z}$ such that $m+s \in 2\mathbb{Z}+1$ and either $m>0,\ s \geq 0$ or $m<0,\ s<0$, we set $\lambda = \lambda_{m,s}$ and $\Lambda = \Lambda_{\lambda,k}$. The Verma module $M(\Lambda)$ has a singular vector $F(r_{\alpha_{m,s}};\Lambda)|\Lambda\rangle$, where $|\Lambda\rangle$ is a highest weight vector and the real root $\alpha_{m,s}$ is given by

$$\alpha_{m,s} = \begin{cases} \alpha + \dfrac{1}{2}s\delta & \text{if } m>0,\ s\geq 0,\ s\in 2\mathbb{Z}, \\ 2\alpha + s\delta & \text{if } m>0,\ s\geq 0,\ s\in 2\mathbb{Z}+1, \\ -\alpha - \dfrac{1}{2}s\delta & \text{if } m<0,\ s<0,\ s\in 2\mathbb{Z}, \\ -2\alpha - s\delta & \text{if } m<0,\ s<0,\ s\in 2\mathbb{Z}+1. \end{cases}$$

If we put $S_{m,s} = F(r_{\alpha_{m,s}};\Lambda)$ then $S_{m,s}$ is explicitly written as

$$S_{m,s} = f_1({f_1}^2)^{\frac{(m-1)+st}{2}} {f_0}^{\frac{m+(s-1)t}{2}} f_1({f_1}^2)^{\frac{(m-1)+(s-2)t}{2}} \cdots \\ \cdots {f_0}^{\frac{m-(s-1)t}{2}} f_1({f_1}^2)^{\frac{(m-1)-st}{2}},$$

if $m>0,\ s\geq 0$, and

$$S_{m,s} = {f_0}^{-\frac{m+(s+1)t}{2}} f_1({f_1}^2)^{-\frac{(m+1)+(s+2)t}{2}} {f_0}^{-\frac{m+(s+3)t}{2}} \cdots \\ \cdots f_1({f_1}^2)^{-\frac{(m+1)-(s+2)t}{2}} {f_0}^{-\frac{(m+1)-(s+1)t}{2}},$$

if $m<0,\ s<0$.

For $z\in\mathbb{C}^\times$, let $\pi_z : \tilde{\mathfrak{g}} \to \mathfrak{g}$ be an evaluation map such that

$$\pi_z(e_1) = e,\ \pi_z(f_1) = f,\ \pi_z(h_1) = h, \\ \pi_z(e_0) = -\frac{1}{2}zF,\ \pi_z(f_0) = \frac{1}{2}z^{-1}E,\ \pi_z(h_0) = -\frac{1}{2}h.$$

In particular we denote $\pi_1 : \tilde{\mathfrak{g}} \to \mathfrak{g}$ by π. To express the image $\pi(S_{m,s})$ explicitly, we set $P(s) = ef + 2s$ and $Q(s) = fe - 2s$ for $s\in\mathbb{C}$. It is easy to see that $[P(s),P(s')]=0$, $[P(s),Q(s')]=0$ and $[Q(s),Q(s')]=0$ for any $s,\ s'\in\mathbb{C}$. We have

Lemma 4.2.

$$\pi(S_{m,s}) = 2^{-\frac{ms}{2}} \prod_{j=1}^{s}\prod_{i=1}^{m} \left\{ \prod_{i+j\in 2\mathbb{Z}} P(\frac{i}{2}+\frac{j}{2}t) \prod_{i+j\in 2\mathbb{Z}+1} Q(-\frac{i-1}{2}-\frac{j}{2}t) \right\} f^m,$$

if $m>0,\ s\geq 0$, *and*

Case 2 (The fusion algebra at an admissible level). Fix the admissible level $k = p/q - 3$ and take the set of the admissible weight

$$\Lambda^k = \{\lambda_{m,s} |\ m + s \equiv 1 \pmod 2,\ 1 \leq m \leq p-1,\ 0 \leq s \leq q-1\}$$

as the index set $\mathcal{S}_k$ of the fusion algebra $\mathcal{A}_k$. Similarly to the generic level case, the fusion algebra at the admissible level k can be also described by using pairs of irreducible modules for $osp(1|2)$ and sl_2. Since the admissible representation $L_{\lambda,k}$ has the resolution

$$\cdots \to M_1 \oplus M_2 \to M_{\lambda,k} \to L_{\lambda,k} \to 0, \tag{19}$$

two different modules $M_{\lambda,k}/M_1$ and $M_{\lambda,k}/M_2$ project onto the same representation $L_{\lambda,k}$. In the generic level case, these two quotient modules corresponds to (V_i^0, V_j) and $(V^1_{-i+q-1}, V_{-j+p-2})$ respectively. Therefore we introduce the equivalence relation $\approx$ by

$$(V_i^0, V_j) \approx (V^1_{-i+q-1}, V_{-j+p-2}) \quad 0 \leq i \leq q-1,\ 0 \leq j \leq p-2,$$

and denote the equivalence class of (V_i^ϵ, V_j) by $[(V_i^\epsilon, V_j)]$. Further we assign the equivalence class $[(V_m^0, V_{s-1})]$ to the element $l_{\lambda_{m,s}} \in \mathcal{A}_k$. This gives a one-to-one correspondence between the $\mathbb{Z}$-basis $\{l_\lambda | \lambda \in \mathcal{S}_k\}$ of $\mathcal{A}_k$ and the set of the symbols $[(V_i^\epsilon, V_j)]$. In this case, to describe the fusion rule, it is convenient to use the so-called Kazhdan-Lusztig fusion functor. Let us recall that

$$\begin{aligned} V_i^\epsilon \dot{\otimes}_k V_j^\eta &\simeq V^{\epsilon+\eta}_{|i-j|} \oplus V^{\epsilon+\eta-1}_{|i-j|+1} \oplus V^{\epsilon+\eta}_{|i-j|+2} \oplus \cdots \oplus V^{\epsilon+\eta}_{\min\{2k-i-j,\ i+j\}}, \\ V_i \dot{\otimes}_k V_j &\simeq V_{|i-j|} \oplus V_{|i-j|+2} \oplus V_{|i-j|+4} \oplus \cdots \oplus V_{\min\{2k-i-j,\ i+j\}}. \end{aligned} \tag{20}$$

Theorem 4.2. *The structure of the fusion algebra at an admissible level $k = p/q - 3$ can be written as follows:*

$$[(V_{i_0}^{\epsilon_0}, V_{j_0})] \circ [(V_{i_1}^{\epsilon_1}, V_{j_1})] = [(V_{i_0}^{\epsilon_0} \dot{\otimes}_{q-1} V_{i_1}^{\epsilon_1}, V_{j_0} \dot{\otimes}_{p-2} V_{j_1})].$$

Remark 4.1. (1) *In the case of $p,\ q \equiv 1 \mod 2$, (i.e. principal admissible case), the structure constants of the fusion algebra are 0 or 1. The fusion algebra of this case coincides with the one given by Ennes and Ramallo* [ER].

(2) *In the case of $p,\ q \equiv 0 \mod 2$ (i.e. non-principal admissible case), some of the structure constants are equal to 2 (see Theorem 4.4).*

Remark 4.2. *As in* [FZ], *one can also introduce super vertex operator algebra structure on $L_{0,k}$ (see* [KWn]*). It is easy to see that the super vertex operator algebra $L_{0,k}$ ($\Re k > -3$) is rational if and only if k is an admissible level.*

a set of distinct points $E = \{w_1, \cdots, w_m\}$, we introduce a $\mathbb{Z}$-multilinear symmetric form $\Phi_m : \bigotimes^m \mathcal{A}_k \to \mathbb{Z}$ by

$$\begin{aligned}&\Phi_m(l_{\lambda_1}, \cdots, l_{\lambda_m})\\ &\quad = \dim H_0(\mathfrak{g}(w_1, \cdots, w_m), L_{\lambda_1,k}(w_1) \otimes \cdots \otimes L_{\lambda_m,k}(w_m)).\end{aligned}$$

It follows from Lemma 4.1 that the form Φ_2 is non-degenerate. Hence there exists an isomorphism $\mathcal{A}_k \simeq \mathcal{A}_k^*$ (restricted dual) and the basis $\{l_\lambda | \lambda \in \mathcal{S}_k\}$ is self-dual. Using the form Φ_3 and the isomorphism $\mathcal{A}_k \simeq \mathcal{A}_k^*$, we introduce a multiplication on $\mathcal{A}_k$. By definition, we have a commutative algebraic structure on the $\mathbb{Z}$-free module $\mathcal{A}_k$, and call it a fusion algebra. Our aim is to present an explicit description of the multiplication rule of this fusion algebra. In the sequel, we sometimes refer to the element $l_\lambda \in \mathcal{A}_k$ as the irreducible representation $L_{\lambda,k}$.

The structure of the fusion algebra at a generic level is simpler than that at an admissible level. We first state our main result in the generic level case.

Case 1 (The fusion algebra at a generic level). First of all, we fix the generic level k such that $k > -3$, and set

$$\begin{aligned}\mathcal{S}_k = \{\lambda_{m,s} | \quad & m + s \equiv 1 \pmod 2, \text{ and}\\ & m > 0,\ s \geq 0 \ \text{ or } \ m < 0,\ s < 0\}.\end{aligned}$$

Remark that $L_{\lambda,k}$ is a generalized Weyl module if and only if $\lambda \in \mathcal{S}_k = \Lambda^k$. To describe the multiplication rule of the fusion algebra explicitly, we use the following notation: For $i, j \in \mathbb{Z}_{\geq 0}$ and $\epsilon \in \mathbb{Z}/2\mathbb{Z}$, let V_i^ϵ be the irreducible $(2i+1)$-dimensional representation of $osp(1|2)$ whose highest weight vector has degree ϵ, and let V_j be the irreducible $(j+1)$-dimensional representation of sl_2. For $\lambda = \lambda_{m,s}$, we assign the symbol

$$\begin{aligned}&(V_s^0, V_{m-1}) && \text{if } m > 0,\ s \geq 0,\\ &(V_{-s-1}^1, V_{-m-1}) && \text{if } m < 0,\ s < 0,\end{aligned}$$

to the element $l_\lambda \in \mathcal{A}_k$. This is a one-to-one correspondence between the $\mathbb{Z}$-basis $\{l_\lambda | \lambda \in \mathcal{S}_k\}$ of $\mathcal{A}_k$ and the set of the symbols (V_i^ϵ, V_j) for $i,\ j \in \mathbb{Z}_{\geq 0}$ and $\epsilon \in \mathbb{Z}/2\mathbb{Z}$. We extend this notation by $\mathbb{Z}$-bilinearlity. Let us give the multiplication formula by using the following Clebsh-Gordan rule for $osp(1|2)$ and sl_2.

$$\begin{aligned}&V_i^\epsilon \otimes V_j^\eta \simeq V_{|i-j|}^{\epsilon+\eta} \oplus V_{|i-j|+1}^{\epsilon+\eta-1} \oplus V_{|i-j|+2}^{\epsilon+\eta} \oplus \cdots \oplus V_{i+j}^{\epsilon+\eta},\\ &V_i \otimes V_j \simeq V_{|i-j|} \oplus V_{|i-j|+2} \oplus V_{|i-j|+4} \oplus \cdots \oplus V_{i+j}.\end{aligned} \tag{18}$$

Theorem 4.1. *The structure of the fusion algebra at a generic level can be written as follows:*

$$(V_{i_0}^{\epsilon_0}, V_{j_0}) \circ (V_{i_1}^{\epsilon_1}, V_{j_1}) = (V_{i_0}^{\epsilon_0} \otimes V_{i_1}^{\epsilon_1}, V_{j_0} \otimes V_{j_1}).$$

Lemma 4.1. *Let w_1, w_2 be distinct points in $\mathbb{C}P^1$. For $k \in \mathbb{C}$ such that $k > -3$ and λ_1, $\lambda_2 \in \mathfrak{h}^*(\simeq \mathbb{C})$, we have*

$$\dim H_0(\mathfrak{g}(w_1, w_2), L_{\lambda_1,k}(w_1) \otimes L_{\lambda_2,k}(w_2)) = \begin{cases} 1 & \text{if } \lambda_1 = \lambda_2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is easy to see that the coinvariant of the theorem does not depend on the choice of the points w_1 and w_2. Hence we can set $w_1 = 0$ and $w_2 = \infty$ without loss of generality. First we prove that

$$\dim H_0(\mathfrak{g}(0, \infty), M_{\lambda_1,k}(0) \otimes L_{\lambda_2,k}(\infty)) = \delta_{\lambda_1,\lambda_2}. \tag{17}$$

Let $\tilde{\mathfrak{b}} = \tilde{\mathfrak{b}}_0$ be the Borel subalgebra of $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_0$ defined in (13), and $\mathbb{C}_{0,\lambda_1,k}$ the 1-dimensional representation of $\tilde{\mathfrak{b}}$ in (15). Note that $\mathfrak{g}(0, \infty) \simeq \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ and then

$$\mathrm{Ind}_{U(\tilde{\mathfrak{b}} \cap \mathfrak{g}(0,\infty))}^{U(\mathfrak{g}(0,\infty))}(\mathbb{C}_{0,\lambda_1,k} \otimes L_{\lambda_2,k}(\infty)) = M_{\lambda_1,k}(0) \otimes L_{\lambda_2,k}(\infty).$$

By Shapiro's lemma, we have

$$\begin{aligned} H_0(\mathfrak{g}(0, \infty), M_{\lambda_1,k}(0) \otimes L_{\lambda_2,k}(\infty)) &= H_0(\tilde{\mathfrak{b}} \cap \mathfrak{g}(0, \infty), \mathbb{C}_{0,\lambda_1,k} \otimes L_{\lambda_2,k}(\infty)) \\ &= H_0(\tilde{\mathfrak{n}}, L_{\lambda_2,k}(\infty))^{(\lambda_1)}, \end{aligned}$$

where $\tilde{\mathfrak{n}} = [\tilde{\mathfrak{b}}, \tilde{\mathfrak{b}}] = \mathbb{C}e \oplus \mathbb{C}E \oplus (\mathfrak{g} \otimes z\mathbb{C}[z])$ and $H_0(\tilde{\mathfrak{n}}, L_{\lambda_2,k}(\infty))^{(\lambda_1)}$ denotes the eigenspace of h with the eigenvalue λ_1. By definition, we have $\dim H_0(\tilde{\mathfrak{n}}, L_{\lambda_2,k}(\infty))^{(\lambda_1)} = \delta_{\lambda_1,\lambda_2}$ and thus we obtain (17). Next we show the lemma by using (17). It follows from Proposition 3.1 and Theorem 3.1 that, in the case of $k > -3$, there exists the following exact sequence:

$$M \to M_{\lambda_1,k}(0) \to L_{\lambda_1,k}(0) \to 0,$$

where M is a Verma module, the direct sum of two Verma modules or $M = (0)$. From this we obtain

$$M \otimes L_{\lambda_2,k}(\infty) \to M_{\lambda_1,k}(0) \otimes L_{\lambda_2,k}(\infty) \to L_{\lambda_1,k}(0) \otimes L_{\lambda_2,k}(\infty) \to 0,$$

and further

$$\begin{aligned} H_0(\mathfrak{g}(0, \infty), M \otimes L_{\lambda_2,k}(\infty)) &\to \\ H_0(\mathfrak{g}(0, \infty), M_{\lambda_1,k}(0) \otimes L_{\lambda_2,k}(\infty)) &\to \\ H_0(\mathfrak{g}(0, \infty), L_{\lambda_1,k}(0) \otimes L_{\lambda_2,k}(\infty)) &\to 0, \end{aligned}$$

The lemma is an immediate consequence of this exact sequence and (17). Q.E.D.

We define fusion algebras through coinvariants [FF], [FM1], [FM2]. First we fix a complex number, say k. Let $\mathcal{S}_k$ be a subset (at most countable) of $\mathfrak{h}^*(\simeq \mathbb{C})$. We take the $\mathbb{Z}$-free module $\mathcal{A}_k$ with the basis $\{l_\lambda\}$ labeled by the elements $\lambda \in \mathcal{S}_k$. For

and introduce the natural $\mathfrak{g} \otimes \mathbb{C}[z_w, z_w^{-1}]$-module structure on it. Let $\tilde{\mathfrak{n}}_-(w)$ be the nilpotent subalgebra of $\mathfrak{g}_w$ given by

$$\tilde{\mathfrak{n}}_-(w) = \mathbb{C}f_w \oplus \mathbb{C}F_w \oplus \mathfrak{g} \otimes z_w^{-1}\mathbb{C}[z_w^{-1}].$$

For each $\epsilon = 0, 1$, we introduce the $\tilde{\mathfrak{n}}_-(w)$-submodule $\tilde{\mathcal{H}}^{(\epsilon)}_{\lambda,\mu}(w)$ of $\tilde{\mathcal{F}}_{\lambda,\mu}(w)$ as follows:

$$\tilde{\mathcal{H}}^{(\epsilon)}_{\lambda,\mu}(w) = \left(\bigoplus_{\substack{(k,j) \in \mathbb{Z}^2, \\ 2k+2j < \epsilon, \\ \text{or } j<0}} \mathbb{C}F_k\theta \otimes z_w^j \right) \oplus \left(\bigoplus_{\substack{(k,j) \in \mathbb{Z}^2, \\ 2k+2j < \epsilon-1, \\ \text{or } j<0}} \mathbb{C}F_k \otimes z_w^j \right)$$

and denote the quotient $\tilde{\mathfrak{n}}_-(w)$-module $\tilde{\mathcal{F}}_{\lambda,\mu}(w)/\tilde{\mathcal{H}}^{(\epsilon)}_{\lambda,\mu}(w)$ by $\tilde{\mathcal{G}}^{(\epsilon)}_{\lambda,\mu}(w)$. By abuse of notation, we use the same notation to denote the images in $\tilde{\mathcal{G}}^{(\epsilon)}_{\lambda,\mu}(w)$.

4.3. Main result. From now on, we construct fusion algebras based on coinvariants of a Lie superalgebra of $\mathfrak{g}$-valued meromorphic functions on $\mathbb{C}P^1$. Let $E = \{w_1, w_2, \cdots, w_m\}$ be a finite subset E of $\mathbb{C}P^1$. We set $\mathfrak{g}_E = \bigoplus_{w \in E} \mathfrak{g}_w$ and define the cocycle $\langle , \rangle_E$ of $\mathfrak{g}_E$ by $\langle , \rangle_E = \sum_{w \in E} \langle , \rangle_w$. Let $\tilde{\mathfrak{g}}_E$ be the central extension of $\mathfrak{g}_E$ by the cocycle $\langle , \rangle_E$. Let $\mathfrak{g}(E)$ be the Lie superalgebra of the $\mathfrak{g}$-valued meromorphic functions over $\mathbb{C}P^1$ which is holomorphic outside of E. Using the Laurent expansion, we define a Lie superalgebra homomorphism

$$T : \mathfrak{g}(E) \longrightarrow \mathfrak{g}_E.$$

It follows from the residue theorem that T lifts to the homomorphism:

$$\tilde{T} : \mathfrak{g}(E) \longrightarrow \tilde{\mathfrak{g}}_E. \tag{16}$$

Let $W_i(w_i)$ be a highest weight representation of $\tilde{\mathfrak{g}}_{w_i}$ for $w_i \in E$. We remark that $\tilde{\mathfrak{g}}_E$ acts on the tensor product $W_1(w_1) \otimes \cdots \otimes W_m(w_m)$ in the natural way. The pull-back with respect to the homomorphism $\tilde{T}$ induces a $\mathfrak{g}(E)$-module structure on this tensor product. In the sequel, our main objects are the following spaces of coinvariants:

$$\begin{aligned} &H_0(\mathfrak{g}(E), W_1(w_1) \otimes \cdots \otimes W_m(w_m)) \\ &\quad = W_1(w_1) \otimes \cdots \otimes W_m(w_m)/\mathfrak{g}(E)(W_1(w_1) \otimes \cdots \otimes W_m(w_m)). \end{aligned}$$

Using Proposition 3.1 and Theorem 3.1, we can prove the following lemma, which is related to the self-duality of fusion algebras.

For $p(z_w),\ q(z_w) \in \mathbb{C}((z_w))$ and $x,\ y \in \mathfrak{g}$, we define the cocycle $\langle , \rangle_w$ of $\mathfrak{g}_w$ by

$$\langle p(z_w) \otimes x, q(z_w) \otimes y \rangle_w = (x, y) \operatorname{Res}_{z_w=0} p'(z_w) q(z_w) dz_w.$$

where $p'(z_w) = \frac{d}{dz_w} p(z_w)$. $\tilde{\mathfrak{g}}_w$ denotes the unique central extension of $\mathfrak{g}_w$ by the cocycle $\langle , \rangle_w$.

Next we introduce a Verma module attached to a point of $\mathbb{C}P^1$. From [K4] there is a one-to-one correspondence between $\mathbb{C}P^1$ and the conjugacy classes of the Borel subalgebras of $\mathfrak{g}$. Therefore, for each point w, we associate a Borel subalgebra $\mathfrak{b}_w$ of $\mathfrak{g}$ as follows: We set

$$F_w = F,\ f_w = f,\ h_w = h + 2wF,\ e_w = e + wf,\ E_w = E + wh + w^2 F, \tag{12}$$

$$\mathfrak{b}_w = \mathbb{C}h_w \oplus \mathbb{C}e_w \oplus \mathbb{C}E_w,$$

for $w \in \mathbb{C}P^1 \setminus \{\infty\}$, and

$$F_\infty = E,\ f_\infty = -e,\ h_\infty = -h,\ e_\infty = f,\ E_\infty = F,$$

$$\mathfrak{b}_\infty = \mathbb{C}h_\infty \oplus \mathbb{C}e_\infty \oplus \mathbb{C}E_\infty.$$

In fact we have $[h_w, e_w] = 2e_w$, $[e_w, e_w]/2 = E_w$ and $[\mathfrak{b}_w, \mathfrak{b}_w] = \mathbb{C}e_w \oplus \mathbb{C}E_w$. For each $w \in \mathbb{C}P^1$, let us fix a Borel subalgebra $\tilde{\mathfrak{b}}_w$ of $\tilde{\mathfrak{g}}_w$ as follows:

$$\tilde{\mathfrak{b}}_w = \mathfrak{b}_w \oplus (\mathfrak{g} \otimes z_w \mathbb{C}[[z_w]]) \oplus \mathbb{C}c. \tag{13}$$

For $\lambda \in \mathfrak{h}^* (\simeq \mathbb{C})$, we define the character $\phi_{w,\lambda}$ of $\mathfrak{b}_w$ by

$$\phi_{w,\lambda}(h_w) = \lambda,\ \phi_{w,\lambda}(e_w) = \phi_{w,\lambda}(E_w) = 0. \tag{14}$$

Let $k \in \mathbb{C}$. Define the 1-dimensional $\tilde{\mathfrak{b}}_w$-module $\mathbb{C}_{\lambda,w,k} = \mathbb{C}\mathbf{1}_{w,\lambda,k}$ as follows:

$$\begin{array}{ll} b\mathbf{1}_{w,\lambda,k} = \phi_{w,\lambda}(b)\mathbf{1}_{w,\lambda,k} & \text{if } b \in \mathfrak{b}_w, \\ b\mathbf{1}_{w,\lambda,k} = 0 & \text{if } b \in \mathfrak{g} \otimes z_w \mathbb{C}[[z_w]], \\ c\mathbf{1}_{w,\lambda,k} = k\mathbf{1}_{w,\lambda,k}. & \end{array} \tag{15}$$

Let us define the Verma module $M_{\lambda,k}(w)$ attached to a point $w \in \mathbb{C}P^1$ as the induced module $\operatorname{Ind}_{U(\tilde{\mathfrak{b}}_w)}^{U(\tilde{\mathfrak{g}}_w)} \mathbb{C}_{\lambda,w,k}$. Remark that our fusion algebras do not depend on a choice of parity of a highest weight vector. Here we do not specify it. It is clear that any highest weight representation of $\tilde{\mathfrak{g}}_w$ is obtained as a quotient of some Verma module. In particular we denote the irreducible quotient of $M_{\lambda,k}(w)$ by $L_{\lambda,k}(w)$.

Next we define loop modules attached to $w \in \mathbb{C}P^1$. For $\lambda,\ \mu \in \mathbb{C}$, set $\mathcal{F}_{\lambda,\mu}(w) = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}F_i \oplus \mathbb{C}F_i\theta$ and define an action of $\mathfrak{g}$ on $\mathcal{F}_{\lambda,\mu}(w)$ by replacing x with x_w $(x = h, e, f)$ in (10). Furthermore, we define the space $\tilde{\mathcal{F}}_{\lambda,\mu}(w)$ by

$$\mathcal{F}_{\lambda,\mu}(w) \otimes \mathbb{C}[z_w, z_w^{-1}],$$

we identify $\mathfrak{h}^* = \mathbb{C}\alpha_1$ with $\mathbb{C}$ via $\frac{1}{2}\lambda\alpha_1 \mapsto \lambda$ for simplicity. Let Λ^k be the subset of $\mathfrak{h}^*(\simeq \mathbb{C})$ such that $\Lambda_{\lambda,k}$ gives an admissible weight. In fact, we have

$$\Lambda^k = \{\lambda_{m,s} |\ m+s \equiv 1 \pmod 2,\ 1 \le m \le p-1,\ 0 \le s \le q-1\}.$$

We remark that, for each admissible weight $\Lambda = \Lambda_{\lambda,k}$ such that $k+3 = p/q$ and $\lambda = \lambda_{m,s}$, the set Π_Λ is given as follows:

$$\begin{aligned}
\Pi_\Lambda &= \{2\alpha + s\delta,\ -\alpha + \frac{1}{2}(q-s)\delta\} \quad \text{if } p,\ q \in 2\mathbb{Z}+1,\ m \in 2\mathbb{Z},\\
\Pi_\Lambda &= \{\alpha + \frac{1}{2}s\delta,\ -2\alpha + (q-s)\delta\} \quad \text{if } p,\ q \in 2\mathbb{Z}+1,\ m \in 2\mathbb{Z}+1,\\
\Pi_\Lambda &= \{2\alpha + s\delta,\ -2\alpha + (q-s)\delta\} \quad \text{if } p,\ q \in 2\mathbb{Z},\ m \in 2\mathbb{Z},\\
\Pi_\Lambda &= \{\alpha + \frac{1}{2}s\delta,\ -\alpha + \frac{1}{2}(q-s)\delta\} \quad \text{if } p,\ q \in 2\mathbb{Z},\ m \in 2\mathbb{Z}+1.
\end{aligned}$$

For an affine Kac-Moody superalgebra $\mathfrak{g}(A,\tau)$, an admissible weight Λ is called principal admissible if the contragredient Lie superalgebra with the Cartan matrix $(\langle \beta_i^\vee, \beta_j \rangle)_{\beta_0,\, \beta_1 \in \Pi_\Lambda}$ is isomorphic to $\mathfrak{g}(A,\tau)$ [KW2]. In the case of $o\hat{s}p(1|2)$, an admissible weight $\Lambda_{\lambda,k}$ with $k = p/q - 3$ is principal admissible if and only if $p,\ q \equiv 1 \pmod 2$.

To calculate Fusion algebra, we need the following representation of $o\hat{s}p(1|2)$ called a loop module, which is neither highest weight nor lowest weight module. Let θ be a Grassmann variable i.e. $\theta^2 = 0$. We fix $\lambda,\ \mu \in \mathbb{C}$, and set $\mathcal{F}_{\lambda,\mu} = \bigoplus_{i\in\mathbb{Z}} \{\mathbb{C}F_i \oplus \mathbb{C}F_i\theta\}$. $\mathfrak{g}$ acts on the space $\mathcal{F}_{\lambda,\mu}$ via

$$\begin{aligned}
h.F_i\theta^\gamma &= (2\mu + 4i - 2\gamma - \lambda)F_i\theta^{\gamma-1},\\
e.F_i\theta^\gamma &= \gamma F_i\theta^{\gamma-1} + (\mu + 2i - \lambda)F_{i+1}\theta^{\gamma+1},\\
f.F_i\theta^\gamma &= \gamma F_{i-1}\theta^{\gamma-1} + (\mu + 2i)F_i\theta^{\gamma+1},
\end{aligned} \tag{10}$$

where $i \in \mathbb{Z}$ and $\gamma = 0, 1$. We define a $\tilde{\mathfrak{g}}$-module structure on the space $\tilde{\mathcal{F}}_{\lambda,\mu} = \mathcal{F}_{\lambda,\mu} \otimes \mathbb{C}[z, z^{-1}]$ in a natural way, and call it a loop module.

4.2. $o\hat{s}p(1|2)$ attached to points on $\mathbb{C}P^1$. We denote by $\mathbb{C}[[t]]$ and $\mathbb{C}((t))$ the completions of $\mathbb{C}[t]$ and $\mathbb{C}[t, t^{-1}]$ by the positive powers of t. For $w \in \mathbb{C}P^1$, let $\mathfrak{g}_w$ be the Lie superalgebra of the $\mathfrak{g}$-valued functions over the formal neighborhood of w i.e.

$$\mathfrak{g}_w \simeq \mathfrak{g} \otimes \mathbb{C}((z_w)),$$

where z_w is the local coordinate in a neighborhood of w defined by

$$z_w = \begin{cases} z - w & \text{if } w \neq \infty,\\ z^{-1} & \text{if } w = \infty. \end{cases} \tag{11}$$

for $m,\ s \in \mathbb{Z}$ such that $m+s \in 2\mathbb{Z}+1$ and either $m>0,\ s \geq 0$ or $m<0,\ s<0$ holds. We set $\Lambda_{\lambda,k} = \frac{1}{2}(k-\lambda)\Lambda_0 + \lambda\Lambda_1$ and $\lambda_{m,s} = m - st - 1$ $(t = k+3)$. Let us denote the Verma module $M(\Lambda_{\lambda,k})$ and its irreducible quotient by $M_{\lambda,k}$ and $L_{\lambda,k}$ respectively.

In this subsection, we consider two classes of irreducible representations. The one is called generalized Weyl modules and the other is called admissible representations. We compute fusion algebras based on the representations in these two classes respectively.

Case 1 **:** We consider the fusion algebra related with irreducible highest weight representations of a generic (not rational) level. In this case, let k be a complex number such that $k \notin \mathbb{Q}$ and $k > -3$, where the order $>$ is defined as in Subsection 2.1. We call such a level k generic. Here we summarize the structure of Verma modules of a generic level. If the Verma module $M_{\lambda,k}$ is reducible then $\lambda = \lambda_{m,s}$ for $m,\ s \in \mathbb{Z}$ such that $m+s \equiv 1 \pmod 2$ and either $m>0,\ s \geq 0$ or $m<0,\ s<0$ holds, and $M_{\lambda,k}$ has the unique singular vector up to scalar. Let $J_{\lambda,k}$ be the submodule of $M_{\lambda,k}$ generated by this singular vector. In this case, $J_{\lambda,k}$ coincides with the maximal proper submodule of $M_{\lambda,k}$ and thus $M_{\lambda,k}/J_{\lambda,k}$ is irreducible i.e.

$$0 \to J_{\lambda,k} \to M_{\lambda,k} \to L_{\lambda,k} \to 0, \tag{9}$$

is an exact sequence. This irreducible quotient $L_{\lambda,k}$ is called a generalized Weyl module. For a generic level k, let Λ^k be the set of the highest weights of the generalized Weyl modules i.e.

$$\Lambda^k = \{\lambda_{m,s} |\ \ m+s \equiv 1 \pmod 2,\ \text{and}\ \ m>0,\ s \geq 0 \ \text{ or } \ m<0,\ s<0\}.$$

Case 2 **:** Similarly to the non-super case, there is an important class of the irreducible highest weight modules whose characters have the modular invariance property. These representations are called admissible representations. Following Kac and Wakimoto [KW1], we recall them briefly. A weight $\Lambda \in \hat{\mathfrak{h}}^*$ is called an admissible weight if it satisfies

$$\langle \Lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{>0} \text{ for any } \alpha^\vee \in \Delta_\Lambda^{+,\vee},$$

and

$$\mathbb{Q}\Pi_\Lambda = \mathbb{Q}\Pi.$$

It is easy to see that, for any admissible weight Λ, the level $k = \langle \Lambda, c \rangle$ is a rational number greater than -3. Hence, for some positive integers $p,\ q$ such that $p \equiv q \pmod 2$ and $\left(\frac{p-q}{2}, q\right) = 1$, we can uniquely express as $k+3 = p/q$. Such a level k is called an admissible level. For a fixed admissible level k, the set of the admissible weights of level k can be described as follows: Here

4.1. Preliminaries. Through this subsection, let $\mathfrak{g}$ be the simple Lie superalgebra $osp(1|2)$. $\mathfrak{g}$ is one of the simplest Kac-Moody Lie superalgebra defined by 1×1 Cartan matrix (2) and $\tau = \{1\}$. The generators e, f and h have the $\mathbb{Z}_2$-gradation and commutation relations

$$\deg h = 0,\ \deg e = \deg f = 1,\ [h,e] = 2e,\ [h,e] = -2f,\ [e,f] = h.$$

Set $E = \frac{1}{2}[e,e]$ and $F = \frac{1}{2}[f,f]$. Then $\{E,\ e,\ h,\ f,\ F\}$ form a bases of $\mathfrak{g}$. Fix a Cartan subalgebra of $\mathfrak{g}$ by $\mathfrak{h} = \mathbb{C}h$, and set $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathbb{C}e \oplus \mathbb{C}E$.

$o\hat{s}p(1|2)$ is the untwisted affine Lie superalgebra:

$$\mathfrak{g} \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

with the $\mathbb{Z}_2$-gradation

$$\deg(x \otimes z^n) = \deg x\ (x \in \mathfrak{g}),\ \deg c = \deg d = 0,$$

and commutation relations

$$[x \otimes z^m, y \otimes z^n] = [x,y] \otimes z^{m+n} + m\delta_{m+n,0}(x,y)c,$$
$$[d, x \otimes z^m] = mx \otimes z^m,\ \ [c, o\hat{s}p(1|2)] = 0,$$

where $x,\ y \in \mathfrak{g}$ and $(,)$ is the non-degenerate invariant bilinear form on $\mathfrak{g}$ normalized by $(h,h) = 4$.

Though an affine Lie superalgebra is not always a Kac-Moody superalgebra, $o\hat{s}p(1|2)$ is a Kac-Moody superalgebra. Indeed its Cartan matrix is

$$\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix},$$

and $\tau = \{1\}$, where we take $I = \{0,1\}$ as the index set of the simple roots. Its generators e_i, f_i and h_i $(i \in I)$ are given by

$$e_1 = e,\ f_1 = f,\ h_1 = h, e_0 = -\frac{1}{2}F \otimes z,\ f_0 = \frac{1}{2}E \otimes z^{-1},\ h_0 = \frac{1}{2}(c-h),$$

where we identify $x \otimes 1$ with x for $x \in \mathfrak{g}$. We fix a Cartan subalgebra of $o\hat{s}p(1|2)$ as follows: $\hat{\mathfrak{h}} = \mathbb{C}h \oplus \mathbb{C}c \oplus \mathbb{C}d$. Let δ and Λ_i $(i \in I)$ be the generator of the positive imaginary roots and the fundamental weights respectively. Notice that $c = 2h_0 + h_1$, $\delta = \alpha_0 + 2\alpha_1$, and hence the dual Coxeter number of $o\hat{s}p(1|2)$ is equal to 3. We denote the subalgebra $\mathfrak{g} \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}c$ by $\tilde{\mathfrak{g}}$. Set $\tilde{\mathfrak{h}} = \mathbb{C}h \oplus \mathbb{C}c$ and $\tilde{\mathfrak{b}}_+ = \mathfrak{b}_+ \oplus \mathfrak{g} \otimes z\mathbb{C}[z]$.

For $\Lambda \in \hat{\mathfrak{h}}^*$, let $M(\Lambda)$ be the Verma module of $o\hat{s}p(1|2)$ with highest weight Λ. It follows from Theorem 1.1 that the Verma module $M(\Lambda)$ is reducible if and only if its highest weight $\Lambda = \frac{1}{2}(k-\lambda)\Lambda_0 + \lambda\Lambda_1$ satisfies

$$\lambda = m - s(k+3) - 1,$$

for any $k \in \mathbb{Z}_{\geq 0}$ and $i \in \mathbb{Z}_{>0}$. Q.E.D.

Theorem 3.1. *For any $\lambda = \lambda_k \in K^g$, there exists the following resolution of the irreducible highest weight module $L(\lambda)$:*

(1) *if $(\lambda_0 + \rho, \beta_2) \neq 0$ then*

$$\cdots \to M(\lambda_{k+i}) \oplus M(\lambda'_{k+i}) \to \cdots$$

$$\to M(\lambda_{k+2}) \oplus M(\lambda'_{k+2}) \to M(\lambda_{k+1}) \oplus M(\lambda'_{k+1}) \to M(\lambda) \to L(\lambda) \to 0,$$

(2) *if $(\lambda_0 + \rho, \beta_2) = 0$ then*

$$0 \to M(\lambda_{k+1}) \to M(\lambda) \to L(\lambda) \to 0.$$

Proof. Here we give the proof in the case that $(\lambda + \rho, \beta_2) \neq 0$. From Proposition 1.1 we obtain a short exact sequence

$$0 \to M(\lambda_k)^1 \to M(\lambda) \to L(\lambda) \to 0.$$

Moreover, from the definition of $N(\lambda_k)^1$, we see that

$$0 \to M(\lambda_{k+1}) \cap M(\lambda'_{k+1}) \to M(\lambda_{k+1}) \oplus M(\lambda'_{k+1}) \to N(\lambda_k)^1 \to 0.$$

is exact for each $k \in \mathbb{Z}_{>0}$. By Lemma 3.5, we have

$$M(\lambda_k)^1 = N(\lambda_k)^1$$

and

$$M(\lambda_{k+1}) \cap M(\lambda'_{k+1}) = N(\lambda_{k+1})^1.$$

Hence we obtain the following exact sequences

$$I_k : \quad 0 \to N(\lambda_k)^1 \to M(\lambda) \to L(\lambda) \to 0,$$

$$J_k : \quad 0 \to N(\lambda_{k+1})^1 \to M(\lambda_{k+1}) \oplus M(\lambda'_{k+1}) \to N(\lambda_k)^1 \to 0.$$

Combining these short exact sequences $\cdots, J_{k+2}, J_{k+1}, J_k, I_k$ by Yoneda product, we obtain the desired results. Q.E.D.

4. Fusion algebras

The main purpose of this section is to describe fusion algebras of admissible representations for $o\hat{s}p(1|2)$ explicitly. We first recall the affine Lie superalgebra $o\hat{s}p(1|2)$ and its admissible representations.

where the maps d_j are given by

$$\begin{array}{rccc} d_0 & : & M(\lambda_{k+i}) \oplus M(\lambda'_{k+i}) & \to & N(\lambda_k)^i, \\ & & (x,y) & \mapsto & x+y, \end{array}$$

for $j=0$ and for $j>0$,

$$\begin{array}{rccc} d_j & : & M(\lambda_{k+i+j}) \oplus M(\lambda'_{k+i+j}) & \to & M(\lambda_{k+i+j-1}) \oplus M(\lambda'_{k+i+j-1}), \\ & & (x,y) & \mapsto & (x+y,-x-y). \end{array}$$

By definition, this is a complex of $\mathfrak{g}$-modules. In particular the map d_0 is surjective and for $j>0$

$$\begin{aligned} \ker d_j &= \{(x,-x) | x \in M(\lambda_{k+i+j}) \cap M(\lambda'_{k+i+j})\}, \\ \operatorname{im} d_{j+1} &= \{(x,-x) | x \in M(\lambda_{k+i+j+1}) + M(\lambda'_{k+i+j+1})\}. \end{aligned}$$

Here we notice that $M(\lambda_{k+i+j})^1$ is the maximal proper submodule of $M(\lambda_{k+i+j})$ and hence $M(\lambda_{k+i+j}) \cap M(\lambda'_{k+i+j}) \subset M(\lambda_{k+i+j})^1$. Since $\lambda_k - \eta = \lambda_{k+i+j} - (\lambda_{k+i+j} - \lambda_k + \eta)$ and $\lambda_{k+i+j} - \lambda_k + \eta \lneq \eta$, by inductive assumption we have

$$(M(\lambda_{k+i+j})^1)_{\lambda_k - \eta} = (N(\lambda_{k+i+j})^1)_{\lambda_k - \eta},$$

for $i \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}_{\geq 0}$. From

$$N(\lambda_{k+i+j})^1 = M(\lambda_{k+i+j+1}) + M(\lambda'_{k+i+j+1}),$$

we obtain

$$(M(\lambda_{k+i+j}) \cap M(\lambda'_{k+i+j}))_{\lambda_k - \eta} \subset (M(\lambda_{k+i+j+1}) + M(\lambda'_{k+i+j+1}))_{\lambda_k - \eta}.$$

Hence $\ker d_j \subset \operatorname{im} d_{j+1}$ and thus the sequence is exact in the homogeneous space of weight $\lambda_k - \eta$ for each $k \in \mathbb{Z}_{\geq 0}$ and $i \in \mathbb{Z}_{>0}$.

By Euler-Poincaré principle, we have

$$\dim (N(\lambda_k)^i)_{\lambda_k - \eta} = \sum_{j=1}^{\infty} (-1)^{j-1} (\dim M(\lambda_{k+i+j-1})_{\lambda_k - \eta} + \dim M(\lambda'_{k+i+j-1})_{\lambda_k - \eta}),$$

and therefore,

$$\sum_{i=1}^{\infty} \dim (N(\lambda_k)^i)_{\lambda_k - \eta} = \sum_{i=1}^{\infty} (\dim M(\lambda_{k+2i-1})_{\lambda_k - \eta} + \dim M(\lambda'_{k+2i-1})_{\lambda_k - \eta}).$$

From the formula (7), we see that

$$\sum_{i=1}^{\infty} \dim (N(\lambda_k)^i)_{\lambda_k - \eta} = \sum_{i=1}^{\infty} \dim (M(\lambda_k)^i)_{\lambda_k - \eta}.$$

Combining this fact with Lemma 3.4, we conclude

$$(N(\lambda_k)^i)_{\lambda_k - \eta} = (M(\lambda_k)^i)_{\lambda_k - \eta}$$

From the singular vectors formula, we see that the diagrams in question are commutative. Q.E.D.

Remark 3.1. *Strictly speaking, to describe the above morphisms precisely, one has to check the parity of each singular vector carefully. But since we do not need such information to compute fusion algebras, we abbreviated them.*

From now on we identify the Verma module $M(\lambda_{k+i})$ and $M(\lambda'_{k+i})$ with their images under the embedding in $M(\lambda_k)$ which is given in the previous Lemma. Set

$$N(\lambda_k)^i = \begin{cases} M(\lambda_{k+i}) + M(\lambda'_{k+i}) & \text{if } (\lambda+\rho,\beta_2) \neq 0, \\ \\ M(\lambda_{k+i}) & \text{if } (\lambda+\rho,\beta_2) = 0, \end{cases} \qquad \text{for} \quad i > 0.$$

This defines a new filtration of the Verma module $M(\lambda_k)$:

$$M(\lambda_k) \supset N(\lambda_k)^1 \supset N(\lambda_k)^2 \supset \cdots .$$

Lemma 3.4. *For any $k \in \mathbb{Z}_{\geq 0}$ and $i \in \mathbb{Z}_{>0}$, we have $N(\lambda_k)^i \subset M(\lambda_k)^i$.*

Proof. From [Jan] we see that there exists a non-degenerate contravariant form on the quotient module $M(\lambda_k)^i/M(\lambda_k)^{i+1}$. Therefore, any singular vector in $M(\lambda_k)^i$ is contained in $M(\lambda_k)^{i+1}$. From the embedding diagram of Verma modules given in Lemma 3.3, we see that $N(\lambda_k)^i \subset M(\lambda_k)^i$, by induction on i. Q.E.D.

Lemma 3.5. *For any $k \in \mathbb{Z}_{\geq 0}$ and $i \in \mathbb{Z}_{>0}$, we have*

$$N(\lambda_k)^i = M(\lambda_k)^i.$$

In particular, $N(\lambda_k)^1 = M(\lambda_k)^1$.

Proof. We only give the proof in the case that $(\lambda+\rho,\beta_2) \neq 0$, the other case can be proved similarly. We prove Lemma 3.5 by induction. We fix a weight η. Suppose that

$$(N(\lambda_k)^i)_{\lambda_k-\alpha} = (M(\lambda_k)^i)_{\lambda_k-\alpha}, \tag{8}$$

for any $k \in \mathbb{Z}_{\geq 0}$, $i \in \mathbb{Z}_{>0}$ and $\alpha < \eta$. We prove that (8) is true in the homogeneous space of weight $\lambda_k - \eta$ for any $k \in \mathbb{Z}_{\geq 0}$, $i \in \mathbb{Z}_{>0}$. Here we remark that the base of the induction i.e. the case of $\alpha = 0$ in (8) is obvious from the definition of $N(\lambda_k)^i$. For each $k \in \mathbb{Z}_{\geq 0}$ and $i \in \mathbb{Z}_{>0}$, let us consider the sequence

$$\cdots \overset{d_{j+1}}{\to} M(\lambda_{k+i+j}) \oplus M(\lambda'_{k+i+j}) \overset{d_j}{\to} \cdots$$

$$\cdots \to M(\lambda_{k+i+1}) \oplus M(\lambda'_{k+i+1}) \overset{d_1}{\to} M(\lambda_{k+i}) \oplus M(\lambda'_{k+i}) \overset{d_0}{\to} N(\lambda_k)^i \to 0$$

(3) *There exists an unique element* λ_0 *in* $\mathfrak{h}^*$ *such that* $\lambda_0 \in W_\lambda \circ \lambda$ *and* $(\lambda+\rho,\beta_i) \geq 0$ *for* $i = 1, 2$.

For each $i \in \mathbb{Z}_{\geq 0}$ and $\epsilon = 0, 1$, we put

$$w_{\epsilon+2i} = r^{\epsilon}_{\beta_1}(r_{\beta_2}r_{\beta_1})^i,\ w'_{\epsilon+2i} = r^{\epsilon}_{\beta_2}(r_{\beta_1}r_{\beta_2})^i,$$
$$\lambda_k = w_k \circ \lambda_0,\ \lambda'_k = w'_k \circ \lambda_0.$$

Without loss of generality, we can assume that $\lambda = \lambda_k$ and $(\lambda+\rho,\beta_1) \neq 0$.

Lemma 3.2. *For* $\lambda = \lambda_k \in K^g$, *we have*

$$\{\lambda - n\alpha | (\alpha, n) \in \bar{R}_\lambda \cup \tilde{R}_\lambda \cup \bar{S}_\lambda\}$$
$$= \begin{cases} \{\lambda_{k+2i-1},\ \lambda'_{k+2i-1} | i \in \mathbb{Z}_{>0}\} & \text{if } (\lambda+\rho,\beta_2) \neq 0, \\ \{\lambda_{k+i} | i \in \mathbb{Z}_{>0}\} & \text{if } (\lambda+\rho,\beta_2) = 0. \end{cases}$$

From this Lemma we obtain the following explicit form of the character sum: For $\lambda = \lambda_k \in K^g$, we have

$$(7) \qquad \begin{aligned} &\sum_{i=1}^{\infty} \operatorname{ch} M(\lambda)^i \\ &= \begin{cases} \displaystyle\sum_{i=1}^{\infty} \operatorname{ch} M(\lambda_{k+2i-1}) + \operatorname{ch} M(\lambda'_{k+2i-1}) & \text{if } (\lambda+\rho,\beta_2) \neq 0, \\ \displaystyle\sum_{i=1}^{\infty} \operatorname{ch} M(\lambda_{k+i}) & \text{if } (\lambda+\rho,\beta_2) = 0. \end{cases} \end{aligned}$$

Lemma 3.3. *For* $\lambda \in K^g$, *there exist the following morphisms between Verma modules and the diagram is commutative:*

$$1) \qquad \begin{array}{ccccccccc} \cdots & \to & M(\lambda_3) & \to & M(\lambda_2) & \to & M(\lambda_1) & \searrow & \\ & \nearrow & & \nearrow & & \nearrow & & & M(\lambda_0) \\ \cdots & \to & M(\lambda'_3) & \to & M(\lambda'_2) & \to & M(\lambda'_1) & \nearrow & \end{array}$$
if $(\lambda+\rho,\beta_2) \neq 0$,

$$2) \quad \cdots \to M(\lambda_3) \to M(\lambda_2) \to M(\lambda_1) \to M(\lambda_0)$$
if $(\lambda+\rho,\beta_2) = 0$.

where each arrow between two Verma modules stands for the injective homomorphism given in Theorem 2.2.

Proof. Let $M(\mu)$ and $M(\nu)$ be two Verma modules jointed by an arrow from $M(\nu)$ to $M(\mu)$. For $\mu, \nu \in K^g$, it follows that there exists a real root $\beta_{\mu,\nu} \in \Delta^+_\lambda$ and a positive integer $n_{\mu,\nu}$ such that $\nu = \mu - n_{\mu,\nu}\beta_{\mu,\nu}$. For each such pair (μ, ν), we fix the following non-trivial injection:

$$\begin{array}{rccc} \iota_{\mu,\nu}: & M(\nu) & \longrightarrow & M(\mu), \\ & v_\nu & \mapsto & F(r_{\beta_{\mu,\nu}};\mu)v_\mu. \end{array}$$

Then we can show (6) in the same way as in [MFF]. Q.E.D.

By an argument similar to [MFF] we have

Proposition 2.2. *For $\lambda \in \mathfrak{h}^*$, $\alpha \in \Delta^{\mathrm{re}} \cap \Delta_0^+$ and $n \in \mathbb{Z}_{>0}$, or $\alpha \in \Delta^{\mathrm{re}} \cap \Delta_1^+$ and $n \in 2\mathbb{Z}_{>0} - 1$ which satisfy the Kac-Kazhdan condition:*

$$2(\lambda + \rho, \alpha) = n(\alpha, \alpha),$$

the monomial $F(r_\alpha; \lambda)$ makes sense as an element of the universal enveloping algebra $U(\mathfrak{g})$.

From Theorem 2.1 and Proposition 2.1, we obtain the following Theorem.

Theorem 2.2. *For $\lambda \in \mathfrak{h}^*$, suppose that there exist a real root α and an integer n such that $(\alpha, n) \in \bar{R}_\lambda \cup \bar{S}_\lambda$. Then there exists a non-trivial injective homomorphism from $M(r_\alpha \circ \lambda)$ to $M(\lambda)$.*

3. Bernstein-Gel'fand-Gel'fand resolution

We construct Bernstein-Gel'fand-Gel'fand (BGG) resolution for Kac-Moody superalgebras of rank 2 in a way similar to [RW] and [Mal]. Through this section let $\mathfrak{g}$ be a symmetrizable Kac-Moody superalgebra of rank 2 and let us fix $\lambda \in K^g$. Let us introduce the set

$$\Delta_{\lambda,1} := \{\alpha \in \Delta_1^{\mathrm{re}} | (\lambda + \rho, \alpha^\vee) \in 2\mathbb{Z} - 1\}, \quad \Delta_{\lambda,0} := \left\{\alpha \in \bar{\Delta}_0^{\mathrm{re}} | (\lambda + \rho, \alpha^\vee) \in \mathbb{Z}\right\} \cup 2\Delta_{\lambda,1},$$
$$\Delta_\lambda := \Delta_{\lambda,0} \cup \Delta_{\lambda,1}, \quad \Delta_\lambda^+ := \Delta_\lambda \cap \Delta^+,$$

where we denote the coroot $2\alpha/(\alpha, \alpha)$ by $\alpha^\vee$ for $\alpha \in \Delta^{\mathrm{re}}$. We put $\Delta_\lambda^{\vee,+} = \{\alpha^\vee | \alpha \in \Delta_\lambda^+\}$ and

$$\Pi_\lambda = \{\alpha \in \Delta_\lambda^+ | \alpha^\vee \in \Delta_\lambda^{\vee,+} \setminus (\Delta_\lambda^{\vee,+} + \Delta_\lambda^{\vee,+})\},$$
$$W_\lambda = \langle r_\alpha | \alpha \in \Pi_\lambda \rangle.$$

In the case that $|\Pi_\lambda| = 1$, the structure of the Verma module $M(\lambda)$ is simple so that we only state the following proposition.

Proposition 3.1. *Let $M(\lambda - n\alpha) \to M(\lambda)$ be the injective homomorphism in Theorem 2.2. Then the image of $M(\lambda - n\alpha)$ coincides with the maximal proper submodule of $M(\lambda)$ i.e. $L(\lambda) \simeq M(\lambda)/M(\lambda - n\alpha)$.*

From now on, we assume that $|\Pi_\lambda| \geq 2$. In this case, one can prove the following Lemma.

Lemma 3.1. *There exists β_1, $\beta_2 \in \Delta_\lambda^+$ such that*

(1) $W_\lambda = \{r_{\beta_1}^\epsilon (r_{\beta_2} r_{\beta_1})^i,\ r_{\beta_2}^\epsilon (r_{\beta_1} r_{\beta_2})^i | i \in \mathbb{Z}_{\geq 0},\ \epsilon = 0, 1\}$,
(2) $\Pi_\lambda = \{\beta_1,\ \beta_2\}$,

Proof.

As usual, we have only to show that $U(\mathfrak{n}_-)$ has no zero divisor. According to [Ray], it follows that

$$[x,x] \neq 0 \quad \forall x \in \mathfrak{g}_1 \setminus \{0\}.$$

Hence we can apply Theorem 2.7 in [AL], since $\mathfrak{n}_-$ has a strictly positively graded structure through the gradation defined by

$$\deg f_i = \begin{cases} 1 & \text{if } i \in \tau, \\ 2 & \text{otherwise.} \end{cases}$$

Q.E.D

2.3. Singular vectors. We state a singular vectors formula in the Verma module $M(\lambda)$ for a symmetrizable Kac-Moody superalgebra $\mathfrak{g}$ which is similar to the one given in [MFF]. Here we notice that, if $(\alpha, n) \in \tilde{R}_\lambda$ then $\alpha \notin \Delta^{\mathrm{re}}$ (see [K2]). Let α be a real root such that $(\alpha, n) \in \bar{R}_\lambda \cup \bar{S}_\lambda$ for some positive integer n. Fix any reduced expression $r_\alpha = r_{i_k} \cdots r_{i_2} r_{i_1}$ and consider the sequence $\{\lambda_0, \cdots, \lambda_{k-1}\} \subset \mathfrak{h}^*$ given by $\lambda_0 = \lambda$ and $\lambda_s = r_{i_s} \circ \lambda_{s-1}$ for $1 \leq s \leq k-1$. We set

$$F(r_\alpha; \lambda) = f_{i_k}^{\gamma_k} \cdots f_{i_2}^{\gamma_2} f_{i_1}^{\gamma_1},$$

where $\gamma_s = \lambda_{s-1}(h_{i_s}) + 1$ and

$$f_i^\gamma = \begin{cases} f_i^\gamma & \text{if } i \notin \tau, \\ f_i (f_i^2)^{\frac{\gamma-1}{2}} & \text{if } i \in \tau. \end{cases}$$

We regard $(f_i^2)^{\frac{\gamma-1}{2}}$ as an even element for any $\gamma \in \mathbb{C}$. Next apply the rule

$$x^t y = y x^t + \sum_{j>0} \binom{t}{j} (\mathrm{ad}x)^j(y) x^{t-j}, \quad \text{for } x \in \mathfrak{g}_0,\ y \in U(\mathfrak{g}),\ t \in \mathbb{C},$$

which naturally holds for $t \in \mathbb{Z}_{>0}$, to rewrite the formal expression $F(r_\alpha; \lambda)$ as in [MFF].

Proposition 2.1. *If $F(r_\alpha; \lambda)$ makes sense as an element of the universal enveloping algebra $U(\mathfrak{g})$, then $F(r_\alpha; \lambda)v_\lambda$ gives a singular vector in the Verma module $M(\lambda)$.*

Proof. We have to show that

$$e_i F(r_\alpha; \lambda) v_\lambda = 0, \tag{6}$$

for any $i \in I$. We notice that , for $i \in \tau$, we have

$$[e_i, f_i^\gamma] = [e_i, f_i (f_i^2)^{\frac{\gamma-1}{2}}] = (f_i^2)^{\frac{\gamma-1}{2}} \{h_i - (\gamma - 1)\}.$$

2. Kac-Moody superalgebras

In what follows, we assume that our Cartan matrix A is symmetrizable, unless otherwise stated.

2.1. Definitions. Let $\mathfrak{g} = \mathfrak{g}(A, \tau)$ be the contragredient Lie superalgebra associated with a Cartan matrix A. We call $\mathfrak{g}$ a Kac-Moody superalgebra if the Cartan matrix A satisfies the following conditions:

(1) $a_{i,j}$ is an integer for $i, j \in I$.
(2) $a_{i,i} = 2$ for $i \in I$.
(3) $a_{i,j} \leq 0$ if $i \neq j$.
(4) $a_{i,j} = 0$ if $a_{j,i} = 0$.
(5) $a_{i,j} \in 2\mathbb{Z}$ if $i \in \tau$.

Let $\Pi = \{\alpha_i | i \in I\}$ be the set of the simple roots and $\tilde{\Pi} = \{\alpha_i,\ i \in I,\ 2\alpha_i,\ i \in \tau\}$ the set of the non-reduced simple roots. For $i \in I$, we define the simple reflection r_i acting on $\mathfrak{h}^*$ as follows:

$$r_i(\lambda) = \lambda - \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i.$$

The Weyl group W is the subgroup of $GL(\mathfrak{h}^*)$ generated by the simple reflections r_i. A root $\beta \in \Delta$ is called a real root if there exists a non-reduced simple root $\alpha \in \tilde{\Pi}$ such that $\beta = w(\alpha)$ for some $w \in W$. We denote the set of the real roots by Δ^{re}. The elements of $\Delta^{\mathrm{im}} = \Delta \setminus \Delta^{\mathrm{re}}$ are called imaginary roots. For a real root $\beta \in \Delta^{\mathrm{re}}$, if $\beta = w(\alpha_i)$ or $\beta = w(2\alpha_i)$ then $r_\beta = w r_i w^{-1}$ gives the reflection relative to β. We use the following notation to denote the shift action of W on $\mathfrak{h}^*$: $r_\alpha \circ \lambda = r_\alpha(\lambda + \rho) - \rho$.

Following [DGK], we introduce a"good" subset K^g of $\mathfrak{h}^*$. For a complex number c, we denote its real part (resp. imaginary part) by $\mathrm{Re}(c)$ (resp. $\mathrm{Im}(c)$). We write $c > 0$ if either $\mathrm{Re}(c) > 0$ or else $\mathrm{Re}(c) = 0$ and $\mathrm{Im}(c) > 0$. Let us consider the following subset C and K^g of $\mathfrak{h}^*$ defined by

$$\begin{aligned} C = \{\lambda \in \mathfrak{h}^* |\ & (\lambda + \rho, \alpha_i) \geq 0 \text{ for any } \alpha_i \in \Pi \text{ and} \\ & (\lambda + \rho, \alpha) \neq 0 \text{ if } \alpha \in \Delta^+ \text{ such that } (\alpha, \alpha) = 0\}, \\ K^g = \bigcup\nolimits_{w \in W} & w \circ (C). \end{aligned}$$

2.2. Injectivity of morphisms between Verma modules. Let $M(\lambda)$ and $M(\mu)$ be Verma modules over a Kac-Moody superalgebra $\mathfrak{g}$ with highest weight λ and μ. The following Theorem is non-trivial for Lie sueralgebras.

Theorem 2.1. *Let $\mathfrak{g} = \mathfrak{g}(A, \tau)$ be a symmetrizable Kac-Moody superalgebra. Then any non-trivial homomorphism $\phi : M(\lambda) \to M(\mu)$ is injective.*

For each $i \in \mathbb{Z}_{>0}$ we set

$$M(\lambda)^i = \{v \in M(\lambda) | \operatorname{ord}_t((v,w)_{\lambda+tz}) \geq i, \text{ for any } w \in M(\lambda)\}.$$

where $\operatorname{ord}_t(P(t))$ denotes the highest power of t which divides $P(t)$. Now we have the following filtration of the Verma module $M(\lambda)$:

$$M(\lambda) \supset M(\lambda)^1 \supset M(\lambda)^2 \supset \cdots .$$

This filtration is called the Jantzen filtration.

Proposition 1.1. *The following properties hold.*

(1) *For any $\gamma \in Q^+$,*

$$\operatorname{ord}_t(\det F_\gamma) = \sum_{k=1}^{\infty} \dim(M(\lambda)^k)_{\lambda-\gamma},$$

where $F_\gamma = F|_{M(\lambda)_{\lambda-\gamma} \times M(\lambda)_{\lambda-\gamma}}$.

(2) *$M(\lambda)^1 = J(\lambda)$, where $J(\lambda)$ is the maximal proper submodule of $M(\lambda)$.*

(3) $\bigcap_{k=1}^{\infty} M(\lambda)^k = (0)$.

From this proposition we obtain a character sum formula of the Jantzen filtration. Set $H_{n,\beta} = \{\lambda \in \mathfrak{h}^* | 2(\lambda+\rho,\beta) = n(\beta,\beta)\}$ for $n \in \mathbb{Z}$, $\beta \in Q$ and

$$\bar{R}_\lambda = \{(\alpha, n) \mid \alpha \in \bar{\Delta}_0^+,\ n \in \mathbb{Z}_{>0},\ \lambda \in H_{n,\alpha}\},$$

$$\tilde{R}_\lambda = \{(\alpha, n) \mid \alpha \in \Delta_0^+ \setminus \bar{\Delta}_0^+,\ n \in \mathbb{Z}_{>0},\ \lambda \in H_{n,\alpha},\ \operatorname{mult}\alpha \neq \operatorname{mult}\tfrac{\alpha}{2}\},$$

$$\bar{S}_\lambda = \{(\alpha, n) \mid \alpha \in \Delta_1^+ \setminus \bar{\Delta}_1^+,\ n \in 2\mathbb{Z}_{>0} - 1,\ \lambda \in H_{n,\alpha}\},$$

Proposition 1.2. *Let $\mathfrak{g}$ be a symmetrizable contragredient Lie superalgebra and let $\lambda \in \mathfrak{h}^*$. Suppose that*

$$(\lambda+\rho,\alpha) \neq 0 \text{ for any } \alpha \in \bar{\Delta}_1^+. \tag{4}$$

Then we have

$$\begin{aligned} \sum_{i=1}^{\infty} \operatorname{ch} M(\lambda)^i = & \sum_{(\alpha,n) \in \bar{R}_\lambda \cup \bar{S}_\lambda} \operatorname{mult}\alpha \ \operatorname{ch} M(\lambda - n\alpha) \\ & + \sum_{(\alpha,n) \in \tilde{R}_\lambda} (\operatorname{mult}\alpha - \operatorname{mult}\frac{\alpha}{2}) \ \operatorname{ch} M(\lambda - n\alpha). \end{aligned} \tag{5}$$

where L is defined as (2) and L_α is given by $L_\alpha = (1+e^{-\alpha})L$.

For $\alpha \in Q^+$ such that $\alpha = \sum_i k_i\alpha_i$, we set $h_\alpha = \sum_i k_i d_i^{-1} h_i$, where $\{d_i\}$ is given in (3). We fix a special element $\rho \in \mathfrak{h}^*$ such that $(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$ for any $i \in I$.

Theorem 1.1. *Let $\mathfrak{g} = \mathfrak{g}(A, \tau)$ be a symmetrizable contragredient Lie superalgebra. Up to a nonzero constant factor depending on the choice of a base of $U(\mathfrak{n}_-)_{-\gamma}$, we have*

$$\begin{aligned} detF_\gamma \ \propto & \prod_{n\in\mathbb{Z}_{>0}} \prod_{\alpha\in\bar{\Delta}_0^+} (h_\alpha + \rho(h_\alpha) - \frac{1}{2}n(\alpha,\alpha))^{(\mathrm{mult}\,\alpha)P(\gamma-n\alpha)} \\ & \times \prod_{n\in\mathbb{Z}_{>0}} \prod_{\alpha\in\Delta_0^+\setminus\bar{\Delta}_0^+} (h_\alpha + \rho(h_\alpha) - \frac{1}{2}n(\alpha,\alpha))^{(\mathrm{mult}\,\alpha - \mathrm{mult}\,\frac{\alpha}{2})P(\gamma-n\alpha)} \\ & \times \prod_{n\in 2\mathbb{Z}_{>0}-1} \prod_{\alpha\in\Delta_1^+\setminus\bar{\Delta}_1^+} (h_\alpha + \rho(h_\alpha) - \frac{1}{2}n(\alpha,\alpha))^{(\mathrm{mult}\,\alpha)P(\gamma-n\alpha)} \\ & \times \prod_{\alpha\in\bar{\Delta}_1^+} (h_\alpha + \rho(h_\alpha))^{(\mathrm{mult}\,\alpha)P_\alpha(\gamma-\alpha)}. \end{aligned}$$

We can show this theorem in the same way as [KK] (see [K2] and [K3]).

Remark 1.1. *Although in* [K4] *the second term of our formula is lacking, the formula in* [K4] *still holds for non-twisted affine type Lie superalgebras.*

Remark 1.2. *A factor with negative power does not appear in the above formula, since such factor in the second term is always canceled by some factor in the third term. In particular, for a Kac-Moody superalgebra $\mathfrak{g}$ (see Section 2), we can show that*

$$\mathrm{mult}\,\alpha \geq \mathrm{mult}\,\frac{\alpha}{2},$$

for any $\alpha \in \Delta_0^+ \setminus \bar{\Delta}_0^+$.

1.2. Jantzen filtration. In this section, we construct a filtration of the Verma module $M(\lambda)$, which is called the Jantzen filtration [Jan]. Let $\mathfrak{g}$ be a symmetrizable contragredient Lie superalgebra and $\mathbb{C}[t]$ the polynomial ring over $\mathbb{C}$ with one indeterminate. From now on we fix a regular element z of $\mathfrak{h}^*$ i.e. $(z, \alpha) \neq 0$ for any $\alpha \in \Delta_+$. We consider the following evaluation map $\mathbb{C}[t]\otimes_{\mathbb{C}} S(\mathfrak{h}) \to \mathbb{C}[t]$:

$$f \otimes g \mapsto f\,\mathrm{ev}_{\lambda+tz}(g),$$

and denote the image of $f\otimes u$ under this evaluation map by $f\cdot u(\lambda+tz)$. Let $(,)_{\lambda+tz}$ be the following composition map

$$(,)_{\lambda+tz} : M(\lambda)\times M(\lambda) \to S(\mathfrak{h}) \hookrightarrow \mathbb{C}[t]\otimes_{\mathbb{C}} S(\mathfrak{h}) \to \mathbb{C}[t],$$

which is given by

$$(xv_\lambda, yv_\lambda) \mapsto \pi(\omega(x)y) \mapsto 1\otimes\pi(\omega(x)y) \mapsto \pi(\omega(x)y)(\lambda+tz).$$

symmetric bilinear form F on $U(\mathfrak{g})$. By the triangular decomposition of $\mathfrak{g}$ and Poincaré-Birkhoff-Witt theorem, we have

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (U(\mathfrak{g})\mathfrak{n}_+ + \mathfrak{n}_- U(\mathfrak{g})).$$

Let $\pi : U(\mathfrak{g}) \to U(\mathfrak{h})$ be the projection with respect to this decomposition. Since $\mathfrak{h}$ is an even commutative subalgebra, we can identify $U(\mathfrak{h})$ with the symmetric algebra $S(\mathfrak{h})$. Then we define a symmetric bilinear form

$$F : U(\mathfrak{g}) \times U(\mathfrak{g}) \to S(\mathfrak{h})$$

by

$$F(x, y) = \pi(\omega(x)y).$$

From the definition of F it is easy to see that $F|_{U(\mathfrak{n}_-)_{-\gamma} \times U(\mathfrak{n}_-)_{-\gamma'}} = 0$ if $\gamma \neq \gamma'$. For $\gamma \in Q^+$ we set $F_\gamma = F|_{U(\mathfrak{n}_-)_{-\gamma} \times U(\mathfrak{n}_-)_{-\gamma}}$. Via the identification $S(\mathfrak{h}) \cong \mathbb{C}[\mathfrak{h}^*]$, we define the evaluation map $\mathrm{ev}_\lambda : S(\mathfrak{h}) \to \mathbb{C}$ by $f \mapsto f(\lambda)$ for each $\lambda \in \mathfrak{h}^*$. Using the identification in (1), we define the contravariant form $(,)$ on $M(\lambda)$ as follows:

$$(,) : M(\lambda) \times M(\lambda) \simeq U(\mathfrak{n}_-) \times U(\mathfrak{n}_-) \to S(\mathfrak{h}) \to \mathbb{C},$$

where the second map is given by the symmetric form F and the last one by ev_λ. We remark two properties of the contravariant form $(,)$:

(1) $(v_\lambda, v_\lambda) = 1$,
(2) $(gx, y) = (x, \omega(g)y)$ for any $g \in \mathfrak{g}$ and $x,\ y \in M(\lambda)$.

We say that the matrix A is symmetrizable if there exists a symmetric matrix $B = (b_{i,j})$ and a diagonal matrix $D = \mathrm{diag}(d_1, \cdots, d_n)$ such that $\det(D) \neq 0$ and

$$A = DB. \tag{3}$$

A contragredient Lie superalgebra with a symmetrizable Cartan matrix A is called symmetrizable. If $\mathfrak{g} = \mathfrak{g}(A, \tau)$ is symmetrizable then there exists a non-degenerate symmetric bilinear form on $\mathfrak{h}^*$ such that $(\alpha_i, \alpha_j) = b_{i,j} = a_{i,j}/d_i$. We may identify $\mathfrak{h}^*$ with $\mathfrak{h}$ by using this non-degenerate bilinear form.

We describe an explicit form of the determinant $\det F_\gamma$ for a contragredient Lie superalgebra. Let $\bar{\Delta}_0^+$ and $\bar{\Delta}_1^+$ be the subset of Δ_0^+ and Δ_1^+ defined by

$$\bar{\Delta}_0^+ = \{\alpha \in \Delta_0^+ | \tfrac{1}{2}\alpha \notin \Delta_1\},$$
$$\bar{\Delta}_1^+ = \{\alpha \in \Delta_1^+ | 2\alpha \notin \Delta_0\}.$$

For each $\gamma \in Q^+$, we define non-negative integer $P(\gamma)$ and $P_\alpha(\gamma)$ $(\alpha \in \Delta_1^+)$ by the following equality:

$$\sum_{\gamma \in Q^+ \cup \{0\}} P(\gamma) e^{-\gamma} = L^{-1},$$
$$\sum_{\gamma \in Q^+ \cup \{0\}} P_\alpha(\gamma) e^{-\gamma} = L_\alpha^{-1},$$

For a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, we denote its universal enveloping algebra by $U(\mathfrak{g})$. The Poincaré-Birkhoff-Witt theorem for $U(\mathfrak{g})$ can be stated as follows: Let $\{a_1, a_2, \cdots\}$ and $\{b_1, b_2, \cdots\}$ be ordered bases of $\mathfrak{g}_0$ and $\mathfrak{g}_1$. The set of the elements

$$\left\{a_{k_1}^{s_1} \cdots a_{k_m}^{s_m} b_{l_1} \cdots b_{l_n} | (k_1 < \cdots < k_m,\ l_1 < \cdots < l_n)\right\}$$

form a base of $U(\mathfrak{g})$. Thus, for a contragredient Lie superalgebra $\mathfrak{g}$, its universal enveloping algebra $U(\mathfrak{g})$ has the Q-gradation inherited from that of $\mathfrak{g}$. We denote each graded subspace by $U(\mathfrak{g})_\gamma$ for each $\gamma \in Q$. For any subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, we identify $U(\mathfrak{a})$ with a subalgebra of $U(\mathfrak{g})$.

A $\mathfrak{g}$-module V is called $\mathfrak{h}$-diagonalizable if V has the weight space decomposition

$$V = \oplus_{\eta \in \mathfrak{h}^*} V_\eta, \qquad V_\eta = \{v \in V | hv = \eta(h)v, \quad \forall h \in \mathfrak{h}\}$$

with finite dimensional weight subspaces.

For any $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-module V, we define its formal character by

$$\mathrm{ch}\, V = \sum_{\eta \in \mathfrak{h}^*} \dim V_\eta \ e^\eta$$

where e^η stands for formal exponentials ([K0]).

For $\lambda \in \mathfrak{h}^*$, let $\mathbb{C}_\lambda = \mathbb{C}\mathbf{1}_\lambda$ be the one-dimensional representation of a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ defined as follows:

$$\begin{array}{ll} h\mathbf{1}_\lambda = \lambda(h)\mathbf{1}_\lambda & (h \in \mathfrak{h}), \\ n\mathbf{1}_\lambda = 0 & (n \in \mathfrak{n}_+). \end{array}$$

We denote the $\mathfrak{g}$-module induced from the $\mathfrak{b}$-module $\mathbb{C}_\lambda$ by $M(\lambda)$ i.e. $M(\lambda) = \mathrm{Ind}_{U(\mathfrak{b})}^{U(\mathfrak{g})} \mathbb{C}_\lambda$ and call it a Verma module with highest weight λ. The element $v_\lambda = 1 \otimes \mathbf{1}_\lambda$ of $M(\lambda)$ is called a highest weight vector. Since the structures of modules described here do not depend on a choice of parity of a highest weight vector, we do not specify it. As $\mathfrak{n}_-$-module, we have

$$M(\lambda) \simeq U(\mathfrak{n}_-), \tag{1}$$

which in turn gives

$$\mathrm{ch}\, M(\lambda) = e^\lambda L^{-1},$$

where

$$L = \frac{\prod_{\beta \in \Delta_0^+} (1 - e^{-\beta})^{\mathrm{mult}\, \beta}}{\prod_{\beta \in \Delta_1^+} (1 + e^{-\beta})^{\mathrm{mult}\, \beta}}. \tag{2}$$

It's known that there exists the unique maximal proper submodule $J(\lambda)$ of $M(\lambda)$ and hence its irreducible quotient $L(\lambda) = M(\lambda)/J(\lambda)$.

The Verma module $M(\lambda)$ carries a symmetric bilinear form, which is called a contravariant form. To define the contravariant form $(,)$, we first introduce the

$I = \{1, 2, \cdots, n\}$. Let $\mathfrak{h}$ be a $(2n - l)$-dimensional vector space over $\mathbb{C}$ and Q the free Abelian group of rank n with basis $\{\alpha_i | i \in I\} \subset \mathfrak{h}^*$. We can choose a subset $\{h_i | i \in I\}$ of $\mathfrak{h}$ such that

(1) $\{h_i | i \in I\}$ is linearly independent,
(2) $\alpha_i(h_j) = a_{j,i}$ for any i, $j \in I$.

Using these data, we define the contragredient Lie superalgebra $\mathfrak{g} = \mathfrak{g}(A, \tau)$ as the Lie superalgebra over the complex number field $\mathbb{C}$ which satisfies the following properties:

(1) $\mathfrak{g}$ has a Q-gradation $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$.

(2) $\mathfrak{g}$ is generated by the elements e_i, f_i $(i \in I)$ and $h \in \mathfrak{h}$ such that

$$\mathfrak{g}_0 = \mathfrak{h}, \quad \mathfrak{g}_{\alpha_i} = \mathbb{C}e_i, \quad \mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i.$$

These generators have the $\mathbb{Z}_2$-gradation

$$\deg e_i = \deg f_i = \begin{cases} 1 & \text{if } i \in \tau, \\ 0 & \text{if } i \notin \tau, \end{cases}$$

and satisfy commutation relations

$$[h, h'] = 0 \ (h, h' \in \mathfrak{h}), \ [e_i, f_j] = \delta_{i,j} h_i,$$

$$[h, e_i] = \alpha_i(h) e_i, \ [h, f_i] = -\alpha_i(h) f_i.$$

(3) Every Q-graded ideal of $\mathfrak{g}$ that intersects $\mathfrak{h}$ trivially is zero.

The matrix A is called the Cartan matrix of $\mathfrak{g}$ and the size of the Cartan matrix A is called the rank of $\mathfrak{g}$.

For a Lie superalgebra $\mathfrak{g}$, we denote its even part (resp. odd part) by $\mathfrak{g}_0$ (resp. $\mathfrak{g}_1$). An element of $\Delta = \{\alpha \in Q | \mathfrak{g}_\alpha \neq (0)\}$ is called a root. For each root $\alpha \in \Delta$, we denote $\dim \mathfrak{g}_\alpha$ by $\operatorname{mult} \alpha$. Let $\Delta_0 = \{\alpha \in \Delta | \mathfrak{g}_\alpha \subset \mathfrak{g}_0\}$ (resp. $\Delta_1 = \{\alpha \in \Delta | \mathfrak{g}_\alpha \subset \mathfrak{g}_1\}$) be the set of the even roots (resp. odd roots). We set $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$, $\Delta^+ = \Delta \cap Q^+$, $\Delta^- = \Delta \cap (-Q^+)$, $\Delta_0^\pm = \Delta_0 \cap Q^\pm$ and $\Delta_1^\pm = \Delta_1 \cap Q^\pm$. We define a partial order $\alpha < \beta$ on the root lattice Q by $\beta - \alpha \in Q^+$. It is easy to see that $\mathfrak{g}$ has the following triangular decomposition;

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

where $\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha$. We remark that $\mathfrak{g}$ admits an anti-involutive automorphism ω defined by

$$\omega(h) = h \ (h \in \mathfrak{h}), \quad \omega(e_i) = f_i, \ \omega(f_i) = e_i \ (i \in I).$$

In any case, from mathematical point of view, one only has to compute the dimension of coinvariants of tensor products of 3 irreducible highest weight modules to obtain fusion algebras, as was explained above. Along this line, the fusion algebras of minimal models for the Virasoro algebra are computed in [FF] and that of WZNW models with fractional level for $\hat{\mathfrak{sl}}_2$ are computed in [FM1]. In both cases, they used the projective resolution of irreducible highest weight modules called Bernstein-Gel'fand-Gel'fand type (BGG) resolution. Furthermore, Feigin and Malikov showed in [FM2] that fusion algebras of these models are isomorphic to the tensor product of the Gothendieck groups of the category of finite dimensional representations of some quantum groups at roots of unity. As a next step, one may ask what happens for the fusion algebras of minimal models for $\mathcal{W}_n$ algebra and that of WZNW models with fractional level for $\hat{\mathfrak{sl}}_n$, at least for $n = 3$. Unfortunately, the method developed in [FF] works well only for the lowest rank case by purely technical reason and it has not succeeded yet even for $n = 3$. We would like to have some examples other than the case $n = 2$. It naturally leads us to compute fusion algebras of minimal models for the $N = 1$ super Virasoro algebra and that of WZNW models with fractional level for $o\hat{s}p(1|2)$, and these are the main subjects of the series of our papers [IK]. In this paper, we compute the fusion algebras of WZNW models with $o\hat{s}p(1|2)$-symmetry (Here and after, we abbreviate them as the fusion algebras for $o\hat{s}p(1|2)$ for brevity).

To be precise, this article is organized as follows.

In section 1, we recall some basic definition including contragredient Lie superalgebras. (It seems that its definition has not fixed yet among the experts.) We also state the Shapovalov determinant formulae for a symmetrizable contragredient Lie superalgebra (Theorem 1.1). As an application, we obtain the character sum formula for the Jantzen filtration on Verma modules. In section 2, we define Kac-Moody superalgebras in the sense of [K0]. Here we show the injectivity of the non-trivial homomorphisms between Verma modules for Kac-Moody superalgebras (Theorem 2.1) and show that an analogous formula to [MFF] holds for symmetrizable Kac-Moody superalgebras (Proposition 2.1). In section 3, we construct Bernstein-Gel'fand-Gel'fand (BGG) resolution for symmetrizable Kac-Moody superalgebras of rank 2 (Theorem 3.1). In section 4, we compute the fusion algebras for $o\hat{s}p(1|2)$ (Theorem 4.1, 4.2) as an application.

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1. Contragredient Lie superalgebras

1.1. Definitions. Following [K2], we define contragredient Lie superalgebras. Let $A = (a_{i,j})_{1 \leq i,j \leq n}$ be a complex $n \times n$ matrix of rank l and τ a subset of the index set

The first crucial step was made by E. Verlinde. In [V], he expresses the structure constants of the fusion algebra of minimal models in terms of the matrix elements of a modular transformation of the character of irreducible highest weight modules over Virasoro algebra, which is now known as a Verlinde's formula. This formula looks as follows. Let $z = 1 - 6\frac{(p-q)^2}{pq}$, with relatively prime positive integers p, q, be the fixed central charge. Let $N_{p,q}$ be the finite subset of $(\mathbb{Z}_{>0})^2$ that parameterizes so-called the minimal model with the central charge z [BPZ]. For each $(r,s) \in N_{p,q}$, we assign the complex number $h_{r,s}$ called conformal weight. Now let $L(z, h_{r,s})$ be the irreducible highest weight modules with highest weight $(z, h_{r,s})$ over the Virasoro algebra. We define the normalized character of $L(z, h_{r,s})$ by

$$\chi_{r,s}(\tau) := \mathrm{tr}_{L(z,h_{r,s})}\, q^{L_0 - \frac{1}{24}z},$$

where we set $q = \exp(2\pi\sqrt{-1}\tau)$ and define the matrix $(S_{(r,s),(r',s')})_{(r,s),(r',s')\in N_{p,q}}$ by

$$\chi_{r,s}(-\frac{1}{\tau}) = \sum_{(r',s')\in N_{p,q}} S_{(r,s),(r',s')}\chi_{r',s'}(\tau).$$

Then for any $(r_i, s_i) \in N_{p,q}$ $(i = 1, 2, 3)$, Verlinde's formula reads as follows:

$$\sum_{(r',s')\in N_{p,q}} \frac{S_{(r_1,s_1),(r',s')}S_{(r_2,s_2),(r',s')}S_{(r_3,s_3),(r',s')}}{S_{(1,1),(r',s')}},$$

where $(1,1) \in N_{p,q}$ and we fix $h_{(1,1)} = 0$ as usual. A variant of such formula also exist for WZNW model in the case when the central charge is a positive integer. Surprisingly, one obtains non-negative structure constants from such complicated expression. A common feature of these models are that, for each fixed central charge, the $\mathbb{C}$-span of the characters of the irreducible highest weight module, which corresponds to the primary fields, forms an $SL(2,\mathbb{Z})$-module. On the other hand, Kac and Wakimoto [KW1] showed that for any affine Lie algebra $\hat{\mathfrak{g}}$ and a rational number k satisfying certain conditions, there exists finite number of weights of level k, say $\lambda_1, \cdots, \lambda_n$, such that the $\mathbb{C}$-span of the character of the irreducible highest weight $\hat{\mathfrak{g}}$-modules with highest weight λ_i $(1 \leq i \leq n)$ forms an $SL(2,\mathbb{Z})$-module. Such weights are called admissible weights. So the following question naturally arise: Is there any interesting theory whose level is a rational number as above? Unfortunately, the naïve generalization does not work. The WZNW models are not well defined at rational level, see e.g. [MW]. But according to [P, KnPZ], the rational level cases plays an important role in $2D$ quantum gravity. Moreover, as was pointed out in [AY], applications of Verlinde's formula give negative structure constants. They introduced a modification of primary fields to obtain non-trivial results of fusion algebra for $\hat{\mathfrak{sl}}_2$. This theory can be regarded as a modified WZNW model and we will do so.

0. Introduction

In [BPZ], 2-dimensional conformal field theory (CFT) was proposed as an universal class of solvable lattice models at secondary phase transition points. It possesses an infinite dimensional symmetry which includes, by definition, the Virasoro algebra. Several examples of CFTs are known. Two famous examples are minimal model, which has Virasoro symmetry, and Wess-Zumino-Novikov-Witten model (WZNW), which has an affine Lie algebra symmetry. Their super-extensions are also important classes of quantum field theory in connection with superstring theory. In these theories, one of the most important problems is to calculate physical quantities, such as correlation functions. For this purpose, one is led to compute the so-called fusion algebra of the model, which is the algebra satisfied by the primary fields. In other words, the structure constants of the fusion algebra are given by the number of independent couplings between two primary fields. These models are parameterized by the value of the central element. An important class of such model consists of finitely many primary fields which form a closed algebra as fusion algebra.

From representation theoretical point of view, primary fields in some sense corresponds to irreducible highest weight modules and the structure constants of the fusion algebra can be computed as the dimension of some coinvariants. Let us explain this picture more concretely for WZNW model. Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra over $\mathbb{C}$ and $\hat{\mathfrak{g}}$ be its affinization. Fix a positive integer k which parameterizes the central charge, called level. In the case of the level k WZNW model with $\hat{\mathfrak{g}}$-symmetry on a smooth projective curve $\mathcal{E}$ over $\mathbb{C}$, the following description is well-known, see [TUY] for detail. For a positive integer m, fix a set E of m distinct points on $\mathcal{E}$. For any $P \in \mathcal{E}$, we define $\tilde{\mathfrak{g}}_P$ as the universal central extension of the Lie algebra of $\mathfrak{g}$-valued functions over the formal neighborhood of P. Moreover, let $L_{\lambda,k}(P)$ be the irreducible highest weight $\tilde{\mathfrak{g}}_P$-module with highest weight (λ, k). For each point $P \in \mathcal{E}$, we associate a dominant integral weight of level k, say (λ_P, k) whose $\mathfrak{g}$-weight part is λ_P. Then the Lie algebra $\mathfrak{g}(E)$ of $\mathfrak{g}$-valued meromorphic functions on $\mathcal{E}$ which are holomorphic outside E acts on the module $\otimes_{P\in E} L_{\lambda,k}(P)$ (hence its dual also) via

$$\mathfrak{g}(E) \hookrightarrow \oplus_{P\in E} \tilde{\mathfrak{g}}_P.$$

The space

$$H^0\left(\mathfrak{g}(E), (\otimes_{P\in E} L_{\lambda_P,k}(P))^*\right)$$

is called *the space of conformal blocks*. In particular its dimension (as a $\mathbb{C}$ vector space) for $m = 3$ gives us a structure constant of the fusion algebra of WZNW model on $\mathcal{E}$ with $\hat{\mathfrak{g}}$-symmetry. Hence we have only to compute its dual

$$H_0\left(\mathfrak{g}(E), \otimes_{P\in E} L_{\lambda_P,k}(P)\right)$$

for $m = 3$. Here and after, we assume that $\mathcal{E} = \mathbb{C}P^1$ unless otherwise stated.

FUSION ALGEBRAS FOR N=1 SUPERCONFORMAL FIELD THEORIES THROUGH COINVARIANTS *I*: $o\hat{s}p(1|2)$-SYMMETRY

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Abstract. Fusion algebras of the super-WZNW model with $o\hat{s}p(1|2)$-symmetry are computed through computation of cohomology of nilpotent subalgebras. Some of the proofs of the necessary facts, which have not appeared in the literatures, are also given.

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