THE CANONICAL ARITHMETIC HEIGHT OF SUBVARIETIES OF AN ABELIAN VARIETY OVER A FINITELY GENERATED FIELD

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Introduction

This paper is the sequel of [2]. In [4], S. Zhang defined the canonical height of subvarieties of an abelian variety over a number field in terms of adelic metrics. In this paper, we generalize it to an abelian variety defined over a finitely generated field over $\mathbb{Q}$. Our way is slightly different from his method. Instead of using adelic metrics directly, we introduce an adelic sequence and an adelic structure (cf. §§3.1).

Let K be a finitely generated field over $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$, and $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ a polarization of K, i.e., B is a projective arithmetic variety whose function field is K, and $\overline{H}_1, \ldots, \overline{H}_d$ are nef C^∞-hermitian line bundles on B. Let A be an abelian variety over K, and L a symmetric ample line bundle on A. Fix a projective arithmetic variety $\mathcal{A}$ over B and a nef C^∞-hermitian $\mathbb{Q}$-line bundle $\overline{\mathcal{L}}$ on $\mathcal{A}$ such that A is the generic fiber of $\mathcal{A} \to B$ and $\mathcal{L}$ is isomorphic to L on A. Then we can assign the naive height $h^{\overline{B}}_{(\mathcal{A},\overline{\mathcal{L}})}(X)$ to a subvariety X of $A_{\overline{K}}$. Indeed, if X is defined over K, $h^{\overline{B}}_{(\mathcal{A},\overline{\mathcal{L}})}(X)$ is given by

$$\frac{\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}\big|_{\mathcal{X}})^{\cdot \dim X+1} \cdot \widehat{c}_1(\pi_{\mathcal{X}}^*(\overline{H}_1)) \cdots \widehat{c}_1(\pi_{\mathcal{X}}^*(\overline{H}_d))\right)}{(\dim X + 1)\deg(L|_X^{\dim X})},$$

where $\mathcal{X}$ is the Zariski closure of X in $\mathcal{A}$ and $\pi_{\mathcal{X}} : \mathcal{X} \to B$ is the canonical morphism. The canonical height $\hat{h}^{\overline{B}}_L(X)$ of X with respect to L and $\overline{B}$ is characterized by the following properties:

(a) $\hat{h}^{\overline{B}}_L(X) \geq 0$ for all subvarieties X of $A_{\overline{K}}$.

(b) There is a constant C such that

$$\left|\hat{h}^{\overline{B}}_L(X) - h^{\overline{B}}_{(\mathcal{A},\overline{\mathcal{L}})}(X)\right| \leq C$$

for all subvarieties X of $A_{\overline{K}}$.

(c) $\hat{h}^{\overline{B}}_L([N](X)) = N^2\hat{h}^{\overline{B}}_L(X)$ for all subvarieties X of $A_{\overline{K}}$ and all non-zero integers N.

The main result of this paper is the following theorem, which is a generalization of [5].

Theorem (cf. Theorem 5.1)**.** *If the polarization $\overline{B}$ is big (i.e., $\overline{H}_1, \ldots, \overline{H}_d$ are nef and big), then, for a subvariety X of $A_{\overline{K}}$, the following are equivalent.*

Date: 19/October/1999, 0:20PM (JP), (Version 1.0).

1991 *Mathematics Subject Classification.* Primary 11G35, 14G25, 14G40; Secondary 11G10, 14K15.

Key words and phrases. height function, abelian variety, finitely generated field, Arakelov Geometry, Bogomolov's conjecture.

(1) *X is a translation of an abelian subvariety by a torsion point.*
(2) *The set $\{x \in X(\overline{K}) \mid \hat{h}_L^{\overline{B}}(x) \leq \epsilon\}$ is Zariski dense in X for every $\epsilon > 0$.*
(3) *The canonical height of X with respect to L and $\overline{B}$ is zero, i.e., $\hat{h}_L^{\overline{B}}(X) = 0$.*

Next let us consider a case where a curve and its Jacobian. Let X be a smooth projective curve of genus $g \geq 2$ over K, and J the Jacobian of X. Let Θ be a symmetric theta divisor on J, and $j : X \to J$ a morphism given by $j(x) = \omega_X - (2g-2)x$. Then, since $j^*(\mathcal{O}_J(\Theta)) = \omega_X^{\otimes 2g(g-1)}$, we can assign the canonical adelic structure $\overline{\omega}_X^a$ to ω_X. As a corollary of the above theorem, we have the following, which is a generalization of [3].

Corollary (cf. Corollary 5.4)**.** *If the polarization $\overline{B}$ is big, then the adelic self intersection number of $\overline{\omega}_X^a$ with respect to $\overline{B}$ is positive, i.e., $\langle \overline{\omega}_X^a \cdot \overline{\omega}_X^a \rangle_{\overline{B}} > 0$.*

1. Preliminaries

For the basic notation of Arakelov Geometry, we follow the paper [2].

Let X be a projective arithmetic variety with $d = \dim X_{\mathbb{Q}}$, and $\overline{L}$ a C^∞-hermitian $\mathbb{Q}$-line bundle on X. First we review several kinds of positivity of $\overline{L}$.

•**ample**: We say $\overline{L}$ is *ample* if L is ample, $c_1(\overline{L})$ is a semipositive form on $X(\mathbb{C})$, and, for a sufficiently large n, $H^0(X, L^{\otimes n})$ is generated by $\{s \in H^0(X, L^{\otimes n}) \mid \|s\|_{\sup} < 1\}$.

•**nef**: We say $\overline{L}$ is *nef* if $c_1(\overline{L})$ is a semipositive form on $X(\mathbb{C})$ and, for all one-dimensional integral closed subschemes Γ of X, $\widehat{\deg}\left(\overline{L}\big|_{\Gamma}\right) \geq 0$.

•**big**: $\overline{L}$ is said to be *big* if $\operatorname{rk}_{\mathbb{Z}} H^0(X, L^{\otimes m}) = O(m^d)$, and there is a non-zero section s of $H^0(X, L^{\otimes n})$ with $\|s\|_{\sup} < 1$ for some positive integer n.

•**$\mathbb{Q}$-effective**: We say $\overline{L}$ is $\mathbb{Q}$-effective, denote by $\overline{L} \gtrsim 0$, if there are a positive integer n and a non-zero section $s \in H^0(X, L^{\otimes n})$ with $\|s\|_{\sup} \leq 1$. Moreover, if U is a non-empty Zariski open set of X with $\operatorname{div}(s) \subseteq X \setminus U$, then we use the notation $\overline{L} \gtrsim_U 0$. Let $\overline{M}$ be another C^∞-hermitian $\mathbb{Q}$-line bundle on X. If $\overline{L} \otimes \overline{M}^{\otimes -1} \gtrsim 0$ (resp. $\overline{L} \otimes \overline{M}^{\otimes -1} \gtrsim_U 0$), then we denote this by $\overline{L} \gtrsim \overline{M}$ (resp. $\overline{L} \gtrsim_U \overline{M}$).

Proposition 1.1. (1) *If $\overline{L}$ is a nef C^∞-hermitian $\mathbb{Q}$-line bundle and $\overline{A}$ is an ample C^∞-hermitian $\mathbb{Q}$-line bundle, then $\overline{L} + \epsilon\overline{A}$ is ample for all positive rational numbers ϵ.*
(2) *If $\overline{L}_1, \ldots, \overline{L}_{d+1}$ are nef C^∞-hermitian $\mathbb{Q}$-line bundles, then*
$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_{d+1})\right) \geq 0.$$
(3) *If $\overline{L}_1, \ldots, \overline{L}_d$ are nef C^∞-hermitian $\mathbb{Q}$-line bundles and $\overline{M}$ is a $\mathbb{Q}$-effective C^∞-hermitian $\mathbb{Q}$-line bundle, then*
$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_d) \cdot \widehat{c}_1(\overline{M})\right) \geq 0.$$
(4) *Let $\overline{L}_1, \ldots, \overline{L}_{d+1}$ and $\overline{M}_1, \ldots, \overline{M}_{d+1}$ be nef C^∞-hermitian line bundles on X. If $\overline{M}_i \gtrsim \overline{L}_i$ for every i, then*
$$\widehat{\deg}\left(\widehat{c}_1(\overline{M}_1) \cdots \widehat{c}_1(\overline{M}_{d+1})\right) \geq \widehat{\deg}\left(\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_{d+1})\right).$$

Proof. (1), (2) and (3) was proved in [2, Proposition 2.3]. (4) follows from the following equation:

$$\widehat{\deg}\left(\widehat{c}_1(\overline{M}_1)\cdots\widehat{c}_1(\overline{M}_{d+1})\right) - \widehat{\deg}\left(\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_{d+1})\right) \\ = \sum_{i=1}^{d+1}\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_{i-1})\cdot(\widehat{c}_1(\overline{M}_i)-\widehat{c}_1(\overline{L}_i))\cdot\widehat{c}_1(\overline{M}_{i+1})\cdots\widehat{c}_1(\overline{M}_{d+1})\right).$$

□

Moreover, the following lemmas will be used in the later sections.

Lemma 1.2. *Let X be a projective arithmetic variety, and $\overline{L}$ a big C^∞-hermitian $\mathbb{Q}$-line bundle on X. Let x be a (not necessarily closed) point of X. Then, there are a positive number n and a non-zero section $s \in H^0(X, L^{\otimes n})$ with $s(x) = 0$ and $\|s\|_{\sup} < 1$.*

Proof. Since $\operatorname{rk}_{\mathbb{Z}} H^0(X, L^{\otimes m}) = O(m^d)$, there are a positive number n_0 and a non-zero section $s_0 \in H^0(X, L^{\otimes n_0})$ with $s_0(x) = 0$. On the other hand, there is a non-zero section $s_1 \in H^0(X, L^{\otimes n_1})$ with $\|s_1\|_{\sup} < 1$ for some positive integer n_1. Let n_2 be a positive integer with

$$\|s_0\|_{\sup}\|s_1\|_{\sup}^{n_2} < 1.$$

Thus, if we set $s = s_0 \otimes s_1^{\otimes n_2} \in H^0(X, L^{\otimes n_0+n_1n_2})$, then we have the desired assertion. □

Lemma 1.3. *Let B be a projective arithmetic variety and K the function field of B. Let X be a projective variety over K, and L an ample line bundle on X. Then, there are a projective arithmetic variety $\mathcal{X}$ over B, and an ample C^∞-hermitian $\mathbb{Q}$-line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$ such that X is the generic fiber of $\mathcal{X} \to B$ and $\mathcal{L}$ coincides with L in* $\operatorname{Pic}(X) \otimes \mathbb{Q}$.

Proof. Choose a sufficiently large integer n such that $\phi_{|L^{\otimes n}|}$ gives rise to an embedding $X \hookrightarrow \mathbb{P}^N_K$. Let $\mathcal{X}$ be the Zariski closure of X in $\mathbb{P}^N_B = \mathbb{P}^N \times B$. Since $\mathcal{O}_{\mathbb{P}^N}(1)$ is relative ample, there is an ample line bundle Q on B such that $\mathcal{A} = \mathcal{O}_{\mathbb{P}^N}(1) \otimes \pi^*(Q)$ is ample, where π is the natural projection $\mathbb{P}^N_B \to B$. We choose a C^∞-hermitian metric of $\mathcal{A}$ such that $\overline{\mathcal{A}} = (\mathcal{A}, \|\cdot\|)$ is ample. Thus, if we set $\overline{\mathcal{L}} = \left(\overline{\mathcal{A}}\big|_{\mathcal{X}}\right)^{\otimes 1/n}$, then we have our assertion. □

Next, let us consider the following relative positivity.

•π-nef (nef with respect to a morphism): Let $\pi : X \to B$ be a morphism of projective arithmetic varieties, and $\overline{L}$ a C^∞-hermitian $\mathbb{Q}$-line bundle on X. We say $\overline{L}$ is *nef with respect to $X \to B$* (or *π-nef*) if the following properties are satisfied:

(1) For any analytic maps $h : M \to X(\mathbb{C})$ from a complex manifold M to $X(\mathbb{C})$ with $\pi(h(M))$ being a point, $c_1(h^*(\overline{L}))$ is semipositive.
(2) For every $b \in B$, the restriction $L|_{X_{\overline{b}}}$ of L to the geometric fiber over b is nef.

Then, we have the following lemma.

Lemma 1.4. *Let $\pi : X \to B$ be a morphism of projective arithmetic varieties with $d = \dim B_{\mathbb{Q}}$ and $e = \dim(X/B)$. Let $\overline{H}_1, \ldots, \overline{H}_d$ be nef C^∞-hermitian $\mathbb{Q}$-line bundles on B. Then, we have the following.*

(1) *Let* $\overline{L}_1, \ldots, \overline{L}_e$ *be* π*-nef* C^∞*-hermitian* $\mathbb{Q}$*-line bundles on* X*, and* $\overline{L}$ *a* C^∞*-hermitian* $\mathbb{Q}$*-line bundle on* X*. If there is a non-empty Zariski open set* U *of* B *with* $\overline{L} \gtrsim_{\pi^{-1}(U)} 0$*, then*

$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_e)\cdot\widehat{c}_1(\overline{L})\cdot\widehat{c}_1(\pi^*\overline{H}_1)\cdots\widehat{c}_1(\pi^*\overline{H}_d)\right) \geq 0.$$

(2) *Let* $\overline{L}_1, \ldots, \overline{L}_{e+1}$ *and* $\overline{L}'_1, \ldots, \overline{L}'_{e+1}$ *be* π*-nef* C^∞*-hermitian* $\mathbb{Q}$*-line bundles on* X*. If there is a Zariski open set* U *of* B *such that* $\overline{L}_i \gtrsim_{\pi^{-1}(U)} \overline{L}'_i$ *for all* i*, then*

$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_{e+1})\cdot\widehat{c}_1(\pi^*\overline{H}_1)\cdots\widehat{c}_1(\pi^*\overline{H}_d)\right) \geq \widehat{\deg}\left(\widehat{c}_1(\overline{L}'_1)\cdots\widehat{c}_1(\overline{L}'_{e+1})\cdot\widehat{c}_1(\pi^*\overline{H}_1)\cdots\widehat{c}_1(\pi^*\overline{H}_d)\right).$$

Proof. (1) By our assumption, there are a positive integer n and a non-zero section $s \in H^0(X, L^{\otimes n})$ such that $\|s\|_{\sup} \leq 1$ and $\operatorname{Supp}(\operatorname{div}(s)) \subseteq X \setminus \pi^{-1}(U)$. Let $\operatorname{div}(s) = a_1\Delta_1 + \cdots + a_r\Delta_r$ be the decomposition as cycles. Then,

$$\begin{aligned} n\widehat{\deg}&\left(\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_e)\cdot\widehat{c}_1(\overline{L})\cdot\widehat{c}_1(\pi^*\overline{H}_1)\cdots\widehat{c}_1(\pi^*\overline{H}_d)\right) \\ &= \sum_{i=1}^r a_i\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1|_{\Delta_i})\cdots\widehat{c}_1(\overline{L}_e|_{\Delta_i})\cdot\widehat{c}_1(\pi^*\overline{H}_1|_{\Delta_i})\cdots\widehat{c}_1(\pi^*\overline{H}_d)|_{\Delta_i}\right) \\ &\quad + \int_{X(\mathbb{C})} -\log(\|s\|)c_1(\overline{L}_1)\wedge\cdots\wedge c_1(\overline{L}_e)\wedge c_1(\pi^*\overline{H}_1)\wedge\cdots\wedge c_1(\pi^*\overline{H}_d). \end{aligned} \tag{1.4.1}$$

First, by the Fubini's theorem,

$$\begin{aligned} &\int_{X(\mathbb{C})} -\log(\|s\|)c_1(\overline{L}_1)\wedge\cdots\wedge c_1(\overline{L}_e)\wedge c_1(\pi^*\overline{H}_1)\wedge\cdots\wedge c_1(\pi^*\overline{H}_d) \\ &\qquad = \int_{B(\mathbb{C})}\left(\int_{X(\mathbb{C})/B(\mathbb{C})} -\log(\|s\|)c_1(\overline{L}_1)\wedge\cdots\wedge c_1(\overline{L}_e)\right)c_1(\overline{H}_1)\wedge\cdots\wedge c_1(\overline{H}_d). \end{aligned}$$

Here, by the property (1) of "π-nef",

$$\int_{X(\mathbb{C})/B(\mathbb{C})} -\log(\|s\|)c_1(\overline{L}_1)\wedge\cdots\wedge c_1(\overline{L}_e)$$

is a non-negative locally integrable function on $B(\mathbb{C})$. Thus, the integral part of (1.4.1) is non-negative. Let b_i be the generic point of $\pi(\Delta_i)$. Then, by the projection formula, we can see

$$\begin{aligned} &\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1|_{\Delta_i})\cdots\widehat{c}_1(\overline{L}_e|_{\Delta_i})\cdot\widehat{c}_1(\pi^*\overline{H}_1|_{\Delta_i})\cdots\widehat{c}_1(\pi^*\overline{H}_d|_{\Delta_i})\right) \\ &= \begin{cases} 0 & \text{if } \operatorname{codim}(\pi(\Delta_i)) \geq 2 \\ \deg(L_1|_{(\Delta_i)_{\bar{b}_i}}\cdots L_e|_{(\Delta_i)_{\bar{b}_i}})\widehat{\deg}\left(\widehat{c}_1(\overline{H}_1|_{\pi(\Delta_i)})\cdots\widehat{c}_1(\overline{H}_d|_{\pi(\Delta_i)})\right) & \text{if } \operatorname{codim}(\pi(\Delta_i)) = 1 \end{cases} \end{aligned}$$

Therefore, we get (1) because

$$\deg(L_1|_{(\Delta_i)_{\bar{b}_i}}\cdots L_e|_{(\Delta_i)_{\bar{b}_i}}) \geq 0 \quad \text{and} \quad \widehat{\deg}\left(\widehat{c}_1(\overline{H}_1|_{\pi(\Delta_i)})\cdots\widehat{c}_1(\overline{H}_d|_{\pi(\Delta_i)})\right) \geq 0.$$

.

(2) Since

$$\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_{e+1}) - \widehat{c}_1(\overline{L}'_1)\cdots\widehat{c}_1(\overline{L}'_{e+1}) \\ = \sum_{i=1}^{e+1}\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_{i-1})\cdot\left(\widehat{c}_1(\overline{L}_i) - \widehat{c}_1(\overline{L}'_i)\right)\cdot\widehat{c}_1(\overline{L}'_{i+1})\cdots\widehat{c}_1(\overline{L}'_{e+1}),$$

(2) is a consequence of (1). □

Finally, let us consider the following lemma.

Lemma 1.5. *Let $\pi : X \to B$ be a morphism of projective arithmetic varieties, and $\overline{L}$ a C^∞-hermitian line bundle on X. Let U be a non-empty Zariski open set of B such that $B \setminus U = \operatorname{Supp}(D)$ for some effective Cartier divisor D on B. If there is a non-zero rational section s of L with $\operatorname{Supp}(\operatorname{div}(s)) \subseteq X \setminus \pi^{-1}(U)$, then there are a positive integer n and a C^∞-metric $\|\cdot\|_{nD}$ of $\mathcal{O}_B(nD)$ with*

$$\pi^*(\mathcal{O}_B(nD), \|\cdot\|_{nD})^{\otimes -1} \precsim_{\pi^{-1}(U)} \overline{L} \precsim_{\pi^{-1}(U)} \pi^*(\mathcal{O}_B(nD), \|\cdot\|_{nD}).$$

Moreover, if D is ample, then we can choose $\|\cdot\|_{nD}$ such that $(\mathcal{O}_B(nD), \|\cdot\|_{nD})$ is ample.

Proof. First, we fix a hermitian metric $\|\cdot\|_D$ of $\mathcal{O}_B(D)$. If D is ample, then we choose $\|\cdot\|_D$ such that $(\mathcal{O}_B(D), \|\cdot\|_D)$ is ample. Find a positive integer n with

$$-nf^*(D) \le \operatorname{div}(s) \le nf^*(D).$$

Let l be a section of $O_Y(nD)$ with $\operatorname{div}(l) = nD$. We set $t_1 = l \otimes s^{-1}$ and $t_2 = l \otimes s$. Then, t_1 and t_2 are global sections of $\mathcal{O}_X(nf^*(D)) \otimes L^{-1}$ and $\mathcal{O}_X(nf^*(D)) \otimes L$ respectively. Choose a sufficiently small positive number c such that if we give a norm of $\mathcal{O}_B(nD)$ by $c\|\cdot\|_D^n$, then $\|t_1\|_{\sup} \le 1$ and $\|t_2\|_{\sup} \le 1$. Thus we get our lemma. □

2. Arithmetic height of subvarieties

Let K be a finitely generated field over $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$, and $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ a polarization of K. Let X be a projective variety over K, and L a nef line bundle on X. Let $\mathcal{X}$ be a projective arithmetic variety over B such that X is the generic fiber of $\mathcal{X} \to B$, and let $\overline{\mathcal{L}}$ be a C^∞-hermitian $\mathbb{Q}$-line bundle on $\mathcal{X}$ such that $\mathcal{L}$ coincides with L in $\operatorname{Pic}(X) \otimes \mathbb{Q}$. The pair $(\mathcal{X}, \overline{\mathcal{L}})$ is called a C^∞-model of (X, L). We assume that $\overline{\mathcal{L}}$ is nef with respect to $\mathcal{X} \to B$. Note that if L is ample, then there is a C^∞-model $(\mathcal{X}, \overline{\mathcal{L}})$ of (X, L) such that $\overline{\mathcal{L}}$ is ample by Lemma 1.3.

Let Y be a subvariety of $X_{\overline{K}}$. We assume that Y is defined over a finite extension field K' of K. Let $B^{K'}$ be the normalization of B in K', and let $\rho^{K'} : B^{K'} \to B$ be the induced morphism. Let $\mathcal{X}^{K'}$ be the main component of $\mathcal{X} \times_B B^{K'}$. We set the induced morphisms as follows.

$$\begin{array}{ccc} \mathcal{X} & \xleftarrow{\tau^{K'}} & \mathcal{X}^{K'} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi^{K'}} \\ B & \xleftarrow{\rho^{K'}} & B^{K'} \end{array}$$

Let $\mathcal{Y}$ be the Zariski closure of Y in $\mathcal{X}^{K'}$. Then the naive height $h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(Y)$ of Y with respect to $(\mathcal{X},\overline{\mathcal{L}})$ and $\overline{B}$ is defined by

$$h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(Y) = \frac{\widehat{\deg}\left(\widehat{c}_1\left(\tau^{K'*}(\overline{\mathcal{L}})\Big|_{\mathcal{Y}}\right)^{\cdot \dim Y+1}\cdot \widehat{c}_1\left(\pi^{K'*}(\rho^{K'*}(\overline{H}_1))\Big|_{\mathcal{Y}}\right)\cdots \widehat{c}_1\left(\pi^{K'*}(\rho^{K'*}(\overline{H}_d))\Big|_{\mathcal{Y}}\right)\right)}{[K':K](\dim Y+1)\deg(L|_Y^{\dim Y})}.$$

Note that the above definition does not depend on the choice of K' by the projection formula. Here we have the following proposition. By this proposition, we may denote by $h^{\overline{B}}_L$ the class of $h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}$ modulo the set of bounded functions. Moreover, we say $h^{\overline{B}}_L$ is the height function associated with L and $\overline{B}$.

Proposition 2.1. *Let $(\mathcal{X}',\overline{\mathcal{L}}')$ be another model of (X,L) over B such that $\overline{\mathcal{L}}'$ is nef with respect to $\mathcal{X}'\to B$. Then, there is a constant C such that*

$$\left|h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(Y) - h^{\overline{B}}_{(\mathcal{X}',\overline{\mathcal{L}}')}(Y)\right| \leq C$$

for all subvarieties Y of $X_{\overline{K}}$.

Proof. Let U be a Zariski open set of B with $\mathcal{X}_U = \mathcal{X}'_U$ and $\mathcal{L}_U = \mathcal{L}'_U$ in $\mathrm{Pic}(\mathcal{X}_U)\otimes\mathbb{Q}$. Let A be an ample line bundle on B and I the defining ideal of $B\setminus U$. Then, there is a non-zero section t of $H^0(B, A^{\otimes m}\otimes I)$ for some positive integer m. Thus, $B\setminus U \subseteq \mathrm{Supp}(\mathrm{div}(s))$. Therefore, shrinking U, we may assume that there is an effective ample Cartier divisor D on B with $\mathrm{Supp}(D) = B\setminus U$.

Let $\mu : \mathcal{Z}\to\mathcal{X}$ and $\mu' : \mathcal{Z}\to\mathcal{X}'$ be birational morphisms of projective arithmetic varieties such that μ and μ' are the identity map over $\mathcal{X}_U$. Then, $h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(Y) = h^{\overline{B}}_{(\mathcal{Z},\mu^*(\overline{\mathcal{L}}))}(Y)$ and $h^{\overline{B}}_{(\mathcal{X}',\overline{\mathcal{L}}')}(Y) = h^{\overline{B}}_{(\mathcal{Z},\mu'^*(\overline{\mathcal{L}}'))}(Y)$ for all subvarieties Y of $X_{\overline{K}}$. Thus, to prove our proposition, we may assume that $\mathcal{X}=\mathcal{X}'$.

First of all, by Lemma 1.5, there is a nef C^∞-hermitian line bundle $\overline{T}$ on B such that

$$\pi^*(\overline{T})^{\otimes -1} \precsim_{\pi^{-1}(U)} \overline{\mathcal{L}}\otimes\overline{\mathcal{L}}'^{\otimes -1} \precsim_{\pi^{-1}(U)} \pi^*(\overline{T}), \tag{2.1.1}$$

where $\pi : \mathcal{X}\to B$ is the canonical morphism. Let Y be a subvariety of $X_{\overline{K}}$. We assume that Y is defined over a finite extension field K' of K. Let $B^{K'}$ be the normalization of B in K', and $\mathcal{X}^{K'}$ the main components of $\mathcal{X}\times_B B^{K'}$. Let $\mathcal{Y}$ be the closure of Y in $\mathcal{X}^{K'}$. Then,

$$h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(Y) = \frac{\widehat{\deg}\left(\widehat{c}_1\left(\overline{\mathcal{L}}^{K'}\Big|_{\mathcal{Y}}\right)^{\cdot \dim Y+1}\cdot\widehat{c}_1\left(\overline{H}_1^{K'}\Big|_{\mathcal{Y}}\right)\cdots\widehat{c}_1\left(\overline{H}_d^{K'}\Big|_{\mathcal{Y}}\right)\right)}{[K':K](\dim Y+1)\deg(L|_Y^{\dim Y})}$$

and

$$h^{\overline{B}}_{(\mathcal{X}',\overline{\mathcal{L}}')}(Y) = \frac{\widehat{\deg}\left(\widehat{c}_1\left(\overline{\mathcal{L}}'^{K'}\Big|_{\mathcal{Y}}\right)^{\cdot \dim Y+1}\cdot\widehat{c}_1\left(\overline{H}_1^{K'}\Big|_{\mathcal{Y}}\right)\cdots\widehat{c}_1\left(\overline{H}_d^{K'}\Big|_{\mathcal{Y}}\right)\right)}{[K':K](\dim Y+1)\deg(L|_Y^{\dim Y})},$$

where $\overline{\mathcal{L}}^{K'}$, $\overline{\mathcal{L}}'^{K'}$ and $\overline{H}_i^{K'}$'s are pullbacks of $\overline{\mathcal{L}}$, $\overline{\mathcal{L}}'$ and $\overline{H}_i$'s to $\mathcal{X}^{K'}$ respectively. Here, by virtue of (2.1.1),

$$\overline{\mathcal{L}}'^{K'} \otimes \overline{T}^{K'\otimes -1} \precsim \overline{\mathcal{L}}^{K'} \precsim \overline{\mathcal{L}}'^{K'} \otimes \overline{T}^{K'}.$$

Therefore, by (2) of Lemma 1.4, we can see that

$$\begin{aligned}\Bigg|\widehat{\deg}\left(\widehat{c}_1\left(\overline{\mathcal{L}}^{K'}\Big|_{\mathcal{Y}}\right)^{\cdot \dim Y+1} \cdot \widehat{c}_1\left(\overline{H}_1^{K'}\Big|_{\mathcal{Y}}\right)\cdots \widehat{c}_1\left(\overline{H}_d^{K'}\Big|_{\mathcal{Y}}\right)\right) \\ -\widehat{\deg}\left(\widehat{c}_1\left(\overline{\mathcal{L}}'^{K'}\Big|_{\mathcal{Y}}\right)^{\cdot \dim Y+1} \cdot \widehat{c}_1\left(\overline{H}_1^{K'}\Big|_{\mathcal{Y}}\right)\cdots \widehat{c}_1\left(\overline{H}_d^{K'}\Big|_{\mathcal{Y}}\right)\right)\Bigg| \\ \leq [K' : K](\dim Y + 1)\deg(L|_Y^{\dim Y})\widehat{\deg}\left(\overline{T}\cdot\overline{H}_1\cdots\overline{H}_d\right).\end{aligned}$$

Thus we get our proposition. □

3. Adelic sequence and adelic structure

3.1. **Adelic sequence, adelic structure and adelic line bundle.** Let K be a finitely generated field over $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$, and $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ a polarization of K. Let X be a projective variety over K, and L a nef line bundle on X.

A sequence of C^∞-models $\{(\mathcal{X}_n, \overline{\mathcal{L}}_n)\}$ of (X, L) is called an *adelic sequence of* (X, L) (with respect to $\overline{B}$) if $\overline{\mathcal{L}}_n$ is nef with respect to $\mathcal{X}_n \to B$ for every n, and there is a non-empty Zariski open set U of B with following properties:

(1) $\mathcal{X}_n|_U = \mathcal{X}_m|_U$ (say $\mathcal{X}_U$) and $\mathcal{L}_n|_U = \mathcal{L}_m|_U$ in $\operatorname{Pic}(\mathcal{X}_U)\otimes\mathbb{Q}$ for all n, m.

(2) For each n, m, there are a projective arithmetic variety $\mathcal{X}_{n,m}$ over B, birational morphisms $\mu^n_{n,m} : \mathcal{X}_{n,m} \to \mathcal{X}_n$ and $\mu^m_{n,m} : \mathcal{X}_{n,m} \to \mathcal{X}_m$, and a nef C^∞-hermitian $\mathbb{Q}$-line bundle $\overline{D}_{n,m}$ on B such that

$$\pi^*_{n,m}(\overline{D}^{\otimes -1}_{n,m}) \precsim_{\pi^{-1}_{n,m}(U)} (\mu^n_{n,m})^*(\overline{\mathcal{L}}_n)\otimes(\mu^m_{n,m})^*(\overline{\mathcal{L}}^{\otimes -1}_m) \precsim_{\pi^{-1}_{n,m}(U)} \pi^*_{n,m}(\overline{D}_{n,m})$$

and that

$$\widehat{\deg}\left(\widehat{c}_1(\overline{D}_{n,m})\cdot\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\right) \longrightarrow 0$$

as $n, m \longrightarrow \infty$, where $\pi_{n,m}$ is the natural morphism $\mathcal{X}_{n,m} \to B$.

The open set U as above is called a *common base* of the sequence $\{(\mathcal{X}_n, \overline{\mathcal{L}}_n)\}$. Note that if U' is a non-empty Zariski open set of U, then U' is also a common base of $\{(\mathcal{X}_n, \overline{\mathcal{L}}_n)\}$.

Let $\{(\mathcal{Y}_n, \overline{\mathcal{M}}_n)\}$ be another adelic sequence of (X, L). We say $\{(\mathcal{X}_n, \overline{\mathcal{L}}_n)\}$ is *equivalent* to $\{(\mathcal{Y}_n, \overline{\mathcal{M}}_n)\}$, denoted by $\{(\mathcal{X}_n, \overline{\mathcal{L}}_n)\} \sim \{(\mathcal{Y}_n, \overline{\mathcal{M}}_n)\}$, if the concatenated sequence

$$(\mathcal{X}_1, \overline{\mathcal{L}}_1), (\mathcal{Y}_1, \overline{\mathcal{M}}_1), \cdots, (\mathcal{X}_n, \overline{\mathcal{L}}_n), (\mathcal{Y}_n, \overline{\mathcal{M}}_n), \cdots$$

is adelic. In other words, if we choose a suitable common base U, then, for each n, there are a projective arithmetic variety $\mathcal{Z}_n$ over B, birational morphisms $\mu_n : \mathcal{Z}_n \to \mathcal{X}_n$ and $\nu_n : \mathcal{Z}_n \to \mathcal{Y}_n$, and a nef C^∞-hermitian $\mathbb{Q}$-line bundle $\overline{D}_n$ on B such that

$$\pi^*_{\mathcal{Z}_n}(\overline{D}^{\otimes -1}_n) \precsim_{\pi^{-1}_{\mathcal{Z}_n}(U)} \mu^*_n(\overline{\mathcal{L}}_n)\otimes\nu^*_n(\overline{\mathcal{M}}^{\otimes -1}_n) \precsim_{\pi^{-1}_{\mathcal{Z}_n}(U)} \pi^*_{\mathcal{Z}_n}(\overline{D}_n)$$

and that

$$\lim_{n\to\infty} \widehat{\deg}\left(\widehat{c}_1(\overline{D}_n)\cdot\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\right) = 0,$$

where $\pi_{\mathcal{Z}_n}$ is the natural morphism $\mathcal{Z}_n \to B$.

An equivalent class of adelic sequences of (X, L) is called an *adelic structure of* L (with respect to $\overline{B}$). Further, a line bundle L with an adelic structure is called an *adelic line bundle* and is often denoted by $\overline{L}$ for simplicity. If an adelic line bundle $\overline{L}$ is given by an adelic sequence $\{(\mathcal{X}_n, \overline{\mathcal{L}}_n)\}$, then we denote this by $\overline{L} = \lim_{n\to\infty}(\mathcal{X}_n, \overline{\mathcal{L}}_n)$. Moreover, we say $\overline{L}$ is nef if $\overline{L} = \lim_{n\to\infty}(\mathcal{X}_n, \overline{\mathcal{L}}_n)$ and $\overline{\mathcal{L}}_n$ is nef for $n \gg 0$.

Let $g : Y \to X$ be a morphism of projective varieties over K, and $\overline{L}$ an adelic line bundle on X. We assume that $\overline{L}$ is given by an adelic sequence $\{(\mathcal{X}_n, \overline{\mathcal{L}}_n)\}$. Let us fix a morphism $g_n : \mathcal{Y}_n \to \mathcal{X}_n$ of projective arithmetic varieties over B for each n with the following properties:

(a) $g_n : \mathcal{Y}_n \to \mathcal{X}_n$ coincides with $g : Y \to X$ over K for every n.
(b) There is a non-empty Zariski open set U of B such that $\mathcal{Y}_n|_U = \mathcal{Y}_m|_U$, $\mathcal{X}_n|_U = \mathcal{X}_m|_U$, and $g_n|_U = g_m|_U$ for all n, m

Then it is not difficult to see that $\{(\mathcal{Y}_n, g_n^*(\overline{\mathcal{L}}_n))\}$ is an adelic sequence of $(Y, g^*(L))$. We denote by $g^*(\overline{L})$ the adelic structure given by $\{(\mathcal{Y}_n, g_n^*(\overline{\mathcal{L}}_n))\}$. Note that this adelic structure does not depend on the choice of the adelic sequence $\{(\mathcal{X}_n, \overline{\mathcal{L}}_n)\}$ and the morphisms $g_n : \mathcal{Y}_n \to \mathcal{X}_n$.

3.2. **Adelic sequence by an endomorphism.** Let K be a finitely generated field over $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$, and $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ a polarization of K. Let X be a projective variety over K, and L an ample line bundle on X. We assume that there is a surjective morphism $f : X \to X$ and an integer $d \geq 2$ with $L^{\otimes d} \simeq f^*(L)$. Let $(\mathcal{X}, \overline{\mathcal{L}})$ be a C^∞-model of (X, L) such that $\overline{\mathcal{L}}$ is nef with respect to $\mathcal{X} \to B$. Note that the existence of a C^∞-model $(\mathcal{X}, \overline{\mathcal{L}})$ of (X, L) with $\mathcal{L}$ being nef with respect to $\mathcal{X} \to B$ is guaranteed by Lemma 1.3. Then, there is a Zariski open set U of B such that f extends to $f_U : \mathcal{X}_U \to \mathcal{X}_U$ and $\mathcal{L}_U^{\otimes d} = f_U^*(\mathcal{L}_U)$ in $\operatorname{Pic}(\mathcal{X}_U) \otimes \mathbb{Q}$. Let $\mathcal{X}_n$ be the normalization of $\mathcal{X}_U \xrightarrow{f_U^n} \mathcal{X}_U \to \mathcal{X}$, and $f_n : \mathcal{X}_n \to \mathcal{X}$ the induced morphism. Then, we have the following proposition.

Proposition 3.2.1. (1) $\left\{(\mathcal{X}_n, f_n^*(\overline{\mathcal{L}})^{\otimes d^{-n}})\right\}$ *is an adelic sequence of* (X, L)*. Moreover, if* $\overline{\mathcal{L}}$ *is nef, then the adelic line bundle* $\lim_{n\to\infty}(\mathcal{X}_n, f_n^*(\overline{\mathcal{L}})^{\otimes d^{-n}})$ *is nef.*

(2) *Let* $f' : X \to X$ *be another surjective morphism with* $L^{\otimes d'} \simeq f'^*(L)$ *for some* $d' \geq 2$*. Let* $(\mathcal{X}', \overline{\mathcal{L}}')$ *be another* C^∞*-model of* (X, L) *such that* $\overline{\mathcal{L}}'$ *is nef with respect to* $\mathcal{X}' \to B$*. Let* U' *be a non-empty Zariski open set of* B *such that* f' *extends to* $f'_{U'} : \mathcal{X}'_{U'} \to \mathcal{X}'_{U'}$ *and* ${\mathcal{L}'_{U'}}^{\otimes d'} = {f'_{U'}}^*(\mathcal{L}'_{U'})$ *in* $\operatorname{Pic}(\mathcal{X}'_{U'}) \otimes \mathbb{Q}$*. Let* $\mathcal{X}'_n$ *be the normalization of* $\mathcal{X}'_{U'} \xrightarrow{{f'_{U'}}^n} \mathcal{X}'_{U'} \to \mathcal{X}'$*, and* $f'_n : \mathcal{X}'_n \to \mathcal{X}'$ *the induced morphism. If* $f \cdot f' = f' \cdot f$*, then*

$$\left\{(\mathcal{X}_n, f_n^*(\overline{\mathcal{L}})^{\otimes d^{-n}})\right\} \sim \left\{(\mathcal{X}'_n, {f'_n}^*(\overline{\mathcal{L}}')^{\otimes d'^{-n}})\right\}.$$

Definition 3.2.2 (f-adelic structure)**.** The adelic sequence $\left\{(\mathcal{X}_n, f_n^*(\overline{\mathcal{L}})^{\otimes d^{-n}})\right\}$ in the above proposition gives rise to the adelic structure on L, which is called the *f-adelic structure of L*. The line bundle L with this adelic structure is denoted by $\overline{L}^f$, i.e., $\overline{L}^f = \lim_{n\to\infty}(\mathcal{X}_n, f_n^*(\overline{\mathcal{L}})^{\otimes d^{-n}})$. Considering a case "$f = f'$" in (2), we can see that $\overline{L}^f$ does not depend on the choice of the C^∞-model $(\mathcal{X}, \overline{\mathcal{L}})$. Moreover, (2) says us that if $f \cdot f' = f' \cdot f$, then $\overline{L}^f = \overline{L}^{f'}$. Further, $\overline{L}^f$ is nef by the second assertion of (1) and Lemma 1.3.

Proof of Proposition 3.2.1. In the same way as in the proof of Proposition 2.1, shrinking U if necessarily, we may assume that there is an effective ample Cartier divisor D on B with $\operatorname{Supp}(D) = B \setminus U$.

(1) For simplicity, we denote $f_n^*(\overline{\mathcal{L}})^{\otimes d^{-n}}$ by $\overline{\mathcal{L}}_n$. From now on, we treat the group structure of the Picard group additively. Note that $\mathcal{X}_0 = \mathcal{X}$ and $\overline{\mathcal{L}}_0 = \overline{\mathcal{L}}$. Let $\mathcal{Y}$ be a projective arithmetic variety over B such that there are birational morphisms $\rho_0 : \mathcal{Y} \to \mathcal{X}_0$ and $\rho_1 : \mathcal{Y} \to \mathcal{X}_1$, which are the identity map over U. We fix $n > m \geq 0$. Let $\mathcal{Z}$ be a projective arithmetic variety over B with the following properties:

(a) $\mathcal{Z}_U = \mathcal{X}_U$.
(b) For each $m \leq i \leq n$, there is a birational morphism $\mu_i : \mathcal{Z} \to \mathcal{X}_i$, which is the identity map over U.
(c) For each $m \leq j < n$, there is a morphism $g_j : \mathcal{Z} \to \mathcal{Y}$ which is an extension of $f_U^j : \mathcal{Z}_U \to \mathcal{Y}_U$.

Here we claim the following.

Claim 3.2.3. (i) *$\mu_{j+1}^*(\overline{\mathcal{L}}_{j+1}) = d^{-j} g_j^*(\rho_1^*(\overline{\mathcal{L}}_1))$ for each $m \leq j < n$.*
(ii) *$\mu_j^*(\overline{\mathcal{L}}_j) = d^{-j} g_j^*(\rho_0^*(\overline{\mathcal{L}}_0))$ for each $m \leq j < n$.*
(iii) $\mu_n^*(\overline{\mathcal{L}}_n) - \mu_m^*(\overline{\mathcal{L}}_m) = \displaystyle\sum_{j=m}^{n-1} d^{-j} g_j^*(\rho_1^*(\overline{\mathcal{L}}_1) - \rho_0^*(\overline{\mathcal{L}}_0))$.

(i) Let us consider the following two morphisms between $\mathcal{Z}$ and $\mathcal{X}_0$:

$$\mathcal{Z} \xrightarrow{g_j} \mathcal{Y} \xrightarrow{\rho_1} \mathcal{X}_1 \xrightarrow{f_1} \mathcal{X}_0 \qquad \text{and} \qquad \mathcal{Z} \xrightarrow{\mu_{j+1}} \mathcal{X}_{j+1} \xrightarrow{f_{j+1}} \mathcal{X}_0.$$

These are same over U. Thus, so are over B. Therefore, $g_j^*\rho_1^*f_1^*(\overline{\mathcal{L}}) = \mu_{j+1}^* f_{j+1}^*(\overline{\mathcal{L}})$, which shows us the assertion of (i).

(ii) In the same way as above, we can see $\mu_j \cdot f_j = \rho_0 \cdot g_j$. Thus we get (ii).

(iii) Since $\mu_n^*(\overline{\mathcal{L}}_n) - \mu_m^*(\overline{\mathcal{L}}_m) = \displaystyle\sum_{j=m}^{n-1} \mu_{j+1}^*(\overline{\mathcal{L}}_{j+1}) - \mu_j^*(\overline{\mathcal{L}}_j)$, this is a consequence of (i) and (ii).

By Lemma 1.5, there is an ample C^∞-hermitian line bundle $\overline{\Delta}$ on B such that

$$-\pi_{\mathcal{Y}}^*(\overline{\Delta}) \precsim_{\pi_{\mathcal{Y}}^{-1}(U)} \rho_1^*(\overline{\mathcal{L}}_1) - \rho_0^*(\overline{\mathcal{L}}_0) \precsim_{\pi_{\mathcal{Y}}^{-1}(U)} \pi_{\mathcal{Y}}^*(\overline{\Delta}).$$

Hence, by (iii) of the above claim, we get

$$-\left(\sum_{j=m}^{n-1} d^{-j}\right)\pi_{\mathcal{Z}}^{*}(\overline{\Delta}) \precsim_{\pi_{\mathcal{Z}}^{-1}(U)} \mu_n^{*}(\overline{\mathcal{L}}_n) - \mu_m^{*}(\overline{\mathcal{L}}_m) \precsim_{\pi_{\mathcal{Z}}^{-1}(U)} \left(\sum_{j=m}^{n-1} d^{-j}\right)\pi_{\mathcal{Z}}^{*}(\overline{\Delta}).$$

Thus, we obtain the first assertion of (1). The second assertion is obvious.

(2) Let us consider the following cases:

Case 1 : $f = f'$.

Case 2 : $\mathcal{X} = \mathcal{X}'$ and $\overline{\mathcal{L}} = \overline{\mathcal{L}}'$.

Clearly, it is sufficient to check (2) under the assumption Case 1 or Case 2.

Case 1 : In this case, we assume $f = f'$. Shrinking U and U', we may assume that $U = U'$, $\mathcal{X}_U = \mathcal{X}'_{U'}$ and $\mathcal{L}_U = \mathcal{L}'_{U'}$ in $\operatorname{Pic}(\mathcal{X}_U)\otimes\mathbb{Q}$. For each $n \geq 0$, let $\mathcal{Z}_n$ be a projective arithmetic variety over B such that there are birational morphisms $\nu_n : \mathcal{Z}_n \to \mathcal{X}_n$ and $\nu'_n : \mathcal{Z}_n \to \mathcal{X}'_n$, which are the identity map over U. We may assume that there is a morphism $g_n : \mathcal{Z}_n \to \mathcal{Z}_0$ such that the following diagrams are commutative:

$$\begin{array}{ccc} \mathcal{Z}_0 & \xleftarrow{\;g_n\;} & \mathcal{Z}_n \\ {\scriptstyle \nu_0}\big\downarrow & & \big\downarrow{\scriptstyle \nu_n} \\ \mathcal{X}_0 & \xleftarrow{\;f_n\;} & \mathcal{X}_n \end{array} \qquad \begin{array}{ccc} \mathcal{Z}_0 & \xleftarrow{\;g_n\;} & \mathcal{Z}_n \\ {\scriptstyle \nu'_0}\big\downarrow & & \big\downarrow{\scriptstyle \nu'_n} \\ \mathcal{X}'_0 & \xleftarrow{\;f'_n\;} & \mathcal{X}'_n \end{array}$$

Then,

$$d^{-n}\nu_n^{*}(f_n^{*}(\overline{\mathcal{L}})) - d^{-n}{\nu'_n}^{*}({f'_n}^{*}(\overline{\mathcal{L}}')) = d^{-n}g_n^{*}\left(\nu_0^{*}(\overline{\mathcal{L}}) - {\nu'_0}^{*}(\overline{\mathcal{L}}')\right).$$

By Lemma 1.5, there is an ample C^{∞}-hermitian line bundle $\overline{\Delta}$ on B such that

$$-\pi_{\mathcal{Z}_0}^{*}(\overline{\Delta}) \precsim_{\pi_{\mathcal{Z}_0}^{-1}(U)} \nu_0^{*}(\overline{\mathcal{L}}) - {\nu'_0}^{*}(\overline{\mathcal{L}}') \precsim_{\pi_{\mathcal{Z}_0}^{-1}(U)} \pi_{\mathcal{Z}_0}^{*}(\overline{\Delta}).$$

Therefore, we have

$$-d^{-n}\pi_{\mathcal{Z}_n}^{*}(\overline{\Delta}) \precsim_{\pi_{\mathcal{Z}_n}^{-1}(U)} d^{-n}\nu_n^{*}(f_n^{*}(\overline{\mathcal{L}})) - d^{-n}{\nu'_n}^{*}({f'_n}^{*}(\overline{\mathcal{L}}')) \precsim_{\pi_{\mathcal{Z}_n}^{-1}(U)} d^{-n}\pi_{\mathcal{Z}_n}^{*}(\overline{\Delta}),$$

which shows us our assertion in this case.

Case 2 : In this case, we assume that $\mathcal{X} = \mathcal{X}'$ and $\overline{\mathcal{L}} = \overline{\mathcal{L}}'$. We denote $f_n^{*}(\overline{\mathcal{L}})^{\otimes d^{-n}}$ and ${f'_n}^{*}(\overline{\mathcal{L}})^{\otimes d'^{-n}}$ by $\overline{\mathcal{L}}_n$ and $\overline{\mathcal{L}}'_n$ respectively. Let $\mathcal{Y}$ be a projective arithmetic variety over B such that there are birational morphisms $\rho : \mathcal{Y} \to \mathcal{X}$, $\rho_1 : \mathcal{Y} \to \mathcal{X}_1$, and $\rho'_1 : \mathcal{Y} \to \mathcal{X}'_1$, which are the identity map over U. We fix $n > 0$. Let $\mathcal{Z}$ be a projective arithmetic variety over B with the following properties:

(a) $\mathcal{Z}_U = \mathcal{X}_U$.

(b) For each $0 \leq i \leq n$, there are birational morphisms $\mu_i : \mathcal{Z} \to \mathcal{X}_i$ and $\mu'_i : \mathcal{Z} \to \mathcal{X}'_i$, which are the identity map over U.

(c) For each $0 \leq j \leq n$, there are morphisms $g_j : \mathcal{Z} \to \mathcal{Y}$ and $g'_j : \mathcal{Z} \to \mathcal{Y}$ which are extensions of $f_U^{j} : \mathcal{Z}_U \to \mathcal{Y}_U$ and ${f'}_U^{j} : \mathcal{Z}_U \to \mathcal{Y}_U$ respectively.

Note that $\mu_0 = \mu'_0$. Then, in the same way as in (1) ((iii) of Claim 3.2.3), we can see

$$\mu_n^*(\overline{\mathcal{L}}_n) - \mu_0^*(\overline{\mathcal{L}}) = \sum_{j=0}^{n-1} d^{-j} g_j^*(\rho_1^*(\overline{\mathcal{L}}_1) - \rho^*(\overline{\mathcal{L}})) \tag{3.2.4}$$

and

$$\mu_n'^*(\overline{\mathcal{L}}'_n) - \mu_0^*(\overline{\mathcal{L}}) = \sum_{j=0}^{n-1} d'^{-j} g_j'^*(\rho_1'^*(\overline{\mathcal{L}}'_1) - \rho^*(\overline{\mathcal{L}})). \tag{3.2.5}$$

Let $\mathcal{Z}_n$ (resp. $\mathcal{Z}'_n$) be the normalization of $\mathcal{Z}_U \xrightarrow{f_U^n} \mathcal{Z}_U \to \mathcal{Z}$ (resp. $\mathcal{Z}_U \xrightarrow{f_U'^n} \mathcal{Z}_U \to \mathcal{Z}$) and let $h_n : \mathcal{Z}_n \to \mathcal{Z}$ (resp. $h'_n : \mathcal{Z}'_n \to \mathcal{Z}$) be the induced morphism. Moreover, let $\mathcal{T}$ be a projective arithmetic variety over B such that there are birational morphisms $\tau : \mathcal{T} \to \mathcal{Z}_n$, $\tau' : \mathcal{T} \to \mathcal{Z}'_n$, $\sigma : \mathcal{T} \to \mathcal{X}_n$, and $\sigma' : \mathcal{T} \to \mathcal{X}'_n$, which are the identity map over U. Now we have a lot of morphisms, so that we summarize them. The following morphisms are birational and the identity map over U.

$$\mathcal{Y} \xrightarrow{\rho} \mathcal{X} \qquad \begin{matrix} \mathcal{Y} \xrightarrow{\rho_1} \mathcal{X}_1 & \mathcal{Z} \xrightarrow{\mu_i} \mathcal{X}_i & \mathcal{T} \xrightarrow{\tau} \mathcal{Z}_n & \mathcal{T} \xrightarrow{\sigma} \mathcal{X}_n \\ \mathcal{Y} \xrightarrow{\rho'_1} \mathcal{X}'_1 & \mathcal{Z} \xrightarrow{\mu'_i} \mathcal{X}'_i & \mathcal{T} \xrightarrow{\tau'} \mathcal{Z}'_n & \mathcal{T} \xrightarrow{\sigma'} \mathcal{X}'_n \end{matrix}$$

Moreover, the following morphisms are extensions of the power of f or f'.

$$\begin{matrix} \mathcal{X}_n \xrightarrow{f_n} \mathcal{X}\ (f^n \text{ over } U) & \mathcal{Z} \xrightarrow{g_j} \mathcal{Y}\ (f^j \text{ over } U) & \mathcal{Z}_n \xrightarrow{h_n} \mathcal{Z}\ (f^n \text{ over } U) \\ \mathcal{X}'_n \xrightarrow{f'_n} \mathcal{X}\ (f'^n \text{ over } U) & \mathcal{Z} \xrightarrow{g'_j} \mathcal{Y}\ (f'^j \text{ over } U) & \mathcal{Z}'_n \xrightarrow{h'_n} \mathcal{Z}\ (f'^n \text{ over } U) \end{matrix}$$

Here, $f_n \cdot \mu_n \cdot h'_n \cdot \tau' = f'_n \cdot \mu'_n \cdot h_n \cdot \tau$ over U because $f \cdot f' = f' \cdot f$. Hence, so is over B as $\mathcal{T} \to \mathcal{X}$. Thus,

$$d'^{-n}\tau'^* h_n'^* \left(\mu_n^*(\overline{\mathcal{L}}_n) - \mu_0^*(\overline{\mathcal{L}})\right) - d^{-n}\tau^* h_n^* \left(\mu_n'^*(\overline{\mathcal{L}}'_n) - \mu_0^*(\overline{\mathcal{L}})\right) = d^{-n}\tau^* h_n^* \mu_0^*(\overline{\mathcal{L}}) - d'^{-n}\tau'^* h_n'^* \mu_0^*(\overline{\mathcal{L}})$$

Moreover, since $\mu_0 \cdot h_n \cdot \tau = f_n \cdot \sigma$ and $\mu_0 \cdot h'_n \cdot \tau' = f'_n \cdot \sigma'$, by the above equation, we have

$$d'^{-n}\tau'^* h_n'^* \left(\mu_n^*(\overline{\mathcal{L}}_n) - \mu_0^*(\overline{\mathcal{L}})\right) - d^{-n}\tau^* h_n^* \left(\mu_n'^*(\overline{\mathcal{L}}'_n) - \mu_0^*(\overline{\mathcal{L}})\right) = \sigma^*(\overline{\mathcal{L}}_n) - \sigma'^*(\overline{\mathcal{L}}'_n) \tag{3.2.6}$$

On the other hand, by Lemma 1.5, we can find ample C^∞-hermitian $\mathbb{Q}$-line bundles $\overline{\Delta}$ and $\overline{\Delta}'$ such that

$$-\pi_{\mathcal{Y}}^*(\overline{\Delta}) \precsim_{\pi_{\mathcal{Y}}^{-1}(U)} \rho_1^*(\overline{\mathcal{L}}_1) - \rho^*(\overline{\mathcal{L}}) \precsim_{\pi_{\mathcal{Y}}^{-1}(U)} \pi_{\mathcal{Y}}^*(\overline{\Delta})$$

and

$$-\pi_{\mathcal{Y}}^*(\overline{\Delta}') \precsim_{\pi_{\mathcal{Y}}^{-1}(U)} \rho_1'^*(\overline{\mathcal{L}}'_1) - \rho^*(\overline{\mathcal{L}}) \precsim_{\pi_{\mathcal{Y}}^{-1}(U)} \pi_{\mathcal{Y}}^*(\overline{\Delta}').$$

Therefore, if we set

$$d_n = \left(d'^{-n} \sum_{j=0}^{n-1} d^{-j}\right) \quad \text{and} \quad d'_n = \left(d^{-n} \sum_{j=0}^{n-1} d'^{-j}\right),$$

then, by (3.2.4) and (3.2.5),

$$-d_n\pi_{\mathcal{T}}(\overline{\Delta}) \precsim_{\pi_{\mathcal{T}}^{-1}(U)} d'^{-n}\tau'^{*}{h'_n}^{*}\left(\mu_n^{*}(\overline{\mathcal{L}}_n) - \mu_0^{*}(\overline{\mathcal{L}})\right) \precsim_{\pi_{\mathcal{T}}^{-1}(U)} d_n\pi_{\mathcal{T}}(\overline{\Delta})$$

and

$$-d'_n\pi_{\mathcal{T}}(\overline{\Delta}') \precsim_{\pi_{\mathcal{T}}^{-1}(U)} d^{-n}\tau^{*}h_n^{*}\left({\mu'_n}^{*}(\overline{\mathcal{L}}'_n) - \mu_0^{*}(\overline{\mathcal{L}})\right) \precsim_{\pi_{\mathcal{T}}^{-1}(U)} d'_n\pi_{\mathcal{T}}(\overline{\Delta}').$$

Hence, using (3.2.6),

$$-\pi_{\mathcal{T}}\left(d_n\overline{\Delta} + d'_n\overline{\Delta}'\right) \precsim_{\pi_{\mathcal{T}}^{-1}(U)} \sigma^{*}(\overline{\mathcal{L}}_n) - \sigma'^{*}(\overline{\mathcal{L}}'_n) \precsim_{\pi_{\mathcal{T}}^{-1}(U)} \pi_{\mathcal{T}}\left(d_n\overline{\Delta} + d'_n\overline{\Delta}'\right).$$

Therefore, we have this case because $\lim_{n\to\infty} d_n = \lim_{n\to\infty} d'_n = 0$. □

4. Adelic intersection number and adelic height

4.1. **Adelic intersection number.** Let K be a finitely generated field over $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$, and $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ a polarization of K.

Proposition 4.1.1. *Let X be an e-dimensional projective variety over K, and let $L_1, \ldots, L_{e+1}$ be nef line bundles on X. Let $\left\{(\mathcal{X}_n^{(i)}, \overline{\mathcal{L}}_n^{(i)})\right\}$ be an adelic sequence of (X, L_i) for each $1 \le i \le e+1$. Let $\mathcal{Z}_n$ be a projective arithmetic variety over B such that there are birational morphisms $\mu_n^{(i)} : \mathcal{Z}_n \to \mathcal{X}_n^{(i)}$ $(i = 1, \ldots, e+1)$. Then, the limit*

$$\widehat{\deg}\left(\widehat{c}_1({\mu_n^{(1)}}^{*}(\overline{\mathcal{L}}_n^{(1)})) \cdots \widehat{c}_1({\mu_n^{(e+1)}}^{*}(\overline{\mathcal{L}}_n^{(e+1)})) \cdot \widehat{c}_1(\pi_{\mathcal{Z}_n}^{*}(\overline{H}_1)) \cdots \widehat{c}_1(\pi_{\mathcal{Z}_n}^{*}(\overline{H}_d))\right)$$

as $n \to \infty$ exists, where $\pi_{\mathcal{Z}_n} : \mathcal{Z}_n \to B$ is the natural morphism. Moreover, if $\left\{(\mathcal{Y}_n^{(i)}, \overline{\mathcal{M}}_n^{(i)})\right\}$ is another adelic sequence of (X, L_i) for each $1 \le i \le e+1$, and $\left\{(\mathcal{X}_n^{(i)}, \overline{\mathcal{L}}_n^{(i)})\right\}$ is equivalent to $\left\{(\mathcal{Y}_n^{(i)}, \overline{\mathcal{M}}_n^{(i)})\right\}$ for each i, then the limit by $\left\{(\mathcal{X}_n^{(i)}, \overline{\mathcal{L}}_n^{(i)})\right\}$ coincides with the limit by $\left\{(\mathcal{Y}_n^{(i)}, \overline{\mathcal{M}}_n^{(i)})\right\}$.

Proof. Let $\mathcal{Z}_{n,m}$ be a projective arithmetic variety over B such that there are birational morphisms $\mathcal{Z}_{n,m} \to \mathcal{Z}_n$ and $\mathcal{Z}_{n,m} \to \mathcal{Z}_m$. By abuse of notation, we denote birational morphisms $\mathcal{Z}_{n,m} \to \mathcal{X}_n^{(i)}$ and $\mathcal{Z}_{n,m} \to \mathcal{X}_m^{(j)}$ by $\mu_n^{(i)}$ and $\mu_m^{(j)}$ respectively. First of all, we can see

$$\begin{aligned}
&\widehat{c}_1({\mu_n^{(1)}}^{*}(\overline{\mathcal{L}}_n^{(1)})) \cdots \widehat{c}_1({\mu_n^{(e+1)}}^{*}(\overline{\mathcal{L}}_n^{(e+1)})) - \widehat{c}_1({\mu_m^{(1)}}^{*}(\overline{\mathcal{L}}_m^{(1)})) \cdots \widehat{c}_1({\mu_m^{(e+1)}}^{*}(\overline{\mathcal{L}}_m^{(e+1)})) \\
&\qquad = \sum_{i=1}^{e+1} \widehat{c}_1({\mu_n^{(1)}}^{*}(\overline{\mathcal{L}}_n^{(1)})) \cdots \left(\widehat{c}_1({\mu_n^{(i)}}^{*}(\overline{\mathcal{L}}_n^{(i)})) - \widehat{c}_1({\mu_m^{(i)}}^{*}(\overline{\mathcal{L}}_m^{(i)}))\right) \cdots \widehat{c}_1({\mu_m^{(e+1)}}^{*}(\overline{\mathcal{L}}_m^{(e+1)}))
\end{aligned}$$

Therefore, it is sufficient to show that, for any positive ϵ, there is a positive integer N such that if $n, m \ge N$, then

$$\widehat{\deg}\left(\Delta_{n,m,i} \cdot \widehat{c}_1(\pi_{\mathcal{Z}_{n,m}}^{*}(\overline{H}_1)) \cdots \widehat{c}_1(\pi_{\mathcal{Z}_{n,m}}^{*}(\overline{H}_d))\right) \le \epsilon,$$

where $\Delta_{n,m,i} = \widehat{c}_1({\mu_n^{(1)}}^{*}(\overline{\mathcal{L}}_n^{(1)})) \cdots \left(\widehat{c}_1({\mu_n^{(i)}}^{*}(\overline{\mathcal{L}}_n^{(i)})) - \widehat{c}_1({\mu_m^{(i)}}^{*}(\overline{\mathcal{L}}_m^{(i)}))\right) \cdots \widehat{c}_1({\mu_m^{(e+1)}}^{*}(\overline{\mathcal{L}}_m^{(e+1)}))$. By the definition of adelic sequences, there are a projective arithmetic variety $\mathcal{X}_{n,m}$ over B, a birational morphism $\nu_{n,m} : \mathcal{X}_{n,m} \to \mathcal{Z}_{n,m}$, and a nef C^∞-hermitian $\mathbb{Q}$-line bundle $\overline{D}_{n,m}$ on B such that

$$-\pi^*_{\mathcal{X}_{n,m}}(\overline{D}_{n,m}) \precsim \nu^*_{n,m}\left(\mu_n^{(i)^*}(\overline{\mathcal{L}}_n^{(i)}) - \mu_m^{(i)^*}(\overline{\mathcal{L}}_m^{(i)})\right) \precsim \pi^*_{\mathcal{X}_{n,m}}(\overline{D}_{n,m}).$$

Here, since $\overline{\mathcal{L}}_n^{(i)}$'s are nef with respect to $\mathcal{X}_n^{(i)} \to B$ and $\overline{H}_j$'s are nef, by using Lemma 1.4 together with the projection formula, we can see

$$\begin{aligned}\left|\widehat{\deg}\left(\Delta_{n,m,i}\cdot \widehat{c}_1(\pi^*_{\mathcal{Z}_{n,m}}(\overline{H}_1))\cdots \widehat{c}_1(\pi^*_{\mathcal{Z}_{n,m}}(\overline{H}_d))\right)\right| \\ \leq \deg(L_1\cdots L_{i-1}\cdot L_{i+1}\cdots L_{e+1})\left|\widehat{\deg}\left(\widehat{c}_1(\overline{D}_{n,m})\cdot \widehat{c}_1(\overline{H}_1)\cdots \widehat{c}_1(\overline{H}_d)\right)\right|.\end{aligned}$$

Thus we get the first assertion. The second one is obvious by the definition of equivalence. □

Definition 4.1.2 (Adelic intersection number)**.** Let $\overline{L}_1, \ldots, \overline{L}_{e+1}$ be adelic line bundles on X. Then, by the above proposition, the limit of intersection numbers does not depend on the choice of adelic sequences representing each $\overline{L}_i$. Thus, we may define the adelic intersection number $\langle \overline{L}_1 \cdots \overline{L}_{e+1}\rangle_{\overline{B}}$ to be the limit in Proposition 4.1.1.

Here let us consider the following two propositions. The second proposition is a property concerning the specialization of adelic intersection number.

Proposition 4.1.3. *Let $\overline{L}_1, \ldots, \overline{L}_{e+1}$ be adelic line bundles on X. Then, we have the following.*

(1) *If $\overline{L}_1, \ldots, \overline{L}_{e+1}$ are nef, then $\langle \overline{L}_1 \cdots \overline{L}_{e+1}\rangle_{\overline{B}} \geq 0$.*
(2) *Let $\overline{H}'_1, \cdots, \overline{H}'_d$ be nef C^∞-hermitian line bundles on B with $\overline{H}'_i \succsim \overline{H}_i$ for all i. If $\overline{L}_1, \ldots, \overline{L}_{e+1}$ are nef, then*

$$\langle \overline{L}_1 \cdots \overline{L}_{e+1}\rangle_{(B;\overline{H}'_1,\cdots,\overline{H}'_d)} \geq \langle \overline{L}_1 \cdots \overline{L}_{e+1}\rangle_{(B;\overline{H}_1,\cdots,\overline{H}_d)}.$$

(3) *Let $g : Y \to X$ be a generically finite morphism of projective varieties over K. Then,*

$$\langle g^*(\overline{L}_1)\cdots g^*(\overline{L}_{e+1})\rangle_{\overline{B}} = \deg(g)\langle \overline{L}_1 \cdots \overline{L}_{e+1}\rangle_{\overline{B}}$$

Proof. (1) is a consequence of (2) of Proposition 1.1. (2) follows from (4) of Proposition 1.1. (3) is a consequence of the projection formula. □

Proposition 4.1.4. *Let $\{(\mathcal{X}_n, \overline{\mathcal{L}}_n)\}$ be an adelic sequence of (X, L) such that $\overline{\mathcal{L}}_n$ is nef for every n, and let $\overline{L}$ be a nef adelic line bundle on X given by the adelic sequence $\{(\mathcal{X}_n, \overline{\mathcal{L}}_n)\}$. Let U be a common base of the adelic sequence $\{(\mathcal{X}_n, \overline{\mathcal{L}}_n)\}$ (cf. the definition of adelic sequences in §§3.1). Let γ be a point of codimension one in $U_{\mathbb{Q}}$ such that $\mathcal{X}_U$ is flat over γ and the fiber X_γ of $\mathcal{X}_U \to U$ over γ is integral. Then, X_γ is a projective variety over the residue field $\kappa(\gamma)$ at γ, and $L_\gamma = \mathcal{L}|_{X_\gamma}$ is a line bundle on X_γ. Let Γ be the Zariski closure of $\{\gamma\}$ in B, and $\mathcal{Z}_n$ the Zariski closure of X_γ in $\mathcal{X}_n$. If $\overline{H}_d$ is big, then we have the following.*

(1) $\left\{(\mathcal{Z}_n, \overline{\mathcal{L}}_n\big|_{\mathcal{Z}_n})\right\}$ *is an adelic sequence of (X_γ, L_γ) with respect to $\left(\Gamma; \overline{H}_1\big|_\Gamma, \ldots, \overline{H}_{d-1}\big|_\Gamma\right)$.*

(2) *If we denote by* $\overline{L}_\gamma$ *the adelic line bundle arising from the adelic sequence* $\left\{(\mathcal{Z}_n, \overline{\mathcal{L}}_n\big|_{\mathcal{Z}_n})\right\}$, *then* $\left\langle \overline{L}^{\cdot \dim X+1} \right\rangle_{(B;\overline{H}_1,\ldots,\overline{H}_d)} = 0$ *implies* $\left\langle \overline{L}_\gamma^{\cdot \dim X_\gamma+1} \right\rangle_{(\Gamma;\overline{H}_1|_\Gamma,\ldots,\overline{H}_{d-1}|_\Gamma)} = 0$.

Proof. First of all, by using Lemma 1.2, we fix a positive integer N and a non-zero section $s \in H^0(B, H_d^{\otimes N})$ with $s(\gamma) = 0$ and $\|s\|_{\sup} \leq 1$. Then, $\operatorname{div}(s) = \Gamma + \Sigma$ for some effective divisor Σ.

(1) To prove (1), it is sufficient to show that

$$\lim_{n,m\to\infty} \widehat{\deg}\left(\widehat{c}_1(\overline{D}_{n,m}\big|_\Gamma) \cdot \widehat{c}_1(\overline{H}_1\big|_\Gamma) \cdots \widehat{c}_1(\overline{H}_{d-1}\big|_\Gamma)\right) = 0,$$

where $\overline{D}_{n,m}$ is a nef C^∞-hermitian $\mathbb{Q}$-line bundle on B appeared in the definition of adelic sequences (cf. §§3.1). First of all,

$$\begin{aligned} N\widehat{\deg}&\left(\widehat{c}_1(\overline{D}_{n,m}) \cdot \widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d)\right) \\ &= \widehat{\deg}\left(\widehat{c}_1(\overline{D}_{n,m}\big|_\Gamma) \cdot \widehat{c}_1(\overline{H}_1\big|_\Gamma) \cdots \widehat{c}_1(\overline{H}_{d-1}\big|_\Gamma)\right) \\ &\quad + \widehat{\deg}\left(\widehat{c}_1(\overline{D}_{n,m}\big|_\Sigma) \cdot \widehat{c}_1(\overline{H}_1\big|_\Sigma) \cdots \widehat{c}_1(\overline{H}_{d-1}\big|_\Sigma)\right) \\ &\quad + \int_{B(\mathbb{C})} -\log(\|s\|) c_1(\overline{D}_{n,m}) \wedge c_1(\overline{H}_1) \wedge \cdots \wedge c_1(\overline{H}_{d-1}). \end{aligned}$$

Here every term is non-negative. Thus, we can see that

$$\lim_{n,m\to\infty} \widehat{\deg}\left(\widehat{c}_1(\overline{D}_{n,m}\big|_\Gamma) \cdot \widehat{c}_1(\overline{H}_1\big|_\Gamma) \cdots \widehat{c}_1(\overline{H}_{d-1}\big|_\Gamma)\right) = 0.$$

(2) We can set $\operatorname{div}(\pi_{\mathcal{X}_n}^*(s)) = \mathcal{Z}_n + \Delta_n$ for some effective divisor Δ_n. Therefore,

$$\begin{aligned} N\widehat{\deg}&\left(\widehat{c}_1(\overline{\mathcal{L}}_n)^{\cdot e+1} \cdot \widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d)\right) \\ &= \widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}_n\big|_{\mathcal{Z}_n})^{\cdot e+1} \cdot \widehat{c}_1(\pi_{\mathcal{X}_n}^*(\overline{H}_1)\big|_{\mathcal{Z}_n}) \cdots \widehat{c}_1(\pi_{\mathcal{X}_n}^*(\overline{H}_{d-1})\big|_{\mathcal{Z}_n})\right) \\ &\quad + \widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}_n\big|_{\Delta_n})^{\cdot e+1} \cdot \widehat{c}_1(\pi_{\mathcal{X}_n}^*(\overline{H}_1)\big|_{\Delta_n}) \cdots \widehat{c}_1(\pi_{\mathcal{X}_n}^*(\overline{H}_{d-1})\big|_{\Delta_n})\right) \\ &\quad + \int_{\mathcal{X}_n(\mathbb{C})} -\log(\|\pi_{\mathcal{X}_n}^*(s)\|) c_1(\overline{\mathcal{L}}_n)^{\wedge e+1} \wedge c_1(\pi_{\mathcal{X}_n}^*(\overline{H}_1)) \wedge \cdots \wedge c_1(\pi_{\mathcal{X}_n}^*(\overline{H}_{d-1})). \end{aligned}$$

Since the last two terms of the above equation are non-negative, we have

$$\begin{aligned} N\widehat{\deg}&\left(\widehat{c}_1(\overline{\mathcal{L}}_n)^{\cdot e+1} \cdot \widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d)\right) \\ &\geq \widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}_n\big|_{\mathcal{Z}_n})^{\cdot e+1} \cdot \widehat{c}_1(\pi_{\mathcal{X}_n}^*(\overline{H}_1)\big|_{\mathcal{Z}_n}) \cdots \widehat{c}_1(\pi_{\mathcal{X}_n}^*(\overline{H}_{d-1})\big|_{\mathcal{Z}_n})\right). \end{aligned}$$

Thus, taking $n \to \infty$,

$$N \left\langle \overline{L}^{\cdot \dim X+1} \right\rangle_{(B;\overline{H}_1,\ldots,\overline{H}_d)} \geq \left\langle \overline{L}_\gamma^{\cdot \dim X_\gamma+1} \right\rangle_{(\Gamma;\overline{H}_1|_\Gamma,\ldots,\overline{H}_{d-1}|_\Gamma)}.$$

Therefore, we get (2). □

4.2. **Adelic height.** Let K be a finitely generated field over $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$, and $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ a polarization of K. Let X be a projective variety over K, and L an ample line bundle on X.

Let $\overline{L}$ be an adelic line bundle given by an adelic sequence $\{(\mathcal{X}_n, \overline{\mathcal{L}}_n)\}$. Let K' be a finite extension of K, B' the normalization of B in K', and let $\rho : B' \to B$ be the induced morphism. Let $\mathcal{X}'_n$ be the main component of $\mathcal{X}_n \times_B B'$. We set the induced morphisms as follows.

$$\begin{array}{ccc} \mathcal{X}_n & \xleftarrow{\tau_n} & \mathcal{X}'_n \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \pi'_n} \\ B & \xleftarrow{\rho} & B' \end{array}$$

Then, $\{\mathcal{X}'_n, \tau_n^*(\overline{\mathcal{L}}_n)\}$ is an adelic sequence of $(X_{K'}, L_{K'})$. We denote by $\overline{L}_{K'}$ the adelic line bundle induced by $\{\mathcal{X}'_n, \tau_n^*(\overline{\mathcal{L}}_n)\}$. With this notation, if $\overline{L}_1, \ldots, \overline{L}_{e+1}$ are adelic line bundles on X, then we can see

$$\langle (\overline{L}_1)_{K'} \cdots (\overline{L}_{e+1})_{K'} \rangle_{\overline{B}_{K'}} = [K' : K] \langle \overline{L}_1 \cdots \overline{L}_{e+1} \rangle_{\overline{B}}, \tag{4.2.1}$$

by virtue of the projection formula, where $\overline{B}_{K'} = (B'; \rho^*(\overline{H}_1), \cdots, \rho^*(\overline{H}_d))$.

Let Y be a subvariety of $X_{\overline{K}}$. We assume that Y is defined over K'. Let $\mathcal{Y}'_n$ be the closure of Y in $\mathcal{X}'_n$. Then, $\{\mathcal{Y}'_n, \tau_n^*(\overline{\mathcal{L}}_n)\big|_{\mathcal{Y}'_n}\}$ is an adelic sequence of $(Y, L_{K'}|_Y)$. We denote by $\overline{L}_{K'}\big|_Y$ the adelic line bundle given by $\{\mathcal{Y}'_n, \tau_n^*(\overline{\mathcal{L}}_n)\big|_{\mathcal{Y}'_n}\}$. We define the height of Y with respect to $\overline{L}$ to be

$$h_{\overline{L}}^{\overline{B}}(Y) = \frac{\left\langle \left(\overline{L}_{K'}\big|_Y\right)^{\cdot \dim Y + 1} \right\rangle_{\overline{B}'}}{[K' : K](\dim Y + 1) \deg\left(L_{K'}|_Y^{\dim Y} \right)}.$$

Note that by virtue of (4.2.1), the above does not depend on the choice of K'. We call $h_{\overline{L}}^{\overline{B}}(Y)$ the *adelic height* of Y with respect to $\overline{L}$ and $\overline{B}$.

Proposition 4.2.2. *Let X be a projective variety over K, and L an ample line bundle on X. We assume that there is a surjective morphism $f : X \to X$ and an integer $d \geq 2$ with $L^{\otimes d} \simeq f^*(L)$. Let $\overline{L}^f$ be the adelic line bundle with the f-adelic structure. Then, we have the following.*

1. $h_{\overline{L}^f}^{\overline{B}}(Y) \geq 0$ *for all subvarieties Y of $X_{\overline{K}}$.*
2. *For a C^∞-model $(\mathcal{X}, \overline{\mathcal{L}})$ of (X, L) with $\overline{\mathcal{L}}$ being nef with respect to $\mathcal{X} \to B$, there is a constant C such that*
$$\left| h_{\overline{L}^f}^{\overline{B}}(Y) - h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}}(Y) \right| \leq C$$
for any subvarieties Y of $X_{\overline{K}}$.
3. $h_{\overline{L}^f}^{\overline{B}}(f(Y)) = d h_{\overline{L}^f}^{\overline{B}}(Y)$ *for any subvarieties Y of $X_{\overline{K}}$.*

Moreover, $h_{\overline{L}^f}^{\overline{B}}$ is characterized by the above properties (1), (2) and (3).

Proof. (1) Since $\overline{L}^f$ is nef by Proposition 3.2.1, (1) is a consequence of (1) of Proposition 4.1.3.

(2) We choose a Zariski open set U of B such that f extends to $f_U : \mathcal{X}_U \to \mathcal{X}_U$ and $\mathcal{L}_U^{\otimes d} = f^*(\mathcal{L}_U)$ in $\operatorname{Pic}(\mathcal{X}_U) \otimes \mathbb{Q}$. Let $\mathcal{X}_n$ be the normalization of $\mathcal{X}_U \xrightarrow{f_U^n} \mathcal{X}_U \to \mathcal{X}$, and $f_n : \mathcal{X}_n \to \mathcal{X}$ the induced morphism. We denote $f_n^*(\overline{\mathcal{L}})^{\otimes d^{-n}}$ by $\overline{\mathcal{L}}_n$. Then, as in proof of (1) of Proposition 3.2.1, there are a projective arithmetic variety $\mathcal{Z}_n$ over B, birational morphisms $\mu_n : \mathcal{Z}_n \to \mathcal{X}_n$ and $\nu_n : \mathcal{Z}_n \to \mathcal{X}$ (which are the identity map over U), and an ample C^∞-hermitian line bundle $\overline{D}$ on B such that

$$-d_n \pi_{\mathcal{Z}_n}^*(\overline{D}) \precsim_{\pi_{\mathcal{Z}_n}^{-1}(U)} \mu_n^*(\overline{\mathcal{L}}_n) - \nu_n^*(\overline{\mathcal{L}}) \precsim_{\pi_{\mathcal{Z}_n}^{-1}(U)} d_n \pi_{\mathcal{Z}_n}^*(\overline{D}),$$

where $d_n = \sum_{j=0}^{n-1} d^{-j}$.

Let Y be a subvariety of $X_{\overline{K}}$. We assume that Y is defined over a finite extension field K' of K. Let B' be the normalization of B in K', and let $\rho : B' \to B$ be the induced morphism. We denote by $\mathcal{X}'$, $\mathcal{X}'_n$ and $\mathcal{Z}'_n$ the main components of $\mathcal{X} \times_B B'$, $\mathcal{X}_n \times_B B'$ and $\mathcal{Z}_n \times_B B'$ respectively. We set the induced morphisms as follows.

$$\begin{array}{ccc} \mathcal{X} & \xleftarrow{\tau} & \mathcal{X}' \\ {\scriptstyle \pi_{\mathcal{X}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{X}'}} \\ B & \xleftarrow{\rho} & B' \end{array} \qquad \begin{array}{ccc} \mathcal{X}_n & \xleftarrow{\tau_n} & \mathcal{X}'_n \\ {\scriptstyle \pi_{\mathcal{X}_n}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{X}'_n}} \\ B & \xleftarrow{\rho} & B' \end{array} \qquad \begin{array}{ccc} \mathcal{Z}_n & \longleftarrow & \mathcal{Z}'_n \\ {\scriptstyle \pi_{\mathcal{Z}_n}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{Z}'_n}} \\ B & \xleftarrow{\rho} & B' \end{array}$$

We also have the induced morphisms $\mu'_n : \mathcal{Z}'_n \to \mathcal{X}'_n$ and $\nu'_n : \mathcal{Z}'_n \to \mathcal{X}'$. Then,

$$-d_n \pi_{\mathcal{Z}'_n}^*(\rho^*\overline{D}) \precsim_{\pi_{\mathcal{Z}_n}^{-1}(U)} {\mu'_n}^*(\tau_n^*\overline{\mathcal{L}}_n) - {\nu'_n}^*(\tau^*\overline{\mathcal{L}}) \precsim_{\pi_{\mathcal{Z}_n}^{-1}(U)} d_n \pi_{\mathcal{Z}'_n}^*(\rho^*\overline{D}),$$

On the other hand, since

$$\begin{aligned} &\widehat{c}_1({\mu'_n}^*(\tau_n^*\overline{\mathcal{L}}_n))^{\dim Y+1} - \widehat{c}_1({\nu'_n}^*(\tau^*\overline{\mathcal{L}}))^{\dim Y+1} \\ &\quad = \sum_{i=1}^{\dim Y+1} \widehat{c}_1({\mu'_n}^*(\tau_n^*\overline{\mathcal{L}}_n))^{i-1} \cdot \left(\widehat{c}_1({\mu'_n}^*(\tau_n^*\overline{\mathcal{L}}_n)) - \widehat{c}_1({\nu'_n}^*(\tau^*\overline{\mathcal{L}}))\right) \widehat{c}_1({\nu'_n}^*(\tau^*\overline{\mathcal{L}}))^{\dim Y-i+1}, \end{aligned}$$

by using Lemma 1.4, we have

$$\begin{aligned} &\Big|\widehat{\deg}\left(\widehat{c}_1(\tau_n^*\overline{\mathcal{L}}_n\big|_{\mathcal{Y}_n})^{\dim Y+1} \cdot \widehat{c}_1(\pi_{\mathcal{Y}_n}^*\rho^*\overline{H}_1) \cdots \widehat{c}_1(\pi_{\mathcal{Y}_n}^*\rho^*\overline{H}_d)\right) \\ &\qquad\qquad - \widehat{\deg}\left(\widehat{c}_1(\tau^*\overline{\mathcal{L}}\big|_{\mathcal{Y}})^{\dim Y+1} \cdot \widehat{c}_1(\pi_{\mathcal{Y}}^*\rho^*\overline{H}_1) \cdots \widehat{c}_1(\pi_{\mathcal{Y}}^*\rho^*\overline{H}_d)\right)\Big| \\ &\qquad\qquad \leq d_n[K' : K](\dim Y + 1)\deg(L|_Y^{\dim Y})\widehat{\deg}\left(\widehat{c}_1(\overline{D}) \cdot \widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d)\right), \end{aligned}$$

where $\mathcal{Y}$ and $\mathcal{Y}_n$ are the Zariski closures of Y in $\mathcal{X}'$ and $\mathcal{X}'_n$ respectively. Thus we get (2).

(3) Clearly, we may assume Y is defined over K. Let $(\mathcal{X}, \overline{\mathcal{L}})$ be a C^∞ model of (X, L). Let us consider a sequence of morphisms of projective arithmetic varieties over B:

$$\mathcal{X} = \mathcal{X}_0 \xleftarrow{f_1} \mathcal{X}_1 \xleftarrow{f_2} \cdots \xleftarrow{f_{n-1}} \mathcal{X}_{n-1} \xleftarrow{f_n} \mathcal{X}_n \xleftarrow{f_{n+1}} \mathcal{X}_{n+1} \xleftarrow{f_{n+2}} \cdots$$

such that X is the generic fiber of $\mathcal{X}_n \to B$ for every n, and that $f_n : \mathcal{X}_n \to \mathcal{X}_{n-1}$ is an extension of f for each n. Let $\mathcal{Y}_n$ be the Zariski closure of Y in $\mathcal{X}_n$. Then, $f_{n+1}(\mathcal{Y}_{n+1})$ is the Zariski closure of $f(Y)$ in $\mathcal{X}_n$. By the definition of the height,

$$
\begin{aligned}
&(4.2.2.1)\quad h^{\overline{B}}_{\overline{L}^f}(Y)\\
&= \lim_{n\to\infty} \frac{\widehat{\deg}\left(\widehat{c}_1(f_{n+1}^* f_n^* \cdots f_1^*(\overline{\mathcal{L}}))^{\cdot \dim Y+1} \cdot \widehat{c}_1(f_{n+1}^*\pi_{\mathcal{X}_n}^*(\overline{H}_1)) \cdots \widehat{c}_1(f_{n+1}^*\pi_{\mathcal{X}_n}^*(\overline{H}_d)) \cdot (\mathcal{Y}_{n+1}, 0)\right)}{(\dim Y+1)\deg(L|_Y^{\dim Y}) d^{(n+1)(\dim Y+1)}}.
\end{aligned}
$$

On the other hand, by the projection formula,

$$
\begin{aligned}
&(4.2.2.2)\\
&\widehat{\deg}\left(\widehat{c}_1(f_{n+1}^* f_n^* \cdots f_1^*(\overline{\mathcal{L}}))^{\cdot \dim Y+1} \cdot \widehat{c}_1(f_{n+1}^*\pi_{\mathcal{X}_n}^*(\overline{H}_1)) \cdots \widehat{c}_1(f_{n+1}^*\pi_{\mathcal{X}_n}^*(\overline{H}_d)) \cdot (\mathcal{Y}_{n+1}, 0)\right)\\
&\quad = \deg(f|_Y)\widehat{\deg}\left(\widehat{c}_1(f_n^* \cdots f_1^*(\overline{\mathcal{L}}))^{\cdot \dim Y+1} \cdot \widehat{c}_1(\pi_{\mathcal{X}_n}^*(\overline{H}_1)) \cdots \widehat{c}_1(\pi_{\mathcal{X}_n}^*(\overline{H}_d)) \cdot (f_{n+1}(\mathcal{Y}_{n+1}), 0)\right).
\end{aligned}
$$

Here, since $L^{\otimes d} \simeq f^*(L)$, we have $L^{\otimes d}\big|_Y \simeq (f|_Y)^*\left(L|_{f(Y)}\right)$, which implies

$$
(4.2.2.3)\qquad d^{\dim Y}\deg(L|_Y^{\dim Y}) = \deg(f|_Y)\deg(L|_{f(Y)}^{\dim Y}).
$$

Moreover,

$$
\begin{aligned}
&(4.2.2.4)\quad h^{\overline{B}}_{\overline{L}^f}(f(Y))\\
&\quad = \lim_{n\to\infty} \frac{\widehat{\deg}\left(\widehat{c}_1(f_n^* \cdots f_1^*(\overline{\mathcal{L}}))^{\cdot \dim Y+1} \cdot \widehat{c}_1(\pi_{\mathcal{X}_n}^*(\overline{H}_1)) \cdots \widehat{c}_1(\pi_{\mathcal{X}_n}^*(\overline{H}_d)) \cdot (f_{n+1}(\mathcal{Y}_{n+1}), 0)\right)}{(\dim Y+1)\deg(L|_{f(Y)}^{\dim Y}) d^{n(\dim Y+1)}}.
\end{aligned}
$$

Therefore, by (4.2.2.1), (4.2.2.2), (4.2.2.3), and (4.2.2.4), we obtain

$$
h^{\overline{B}}_{\overline{L}^f}(f(Y)) = d h^{\overline{B}}_{\overline{L}^f}(Y).
$$

Finally, the last assertion is obvious. For, by (2) and (3), we can see

$$
h^{\overline{B}}_{\overline{L}^f}(Y) = \lim_{n\to\infty} \frac{h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(f^{\cdot n}(Y))}{d^n}.
$$

□

5. The canonical height of subvarieties of an abelian variety over finitely generated fields

Let K be a finitely generated field over $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$, and $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ a polarization of K. Let A be an abelian variety over K, and L a symmetric ample line bundle on A. Since $[2]^*(L) \simeq L^{\otimes 4}$, we have an adelic line bundle $\overline{L}^{[2]}$ with the [2]-adelic structure. Let $f : A \to A$ be an endomorphism with $f^*(L) \simeq L^{\otimes d}$ for some $d \geq 2$. Then, since $f \cdot [2] = [2] \cdot f$, by (2) of Proposition 3.2.1, $\overline{L}^f = \overline{L}^{[2]}$. Thus, the adelic structure does not depend on the choice of the endomorphism. In this sense, we have the line bundle $\overline{L}^{can}$ with the canonical adelic structure.

Let X be a subvariety of $A_{\overline{K}}$. We denote by $\hat{h}^{\overline{B}}_{L}(X)$ the adelic height $h^{\overline{B}}_{\overline{L}^{can}}(X)$ of X with respect to the line bundle $\overline{L}^{can}$ with the canonical adelic structure. Then, by Proposition 4.2.2, we can see the following:

(a) $\hat{h}^{\overline{B}}_{L}(X) \geq 0$ for all subvarieties X of $A_{\overline{K}}$.

(b) For a C^{∞}-model $(\mathcal{A}, \overline{\mathcal{L}})$ of (A, L) with $\overline{\mathcal{L}}$ being nef with respect to $\mathcal{A} \to B$, there is a constant C such that

$$\left| \hat{h}^{\overline{B}}_{L}(X) - h^{\overline{B}}_{(\mathcal{A}, \overline{\mathcal{L}})}(X) \right| \leq C$$

for all subvarieties X of $A_{\overline{K}}$.

(c) $\hat{h}^{\overline{B}}_{L}([N](X)) = N^2 \hat{h}^{\overline{B}}_{L}(X)$ for all subvarieties X of $A_{\overline{K}}$ and all non-zero integers N.

The purpose of this section is to prove the following theorem.

Theorem 5.1. *Let A be an abelian variety over K, and L a symmetric ample line bundle on A. Let X be a subvariety of $A_{\overline{K}}$. If the polarization $\overline{B}$ is big, then the following are equivalent.*

(1) X is a translation of an abelian subvariety by a torsion point.

(2) The set $\{x \in X(\overline{K}) \mid \hat{h}^{\overline{B}}_{L}(x) \leq \epsilon\}$ is Zariski dense in X for every $\epsilon > 0$.

(3) $\hat{h}^{\overline{B}}_{L}(X) = 0$.

Proof. Let us begin with the following two lemmas.

Lemma 5.2. *Let A be an abelian subvariety over K, C an abelian subvariety of A, and $\rho : A \to A' = A/C$ the natural homomorphism. Let X be a subvariety of A such that $X = \rho^{-1}(\rho(X))$. Let L and L' be symmetric ample line bundles on A and A' respectively. If $\hat{h}^{\overline{B}}_{L}(X) = 0$, then $\hat{h}^{\overline{B}}_{L'}(Y) = 0$, where $Y = \rho(X)$.*

Proof. Replacing L by $L^{\otimes n}$ $(n > 0)$, we may assume that $L \otimes \rho^*(L')^{\otimes -1}$ is generated by global sections. Let $(\mathcal{A}, \overline{\mathcal{L}})$ and $(\mathcal{A}', \overline{\mathcal{L}}')$ be C^{∞}-models of (A, L) and (A', L') over B with the following properties:

(1) $\overline{\mathcal{L}}$ and $\overline{\mathcal{L}}'$ are nef and big.

(2) There is a morphism $\mathcal{A} \to \mathcal{A}'$ over B as an extension of $\rho : A \to A'$. (By abuse of notation, the extension is also denoted by ρ.)

Let $\pi : \mathcal{A} \to B$ be the canonical morphism. Replacing $\mathcal{L}$ by $\mathcal{L} \otimes \pi^*(Q)$ for some ample line bundle Q on B, we may assume that $\pi_*(\mathcal{L} \otimes \rho^*(\mathcal{L}')^{\otimes -1})$ is generated by global sections. Thus, there are sections $s_1, \ldots, s_r$ of $H^0(\mathcal{L} \otimes \rho^*(\mathcal{L}')^{\otimes -1})$ such that $\{s_1, \ldots, s_r\}$ generates $L \otimes \rho^*(L')^{\otimes -1}$ on A. Moreover, replacing the metric of $\overline{\mathcal{L}}$, we may assume that $s_1, \ldots, s_r$ are small sections, i.e., $\|s_i\|_{\sup} < 1$ for all i.

Let $\mathcal{A}_n$ (resp. $\mathcal{A}'_n$) be the normalization of $A \overset{[2^n]}{\longrightarrow} A \hookrightarrow \mathcal{A}$ (resp. $A' \overset{[2^n]}{\longrightarrow} A' \hookrightarrow \mathcal{A}'$). Then, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A} & \xleftarrow{\ f_n\ } & \mathcal{A}_n \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho_n} \\ \mathcal{A}' & \xleftarrow[\ f'_n\ ]{} & \mathcal{A}'_n, \end{array}$$

where f_n and f'_n are extension of $[2^n]$. Here the adelic structure of $\overline{L}$ (resp. $\overline{L}'$) is induced by $\{4^{-n} f_n^*(\overline{\mathcal{L}})\}$ (resp. $\{4^{-n} {f'_n}^*(\overline{\mathcal{L}}')\}$). Let $\mathcal{X}_n$ (resp. $\mathcal{Y}_n$) be the Zariski closure of X in $\mathcal{A}_n$ (resp. Y in $\mathcal{A}'_n$). Then, since $f_n^*(s_1), \ldots, f_n^*(s_r)$ generate $f_n^*(L \otimes \rho^*(L')^{\otimes -1})$ on A, we can find $f_n^*(s_i)$ such that $f_n^*(s_i) \neq 0$ on $\mathcal{X}_n$. This means that $f_n^*(\overline{\mathcal{L}})\big|_{\mathcal{X}_n} \otimes \rho_n^*({f'_n}^*(\overline{\mathcal{L}}'))^{\otimes -1}\Big|_{\mathcal{X}_n}$ is effective. Therefore, if we denote $\dim X$ and $\dim Y$ by e and e' respectively, then, by virtue of (4) of Proposition 1.1 together with the projection formula,

$$
\begin{aligned}
&\widehat{\deg}\left(\widehat{c}_1(f_n^*(\overline{\mathcal{L}})\big|_{\mathcal{X}_n})^{\cdot e+1} \cdot \widehat{c}_1(\pi_{\mathcal{X}_n}^*\overline{H}_1)\cdots\widehat{c}_1(\pi_{\mathcal{X}_n}^*\overline{H}_d)\right) \\
&\quad \geq \widehat{\deg}\left(\widehat{c}_1(\rho_n^*{f'_n}^*(\overline{\mathcal{L}}')\Big|_{\mathcal{X}_n})^{\cdot e'+1} \cdot \widehat{c}_1(f_n^*(\overline{\mathcal{L}})\big|_{\mathcal{X}_n})^{\cdot e-e'} \cdot \widehat{c}_1(\rho_n^*\pi_{\mathcal{Y}_n}^*\overline{H}_1)\cdots\widehat{c}_1(\rho_n^*\pi_{\mathcal{Y}_n}^*\overline{H}_d)\right) \\
&\qquad = 4^{n(e-e')}\deg(L|_C^{e-e'})\widehat{\deg}\left(\widehat{c}_1({f'_n}^*(\overline{\mathcal{L}}')\Big|_{\mathcal{Y}_n})^{\cdot e'+1} \cdot \widehat{c}_1(\pi_{\mathcal{Y}_n}^*\overline{H}_1)\cdots\widehat{c}_1(\pi_{\mathcal{Y}_n}^*\overline{H}_d)\right).
\end{aligned}
$$

Hence,

$$
\hat{h}_{\overline{L}}^{\overline{B}}(X) \geq \frac{(e'+1)\deg(L'|_Y^{e'})\deg(L|_C^{e-e'})}{(e+1)\deg(L|_X^{e})}\hat{h}_{\overline{L}'}^{\overline{B}}(Y).
$$

Thus we get our assertion. □

Lemma 5.3. *Let A and S be algebraic varieties over a field of characteristic zero, and let $f : A \to S$ be an abelian scheme. Let X be a subvariety of A such that $f|_X : X \to B$ is proper and flat. Let s be a point of S. If $X_{\bar{s}}$ is a translation of an abelian subvariety of $A_{\bar{s}}$, then there is a Zariski open set U of S such that (1) $s \in U$ and (2) $X_{\bar{t}}$ is a translation of an abelian subvariety of $A_{\bar{t}}$ for all $t \in U$. In particular, the geometric generic fiber $X_{\bar{\eta}}$ is a translation of an abelian subvariety.*

Proof. Since $X_{\bar{s}}$ is smooth and $q(X_{\bar{s}}) = \dim(X/S)$, there is a Zariski open set U of S such that $s \in U$, X_U is smooth over U, and that $q(X_{\bar{t}}) \leq \dim(X/S)$ for all $t \in U$. By Ueno's theorem (cf. [1, Theorem 10.12]), $q(X_{\bar{t}}) \geq \dim(X/S)$ and the equality holds if and only if $X_{\bar{t}}$ is a a translation of an abelian subvariety. Thus we get our lemma. □

Let us start the proof of Theorem 5.1. First of all, we may assume that X is defined over K.

"(1) $\Longrightarrow$ (2)" is obvious. "(2) $\Longrightarrow$ (1)" is nothing more than Bogomolov's conjecture solved in [2].

"(1) $\Longrightarrow$ (3)": We set $X = A' + x$, where A' is an abelian subvariety of $A_{\overline{K}}$ and x is a torsion point. Let N be a positive integer with $Nx = 0$ and $N \geq 2$. Then, $[N](X) = A' = [N](A')$. Thus, by Proposition 4.2.2,

$$
\hat{h}_L^{\overline{B}}(X) = (1/N^2)\hat{h}_L^{\overline{B}}([N](X)) = (1/N^2)\hat{h}_L^{\overline{B}}([N](A')) = \hat{h}_L^{\overline{B}}(A').
$$

On the other hand,

$$
\hat{h}_L^{\overline{B}}(A') = \hat{h}_L^{\overline{B}}([N](A')) = N^2\hat{h}_L^{\overline{B}}(A').
$$

Therefore, $\hat{h}_L^{\overline{B}}(X) = \hat{h}_L^{\overline{B}}(A') = 0$.

"(3) $\Longrightarrow$ (1)": Let $\overline{H}$ be an ample C^∞-hermitian line bundle on B. Then, there is a positive integer n such that $\overline{H}_i^{\otimes n} \otimes \overline{H}^{\otimes -1} \gtrsim 0$. for all i. Then, by using (4) of Proposition 1.1, we can see that an adelic sequence with respect to $(B; \overline{H}_1, \ldots, \overline{H}_d)$ is an adelic sequence with respect to $(B; \overline{H}, \ldots, \overline{H})$, and that

$$0 \leq \hat{h}_L^{(B;\overline{H},\ldots,\overline{H})}(X) \leq n^d \hat{h}_L^{(B;\overline{H}_1,\ldots,\overline{H}_d)}(X).$$

Thus, we may assume that $\overline{H}_1, \ldots, \overline{H}_d$ are ample. We prove the assertion "(3) $\Longrightarrow$ (1)" by induction on $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$. If $d = 0$, then this was proved by Zhang [5]. We assume $d > 0$. Then, by the above lemma together with hypothesis of induction and Proposition 4.1.4, X is a translation of an abelian subvariety C. Let us consider $\pi : A \to A' = A/C$. Then, $\pi(X)$ is a point, say P. Then, by Lemma 5.2, $\hat{h}_{L'}^{\overline{B}}(P) = 0$ for a symmetric ample line bundle L' on A'. Thus, P is a torsion point by [2, Proposition 3.4.1]. Therefore, we can see that X is a translation of C by a torsion point. □

Let X be a smooth projective curve of genus $g \geq 2$ over K. Let J be the Jacobian of X and L_Θ a line bundle given by a symmetric theta divisor Θ on J, i.e., $L_\Theta = \mathcal{O}_J(\Theta)$. Let $j : X \to J$ be a morphism given by $j(x) = \omega_X - (2g-2)x$. Then, it is well known that $j^*(L_\Theta) = \omega_X^{\otimes 2g(g-1)}$. Let $\overline{L}_\Theta^{can}$ be the canonical adelic structure of L_Θ. Thus, we have the adelic line bundle $j^*(\overline{L}_\Theta^{can})$ on X. In terms of this, we can give the canonical adelic structure on ω_X. We denote this by $\overline{\omega}_X^a$. Then, as a corollary of Theorem 5.1 and (3) of Proposition 4.1.3, we have the following.

Corollary 5.4. *If the polarization $\overline{B}$ is big, then $\langle \overline{\omega}_X^a \cdot \overline{\omega}_X^a \rangle_{\overline{B}} > 0$.*

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