The Structure of Sectors Associated with the Longo-Rehren Inclusions I. General Theory

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Dedicated to John E. Roberts on the occasion of his sixtieth birth day

Abstract

We investigate the structure of the Longo-Rehren inclusion for a finite closed system of endomorphisms of factors, whose categorical structure is known to be the same as the asymptotic inclusion of A. Ocneanu. In particular, we obtain a precise description of the sectors associated with the Longo-Rehren inclusions in terms of half braidings, which do not necessarily satisfy the usual condition of braidings. In doing so, we give new proofs to most of the known statements concerning asymptotic inclusions. We construct a complete system of matrix units of the tube algebra using the half braidings, which will be used in the second part to describe concrete examples of the Longo-Rehren inclusions arising from the Cuntz algebra endomorphisms. We also discuss the case where the original system has a braiding, and generalize Ocneanu and Evans-Kawahigashi's method for the analysis of the asymptotic inclusions of the Hecke algebra subfactors.

1 Introduction

The notion of the asymptotic inclusions of subfactors was introduced by A. Ocneanu. He announced a description of the asymptotic inclusions in terms of topological quantum field theory (TQFT), and pointed out that it may be considered as a subfactor analogue of the quantum double construction. Detailed accounts of his theory are given by D. E. Evans and Y. Kawahigashi in [7][9]. The asymptotic inclusions also play important roles in S. Popa's classification of amenable subfactors [29]

Inspired by examples arising from local quantum field theory, R. Longo and K. H. Rehren introduced new construction of a subfactor, called the Longo-Rehren inclusion, for

a finite system of irreducible endomorphisms of infinite factors with some conditions [24]. T. Masuda proved that the asymptotic inclusions and the Longo-Rehren inclusions are essentially the same objects [25]. Namely, he generalized the Longo-Rehren construction to a system of bimodules. Then he showed that if the system comes from an inclusion of the AFD II_1 factors, the Longo-Rehren inclusion is isomorphic to the dual inclusion of the asymptotic inclusion.

In the same paper [24], Longo and Rehren introduced induction procedure of endomorphisms from a subfactor. F. Xu successfully employed essentially the same procedure in his analysis of subfactors associated with the conformal inclusions [35]. Following these two works, J. Böckenhauer and D. E. Evans proposed in [2] systematic use of the induction procedure, which is now called the α-induction. We refer to [3][4][5][20] for further development of the material. In this paper, we discuss detailed and direct analysis of the Longo-Rehren inclusions based on (the dual form of) the induction procedure using the notion of half braiding, which is weaker than the usual notion of braidings. We will give new proofs to most of the statements in [7] by using algebraic computation of sectors avoiding subtle and ingenious topological arguments. The category theoretical meaning of our argument, especially the relationship to the center construction in [19], will be discussed by M. Müger in [26].

The original aim of this work is to describe concrete examples of the asymptotic inclusions of subfactors arising from the Cuntz algebra endomorphisms [14], which will be the main subject of the second part. For this purpose, we write down various explicit formulae with precise normalization constants. Indeed, in the second part, we give the dual principal graphs and the S and T matrices of the asymptotic inclusions for several subfactors including the E_6 subfactor. We also realize the Haagerup subfactor using an endomorphism of the Cuntz algebra $\mathcal{O}_4$ and compute the T matrix of the asymptotic inclusion. These are the first examples of computation of the asymptotic inclusions of subfactors neither coming from Hopf algebras nor with braidings.

This paper is organized as follows: In Section 2, we first collect basics of sectors and introduce the Longo-Rehren inclusion $A \supset B$ for a finite closed system Δ of endomorphisms of an infinite factor. Then, using a criterion for intermediate subfactors obtained in [18], we establish a Galois type correspondence for the Longo-Rehren inclusions.

In Section 3, we translate Ocneanu's notion of tube algebras introduced in [7] (and related notions such as S and T matrices) into the language of sectors. We also introduce a faithful positive functional φ_Δ of the tube algebra, which turns out to be a useful tool for later analysis.

Section 4 is the main body of the present work. First we introduce the notion of half-braidings, which satisfy only a half of the defining conditions of the usual notion of braidings. Then, we show that every $B-B$ sector associated with the Longo-Rehren inclusion can be obtained by (the dual form of) the induction procedure with a half braiding, and that the half braiding turns into a genuine braiding for the induced sectors. We also construct a complete system of matrix units of the tube algebra from half braidings, which gives a new proof to the fact that there exists a one-to-one correspondence between the set of minimal central projections of the tube algebra and the set of irreducible $M_\infty - M_\infty$ bimodules [7].

In Section 5, we show that S and T matrices introduced in Section 3 coincide with

the ones associated with the braiding of $B-B$ sectors introduced in Section 4 through Rehren's argument [10][30]. Not only this gives a proof to the fact that S and T satisfy the usual modular group relation and Verlinde formula, it also provides a concrete formula of S and T in terms of the matrix units of the tube algebra, which will be used in the second part in order to obtain S and T matrices for concrete examples.

In Section 6, we explicitly construct a half-braiding corresponding to the restriction of the canonical endomorphism to B, which is necessary to obtain the α-inductions of the sectors with respect to the braiding obtained in Section 4.

In Section 7, we investigate the structure of the Longo-Rehren inclusion when the original system Δ has a braiding. As it has already been obtained in [8][20][28], we first show that if the braiding is non-degenerate, the sectors associated with the Longo-Rehren inclusion is isomorphic to the direct product of the original system and its opposite. Then, we consider the case where the braiding of Δ is not necessarily non-degenerate, and Δ is embedded in a system $\hat{\Delta}$ with a non-degenerate braiding. We obtain several results applying the Galois correspondence established in Section 2 and the α-induction obtained in Section 6 to this situation. Namely, we give a proof to Ocneanu's announcement of the "bipermutant theorem" in [28]. We also show that the method employed in [8][28] to obtain the $M_\infty - M_\infty$ bimodules for the Hecke algebra subfactors works in general as far as Δ is embedded in $\hat{\Delta}$ with a non-degenerate braiding. This allows us to treat the general $SU(n)$ Hecke algebra subfactors while in [8][28] they treated only $SU(2)$ and $SU(3)$ cases for some technical reason.

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2 Preliminaries

First, we collect basics of sectors and Longo-Rehren inclusions.

2-1. Sectors. Let M, N be infinite factors and $Mor(N,M)_0$ the set of unital normal $*$-homomorphisms from N to M whose image has a finite index. For $\rho \in Mor(N,M)_0$, E_ρ denotes the minimal conditional expectation from M to $\rho(N)$. We set

$$d(\rho) = [M : \rho(N)]_0^{\frac{1}{2}},$$

which is called the statistical dimension of ρ, where $[M:\rho(N)]_0$ is the minimum index of $M \supset \rho(N)$ [12]. The set of $M-N$ sectors $Sect(M,N)$ is defined to be the unitary equivalence classes of $Mor(N,M)_0$. $[\rho]$ denotes the class of $\rho \in Mor(N,M)_0$ in $Sect(M,N)$. We use the notation $End(M)_0$ and $Sect(M)$ for $Mor(M,M)_0$ and $Sect(M,M)$. The sectors have 3 natural operations, the sum $[\rho]\oplus[\sigma]$, the product $[\rho][\sigma]$, and the conjugation $\overline{[\rho]}$ [13][22]. For simplicity, $\overline{\rho} \in Mor(M,N)_0$ denotes one of the representatives of $\overline{[\rho]}$. $\overline{\rho}$ is given by $\rho^{-1}\cdot\gamma$ where $\gamma : M \longrightarrow \rho(N)$ is the canonical endomorphism of Longo.

$\rho \in Mor(N,M)_0$ is called irreducible if the relative commutant $M \cap \rho(N)'$ is trivial. For $\rho, \sigma \in Mor(N,M)_0$ the intertwiner space (ρ,σ) is defined by

$$(\rho,\sigma) := \{V \in M; V\rho(x) = \sigma(x)V, \quad x \in N\}.$$

If ρ is irreducible, (ρ,σ) is a Hilbert space with the inner product $\langle V,W\rangle 1 = W^*V$. It is known that for every $\rho \in Mor(N,M)_0$ there exist a pair of isometries $R_\rho \in (id, \overline{\rho}\rho)$, $\overline{R}_\rho \in (id, \rho\overline{\rho})$ satisfying

$$\overline{R}_\rho^*\rho(R_\rho) = \frac{1}{d(\rho)}, \quad R_\rho^*\overline{\rho}(\overline{R}_\rho) = \frac{1}{d(\rho)}. \tag{2.1}$$

One can choose $\overline{R}_\rho$ equal to either $R_{\overline{\rho}}$ or $-R_{\overline{\rho}}$. The standard left inverse ϕ_ρ of ρ and E_ρ are obtained by

$$\phi_\rho(x) = R_\rho^*\overline{\rho}(x)R_\rho, \quad E_\rho = \rho \cdot \phi_\rho. \tag{2.2}$$

2-2. Q-systems. The canonical endomorphisms can be characterized as follows [23]:

Theorem 2.1 *Let M be an infinite factor and γ a unital endomorphism of M. Then,*

(i) *If there exist isometries $V \in (id,\gamma)$, $W \in (\gamma,\gamma^2)$ satisfying*

$$V^*W = W^*\gamma(V) = \frac{1}{d} \in \mathbf{R}_{>0},$$

$$W^*\gamma(W) = WW^*, \quad \gamma(W)W = W^2,$$

then there exists a subfactor N such that γ is the canonical endomorphism for the inclusion $M \supset N$. N is given by

$$N = \{x; Wx = \gamma(x)W, \quad Wx^* = \gamma(x^*)W\}.$$

(ii) *The conditional expectation $E : M \longrightarrow N$ is given by*

$$E(x) = W^*\gamma(x)W.$$

Every $x \in M$ satisfies $x = d^2V^*E(Vx)$.

(iii) *The conditional expectation $E_1 : N \longrightarrow \gamma(M)$ is given by*

$$E_1(y) = \gamma(V^*yV).$$

Every $y \in M$ satisfies $y = d^2W^*E_1(Wy)$.

The triplet (γ, V, W) satisfying the above condition is called a $Q-system$ [23].

2-3. The Longo-Rehren inclusions. Let $\Delta = \{\rho_\xi\}_{\xi\in\Delta_0}$ be a finite system of irreducible endomorphisms in $End(M)_0$ that is closed under the sector operations in the following sense:

(i) $[\rho_\xi] = [\rho_\eta]$ if and only if $\xi = \eta$.

(ii) There exists $e \in \Delta_0$ such that $\rho_e = id$.

(iii) For every $\xi \in \Delta_0$ there exists $\overline{\xi} \in \Delta_0$ such that $\overline{[\rho_\xi]} = [\rho_{\overline{\xi}}]$.

(iv) There exists a non-negative integer $N_{\xi,\eta}^{\zeta}$ such that $[\rho_\xi][\rho_\eta] = \bigoplus_\zeta N_{\xi,\eta}^{\zeta}[\rho_\zeta]$.

We denote $R_\xi := R_{\rho_\xi}$, $\overline{R}_\xi := \overline{R}_{\rho_\xi}$, $\phi_\xi = \phi_{\rho_\xi}$, and $d(\xi) = d(\rho_\xi)$ for simplicity. Set $\lambda = \sum_\xi d(\xi)^2$, which we call the *global index* of Δ. Let M^{op} be the opposite algebra of M and $j : M \longrightarrow M^{op}$ the natural anti-linear isomorphism. (When M^{op} is identified with M', j may be identified with $x \mapsto J_M x J_M$, where J_M is the modular conjugation of M.) We set $A := M \otimes M^{op}$, $\rho_\xi^{op} = j \cdot \rho_\xi \cdot j$, and $\hat{\rho}_\xi = \rho_\xi \otimes \rho_\xi^{op}$. Let $\{T(\begin{smallmatrix}\zeta\\ \xi,\eta\end{smallmatrix})_i\}_{i=1}^{N_{\xi,\eta}^{\zeta}}$ be an orthonormal basis of $(\rho_\zeta, \rho_\xi\rho_\eta)$, and set

$$T_{\xi,\eta}^{\zeta} = \sum_{i=1}^{N_{\xi,\eta}^{\zeta}} T(\begin{smallmatrix}\zeta\\ \xi,\eta\end{smallmatrix})_i \otimes j(T(\begin{smallmatrix}\zeta\\ \xi,\eta\end{smallmatrix})_i),$$

which does not depend on the choice of the basis. Longo and Rehren showed the following [24]:

Theorem 2.2 *Let γ be the direct sum of $\{\hat{\rho}_\xi\}_{\xi\in\Delta_0}$; namely we choose a system of isometries $\{V_\xi\}_{\xi\in\Delta_0} \subset A$ satisfying $\sum_\xi V_\xi V_\xi^* = 1$, and define*

$$\gamma(x) = \sum_\xi V_\xi \hat{\rho}_\xi(x) V_\xi^*.$$

Let $V \in (id, \gamma)$, $W \in (\gamma, \gamma^2)$ be isometries defined by

$$\begin{aligned} V &= V_e, \\ W &= \sum_{\xi,\eta,\zeta} \sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}} V_\xi \hat{\rho}_\xi(V_\eta) T_{\xi,\eta}^{\zeta} V_\zeta^* = \sum_{\xi,\eta,\zeta} \sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}} \gamma(V_\eta) V_\xi T_{\xi,\eta}^{\zeta} V_\zeta^*. \end{aligned}$$

Then, (γ, V, W) is a Q-system.

In consequence, there exists a subfactor B such that $\gamma : A \longrightarrow B$ is the canonical endomorphism.

The above inclusion $A \supset B$ is called the *Longo-Rehren inclusion*. Let $E_\Delta : A \longrightarrow B$ be the conditional expectation, which is given by $E_\Delta(x) = W^*\gamma(x)W$. Using the definition of W, we see that E_Δ is given by

$$E_\Delta(x) = \sum_{\xi\in\Delta_0} Z_\xi^* \hat{\rho}_\xi(x) Z_\xi,$$

$$Z_\xi = \sum_{\eta,\zeta\in\Delta_0} \sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}} \hat{\rho}_\xi(V_\eta) T_{\xi,\eta}^{\zeta} V_\zeta^*. \tag{2.3}$$

2-4. The Galois correspondence. In this subsection, we establish the Galois correspondence for the Longo-Rehren inclusions. The results obtained here will be used only in Section 7.

Let $P \supset Q$ be an inclusion of factors of finite index and $\gamma : P \longrightarrow Q$ be the canonical endomorphism. Let $[\gamma|_Q] = \bigoplus_{i\in I} n_i[\sigma_i]$ be the irreducible decomposition of the restriction of γ to Q. We set

$$\mathcal{H}_i = \{X \in P; Xx = \sigma_i(x)X, \quad \forall x \in Q\}.$$

Then, $\mathcal{H}_i$ is a Hilbert space whose dimension is n_i thanks to Section 3 of [18]. P is uniquely decomposed as

$$P = \sum_{i\in I} Q\mathcal{H}_i,$$

that can be regarded as generalization of crossed product decomposition. Corollary 3.10 of [18] implies the following:

Lemma 2.3 *Let P, Q, and γ be as above. We assume that each σ_i appears in $\gamma|_Q$ with multiplicity one, i.e. $n_i = 1$. Then there exists a one-to-one correspondence in the lattice of intermediate subfactors between $P \supset Q$ and the lattice of subsets $J \subset I$ satisfying the following:*

(i) *If $i \in J$, so does $\bar{i}$.*

(ii) *If $i, j \in J$ and σ_k is contained in $\sigma_i\sigma_j$, then $k \in J$.*

The correspondence is given as follows: For $J \subset I$ satisfying the above condition, the corresponding intermediate subfactor R_J is given by

$$R_J = \sum_{j\in J} Q\mathcal{H}_i.$$

For an intermediate subfactor $P \supset R \supset Q$, the corresponding subset $J_R \subset I$ is given by

$$J_R = \{i \in I; \mathcal{H}_i \subset R\}.$$

Moreover, if $\gamma_R : R \longrightarrow Q$ is the canonical endomorphism, then the irreducible decomposition of γ_R restricted to Q is given by

$$[\gamma_R|_Q] = \bigoplus_{i\in J_R} [\sigma_i].$$

Let $\Delta = \{\rho_\xi\}_{\xi\in\Delta_0}$ be as in the previous subsection and $A \supset B$ be the corresponding Longo-Rehren inclusion. We denote by A_1 the basic extension of $A \supset B$. We say that a subset $\Pi \subset \Delta$ is a closed subsystem if Π is closed under the conjugation and the irreducible decomposition of the products. Applying Lemma 2.3 to $A_1 \supset A$, we can easily obtain the following:

Proposition 2.4 *Let $A \supset B$ be the Longo-Rehren inclusion of Δ. Then, there exists a lattice anti-isomorphism between the set of the intermediate subfactors of $A \supset B$ and the set of closed subsystem of Δ. For a given closed subsystem $\Pi \subset \Delta$, the corresponding intermediate subfactor B_Π is characterized as follows: If $\gamma_{B_\Pi} : A \longrightarrow B_\Pi$ is the canonical endomorphism, then γ_{B_Π} is decomposed as*

$$[\gamma_{B_\Pi}] = \bigoplus_{\xi \in \Pi_0} [\hat{\rho}_\xi].$$

In the above proposition, it is not so clear whether $A \supset B_\Pi$ really gives the Longo-Rehren inclusion of Π because in general, the canonical endomorphism does not determine a subfactor up to inner conjugacy, or even more strongly, up to conjugacy (See, for example, 5.1 of [16]). The rest of this section is devoted to the proof of this fact.

Theorem 2.5 *Let λ_Π be the global index of Π, i.e. $\lambda_\Pi = \sum_{\xi \in \Pi_0} d(\xi)^2$. We set $P_\Pi := \sum_{\xi \in \Pi_0} V_\xi V_\xi^*$, which is a projection in the relative commutant $A \cap \gamma(A)'$. We define a normal completely positive map E_Π from A to itself by*

$$E_\Pi(x) := \frac{\lambda}{\lambda_\Pi} W^* P_\Pi \gamma(x) W.$$

Then,

(i) *$B_\Pi = B P_\Pi B$ and E_Π is the conditional expectation from A to B_Π.*

(ii) *$A \supset B_\Pi$ is the Longo-Rehren inclusion for Π.*

Proof. (i) Since P_Π commutes with $\gamma(A)$, E_Π defines a completely positive map, which can be shown to be unital by direct computation. First we claim that $E_\Pi(xy) = E_\Pi(x)y$ (and $E_\Pi(yx) = yE_\Pi(x)$) holds for all $x \in A$ and all $y \in (B \cup \{P_\Pi\})''$. Indeed, since we have $B = W^*\gamma(A)$, it suffices to show the statement for the following three cases $y \in \gamma(A)$, $y = W^*$ and $y = P_\Pi$. In the first two cases, the facts

$$W \in (\gamma, \gamma^2), \quad \gamma(W^*)W = WW^*$$

imply the statement. Since Π is a closed system, we have

$$P_\Pi \gamma(P_\Pi) W = P_\Pi W P_\Pi = \gamma(P_\Pi) W P_\Pi.$$

Thus,

$$E_\Pi(xP_\Pi) = \frac{\lambda}{\lambda_\Pi} W^* \gamma(x) P_\Pi \gamma(P_\Pi) W = \frac{\lambda}{\lambda_\Pi} W^* \gamma(x) P_\Pi W P_\Pi = E_\Pi(x) P_\Pi.$$

Next we show that E_Π is a norm-one projection onto $BP_\Pi B$ that is the von Neumann algebra generated by B and P_Π. Indeed, using the fact that $A = BVV^*B$ and

$$E_\Pi(VV^*) = \frac{\lambda}{\lambda_\Pi} W^* \gamma(V) P_\Pi \gamma(V^*) W = \frac{1}{\lambda_\Pi} P_\Pi,$$

we get $E_\Pi(A) = BP_\Pi B$. Thanks to the above claim, this shows that E_Π is a norm-one projection onto $BP_\Pi B$ and the image is closed under product, and so it is a von Neumann algebra. Let γ' be the canonical endomorphism from A into the image of E_Π. Then, Proposition 2.9, (i) of [18] and the definition of E_Π imply

$$[\gamma'] = \bigoplus_{\xi \in \Pi_0} [\hat{\rho}_\xi].$$

Thus, Proposition 2.4 implies that the image is nothing but B_Π.

(ii) We take an isometry $R_\Pi \in B_\Pi$ satisfying $R_\Pi R_\Pi^* = P_\Pi$. Then, we have

$$E_\Pi(x) = R_\Pi^* E_\Pi(R_\Pi x R_\Pi^*) R_\Pi = \sum_{\xi \in \Pi_0} Z_\xi'^* \hat{\rho}_\xi(x) Z_\xi',$$

where

$$Z_\xi' = \sqrt{\frac{\lambda}{\lambda_\Pi}} \hat{\rho}_\xi(R_\Pi^*) Z_\xi R_\Pi.$$

We set $V_\xi' := R_\Pi^* V_\xi$ for $\xi \in \Pi_0$. Then $\{V_\xi'\}_{\xi \in \Pi_0}$ are isometries satisfying $\sum_{\xi \in \Pi_0} V_\xi' V_\xi'^* = 1$. Moreover, we have

$$Z_\xi' = \sum_{\eta, \zeta \in \Pi_0} \sqrt{\frac{d(\xi) d(\eta)}{\lambda_\Pi d(\zeta)}} \hat{\rho}_\xi(V_\eta') T_{\xi,\eta}^{\zeta} V_\zeta'^*.$$

Comparing this with (2.3), we conclude that E_Π comes from the Longo-Rehren inclusion with the canonical endomorphism

$$\gamma_\Pi(x) = \sum_{\xi \in \Pi_0} V_\xi' \hat{\rho}_\xi(x) V_\xi'^*.$$

Q.E.D.

3 Tube Algebras

The notion of tube algebras was introduced by Ocneanu [7] for a system of bimodules. Here, we will translate his definition into the language of sectors, and give explicit formulae of the tube algebra operations and the S and T matrices for the computation of concrete examples. We will freely use the Frobenius reciprocity of sectors formulated in [15].

Let $\Delta = \{\rho_\xi\}_{\xi \in \Delta_0}$ be a finite system of endomorphisms of M as in the previous section. The tube algebra of this system is, as a linear space, defined by

$$\text{Tube } \Delta := \bigoplus_{\xi, \eta, \zeta} (\rho_\xi \cdot \rho_\zeta, \rho_\zeta \cdot \rho_\eta).$$

Since the same operator may belong to two distinct intertwiner spaces, to avoid possible confusion we use the following notation; when $X \in (\rho_\xi \cdot \rho_\zeta, \rho_\zeta \cdot \rho_\eta)$ is regarded as an

element of Tube Δ, it is denoted by $(\xi\zeta|X|\zeta\eta)$. We introduce $*$-algebra structure into Tube Δ as follows:

$$(\xi\zeta|X|\zeta\eta)(\xi'\zeta'|Y|\zeta'\eta') := \delta_{\eta,\xi'} \sum_{\nu \prec \zeta\zeta'} \sum_{i=1}^{N^{\nu}_{\zeta,\zeta'}} (\xi\nu|T(^{\nu}_{\zeta,\zeta'})^*_i \rho_\zeta(Y) X \rho_\xi(T(^{\nu}_{\zeta,\zeta'})_i)|\nu\eta').$$

$$(\xi\zeta|X|\zeta\eta)^* := d(\zeta)(\eta\overline{\zeta}|\rho_{\overline{\zeta}}(\rho_\xi(\overline{R}^*_\zeta)X^*)R_\zeta|\overline{\zeta}\xi).$$

It is routine to show, by using the Frobenius reciprocity, that Tube Δ is a $*$-algebra with these operations. Moreover, Tube Δ is a C*-algebra. To show it, we need the following:

Lemma 3.1 (i) *For $X, Y \in (\rho_\mu, \rho_\xi\rho_\eta)$, we have*

$$\phi_\xi(XY^*) = \langle X, Y\rangle \frac{d(\mu)}{d(\xi)d(\eta)}.$$

(ii) *For $X, Y \in (\rho_\xi\rho_\zeta, \rho_\zeta\rho_\eta)$, we have*

$$d(\xi)\phi_\xi(Y^*X) = d(\eta)\phi_\zeta(XY^*).$$

Proof. (i) Thanks to the polarization identity, we may assume that $Y = X$ is an isometry. In this case, XX^* is a minimal projection in the relative commutant $M \cap \rho_\xi\rho_\eta(M)'$ and the statement follows from the local index formula for the minimal expectation. (See Section 2 of [15].) (ii) It suffices to show the statement for X and Y of the following form:

$$X = X_1X_2^*, \quad X_1 \in (\rho_\mu, \rho_\zeta\rho_\eta),\ X_2 \in (\rho_\mu, \rho_\xi\rho_\zeta),$$
$$Y = Y_1Y_2^*, \quad Y_1 \in (\rho_\nu, \rho_\zeta\rho_\eta),\ Y_2 \in (\rho_\nu, \rho_\xi\rho_\zeta).$$

For these X and Y, we have

$$\phi_\xi(Y^*X) = \phi_\xi(Y_2Y_1^*X_1X_2^*) = \delta_{\mu,\nu}\langle X_1, Y_1\rangle\phi_\xi(Y_2X_2^*),$$

$$\phi_\zeta(XY^*) = \phi_\zeta(X_1X_2^*Y_2Y_1^*) = \delta_{\mu,\nu}\langle Y_2, X_2\rangle\phi_\zeta(X_1Y_1^*).$$

Thus, the statement follows from (i). Q.E.D.

We define two linear functionals on Tube Δ by

$$\mathrm{Tr}((\xi\zeta|\mathrm{X}|\zeta\eta)) := \mathrm{d}(\xi)\delta_{\xi,\eta}\delta_{\zeta,\mathrm{e}}\mathrm{X},$$

$$\varphi_\Delta((\xi\zeta|X|\zeta\eta)) := d(\xi)^2\delta_{\xi,\eta}\delta_{\zeta,e}X.$$

Note that $X \in (\rho_\xi, \rho_\xi) = \mathbf{C}$ in the above unless the right-hand sides vanish.

Proposition 3.2 Tr *is a faithful positive trace and φ_Δ is a faithful positive functional. In consequence, since* Tube Δ *is a finite dimensional $*$-algebra with a faithful positive functional, it is a C*-algebra.*

Proof. First of all, note that unless $\xi = \xi'$, $\eta = \eta'$, $\zeta = \zeta'$, we have

$$\mathrm{Tr}((\xi\zeta|\mathrm{X}|\zeta\eta)(\xi'\zeta'|\mathrm{Y}|\zeta'\eta')^*) = \mathrm{Tr}((\xi'\zeta'|\mathrm{Y}|\zeta'\eta')^*(\xi\zeta|\mathrm{X}|\zeta\eta)) = 0.$$

Using (2.1) and the definitions of the product and the $*$-operation, we get

$$\begin{aligned}\mathrm{Tr}((\xi\zeta|\mathrm{X}|\zeta\eta)(\xi\zeta|\mathrm{Y}|\zeta\eta)^*) &= d(\xi)d(\zeta)\overline{R}_\zeta^*\rho_\zeta(\rho_{\overline{\zeta}}(\rho_\xi(\overline{R}_\zeta^*)Y^*)R_\zeta)X\rho_\xi(\overline{R}_\zeta)\\ &= d(\xi)\rho_\xi(\overline{R}_\zeta^*)Y^*X\rho_\xi(\overline{R}_\zeta).\end{aligned}$$

Since the right-hand side is already a scalar, we can apply ϕ_ξ and get

$$\mathrm{Tr}((\xi\zeta|\mathrm{X}|\zeta\eta)(\xi\zeta|\mathrm{Y}|\zeta\eta)^*) = \mathrm{d}(\xi)\overline{\mathrm{R}}_\zeta^*\phi_\xi(\mathrm{Y}^*\mathrm{X})\overline{\mathrm{R}}_\zeta = \mathrm{d}(\xi)\phi_\xi(\mathrm{Y}^*\mathrm{X}).$$

In a similar way, we obtain the following using (2.1) and (2.2):

$$\begin{aligned}\mathrm{Tr}((\xi\zeta|\mathrm{Y}|\zeta\eta)^*(\xi\zeta|\mathrm{X}|\zeta\eta)) &= d(\eta)d(\zeta)R_\zeta^*\rho_{\overline{\zeta}}(X)\rho_{\overline{\zeta}}(\rho_\xi(\overline{R}_\zeta^*)Y^*)R_\zeta\rho_\eta(R_\zeta)\\ &= d(\eta)d(\zeta)R_\zeta^*\rho_{\overline{\zeta}}(X\rho_\xi(\overline{R}_\zeta^*)Y^*\rho_\zeta\rho_\eta(R_\zeta))R_\zeta\\ &= d(\eta)d(\zeta)\phi_\zeta(X\rho_\xi(\overline{R}_\zeta^*\rho_\zeta(R_\zeta))Y^*)\\ &= d(\eta)\phi_\zeta(XY^*),\end{aligned}$$

Thanks to Lemma 3.1, Tr is a trace. Let $x = \sum_{\xi,\eta,\zeta}(\xi\zeta|X_{\xi,\zeta,\eta}|\zeta\eta) \in \mathrm{Tube}\ \Delta$. Then,

$$\mathrm{Tr}(\mathrm{x}^*\mathrm{x}) = \sum_{\xi,\eta,\zeta}\mathrm{d}(\eta)\phi_\zeta(\mathrm{X}_{\xi,\zeta,\eta}\mathrm{X}_{\xi,\zeta,\eta}{}^*).$$

Since the left inverses $\{\phi_\xi\}_{\xi\in\Delta_0}$ are faithful and positive, so is Tr. The statement for φ_Δ can be proven in a similar way. Q.E.D.

We denote by $\mathcal{A}_{\xi,\eta}$ the linear subspace $\bigoplus_{\zeta\in\Delta_0}(\rho_\xi\rho_\zeta, \rho_\zeta\rho_\eta) \subset \mathrm{Tube}\ \Delta$. Then, we have

$$\mathcal{A}_{\xi,\eta}\mathcal{A}_{\xi',\eta'} \subset \delta_{\xi',\eta}\mathcal{A}_{\xi,\eta'}, \quad \mathcal{A}_{\xi,\eta}^* = \mathcal{A}_{\eta,\xi}.$$

Thus, $\mathcal{A}_\xi := \mathcal{A}_{\xi,\xi}$ is a $*$-subalgebra of Tube Δ. Let $\mathcal{A}_\Delta$ be the direct sum of $\mathcal{A}_\xi$, which includes the center of Tube Δ. The above proof shows that φ_Δ restricted to $\mathcal{A}_\Delta$ is a faithful trace. We introduce an inner product into $\mathcal{A}_\Delta$ by

$$(X, Y) := \varphi_\Delta(Y^*X) = \varphi_\Delta(XY^*),$$

or more explicitly by

$$((\xi\zeta|X|\zeta\xi), (\xi'\zeta'|Y|\zeta'\xi')) = \delta_{\xi,\xi'}\delta_{\zeta,\zeta'}d(\xi)^2\phi_\zeta(XY^*) = \delta_{\xi,\xi'}\delta_{\zeta,\zeta'}d(\xi)^2\phi_\xi(Y^*X).$$

Now, we introduce S and T matrices. We define a linear operator S_0 on $\mathcal{A}_\Delta$ by

$$S_0((\xi\eta|X|\eta\xi)) = d(\xi)(\overline{\eta}\xi|R_\eta^*\rho_{\overline{\eta}}(X\rho_\xi(\overline{R}_\eta))|\xi\overline{\eta}).$$

We set

$$\mathrm{t} = \sum_\xi d(\xi)(\xi\overline{\xi}|R_\xi\overline{R}_\xi^*|\overline{\xi}\xi) \in \mathrm{Tube}\ \Delta.$$

Theorem 3.3 *Let S_0 and $\mathbf{t}$ be as above. Then,*

(i) *S_0 is a unitary.*

(ii) *$\mathbf{t}$ is a unitary in* Tube Δ *with*

$$\mathbf{t}^* = \sum_{\xi} (\xi\xi|1|\xi\xi).$$

(iii) *$\mathbf{t}$ belongs to the center of* Tube Δ.

Proof. (i) Using the definitions of S_0 and the inner product, we get

$$\begin{aligned}
&(S_0((\xi\eta|X|\eta\xi)), S_0((\xi'\eta'|Y|\eta'\xi'))) \\
&\quad = d(\xi)^2 d(\eta)^2 \delta_{\xi,\xi'}\delta_{\eta,\eta'}\phi_{\overline{\eta}}(\rho_{\overline{\eta}}(\rho_\xi(\overline{R}_\eta^*)Y^*)R_\eta R_\eta^* \rho_{\overline{\eta}}(X\rho_\xi(\overline{R}_\eta))) \\
&\quad = d(\xi)^2 d(\eta)^2 \delta_{\xi,\xi'}\delta_{\eta,\eta'}\rho_\xi(\overline{R}_\eta^*)Y^*\phi_{\overline{\eta}}(R_\eta R_\eta^*)X\rho_\xi(\overline{R}_\eta) \\
&\quad = d(\xi)^2 \delta_{\xi,\xi'}\delta_{\eta,\eta'}\rho_\xi(\overline{R}_\eta^*)Y^*X\rho_\xi(\overline{R}_\eta).
\end{aligned}$$

Since this is already a scalar, we can apply ϕ_ξ to it. Thus, we get

$$(S_0((\xi\eta|X|\eta\xi)), S_0((\xi'\eta'|Y|\eta'\xi'))) = d(\xi)^2\delta_{\xi,\xi'}\delta_{\eta,\eta'}\phi_\xi(Y^*X) = ((\xi\eta|X|\eta\xi),(\xi'\eta'|Y|\eta'\xi')).$$

(ii) Using (2.1), we obtain

$$\mathbf{t}^* = \sum_{\xi} d(\xi)^2 (\xi\xi|\rho_\xi(\rho_\xi(\overline{R_{\overline{\xi}}}^*)\overline{R}_\xi R_\xi^*)R_{\overline{\xi}}|\xi\xi) = \sum_{\xi}(\xi\xi|1|\xi\xi).$$

Since $\sum_\xi (\xi e|1|e\xi)$ is the unit of Tube Δ, we get

$$\mathbf{t}^*\mathbf{t} = \sum_{\xi} d(\xi)(\xi\xi|1|\xi\xi)(\xi\overline{\xi}|R_\xi \overline{R}_\xi^*|\overline{\xi}\xi) = d(\xi)\sum_{\xi}(\xi e|\overline{R}_\xi^*\rho_\xi(R_\xi)|e\xi) = 1.$$

(iii) It suffices to show that $\mathbf{t}^*$ commutes with $(\xi\zeta|X|\zeta\eta)$ of the following form:

$$X = X_1 X_2^*, \quad X_1 \in (\rho_\mu, \rho_\zeta\rho_\eta),\ X_2 \in (\rho_\mu, \rho_\xi\rho_\zeta).$$

We may further assume $X_1 = T(\begin{smallmatrix}\mu\\ \zeta,\eta\end{smallmatrix})_1$, $X_2 = T(\begin{smallmatrix}\mu\\ \xi,\zeta\end{smallmatrix})_1$. Then we get

$$\begin{aligned}
\mathbf{t}^*(\xi\zeta|X|\zeta\eta) &= (\xi\xi|1|\xi\xi)(\xi\zeta|X|\zeta\eta) = \sum_{\nu\prec\xi\zeta}\sum_{i=1}^{N_{\xi,\zeta}^\nu}(\xi\nu|T(\begin{smallmatrix}\nu\\ \xi,\zeta\end{smallmatrix})_i^*\rho_\xi(X)\rho_\xi(T(\begin{smallmatrix}\nu\\ \xi,\zeta\end{smallmatrix})_i)|\nu\eta) \\
&= (\xi\mu|X_2^*\rho_\xi(X_1)|\mu\eta),
\end{aligned}$$

$$\begin{aligned}
(\xi\zeta|X|\zeta\eta)\mathbf{t}^* &= (\xi\zeta|X|\zeta\eta)(\eta\eta|1|\eta\eta) = \sum_{\nu\prec\zeta\eta}\sum_{i=1}^{N_{\zeta,\eta}^\nu}(\xi\nu|T(\begin{smallmatrix}\nu\\ \zeta,\eta\end{smallmatrix})_i^* X\rho_\xi(T(\begin{smallmatrix}\nu\\ \zeta,\eta\end{smallmatrix})_i)|\nu\eta) \\
&= (\xi\mu|X_2^*\rho_\xi(X_1)|\mu\eta).
\end{aligned}$$

Q.E.D.

In Section 5, we show that S_0 leaves the center $Z(\text{Tube }\Delta)$ invariant. Therefore, the restriction of S_0 to $Z(\text{Tube }\Delta)$, denoted by S, is well-defined. We denote by T the multiplication operator of $\mathbf{t}$ on $Z(\text{Tube }\Delta)$. Note that S and T are unitaries.

Before closing this section, we show a simple but non-trivial example of Tube Δ. Let G be a finite group. A G-kernel in M is a homomorphism from G to $Out(M)$. We take a lift $\alpha : G \longrightarrow Aut(M)$ of an injective G-kernel. Then, since $[\alpha_g][\alpha_h] = [\alpha_{gh}]$, there exists a unitary $u(g,h) \in M$ for each pair $g, h \in G$ satisfying

$$\alpha_g \cdot \alpha_h = Ad(u(g,h)) \cdot \alpha_{gh}.$$

Therefore, using associativity of $\alpha_g \cdot \alpha_h \cdot \alpha_k$, we get

$$Ad(u(g,h)u(gh,k)) \cdot \alpha_{ghk} = Ad(\alpha_g(u(h,k))u(g,hk)) \cdot \alpha_{ghk}.$$

This implies that there exists a scalar $\omega(g,h,k) \in \mathbf{T}$ satisfying

$$u(g,h)u(gh,k) = \omega(g,h,k)\alpha_g(u(h,k))u(g,hk).$$

ω satisfies the 3 cocycle relation and the cohomology class $[\omega] \in H^3(G, \mathbf{T})$ depends only on the G-kernel [31]. Now we consider the case $\Delta = \{\alpha_g\}_{g \in G}$. Let

$$c(g,h) := (gh|u(h,g^h)u(g,h)^*|hg^h)$$

where $g^h = h^{-1}gh$. Then, clearly $\{c(g,h)\}_{g,h \in G}$ forms a basis of Tube Δ. By applying the definition of the product introduced above, we get

$$\begin{aligned} c(g,h)c(k,l) &= \delta_{k,g^h}(g(hl)|u(h,l)^*\alpha_h(u(l,g^{hl})u(g^h,l)^*)u(h,g^h)u(g,h)^*\alpha_g(u(h,l))|(hl)g^{hl}) \\ &= \delta_{k,g^h}\omega(h,g^h,l)\overline{\omega(g,h,l)\omega(h,l,g^{hl})}c(g,hl). \end{aligned}$$

Thus Tube Δ in this case is nothing but the quantum double of G with 3 cocycle ω discussed in [6],[32].

4 Half Braidings and $B - B$ Sectors

Let $\Delta = \{\rho_\xi\}_{\xi \in \Delta_0}$ be as before and $A \supset B$ the associated Longo-Rehren subfactor. We keep using the same notation as in the previous two sections. In [7], it is shown that the center of the Tube Δ can be identified with the Hilbert space $H_{S^1 \times S^1}$ associated with the corresponding TQFT, and that the minimal central projections of Tube Δ parameterize the $M_\infty - M_\infty$ bimodules associated with the asymptotic inclusion. In this section, we directly show the relationship between the half braidings of direct sums of sectors in Δ and the matrix units of Tube Δ, and consequently show the relationship between the center of Tube Δ and the sectors generated by $\gamma_{|B}$ without using TQFT.

Let $\iota : B \hookrightarrow A$ be the inclusion map and consider $[\iota]$ to be a $A - B$ sector. Then the conjugate sector of $[\iota]$ is $[\gamma] \in Sect(B, A)$. $[\iota]$ and $\overline{[\iota]}$ generate 4 kinds of sectors, $A - A$, $A - B$, $B - A$, and $B - B$ sectors, and we call them *sectors associated with* $A \supset B$. The induction-reduction graph of the $A - A$ and $A - B$ sectors can be obtained by simple manipulation of the Frobenius reciprocity (c.f. Section 4 of [7]).

Theorem 4.1 *Let $A \supset B$ be the Longo-Rehren inclusion of $\Delta = \{\rho_\xi\}_{\xi\in\Delta_0}$ and ι the inclusion map. Then,*

(i) *Every irreducible $A-A$ sector associated with $A \supset B$ is given by $[\rho_\xi \otimes \rho_\eta^{op}]$ for some $\xi, \eta \in \Delta_0$.*

(ii) *For every ξ, $[\rho_\xi \otimes id][\iota] = [id \otimes \rho_{\overline{\xi}}^{op}][\iota]$ is irreducible. In consequence, we have*

$$[\rho_\xi \otimes \rho_\eta^{op}][\iota] = \bigoplus_\zeta N_{\xi,\overline{\eta}}^\zeta [\rho_\zeta \otimes id][\iota].$$

(iii) *Every irreducible $A - B$ sector associated with $A \supset B$ is given by $[\rho_\xi \otimes id][\iota]$ for some ξ. $[\rho_\xi \otimes id][\iota] = [\rho_\eta \otimes id][\iota]$ if and only if $\xi = \eta$.*

Proof. (i) Since $[\iota\overline{\iota}] = \bigoplus_\xi [\rho_\xi \otimes \rho_\xi^{op}]$, this is obvious. (ii) Thanks to the Frobenius reciprocity, we have

$$\dim((\rho_\xi \otimes id^{op}) \cdot \iota, (\rho_\xi \otimes id^{op}) \cdot \iota) = \dim(\rho_{\overline{\xi}}\rho_\xi \otimes id^{op}, \iota\overline{\iota}) = 1$$

because $\rho_{\overline{\xi}}\rho_\xi$ contains identity only once. Thus $[(\rho_\xi \otimes id^{op}) \cdot \iota]$ is an irreducible $A - B$ sector. In the same way, we can show that $[(id \otimes \rho_{\overline{\xi}}^{op}) \cdot \iota]$ is irreducible too. Again by the Frobenius reciprocity, we get

$$\dim((\rho_\xi \otimes id^{op}) \cdot \iota, (id \otimes \rho_{\overline{\xi}}^{op}) \cdot \iota) = \dim(\rho_\xi \otimes \rho_\xi^{op}, \iota\overline{\iota}) = 1,$$

which shows $[(\rho_\xi \otimes id^{op}) \cdot \iota] = [(id \otimes \rho_{\overline{\xi}}^{op}) \cdot \iota]$. Using these facts, we obtain

$$[\rho_\xi \otimes \rho_\eta^{op}][\iota] = [\rho_\xi \otimes id^{op}][id \otimes \rho_\eta^{op}][\iota] = [\rho_\xi \rho_{\overline{\eta}} \otimes id][\iota] = \bigoplus_\zeta N_{\xi,\overline{\eta}}^\zeta [\rho_\zeta \otimes id][\iota].$$

(iii) The first statement is a direct consequence of (i) and (ii). $[\rho_\xi \otimes id][\iota] = [\rho_\eta \otimes id][\iota]$ if and only if

$$1 = \dim((\rho_\xi \otimes id^{op}) \cdot \iota, (\rho_\eta \otimes id^{op}) \cdot \iota) = \dim(\rho_{\overline{\eta}}\rho_\xi \otimes id^{op}, \iota\overline{\iota}).$$

Since $\rho_{\overline{\eta}}\rho_\xi$ contains identity if and only if $\xi = \eta$, we get the result. Q.E.D.

Remark. (ii) and (iii) give one of the principal graphs of $A \supset B$.

To determine the $B-B$ sectors and the induction-reduction graph between the $A-B$ sectors and the $B-B$ sectors, we use the following observation due to Xu [35], which is, in fact, the dual form of the induction procedure of Longo-Rehren [24]: if $\rho \in End(A)_0$ and $U \in A$ is a unitary satisfying

$$U \in (\rho \cdot \gamma, \gamma \cdot \rho), \quad WU = \gamma(U)U\rho(W), \tag{4.1}$$

where W is as in subsection 2-3, then the restriction of $Ad(U) \cdot \rho$ to B is an endomorphism of B. Indeed, the above two conditions imply

$$Ad(U) \cdot \rho(\gamma(A)) = \gamma(\rho(A)), \quad Ad(U) \cdot \rho(W) = \gamma(U^*)W.$$

Since $B = \gamma(A)W$, we have $Ad(U) \cdot \rho(B) \subset B$. In [35], Xu used a braiding operator for U. However, we need only two properties above.

Definition 4.2 *Let $\Delta = \{\rho_\xi\}_{\xi\in\Delta_0}$ be as before.*

(i) *Let σ be a finite direct sum of endomorphisms from sectors in Δ. A system of unitary operators $\{\mathcal{E}_\sigma(\xi)\}_{\xi\in\Delta_0}$ is called a half braiding of σ with respect to Δ if it satisfies the following:*

(1) $\mathcal{E}_\sigma(\xi) \in (\sigma\rho_\xi, \rho_\xi\sigma)$.

(2) *For every $X \in (\rho_\zeta, \rho_\xi\rho_\eta)$, the following holds:*

$$X\mathcal{E}_\sigma(\zeta) = \rho_\xi(\mathcal{E}_\sigma(\eta))\mathcal{E}_\sigma(\xi)\sigma(X).$$

Two half braidings $\{\mathcal{E}_\sigma(\xi)\}_{\xi\in\Delta_0}$ and $\{\mathcal{E}'_\sigma(\xi)\}_{\xi\in\Delta_0}$ are equivalent if there exists a unitary $u \in (\sigma, \sigma)$ such that

$$\mathcal{E}_\sigma(\xi) = \rho_\xi(u)\mathcal{E}'_\sigma(\xi)u^*$$

holds for all $\xi \in \Delta_0$.

(ii) *A system of unitary operators $\{\mathcal{E}(\xi,\eta)\}_{\xi,\eta\in\Delta_0}$ is a braiding of Δ if $\{\mathcal{E}(\xi,\eta)\}_{\eta\in\Delta_0}$ and $\{\mathcal{E}(\eta,\xi)^*\}_{\eta\in\Delta_0}$ are half braidings of ξ with respect to Δ for each fixed ξ.*

Remark.

(i) The second condition in (i) is called the braiding fusion equation. Readers are encouraged to draw diagrams as in [35] when we use this condition.

(ii) Let μ be a finite direct sum of sectors in Δ and $\{X(\xi)_i\}$ an orthonormal basis of (ρ_ξ, μ). For a half braiding $\{\mathcal{E}_\sigma(\xi)\}$ of σ with respect to Δ, we set

$$\mathcal{E}_\sigma(\mu) = \sum_{\xi,i} X(\xi)_i\mathcal{E}_\sigma(\xi)\sigma(X(\xi)_i^*).$$

Then, the two conditions of the definition of half braidings hold with respect to arbitrary finite direct sums of sectors in Δ instead of irreducibles ρ_ξ, ρ_η, and ρ_ζ above. Moreover, it is routine to show the following for finite direct sums μ_1, μ_2 of sectors in Δ:

$$\mathcal{E}_\sigma(\mu_1\mu_2) = \mu_1(\mathcal{E}_\sigma(\mu_2))\mathcal{E}_\sigma(\mu_1).$$

In the same way, starting with the braiding of irreducibles in the above sense, we can construct, in a natural way, a braiding of the category generated by Δ.

For a half braiding $\{\mathcal{E}_\sigma(\xi)\}_{\xi\in\Delta_0}$, we define unitaries $U(\sigma, \mathcal{E}_\sigma) \in A$ and $U(\sigma^{op}, \mathcal{E}_\sigma) \in A$ by

$$U(\sigma, \mathcal{E}_\sigma) = \sum_\xi V_\xi(\mathcal{E}_\sigma(\xi) \otimes 1)(\sigma \otimes id^{op})(V_\xi^*),$$

$$U(\sigma^{op}, \mathcal{E}_\sigma) = \sum_\xi V_\xi(1 \otimes j(\mathcal{E}_\sigma(\xi)))(id \otimes \sigma^{op})(V_\xi^*).$$

Lemma 4.3 *$U(\sigma, \mathcal{E}_\sigma)$ and $U(\sigma^{op}, \mathcal{E}_\sigma)$ satisfy (4.1) for $\rho = \sigma \otimes id^{op}$ and $\rho = id \otimes \sigma^{op}$ respectively.*

Proof. It suffices to show the statement only for $U(\sigma, \mathcal{E}_\sigma)$ because that for $U(\sigma^{op}, \mathcal{E}_\sigma)$ can be shown in the same way. It is easy to show that $U(\sigma, \mathcal{E}_\sigma)$ is a unitary belonging to $(\rho \cdot \gamma, \gamma \cdot \rho)$. Using the braiding fusion equation, we get

$$\begin{aligned} WU(\sigma, \mathcal{E}_\sigma) &= \sum_{\xi,\eta,\zeta} \sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}} \gamma(V_\eta) V_\xi T^\zeta_{\xi,\eta} (\mathcal{E}_\sigma(\zeta) \otimes 1)(\sigma \otimes id^{op})(V_\zeta^*) \\ &= \sum_{\xi,\eta,\zeta} \sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}} \gamma(V_\eta) V_\xi (\rho_\xi(\mathcal{E}_\sigma(\eta))\mathcal{E}_\sigma(\xi) \otimes 1)(\sigma \otimes id^{op})(T^\zeta_{\xi,\eta} V_\zeta^*), \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\gamma(U(\sigma, \mathcal{E}_\sigma))U(\sigma, \mathcal{E}_\sigma)(\sigma \otimes id^{op})(W) \\ &= \sum_\eta \gamma(V_\eta(\mathcal{E}_\sigma(\eta) \otimes 1)(\sigma \otimes id^{op})(V_\eta^*))U(\sigma, \mathcal{E}_\sigma)(\sigma \otimes id^{op})(W) \\ &= \sum_\eta \gamma(V_\eta(\mathcal{E}_\sigma(\eta) \otimes 1))U(\sigma, \mathcal{E}_\sigma)(\sigma \otimes id^{op})(\gamma(V_\eta^*)W) \\ &= \sum_{\xi,\eta,\zeta} \sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}} \gamma(V_\eta(\mathcal{E}_\sigma(\eta) \otimes 1))U(\sigma, \mathcal{E}_\sigma)(\sigma \otimes id^{op})(V_\xi T^\zeta_{\xi,\eta} V_\zeta^*) \\ &= \sum_{\xi,\eta,\zeta} \sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}} \gamma(V_\eta) V_\xi (\rho_\xi(\mathcal{E}_\sigma(\eta))\mathcal{E}_\sigma(\xi) \otimes 1)(\sigma \otimes id^{op})(T^\zeta_{\xi,\eta} V_\zeta^*). \end{aligned}$$

Therefore the statement holds. Q.E.D.

Definition 4.4 *Let Δ be as above.*

(i) *For a finite direct sum σ of sectors in Δ and a half braiding $\mathcal{E}_\sigma$ of σ, we define $\widetilde{(\sigma, \mathcal{E}_\sigma)} \in End(B)$ by the restriction of $Ad(U(\sigma, \mathcal{E}_\sigma)) \cdot (\sigma \otimes id^{op})$ to B. Note that $\widetilde{(\sigma, \mathcal{E}_\sigma)} \in End(B)$ is characterized by*

$$\widetilde{(\sigma, \mathcal{E}_\sigma)}(\gamma(x)) = \gamma((\sigma \otimes id^{op})(x)), \quad x \in A,$$

$$\widetilde{(\sigma, \mathcal{E}_\sigma)}(W) = \gamma(U(\sigma, \mathcal{E}_\sigma)^*)W.$$

We also define $\widetilde{(\sigma^{op}, \mathcal{E}_\sigma)}$ in an obvious way. In general, σ might have several half braidings. To distinguish them, we put a parameter α as $\{\mathcal{E}^\alpha_\sigma(\xi)\}_{\xi \in \Delta_0}$. For simplicity, we sometimes use the following notation:

$$U^\alpha(\sigma) = U(\sigma, \mathcal{E}^\alpha_\sigma), \quad \tilde{\sigma}^\alpha = \widetilde{(\sigma, \mathcal{E}^\alpha_\sigma)},$$

$$U^\alpha(\sigma^{op}) = U(\sigma^{op}, \mathcal{E}^\alpha_\sigma), \quad \tilde{\sigma}^{op,\alpha} = \widetilde{(\sigma^{op}, \mathcal{E}^\alpha_\sigma)}.$$

(ii) *We define $D(\Delta)$ to be the set of endomorphisms $\rho \in End(B)_0$ such that $[\iota][\rho]$ is a finite direct sum of sectors in $\{[\rho_\xi \otimes id^{op}][\iota]\}_{\xi \in \Delta_0}$. We call $D(\Delta)$ the quantum double of Δ.*

Lemma 4.5 *Let $D(\Delta)$ be as above. Then,*

(i) *$D(\Delta)$ is closed under the sector operations, i.e. sum, product, decomposition, and conjugation.*

(ii) *For every $\rho \in D(\Delta)$, there exist $\sigma \in End(M)_0$ that is a finite direct sum of sectors in Δ and a half braiding $\{\mathcal{E}_\sigma(\xi)\}_{\xi\in\Delta_0}$ of σ with respect to Δ such that*

$$[\rho] = [(\widetilde{\sigma, \mathcal{E}_\sigma})].$$

Proof. (i) It is easy to see that $D(\Delta)$ is closed under the first 3 operations. Let $[\rho] \in D(\Delta)$. By definition, we may assume $[\iota][\rho] = \bigoplus_\xi n_\xi[\rho_\xi \otimes id^{op}][\iota]$, where n_ξ is a non-negative integer. The Frobenius reciprocity implies

$$n_\xi = \dim((\rho_\xi \otimes id^{op}) \cdot \iota, \iota \cdot \rho) = \dim(\iota \cdot \overline{\rho}, (\rho_{\overline{\xi}} \otimes id^{op}) \cdot \iota).$$

By comparing dimensions, we obtain $[\iota][\overline{\rho}] = \bigoplus_\xi n_\xi[(\rho_{\overline{\xi}} \otimes id^{op})][\iota]$, which shows $[\overline{\rho}] \in D(\Delta)$. (ii) Since $\bar{\rho} \in D(\Delta)$, we have

$$[\rho][\bar{\iota}] = \bigoplus_\xi n_\xi[\bar{\iota}][(\rho_\xi \otimes id^{op})].$$

We set $\sigma = \bigoplus_\xi n_\xi \rho_\xi$. Then, $[\rho][\bar{\iota}] = [\bar{\iota}][(\sigma \otimes id^{op})]$. Therefore, perturbing ρ by an inner automorphism of B if necessary, we may and do assume $\rho \cdot \gamma = \gamma \cdot (\sigma \otimes id^{op})$. Note that $B = \gamma(A)W$ and for each $x \in B$ there exists a unique element $y \in A$ such that $x = \gamma(y)W$. Indeed, if $x = \gamma(y)W$,

$$V^*x = V^*\gamma(y)W = yV^*W = \sqrt{\lambda}y.$$

So there exists a unique element $U \in A$ satisfying $\rho(W) = \gamma(U^*)W$. Since U is given by $\frac{1}{\sqrt{\lambda}}\rho(W^*)V$, we have

$$U \in ((\sigma \otimes id^{op}) \cdot \gamma, \gamma \cdot (\sigma \otimes id^{op})),$$

and so

$$V_\eta^* U(\sigma \otimes id^{op})(V_\xi) \in ((\sigma \cdot \rho_\xi \otimes \rho_\xi^{op}), (\rho_\eta \cdot \sigma \otimes \rho_\eta^{op})).$$

Note that $((\sigma \cdot \rho_\xi \otimes \rho_\xi^{op}), (\rho_\eta \cdot \sigma \otimes \rho_\eta^{op}))$ is non-trivial only if $\xi = \eta$. Thus, there exists $u(\xi) \in (\sigma \cdot \rho_\xi, \rho_\xi \cdot \sigma)$ such that

$$U = \sum_{\xi\in\Delta_0} V_\xi(u(\xi) \otimes 1)(\sigma \otimes id^{op})(V_\xi^*).$$

We show that $\{u(\xi)\}_{\xi\in\Delta}$ is a half braiding of σ with respect to Δ. First we show that $u(\xi)$ is a unitary. $\frac{1}{\sqrt{\lambda}} = W^*\gamma(V)$ implies

$$\begin{aligned} \frac{1}{\sqrt{\lambda}} &= \rho(W^*)\rho(\gamma(V)) = W^*\gamma(U(\sigma \otimes id^{op})(V)) \\ &= W^*\gamma(V(u(e) \otimes 1)) = \frac{1}{\sqrt{\lambda}}\gamma(u(e) \otimes 1), \end{aligned}$$

and so $u(e) = 1$. Since we have

$$\rho(\gamma(W^*)W) = \gamma((\sigma \otimes id^{op})(W^*)U^*)W,$$
$$\rho(WW^*) = \gamma(U^*)WW^*\gamma(U) = \gamma(U^*W^*\gamma(U))W,$$
$$\rho(\gamma(W)W) = \gamma((\sigma \otimes id^{op})(W)U^*)W,$$
$$\rho(W^2) = \gamma(U^*)W\gamma(U^*)W = \gamma(U^*\gamma(U^*)W)W,$$

$\gamma(W^*)W = WW^*$ and $\gamma(W)W = W^2$ imply

$$(\sigma \otimes id^{op})(W^*)U^* = U^*W^*\gamma(U),$$
$$(\sigma \otimes id^{op})(W)U^* = U^*\gamma(U^*)W.$$

Therefore, we get

$$U^*U(\sigma \otimes id^{op})(W) = (\sigma \otimes id^{op})(W)U^*U,$$

and so

$$\begin{aligned}&(\sigma \otimes id^{op})(V_\xi^*)U^*U(\sigma \otimes id^{op})(W)(\sigma \otimes id^{op})(V_\zeta)\\ =\ &(\sigma \otimes id^{op})(V_\xi^*)(\sigma \otimes id^{op})(W)U^*U(\sigma \otimes id^{op})(V_\zeta).\end{aligned}$$

The left-hand side is

$$\begin{aligned}&(u(\xi)^*u(\xi) \otimes 1)(\sigma \otimes id^{op})(V_\xi^*WV_\zeta)\\ =\ &\sum_{\eta \in \Delta_0} \sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}}(u(\xi)^*u(\xi) \otimes 1)(\sigma \cdot \rho_\xi \otimes \rho_\xi^{op})(V_\eta)(\sigma \otimes id^{op})(T^\zeta_{\xi,\eta})\\ =\ &\sum_{\eta \in \Delta_0} \sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}}(\sigma \cdot \rho_\xi \otimes \rho_\xi^{op})(V_\eta)(u(\xi)^*u(\xi) \otimes 1)(\sigma \otimes id^{op})(T^\zeta_{\xi,\eta}),\end{aligned}$$

where we use $u(\xi)^*u(\xi) \in (\sigma \cdot \rho_\xi, \sigma \cdot \rho_\xi)$. On the other hand, the right-hand side is

$$\begin{aligned}&(\sigma \otimes id^{op})(V_\xi^*WV_\zeta)(u(\zeta)^*u(\zeta) \otimes 1)\\ =\ &\sum_{\eta \in \Delta_0} \sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}}(\sigma \cdot \rho_\xi \otimes \rho_\xi^{op})(V_\eta)(\sigma \otimes id^{op})(T^\zeta_{\xi,\eta})(u(\zeta)^*u(\zeta) \otimes 1).\end{aligned}$$

So we have

$$(u(\xi)^*u(\xi) \otimes 1)(\sigma \otimes id^{op})(T^\zeta_{\xi,\eta}) = (\sigma \otimes id^{op})(T^\zeta_{\xi,\eta})(u(\zeta)^*u(\zeta)^* \otimes 1).$$

Setting $\xi = e$, $\eta = \zeta$, we get $u(\eta)^*u(\eta) = 1$, which shows that U is an isometry. $1 = W^*W$ implies

$$1 = W^*\gamma(UU^*)W = E_\Delta(UU^*),$$

where E_Δ is the conditional expectation from A to B. Therefore, U is unitary and so is $u(\xi)$ for every $\xi \in \Delta_0$. What remains is to show that $\{u(\xi)\}_{\xi\Delta_0}$ satisfies the braiding fusion equation. To do so we compute the both sides of

$$(\sigma \otimes id^{op})(V_\xi)^*(\sigma \otimes id^{op})(W)U^*V_\zeta = (\sigma \otimes id^{op})(V_\xi)^*U^*\gamma(U^*)WV_\zeta.$$

The left-hand side is

$$(\sigma \otimes id^{op})(V_\xi^* W(\sigma \otimes id^{op})(V_\zeta))(u(\zeta)^* \otimes 1)$$
$$= \sum_{\eta \in \Delta_0} \sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}}(\sigma \cdot \rho_\xi \otimes \rho_\xi^{op})(V_\eta)(\sigma \otimes id^{op})(T^\zeta_{\xi,\eta})(u(\zeta)^* \otimes 1).$$

The right-hand side is

$$(u(\xi)^* \otimes 1)(\rho_\xi \otimes \rho_\xi^{op})(U^*)V_\xi^* W V_\zeta$$
$$= \sum_{\eta \in \Delta_0} \sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}}(u(\xi)^* \otimes 1)(\rho_\xi \otimes \rho_\xi^{op})(U^* V_\eta)T^\zeta_{\xi,\eta}$$
$$= \sum_{\eta \in \Delta_0} \sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}}(u(\xi)^* \otimes 1)(\rho_\xi \cdot \sigma \otimes \rho_\xi^{op})(V_\eta)(\rho_\xi(u(\eta)^*) \otimes 1)T^\zeta_{\xi,\eta}$$
$$= \sum_{\eta \in \Delta_0} \sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}}(\sigma \cdot \rho_\xi \otimes \rho_\xi^{op})(V_\eta)(u(\xi)^* \rho_\xi(u(\eta)^*) \otimes 1)T^\zeta_{\xi,\eta},$$

where we use $u(\xi) \in (\sigma \cdot \rho_\xi, \rho_\xi \cdot \sigma)$. Thus, we get

$$T^\zeta_{\xi,\eta}(u(\zeta) \otimes 1) = (\rho_\xi(u(\eta))u(\xi) \otimes 1)(\sigma \otimes id^{op})(T^\zeta_{\xi,\eta}).$$

Multiplying by $1 \otimes j(T^\zeta_{\xi,\eta})^*_i)$ from left, we obtain

$$T^\zeta_{\xi,\eta})_i u(\zeta) = \rho_\xi(u(\eta))u(\xi)\sigma(T^\zeta_{\xi,\eta})_i),$$

which shows that $\{u(\xi)\}_{\xi \in \Delta_0}$ is a half braiding. Q.E.D.

The following is one of the main results in this paper:

Theorem 4.6 *Let $A \supset B$ be the Longo-Rehren inclusion for $\Delta = \{\rho_\xi\}_{\xi \in \Delta_0}$.*

(i) *For every $B-B$ sector ρ associated with $A \supset B$, there exist $\sigma \in End(M)_0$ that is a finite direct sum of sectors in Δ and a half braiding $\{\mathcal{E}_\sigma(\xi)\}_{\xi \in \Delta_0}$ of σ with respect to Δ such that*

$$[\rho] = [(\widetilde{\sigma, \mathcal{E}_\sigma})].$$

(ii) *Let σ, μ be finite direct sums of sectors in Δ and $\{\mathcal{E}^\alpha_\sigma(\xi)\}_{\xi \in \Delta_0}$, $\{\mathcal{E}^\beta_\mu(\xi)\}_{\xi \in \Delta_0}$ half braidings of σ and μ respectively. Then,*

$$(\tilde{\sigma}^\alpha, \tilde{\mu}^\beta) = \{\gamma(X \otimes 1)\,; X \in (\sigma, \mu), \quad \mathcal{E}^\beta_\mu(\xi)X = \rho_\xi(X)\mathcal{E}^\alpha_\sigma(\xi),\ \forall \xi \in \Delta_0\}.$$

In particular, when $\tilde{\sigma}^\alpha$ and $\tilde{\mu}^\beta$ are irreducible, $[\tilde{\sigma}^\alpha] = [\tilde{\mu}^\beta]$ only if $[\sigma] = [\mu]$. When $\mu = \sigma$, $[\tilde{\sigma}^\alpha] = [\tilde{\sigma}^\beta]$ if and only if $\{\mathcal{E}^\alpha_\sigma(\xi)\}_{\xi \in \Delta_0}$ and $\{\mathcal{E}^\beta_\mu(\xi)\}_{\xi \in \Delta_0}$ are equivalent.

(iii) *Let $\tilde{\sigma}^\alpha$, $\tilde{\mu}^\beta$ be as above and set $\mathcal{E}(\tilde{\sigma}^\alpha, \tilde{\mu}^\beta) = \gamma(\mathcal{E}^\alpha_\sigma(\mu) \otimes 1)$. Then, $\{\mathcal{E}(\tilde{\sigma}^\alpha, \tilde{\mu}^\beta)\}$ is a braiding of $D(\Delta)$.*

(iv) *For σ as above, we set*

$$\mathcal{E}^{\overline{\alpha}}_{\overline{\sigma}}(\xi) = d(\sigma)R^*_\sigma\overline{\sigma}(\mathcal{E}^\alpha_\sigma(\xi)^*\rho_\xi(\overline{R}_\sigma)).$$

Then, $\{\mathcal{E}^{\overline{\alpha}}_{\overline{\sigma}}(\xi)\}_{\xi\in\Delta_0}$ is a half braiding of $\overline{\sigma}$ such that $\overline{[\widetilde{\sigma}^\alpha]} = [\widetilde{\overline{\sigma}}^{\overline{\alpha}}]$. When σ is not irreducible, the above definition of $\mathcal{E}^{\overline{\alpha}}_{\overline{\sigma}}(\xi)$ does depend on the choice of R_σ, $\overline{R}_\sigma$; however, different choice of them only amounts to an equivalent half braiding.

Proof. (i) If $[\rho]$ is a $B-B$ sector associated with $A \supset B$, we have $\rho \in D(\Delta)$ thanks to Theorem 4.1. Therefore, this follows from the previous lemma.

(ii) Let $X \in B$. Then, there exists $Y \in A$ such that $X = \gamma(Y)W$. Y is given by $Y = \frac{1}{\sqrt{\lambda}}V^*X$. By the definition of $\widetilde{\sigma}^\alpha$ and $\widetilde{\mu}^\beta$, we can see that X belongs to $(\widetilde{\sigma}^\alpha, \widetilde{\mu}^\beta)$ if and only if X satisfies the following two conditions:

$$X \in (\gamma\cdot(\sigma\otimes id^{op}), \gamma\cdot(\mu\otimes id^{op})),$$

$$X\widetilde{\sigma}^\alpha(W) = \widetilde{\mu}^\beta(W)X.$$

It is easy to show that the first condition is equivalent to

$$Y \in (\gamma\cdot(\sigma\otimes id^{op}), \mu\otimes id^{op}).$$

Therefore, if this is the case, $YV_\xi \in (\rho_\xi\cdot\sigma\otimes\rho^{op}_\xi, \mu\otimes id^{op})$ is non-zero only if $\xi = e$. Since $(\sigma\otimes id^{op}, \mu\otimes id^{op}) = (\sigma,\mu)\otimes\mathbf{C}$, there exists $Z \in (\sigma,\mu)$ such that $Y = (Z\otimes 1)V^*$, and so $X = \gamma((Z\otimes 1)V^*)W = \sqrt{\lambda}\gamma(Z\otimes 1)$. For such X, the second condition is equivalent to

$$\gamma((Z\otimes 1)U^\alpha(\sigma)^*)W = \gamma(U^\beta(\mu)^*)W\gamma(Z\otimes 1) = \gamma(U^\beta(\mu)^*\gamma(Z\otimes 1))W,$$

and so equivalent to

$$\gamma(Z\otimes 1)U^\alpha(\sigma) = U^\beta(\mu)(Z\otimes 1).$$

It is routine to show that this is equivalent to

$$\rho_\xi(Z)\mathcal{E}^\alpha_\sigma(\xi) = \mathcal{E}^\beta_\mu(\xi)Z$$

for all $\xi \in \Delta_0$.

(iii) Let $\mathcal{E}^{\alpha\beta}_{\sigma\cdot\mu}(\xi) = \mathcal{E}^\alpha_\sigma(\xi)\sigma(\mathcal{E}^\beta_\mu(\xi))$. Then, by direct computation we can show that $\{\mathcal{E}^{\alpha\beta}_{\sigma\cdot\mu}(\xi)\}_{\xi\in\Delta_0}$ is a half braiding of $\sigma\cdot\mu$ and

$$\widetilde{\sigma}^\alpha\cdot\widetilde{\mu}^\beta = \widetilde{(\sigma\cdot\mu)}^{\alpha\beta}.$$

Thanks to (ii), we have

$$\begin{aligned}&(\widetilde{\sigma}^\alpha\cdot\widetilde{\mu}^\beta, \widetilde{\mu}^\beta\cdot\widetilde{\sigma}^\alpha)\\ &= \{\gamma(X\otimes 1);\ X\in(\sigma\cdot\mu,\mu\cdot\sigma),\quad \mathcal{E}^\beta_\mu(\xi)\mu(\mathcal{E}^\alpha_\sigma(\xi))X = \rho_\xi(X)\mathcal{E}^\alpha_\sigma(\xi)\sigma(\mathcal{E}^\beta_\mu(\xi)),\ \forall\xi\in\Delta_0\}.\end{aligned}$$

Therefore, to show that $\gamma(\mathcal{E}^\alpha_\sigma(\mu)\otimes 1) \in (\widetilde{\sigma}^\alpha\cdot\widetilde{\mu}^\beta, \widetilde{\mu}^\beta\cdot\widetilde{\sigma}^\alpha)$, it suffices to show

$$\mathcal{E}^\beta_\mu(\xi)\mu(\mathcal{E}^\alpha_\sigma(\xi))\mathcal{E}^\alpha_\sigma(\mu) = \rho_\xi(\mathcal{E}^\alpha_\sigma(\mu))\mathcal{E}^\alpha_\sigma(\xi)\sigma(\mathcal{E}^\beta_\mu(\xi)).$$

This can be shown by the braiding fusion equation of the half braiding of σ. A similar argument using

$$\begin{aligned}&(\tilde{\nu}^{\delta}, \tilde{\sigma}^{\alpha}\cdot\tilde{\mu}^{\beta})\\ &\quad = \{\gamma(X\otimes 1);\ X\in(\nu,\sigma\cdot\mu),\quad \mathcal{E}^{\alpha}_{\sigma}(\xi)\sigma(\mathcal{E}^{\beta}_{\mu}(\xi))X=\rho_{\xi}(X)\mathcal{E}^{\delta}_{\nu}(\xi),\ \forall\xi\in\Delta_0\},\end{aligned}$$

shows the braiding fusion equation of $\{\mathcal{E}(\tilde{\sigma}^{\alpha},\tilde{\mu}^{\beta})\}$. (iv) The proof of Lemma 4.4, (i) shows that there exists a half braiding $\{\mathcal{E}^{\overline{\alpha}}_{\overline{\sigma}}(\xi)\}_{\xi\in\Delta_0}$ such that $\overline{[\tilde{\sigma}^{\alpha}]}=[\tilde{\overline{\sigma}}^{\overline{\alpha}}]$. Let

$$R\in(id_B,\tilde{\overline{\sigma}}^{\overline{\alpha}}\cdot\tilde{\sigma}^{\alpha}),\quad \overline{R}\in(id_B,\tilde{\sigma}^{\alpha}\cdot\tilde{\overline{\sigma}}^{\overline{\alpha}})$$

be isometries satisfying

$$R^{*}\tilde{\overline{\sigma}}^{\overline{\alpha}}(\overline{R})=\overline{R}^{*}\tilde{\sigma}^{\alpha}(R)=\frac{1}{d(\sigma)}.$$

Thanks to (ii), we have

$$\begin{aligned}&(id_B,\tilde{\sigma}^{\alpha}\cdot\tilde{\overline{\sigma}}^{\overline{\alpha}})\\ &\quad = \{\gamma(X\otimes 1);\ X\in(id,\sigma\cdot\overline{\sigma}),\quad \mathcal{E}^{\alpha}_{\sigma}(\xi)\sigma(\mathcal{E}^{\overline{\alpha}}_{\overline{\sigma}}(\xi))X=\rho_{\xi}(X),\ \forall\xi\in\Delta_0\},\end{aligned}$$

$$\begin{aligned}&(id_B,\tilde{\overline{\sigma}}^{\overline{\alpha}}\cdot\tilde{\sigma}^{\alpha})\\ &\quad = \{\gamma(X\otimes 1);\ X\in(id,\overline{\sigma}\cdot\sigma),\quad \mathcal{E}^{\overline{\alpha}}_{\overline{\sigma}}(\xi)\overline{\sigma}(\mathcal{E}^{\alpha}_{\sigma}(\xi))X=\rho_{\xi}(X),\ \forall\xi\in\Delta_0\}.\end{aligned}$$

Therefore, there exist isometries

$$R_{\sigma}\in(id,\overline{\sigma}\cdot\sigma),\quad \overline{R}_{\sigma}\in(id,\sigma\cdot\overline{\sigma})$$

satisfying

$$\begin{aligned}R^{*}_{\sigma}\overline{\sigma}(\overline{R}_{\sigma})=\overline{R}^{*}_{\sigma}\sigma(R_{\sigma})&=\frac{1}{d(\sigma)},\\ \mathcal{E}^{\alpha}_{\sigma}(\xi)\sigma(\mathcal{E}^{\overline{\alpha}}_{\overline{\sigma}}(\xi))\overline{R}_{\sigma}&=\rho_{\xi}(\overline{R}_{\sigma}),\\ \mathcal{E}^{\overline{\alpha}}_{\overline{\sigma}}(\xi)\overline{\sigma}(\mathcal{E}^{\alpha}_{\sigma}(\xi))R_{\sigma}&=\rho_{\xi}(R_{\sigma}),\end{aligned}$$

such that $R=\gamma(R_{\sigma}\otimes 1)$, $\overline{R}=\gamma(\overline{R}_{\sigma}\otimes 1)$. Thus we get

$$d(\sigma)R^{*}_{\sigma}\overline{\sigma}(\mathcal{E}^{\alpha}_{\sigma}(\xi)^{*}\rho_{\xi}(\overline{R}_{\sigma}))=d(\sigma)R^{*}_{\sigma}\overline{\sigma}(\sigma(\mathcal{E}^{\overline{\alpha}}_{\overline{\sigma}}(\xi))\overline{R}_{\sigma})=d(\sigma)\mathcal{E}^{\overline{\alpha}}_{\overline{\sigma}}(\xi)R^{*}_{\sigma}\overline{\sigma}(\overline{R}_{\sigma})=\mathcal{E}^{\overline{\alpha}}_{\overline{\sigma}}(\xi).$$

In general, the pair R_{σ} and $\overline{R}_{\sigma}$ satisfying (2.1) is not unique. However, if R'_{σ} and $\overline{R}'_{\sigma}$ also satisfies the relation, there exists a unitary $u\in(\overline{\sigma},\overline{\sigma})$ such that $R'_{\sigma}=uR_{\sigma}$, $\overline{R}'_{\sigma}=\sigma(u)\overline{R}_{\sigma}$. This change only amounts to an equivalent half braiding of $\overline{\sigma}$. Q.E.D.

Remark.

(i) The set of $B-B$ sectors associated with $A\supset B$ may be smaller than $D(\Delta)$ in general. A typical example is the one discussed at the end of Section 3. In this case, since $[\gamma]=\bigoplus_{g}[\alpha_g\otimes\alpha^{op}_g]$, B is the fixed point subalgebra of A under a G action $\{\alpha_g\otimes\alpha^{op}_g\}_{g\in G}$ with an appropriate unitary perturbation [13]. (Note that even if

the class $[\omega]$ is not trivial, the obstruction of $\{\alpha_g \otimes \alpha_g^{op}\}$ disappears.) Thus the category of $B-B$ sectors associated with the inclusion is isomorphic to the dual of G while $D(\Delta)$ is isomorphic to the (twisted) quantum double of G. Therefore, strictly speaking the right object we should deal with is $D(\Delta)$ rather than the $B-B$ sectors contained in some powers of the restriction $\gamma|_B$ of the canonical endomorphism to B.

(ii) If there exists a braiding $\{\mathcal{E}(\xi,\eta)\}$ such that $\mathcal{E}^\alpha_\sigma(\xi) = \mathcal{E}(\sigma,\xi)$, we have $\mathcal{E}^{\bar\alpha}_{\bar\sigma}(\xi) = \mathcal{E}(\bar\sigma,\xi)$. However, in general even when σ is self-conjugate, $\mathcal{E}^\alpha_\sigma$ and $\mathcal{E}^{\bar\alpha}_\sigma$ are not necessarily equivalent and $\tilde\sigma^\alpha$ may not be self-conjugate.

Now, we investigate the relationship between Tube Δ and $B-B$ sectors. Let $[\sigma] = \bigoplus_\xi n_\xi[\rho_\xi]$ and $\{\mathcal{E}^\alpha_\sigma(\xi)\}_{\xi\in\Delta_0}$ be a half braiding of σ. We assume that $\tilde\sigma^\alpha$ is irreducible. We fix an orthonormal basis $\{W_\sigma(\xi)_i\}_{i=1}^{n_\xi} \subset (\rho_\xi,\sigma)$ and set

$$\mathcal{E}^\alpha_\sigma(\xi)_{(\eta,i),(\zeta,j)} = \rho_\xi(W_\sigma(\zeta)^*_j)\mathcal{E}^\alpha_\sigma(\xi)W_\sigma(\eta)_i \in (\rho_\eta\cdot\rho_\xi, \rho_\xi\cdot\rho_\zeta).$$

Then, we have

$$\mathcal{E}^\alpha_\sigma(\xi) = \sum_{\eta,\zeta}\sum_{i=1}^{n_\eta}\sum_{j=1}^{n_\zeta} \rho_\xi(W_\sigma(\zeta)_j)\mathcal{E}^\alpha_\sigma(\xi)_{(\eta,i),(\zeta,j)}W_\sigma(\eta)^*_i.$$

In [8], Evans and Kawahigashi constructed central elements of Tube Δ from braidings. Generalizing their argument, we define

$$e(\tilde\sigma^\alpha)_{(\eta,i),(\zeta,j)} := \frac{d(\sigma)}{\lambda\sqrt{d(\eta)d(\zeta)}}\sum_\xi d(\xi)(\eta\xi|\mathcal{E}^\alpha_\sigma(\xi)_{(\eta,i),(\zeta,j)}|\xi\zeta) \in \text{Tube }\Delta.$$

Lemma 4.7 *Let $e(\tilde\sigma^\alpha)_{(\eta,i),(\zeta,j)}$ be as above and $X \in (\rho_\mu\cdot\rho_\tau, \rho_\tau\cdot\rho_\nu)$. Then, the following holds:*

(i) $e(\tilde\sigma^\alpha)_{(\eta,i),(\zeta,j)}{}^* = e(\tilde\sigma^\alpha)_{(\zeta,j),(\eta,i)}$.

(ii)

$$e(\tilde\sigma^\alpha)_{(\eta,i),(\zeta,j)}(\mu\tau|X|\tau\nu) = \delta_{\zeta,\mu}\frac{d(\tau)\sqrt{d(\nu)}}{\sqrt{d(\zeta)}}\sum_k \phi_\tau(X\,\mathcal{E}^\alpha_\sigma(\tau)_{(\zeta,j),(\nu,k)}{}^*)e(\tilde\sigma^\alpha)_{(\eta,i),(\nu,k)}.$$

(iii)

$$(\mu\tau|X|\tau\nu)e(\tilde\sigma^\alpha)_{(\eta,i),(\zeta,j)} = \delta_{\eta,\nu}\frac{d(\tau)\sqrt{d(\mu)}}{\sqrt{d(\eta)}}\sum_k \phi_\mu(\mathcal{E}^\alpha_\sigma(\tau)_{(\mu,k),(\eta,i)}{}^*X)e(\tilde\sigma^\alpha)_{(\mu,k),(\zeta,j)}.$$

Proof. (i) Applying the definition of the $*$-operation of Tube Δ, we have

$$\begin{aligned} e(\tilde\sigma^\alpha)_{(\eta,i),(\zeta,j)}{}^* &= \frac{d(\sigma)}{\lambda\sqrt{d(\eta)d(\zeta)}}\sum_\xi d(\xi)^2(\zeta\bar\xi|\rho_{\bar\xi}(\rho_\eta(\overline{R}^*_\xi)W_\sigma(\eta)^*_i\mathcal{E}^\alpha_\sigma(\xi)^*\rho_\xi(W_\sigma(\zeta)_j))R_\xi|\bar\xi\eta) \\ &= \frac{d(\sigma)}{\lambda\sqrt{d(\eta)d(\zeta)}}\sum_\xi d(\xi)^2(\zeta\bar\xi|\rho_{\bar\xi}(W_\sigma(\eta)^*_i\sigma(\overline{R}^*_\xi)\mathcal{E}^\alpha_\sigma(\xi)^*)R_\xi W_\sigma(\zeta)_j|\bar\xi\eta). \end{aligned}$$

Using the braiding fusion equation, we get $\mathcal{E}^\alpha_\sigma(\xi)\sigma(\overline{R}_\xi) = \rho_\xi(\mathcal{E}^\alpha_\sigma(\overline{\xi})^*)\overline{R}_\xi$, and so

$$\begin{aligned}\rho_{\overline{\xi}}(W_\sigma(\eta)_i^*\sigma(\overline{R}_\xi^*)\mathcal{E}^\alpha_\sigma(\xi)^*)R_\xi W_\sigma(\zeta)_j &= \rho_{\overline{\xi}}(W_\sigma(\eta)_i^*\overline{R}_\xi^*\rho_\xi(\mathcal{E}^\alpha_\sigma(\overline{\xi})))R_\xi W_\sigma(\zeta)_j\\ &= \rho_{\overline{\xi}}(W_\sigma(\eta)_i^*\overline{R}_\xi^*)R_\xi\mathcal{E}^\alpha_\sigma(\overline{\xi})W_\sigma(\zeta)_j\\ &= \frac{1}{d(\xi)}\rho_{\overline{\xi}}(W_\sigma(\eta)_i^*)\mathcal{E}^\alpha_\sigma(\overline{\xi})W_\sigma(\zeta)_j.\end{aligned}$$

Therefore, we obtain the result.

(ii) By direct computation, we get

$$\begin{aligned}&e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\zeta,j)}(\mu\tau|X|\tau\nu)\\ &= \frac{d(\sigma)}{\lambda\sqrt{d(\eta)d(\zeta)}}\sum_\xi d(\xi)(\eta\xi|\rho_\xi(W_\sigma(\zeta)_j^*)\mathcal{E}^\alpha_\sigma(\xi)W_\sigma(\eta)_i|\xi\zeta)(\mu\tau|X|\tau\nu)\\ &= \frac{d(\sigma)}{\lambda\sqrt{d(\eta)d(\zeta)}}\delta_{\mu,\zeta}\sum_\xi\sum_{\theta\prec\xi\tau}\sum_{k=1}^{N^\theta_{\xi,\tau}} d(\xi)(\eta\theta|T^\theta_{\xi,\tau})^*_k\rho_\xi(XW_\sigma(\zeta)_j^*)\mathcal{E}^\alpha_\sigma(\xi)W_\sigma(\eta)_i\rho_\eta(T^\theta_{\xi,\tau})_k)|\theta\nu)\\ &= \frac{d(\sigma)}{\lambda\sqrt{d(\eta)d(\zeta)}}\delta_{\mu,\zeta}\sum_\theta\sum_{\xi\prec\theta\overline{\tau}}\sum_{k=1}^{N^\theta_{\xi,\tau}} d(\xi)(\eta\theta|T^\theta_{\xi,\tau})^*_k\rho_\xi(XW_\sigma(\zeta)_j^*)\mathcal{E}^\alpha_\sigma(\xi)\sigma(T^\theta_{\xi,\tau})_k)W_\sigma(\eta)_i|\theta\nu),\end{aligned}$$

where we use the Frobenius reciprocity. Using the braiding fusion equation, we get

$$\begin{aligned}&\sum_{\xi\prec\theta\overline{\tau}}\sum_{k=1}^{N^\theta_{\xi,\tau}} d(\xi)T^\theta_{\xi,\tau})^*_k\rho_\xi(XW_\sigma(\zeta)_j^*)\mathcal{E}^\alpha_\sigma(\xi)\sigma(T^\theta_{\xi,\tau})_k)W_\sigma(\eta)_i\\ &= \sum_{\xi\prec\theta\overline{\tau}}\sum_{k=1}^{N^\theta_{\xi,\tau}} d(\xi)T^\theta_{\xi,\tau})^*_k\rho_\xi(XW_\sigma(\zeta)_j^*\mathcal{E}^\alpha_\sigma(\tau)^*)T^\theta_{\xi,\tau})_k\mathcal{E}^\alpha_\sigma(\theta)W_\sigma(\eta)_i.\end{aligned}$$

Thus we concentrate on computation of the quantity

$$\sum_{\xi\prec\theta\overline{\tau}}\sum_{k=1}^{N^\theta_{\xi,\tau}} d(\xi)T^\theta_{\xi,\tau})^*_k\rho_\xi(XW_\sigma(\zeta)_j^*\mathcal{E}^\alpha_\sigma(\tau)^*)T^\theta_{\xi,\tau})_k.$$

Set

$$T'^\theta_{\xi,\tau})_k = \sqrt{\frac{d(\theta)d(\tau)}{d(\xi)}}T^\xi_{\theta,\overline{\tau}})^*_k\rho_\theta(R_\tau).$$

Then by the Frobenius reciprocity, $\{T'^\theta_{\xi,\tau})_k\}$ is an orthonormal basis of $(\rho_\theta, \rho_\xi\cdot\rho_\tau)$ too. Therefore, we can replace $T^\theta_{\xi,\tau})_k$ with $T'^\theta_{\xi,\tau})_k$ in the above, and get

$$\begin{aligned}&d(\theta)d(\tau)\sum_{\xi\prec\theta\overline{\tau}}\sum_{k=1}^{N^\theta_{\xi,\tau}}\rho_\theta(R_\tau^*)T^\xi_{\theta,\overline{\tau}})_k\rho_\xi(XW_\sigma(\zeta)_j^*\mathcal{E}^\alpha_\sigma(\tau)^*)T^\xi_{\theta,\overline{\tau}})^*_k\rho_\theta(R_\tau)\\ &= d(\theta)d(\tau)\rho_\theta(R_\tau^*\rho_{\overline{\tau}}(XW_\sigma(\zeta)_j^*\mathcal{E}^\alpha_\sigma(\tau)^*)R_\tau)\\ &= d(\theta)d(\tau)\rho_\theta(\phi_\tau(XW_\sigma(\zeta)_j^*\mathcal{E}^\alpha_\sigma(\tau)^*))\end{aligned}$$

$$= d(\theta)d(\tau)\sum_{\omega}\sum_{k=1}^{n_\omega}\rho_\theta(\phi_\tau(X\mathcal{E}^\alpha_\sigma(\tau)_{(\zeta,j),(\omega,k)}{}^*\rho_\tau(W_\sigma(\omega)^*_k)))$$
$$= d(\theta)d(\tau)\sum_{\omega}\sum_{k=1}^{n_\omega}\rho_\theta(\phi_\tau(X\mathcal{E}^\alpha_\sigma(\tau)_{(\zeta,j),(\omega,k)}{}^*)W_\sigma(\omega)^*_k).$$

Since $\phi_\tau(X\mathcal{E}^\alpha_\sigma(\tau)_{(\zeta,j),(\omega,k)}{}^*) \in (\rho_\omega, \rho_\nu)$, this term survives only when $\omega = \nu$, and so we get

$$d(\theta)d(\tau)\sum_{k=1}^{n_\nu}\phi_\tau(X\mathcal{E}^\alpha_\sigma(\tau)_{(\zeta,j),(\nu,k)}{}^*)\rho_\theta(W_\sigma(\nu)^*_k).$$

Therefore, we obtain the result.

(iii) By direct computation, we get

$$(\mu\tau|X|\tau\nu)e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\zeta,j)}$$
$$= \frac{d(\sigma)}{\lambda\sqrt{d(\eta)d(\zeta)}}\delta_{\nu,\eta}\sum_{\xi}d(\xi)(\mu\tau|X|\tau\nu)(\eta\xi|\rho_\xi(W_\sigma(\zeta)^*_j)\mathcal{E}^\alpha_\sigma(\xi)W_\sigma(\eta)_i|\xi\zeta)$$
$$= \frac{d(\sigma)}{\lambda\sqrt{d(\eta)d(\zeta)}}\delta_{\nu,\eta}\sum_{\theta\prec\tau\xi,\xi}\sum_{k=1}^{N^\theta_{\tau,\xi}}d(\xi)(\mu\theta|T(^\theta_{\tau,\xi})^*_k\rho_\tau(\rho_\xi(W_\sigma(\zeta)^*_j)\mathcal{E}^\alpha_\sigma(\xi)W_\sigma(\eta)_i)X\rho_\mu(T(^\theta_{\tau,\xi})_k)|\theta\zeta)$$
$$= \frac{d(\sigma)}{\lambda\sqrt{d(\eta)d(\zeta)}}\delta_{\nu,\eta}\sum_{\theta}\sum_{\xi\prec\overline{\tau}\theta}\sum_{k=1}^{N^\theta_{\tau,\xi}}d(\xi)(\mu\theta|\rho_\theta(W_\sigma(\zeta)^*_j)T(^\theta_{\tau,\xi})^*_k\rho_\tau(\mathcal{E}^\alpha_\sigma(\xi)W_\sigma(\eta)_i)X\rho_\mu(T(^\theta_{\tau,\xi})_k)|\theta\zeta)$$

Using the braiding fusion equation, we get

$$\sum_{\xi\prec\overline{\tau}\theta}\sum_{k=1}^{N^\theta_{\tau,\xi}}d(\xi)\rho_\theta(W_\sigma(\zeta)^*_j)T(^\theta_{\tau,\xi})^*_k\rho_\tau(\mathcal{E}^\alpha_\sigma(\xi)W_\sigma(\eta)_i)X\rho_\mu(T(^\theta_{\tau,\xi})_k)$$
$$= \sum_{\xi\prec\overline{\tau}\theta}\sum_{k=1}^{N^\theta_{\tau,\xi}}d(\xi)\rho_\theta(W_\sigma(\zeta)^*_j)\mathcal{E}^\alpha_\sigma(\theta)\sigma(T(^\theta_{\tau,\xi})^*_k)\mathcal{E}^\alpha_\sigma(\tau)^*\rho_\tau(W_\sigma(\eta)_i)X\rho_\mu(T(^\theta_{\tau,\xi})_k).$$

Thus, we concentrate on computation of the quantity

$$\sum_{\xi\prec\overline{\tau}\theta}\sum_{k=1}^{N^\theta_{\tau,\xi}}d(\xi)\sigma(T(^\theta_{\tau,\xi})^*_k)\mathcal{E}^\alpha_\sigma(\tau)^*\rho_\tau(W_\sigma(\eta)_i)X\rho_\mu(T(^\theta_{\tau,\xi})_k).$$

Set

$$T'(^\theta_{\tau,\xi})_k = \sqrt{\frac{d(\theta)d(\tau)}{d(\xi)}}\rho_\tau(T(^\xi_{\overline{\tau}\theta})^*_k)\overline{R}_\tau.$$

Then, thanks to the Frobenius reciprocity, $\{T'(^\theta_{\tau,\xi})_k\}$ is an orthonormal basis of $(\rho_\theta, \rho_\tau\cdot\rho_\xi)$. Therefore, we can replace $T(^\theta_{\tau,\xi})_k$ with $T'(^\theta_{\tau,\xi})_k$ in the above, and so we get

$$\sum_{\xi\prec\overline{\tau}\theta}\sum_{k=1}^{N^\theta_{\tau,\xi}}d(\xi)\sigma(T(^\theta_{\tau,\xi})^*_k)\mathcal{E}^\alpha_\sigma(\tau)^*\rho_\tau(W_\sigma(\eta)_i)X\rho_\mu(T(^\theta_{\tau,\xi})_k)$$

$$
\begin{aligned}
&= d(\theta)d(\tau)\sum_{\xi\prec\overline{\tau}\theta}\sum_{k=1}^{N^{\theta}_{\tau,\xi}}\sigma(\overline{R}^{*}_{\tau}\rho_{\tau}(T(\tfrac{\xi}{\overline{\tau}\theta})_k))\mathcal{E}^{\alpha}_{\sigma}(\tau)^{*}\rho_{\tau}(W_{\sigma}(\eta)_i)X\rho_{\mu}(\rho_{\tau}(T(\tfrac{\xi}{\overline{\tau}\theta})^{*}_k)\overline{R}_{\tau})\\
&= d(\theta)d(\tau)\sum_{\xi\prec\overline{\tau}\theta}\sum_{k=1}^{N^{\theta}_{\tau,\xi}}\sigma(\overline{R}^{*}_{\tau})\mathcal{E}^{\alpha}_{\sigma}(\tau)^{*}\rho_{\tau}(\sigma(T(\tfrac{\xi}{\overline{\tau}\theta})_k)W_{\sigma}(\eta)_i\rho_{\eta}(T(\tfrac{\xi}{\overline{\tau}\theta})^{*}_k))X\rho_{\mu}(\overline{R}_{\tau})\\
&= d(\theta)d(\tau)\sum_{\xi\prec\overline{\tau}\theta}\sum_{k=1}^{N^{\theta}_{\tau,\xi}}\sigma(\overline{R}^{*}_{\tau})\mathcal{E}^{\alpha}_{\sigma}(\tau)^{*}\rho_{\tau}(W_{\sigma}(\eta)_i\rho_{\eta}(T(\tfrac{\xi}{\overline{\tau}\theta})_k)\rho_{\eta}(T(\tfrac{\xi}{\overline{\tau}\theta})^{*}_k))X\rho_{\mu}(\overline{R}_{\tau})\\
&= d(\theta)d(\tau)\sigma(\overline{R}^{*}_{\tau})\mathcal{E}^{\alpha}_{\sigma}(\tau)^{*}\rho_{\tau}(W_{\sigma}(\eta)_i)X\rho_{\mu}(\overline{R}_{\tau})\\
&= d(\theta)d(\tau)\sum_{\omega}\sum_{k=1}^{n_{\omega}}\sigma(\overline{R}^{*}_{\tau})W_{\sigma}(\omega)_k\mathcal{E}^{\alpha}_{\sigma}(\tau)_{(\omega,k),(\eta,i)}{}^{*}X\rho_{\mu}(\overline{R}_{\tau})\\
&= d(\theta)d(\tau)\sum_{\omega}\sum_{k=1}^{n_{\omega}}W_{\sigma}(\omega)_k\rho_{\omega}(\overline{R}^{*}_{\tau})\mathcal{E}^{\alpha}_{\sigma}(\tau)_{(\omega,k),(\eta,i)}{}^{*}X\rho_{\mu}(\overline{R}_{\tau})
\end{aligned}
$$

Note that $\rho_{\omega}(\overline{R}^{*}_{\tau})\mathcal{E}^{\alpha}_{\sigma}(\tau)_{(\omega,k),(\eta,i)}{}^{*}X\rho_{\mu}(\overline{R}_{\tau})\in(\rho_{\mu},\rho_{\omega})$ and this survives only if $\omega=\mu$. Thus, we get

$$
d(\theta)d(\tau)\sum_{k=1}^{n_{\mu}}W_{\sigma}(\mu)_k\rho_{\mu}(\overline{R}^{*}_{\tau})\mathcal{E}^{\alpha}_{\sigma}(\tau)_{(\mu,k),(\eta,i)}{}^{*}X\rho_{\mu}(\overline{R}_{\tau}).
$$

Since $\rho_{\mu}(\overline{R}^{*}_{\tau})\mathcal{E}^{\alpha}_{\sigma}(\tau)_{(\mu,k),(\eta,i)}{}^{*}X\rho_{\mu}(\overline{R}_{\tau})$ is already a scalar, to evaluate it we can apply ϕ_{μ} and get

$$
\begin{aligned}
&d(\theta)d(\tau)\sum_{k=1}^{n_{\mu}}W_{\sigma}(\mu)_k\rho_{\mu}(\overline{R}^{*}_{\tau})\mathcal{E}^{\alpha}_{\sigma}(\tau)_{(\mu,k),(\eta,i)}{}^{*}X\rho_{\mu}(\overline{R}_{\tau})\\
&\quad= d(\theta)d(\tau)\sum_{k=1}^{n_{\mu}}W_{\sigma}(\mu)_k\overline{R}^{*}_{\tau}\phi_{\mu}(\mathcal{E}^{\alpha}_{\sigma}(\tau)_{(\mu,k),(\eta,i)}{}^{*}X)\overline{R}_{\tau}\\
&\quad= d(\theta)d(\tau)\sum_{k=1}^{n_{\mu}}\phi_{\mu}(\mathcal{E}^{\alpha}_{\sigma}(\tau)_{(\mu,k),(\eta,i)}{}^{*}X)W_{\sigma}(\mu)_k
\end{aligned}
$$

This proves the statement. Q.E.D.

Remark. To show the above lemma, irreducibility of $\tilde{\sigma}^{\alpha}$ is not necessary.

Corollary 4.8 *Let $\tilde{\sigma}^{\alpha}\in D(\Delta)$ be an irreducible sector as above. Then, $z(\tilde{\sigma}^{\alpha}):=\sum_{\eta,i}e(\tilde{\sigma}^{\alpha})_{(\eta,i),(\eta,i)}$ belongs to the center of* Tube Δ.

Proof. By Lemma 4.6, we have

$$
z(\tilde{\sigma}^{\alpha})(\mu\tau|X|\tau\nu)=\frac{d(\tau)\sqrt{d(\nu)}}{\sqrt{d(\mu)}}\sum_{i,j}\phi_{\tau}(X\mathcal{E}^{\alpha}_{\sigma}(\tau)^{*}_{(\mu,i),(\nu,j)})e(\tilde{\sigma}^{\alpha})_{(\mu,i),(\nu,j)},
$$

$$
(\mu\tau|X|\tau\nu)z(\tilde{\sigma}^{\alpha})=\frac{d(\tau)\sqrt{d(\mu)}}{\sqrt{d(\nu)}}\sum_{i,j}\phi_{\tau}(\mathcal{E}^{\alpha}_{\sigma}(\tau)^{*}_{(\mu,i),(\nu,j)}X)e(\tilde{\sigma}^{\alpha})_{(\mu,i),(\nu,j)}.
$$

Thus, the result follows from Lemma 3.1. Q.E.D.

To obtain another main result in this section, we need one more technical lemma.

Lemma 4.9 *With the same notation as above, the following hold:*

(i) *Let $D(\Delta)_0$ be a complete system of representatives of the equivalence classes of the irreducibles in $D(\Delta)$. Then,*

$$\sum_{\widetilde{\sigma}^\alpha \in D(\Delta)_0} d(\widetilde{\sigma}^\alpha)^2 = \lambda^2.$$

(ii) $\varphi_\Delta(z(\widetilde{\sigma}^\alpha)) = \frac{d(\sigma)^2}{\lambda}$.

(iii) $\varphi_\Delta(1) = \lambda$.

Proof. (i) We note that if ρ is an irreducible $B-B$ sector such that $\iota\cdot\rho$ contains $(\rho_\xi \otimes id^{op})\cdot\iota$ for some ξ, then ρ belongs to $D(\Delta)$. Indeed, thanks to the Frobenius reciprocity ρ is contained in $\bar{\iota}\cdot(\rho_\xi\otimes id^{op})\cdot\iota \in D(\Delta)$, and so $\rho\in D(\Delta)$. For $\xi\in\Delta_0$ and $\theta\in D(\Delta)_0$, we denote by $\Gamma_{\xi,\theta}$ the multiplicity of $(\rho_\xi\otimes id^{op})\cdot\iota$ in $\iota\cdot\theta$. We set $v_\xi = d(\xi)$, $w_\theta = d(\theta)$, $\vec{v} = (v_\xi)$, $\vec{w} = (w_\theta)$. Then, we have

$$\Gamma\vec{v} = \lambda\vec{w}, \quad {}^t\Gamma\vec{w} = \vec{v}.$$

Thus, we get

$$\sum_{\theta\in D(\Delta)_0} d(\theta)^2 = \langle\vec{v},\vec{v}\rangle = \langle\vec{v},{}^t\Gamma\vec{w}\rangle = \langle\Gamma\vec{v},\vec{w}\rangle = \lambda\langle\vec{w},\vec{w}\rangle = \lambda^2.$$

(ii) and (iii) can be easily obtained from the definitions of $z(\widetilde{\sigma}^\alpha)$ and φ_Δ, and the fact

$$1 = \sum_\xi (\xi e|1|e\xi).$$

Q.E.D.

Theorem 4.10 *Let $e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\zeta,j)}$ be as above. Then,*

(i) $\{e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\zeta,j)}\}_{(\eta,i),(\zeta,j)}$ *is a system of matrix units of a simple component of* Tube Δ.

(ii) $\{z(\widetilde{\sigma}^\alpha)\}_{\widetilde{\sigma}^\alpha\in D(\Delta)_0}$ *are mutually orthogonal minimal central projections satisfying*

$$\sum_{\widetilde{\sigma}^\alpha\in D(\Delta)_0} z(\widetilde{\sigma}^\alpha) = 1.$$

Proof. (i) Using (ii) and (iii) of Lemma 4.6, we can compute $e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\zeta,j)}e(\widetilde{\sigma}^\alpha)_{(\eta',i'),(\zeta',j')}$ in two ways. First, using (i) we get

$$\begin{aligned}
&e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\zeta,j)}e(\widetilde{\sigma}^\alpha)_{(\eta',i'),(\zeta',j')}\\
&= \frac{d(\sigma)}{\lambda\sqrt{d(\eta')d(\zeta')}}\sum_\xi d(\xi)e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\zeta,j)}(\eta'\xi|\rho_\xi(W_\sigma(\zeta')^*_{j'})\mathcal{E}^\alpha_\sigma(\xi)W_\sigma(\eta')_{i'}|\xi\zeta')\\
&= \frac{d(\sigma)}{\lambda d(\zeta)}\delta_{\zeta,\eta'}\sum_\xi\sum_{k=1}^{n_{\zeta'}} d(\xi)^2\phi_\xi(\rho_\xi(W_\sigma(\zeta')^*_{j'})\mathcal{E}^\alpha_\sigma(\xi)W_\sigma(\zeta)_{i'}\mathcal{E}^\alpha_\sigma(\xi)_{(\zeta,j),(\zeta',k)}{}^*)e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\zeta',k)}\\
&= \frac{d(\sigma)}{\lambda d(\zeta)}\delta_{\zeta,\eta'}\sum_{k=1}^{n_{\zeta'}} W_\sigma(\zeta')^*_{j'}\left(\sum_\xi d(\xi)^2\phi_\xi(\mathcal{E}^\alpha_\sigma(\xi)W_\sigma(\zeta)_{i'}W_\sigma(\zeta)^*_j\mathcal{E}^\alpha_\sigma(\xi)^*)\right)W_\sigma(\zeta')_k e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\zeta',k)}
\end{aligned}$$

Secondly, using (ii) we get

$$\begin{aligned}
&e(\widetilde{\sigma}^{\alpha})_{(\eta,i),(\zeta,j)}e(\widetilde{\sigma}^{\alpha})_{(\eta',i'),(\zeta',j')}\\
&= \frac{d(\sigma)}{\lambda\sqrt{d(\eta)d(\zeta)}}\sum_{\xi} d(\xi)(\eta\xi|\rho_{\xi}(W_{\sigma}(\zeta)_j^*)\mathcal{E}_{\sigma}^{\alpha}(\xi)W_{\sigma}(\eta)_i|\xi\zeta)e(\widetilde{\sigma}^{\alpha})_{(\eta',i'),(\zeta',j')}\\
&= \frac{d(\sigma)}{\lambda d(\zeta)}\delta_{\zeta,\eta'}\sum_{\xi}\sum_{k=1}^{n_\eta} d(\xi)^2\phi_{\eta}(\mathcal{E}_{\sigma}^{\alpha}(\xi)_{(\eta,k),(\zeta,i')}{}^*\mathcal{E}_{\sigma}^{\alpha}(\xi)_{(\eta,i),(\zeta,j)})e(\widetilde{\sigma}^{\alpha})_{(\eta,k),(\zeta',j')}
\end{aligned}$$

Thanks to Lemma 3.1, this is equal to

$$\begin{aligned}
&\frac{d(\sigma)}{\lambda d(\eta)}\delta_{\zeta,\eta'}\sum_{\xi}\sum_{k=1}^{n_\eta} d(\xi)^2\phi_{\xi}(\mathcal{E}_{\sigma}^{\alpha}(\xi)_{(\eta,i),(\zeta,j)}\mathcal{E}_{\sigma}^{\alpha}(\xi)_{(\eta,k),(\zeta,i')}{}^*)e(\widetilde{\sigma}^{\alpha})_{(\eta,k),(\zeta',j')}\\
&= \frac{d(\sigma)}{\lambda d(\eta)}\delta_{\zeta,\eta'}\sum_{k=1}^{n_\eta} W_{\sigma}(\zeta)_j^*\left(\sum_{\xi} d(\xi)^2\phi_{\xi}(\mathcal{E}_{\sigma}^{\alpha}(\xi)W_{\sigma}(\eta)_iW_{\sigma}(\eta)_k^*\mathcal{E}_{\sigma}^{\alpha}(\xi)^*)\right)W_{\sigma}(\zeta)_{i'}e(\widetilde{\sigma}^{\alpha})_{(\eta,k),(\zeta',j')}.
\end{aligned}$$

We set

$$\langle\langle W_{\sigma}(\zeta)_i, W_{\sigma}(\zeta)_j\rangle\rangle = \sum_{\xi} d(\xi)^2\phi_{\xi}(\mathcal{E}_{\sigma}^{\alpha}(\xi)W_{\sigma}(\eta)_iW_{\sigma}(\eta)_j^*\mathcal{E}_{\sigma}^{\alpha}(\xi)^*) \in (\sigma,\sigma),$$

and show that this is a scalar. To do so, since $\widetilde{\sigma}^{\alpha}$ is irreducible, it suffices to show

$$\mathcal{E}_{\sigma}^{\alpha}(\eta)\langle\langle W_{\sigma}(\zeta)_i, W_{\sigma}(\zeta)_j\rangle\rangle\mathcal{E}_{\sigma}^{\alpha}(\eta)^* = \rho_{\eta}(\langle\langle W_{\sigma}(\zeta)_i, W_{\sigma}(\zeta)_j\rangle\rangle)$$

for all η thanks to Theorem 4.6. The braiding fusion equation gives

$$\rho_{\xi}(\mathcal{E}_{\sigma}^{\alpha}(\eta))\mathcal{E}_{\sigma}^{\alpha}(\xi)\sigma(T(^{\mu}_{\xi,\eta})_p) = T(^{\mu}_{\xi,\eta})_p\mathcal{E}_{\sigma}^{\alpha}(\mu),$$

and so, we have

$$\rho_{\xi}(\mathcal{E}_{\sigma}^{\alpha}(\eta))\mathcal{E}_{\sigma}^{\alpha}(\xi) = \sum_{\mu\prec\xi\eta}\sum_{p=1}^{N_{\xi,\eta}^{\mu}} T(^{\mu}_{\xi,\eta})_p\mathcal{E}_{\sigma}^{\alpha}(\mu)\sigma(T(^{\mu}_{\xi,\eta})_p^*).$$

Thus,

$$\begin{aligned}
&\mathcal{E}_{\sigma}^{\alpha}(\eta)\langle\langle W_{\sigma}(\zeta)_i, W_{\sigma}(\zeta)_j\rangle\rangle\mathcal{E}_{\sigma}^{\alpha}(\eta)^*\\
&= \sum_{\xi} d(\xi)^2\phi_{\xi}(\rho_{\xi}(\mathcal{E}_{\sigma}^{\alpha}(\eta))\mathcal{E}_{\sigma}^{\alpha}(\xi)W_{\sigma}(\zeta)_iW_{\sigma}(\zeta)_j^*\mathcal{E}_{\sigma}^{\alpha}(\xi)^*\rho_{\xi}(\mathcal{E}_{\sigma}^{\alpha}(\eta))^*)\\
&= \sum_{\xi,\mu,\nu}\sum_{p,q} d(\xi)^2\phi_{\xi}(T(^{\mu}_{\xi,\eta})_p\mathcal{E}_{\sigma}^{\alpha}(\mu)\sigma(T(^{\mu}_{\xi,\eta})_p^*)W_{\sigma}(\zeta)_iW_{\sigma}(\zeta)_j^*\sigma(T(^{\nu}_{\xi,\eta})_q)\mathcal{E}_{\sigma}^{\alpha}(\nu)^*T(^{\nu}_{\xi,\eta})_q^*)\\
&= \sum_{\xi,\mu}\sum_{p} d(\xi)^2\phi_{\xi}(T(^{\mu}_{\xi,\eta})_p\mathcal{E}_{\sigma}^{\alpha}(\mu)W_{\sigma}(\zeta)_iW_{\sigma}(\zeta)_j^*\mathcal{E}_{\sigma}^{\alpha}(\mu)^*T(^{\mu}_{\xi,\eta})_p^*)\\
&= \sum_{\xi,\mu}\sum_{p} d(\xi)^2R_{\xi}^*\rho_{\overline{\xi}}(T(^{\mu}_{\xi,\eta})_p\mathcal{E}_{\sigma}^{\alpha}(\mu)W_{\sigma}(\zeta)_iW_{\sigma}(\zeta)_j^*\mathcal{E}_{\sigma}^{\alpha}(\mu)^*T(^{\mu}_{\xi,\eta})_p^*)R_{\xi}\\
&= \sum_{\xi,\mu}\sum_{p} \frac{d(\xi)d(\mu)}{d(\eta)}T(^{\eta}_{\overline{\xi},\mu})_p^*\rho_{\overline{\xi}}(\mathcal{E}_{\sigma}^{\alpha}(\mu)W_{\sigma}(\zeta)_iW_{\sigma}(\zeta)_j^*\mathcal{E}_{\sigma}^{\alpha}(\mu)^*)T(^{\eta}_{\overline{\xi},\mu})_p,
\end{aligned}$$

where we use the Frobenius reciprocity in the last equality. Set

$$T'({}^{\eta}_{\overline{\xi},\mu})_p = \sqrt{\frac{d(\eta)d(\mu)}{d(\xi)}} T({}^{\overline{\xi}}_{\eta,\overline{\mu}})^*_p \rho_\eta(R_\mu).$$

Then, $\{T'({}^{\eta}_{\overline{\xi},\mu})_p\}_p$ is an orthonormal basis of $(\rho_\eta, \rho_{\overline{\xi}} \cdot \rho_\mu)$ too. Thus, we can replace $T({}^{\eta}_{\overline{\xi},\mu})_p$ with $T'({}^{\eta}_{\overline{\xi},\mu})_p$ in the above, and so we get

$$\begin{aligned}
&\sum_{\xi,\mu}\sum_p d(\mu)^2 \rho_\eta(R^*_\mu) T({}^{\overline{\xi}}_{\eta,\overline{\mu}})_p \rho_{\overline{\xi}}(\mathcal{E}^\alpha_\sigma(\mu) W_\sigma(\zeta)_i W_\sigma(\zeta)^*_j \mathcal{E}^\alpha_\sigma(\mu)^*) T({}^{\overline{\xi}}_{\eta,\overline{\mu}})^*_p \rho_\eta(R_\mu) \\
&= \sum_{\xi,\mu}\sum_p d(\mu)^2 \rho_\eta(R^*_\mu)\rho_\eta \cdot \rho_{\overline{\mu}}(\mathcal{E}^\alpha_\sigma(\mu) W_\sigma(\zeta)_i W_\sigma(\zeta)^*_j \mathcal{E}^\alpha_\sigma(\mu)^*) \rho_\eta(R_\mu) \\
&= \sum_{\xi,\mu}\sum_p d(\mu)^2 \rho_\eta(\phi_\mu(\mathcal{E}^\alpha_\sigma(\mu) W_\sigma(\zeta)_i W_\sigma(\zeta)^*_j \mathcal{E}^\alpha_\sigma(\mu)^*)) \\
&= \rho_\mu(\langle\langle W_\sigma(\zeta)_i, W_\sigma(\zeta)_j\rangle\rangle).
\end{aligned}$$

Therefore, $\langle\langle W_\sigma(\zeta)_i, W_\sigma(\zeta)_j\rangle\rangle$ is a scalar and we get

$$\begin{aligned}
&e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\zeta,j)} e(\widetilde{\sigma}^\alpha)_{(\eta',i'),(\zeta',j')} \\
&= \frac{d(\sigma)}{\lambda d(\zeta)} \delta_{\zeta,\eta'} \langle\langle W_\sigma(\zeta)_{i'}, W_\sigma(\zeta)_j\rangle\rangle e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\zeta',j')} \\
&= \frac{d(\sigma)}{\lambda d(\eta)} \delta_{\zeta,\eta'}\delta_{j,i'} \sum_k \langle\langle W_\sigma(\eta)_i, W_\sigma(\eta)_k\rangle\rangle e(\widetilde{\sigma}^\alpha)_{(\eta,k),(\zeta',j')}.
\end{aligned}$$

This shows that there exists a non-negative number c such that

$$\langle\langle W_\sigma(\xi)_i, W_\sigma(\xi)_j\rangle\rangle = d(\xi) c \langle W_\sigma(\xi)_i, W_\sigma(\xi)_j\rangle.$$

To determine c, we compute the following quantity by using the definition of $\langle\langle\cdot,\cdot\rangle\rangle$:

$$\sum_{\eta,i} \langle\langle W_\sigma(\eta)_i, W_\sigma(\eta)_i\rangle\rangle = c \sum_\eta n_\eta d(\eta) = c d(\sigma),$$

Indeed, it is

$$\sum_{\xi,\eta,i} d(\xi)^2 \phi_\xi(\mathcal{E}^\alpha_\sigma(\xi) W_\sigma(\eta)_i W_\sigma(\eta)^*_i \mathcal{E}^\alpha_\sigma(\xi)^*) = \sum_\xi d(\xi)^2 = \lambda.$$

This implies $c = \lambda/d(\sigma)$, and that $\{e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\zeta,j)}\}_{(\eta,i),(\zeta,j)}$ is a system of matrix units. (ii) and (iii) of Lemma 4.6 show that $e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\eta,i)}$ is a minimal projection. Since $z(\widetilde{\sigma}^\alpha) = \sum_{\eta,i} e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\eta,i)}$ is a central element due to Corollary 4.7, it should be a unit of some simple component of Tube Δ and $\{e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\zeta,j)}\}_{(\eta,i),(\zeta,j)}$ is a full system of matrix units of the simple component.

(ii) Note that different choice of the basis $\{W_\sigma(\eta)_i\}$ makes matrix units of the same simple component of Tube Δ. The converse is also true; it is routine to see that if two irreducibles $\widetilde{\sigma}^\alpha$, $\widetilde{\mu}^\beta$ define matrix units of the same simple component, then $[\sigma] = [\mu]$ and the half-braidings are equivalent. This observation shows that $\{z(\widetilde{\sigma}^\alpha)\}_{\widetilde{\sigma}^\alpha \in D(\Delta)_0}$ are

mutually orthogonal. To show $\sum_{\widetilde{\sigma}^\alpha \in D(\Delta)_0} z(\widetilde{\sigma}^\alpha) = 1$, we use the fact that φ_Δ is a faithful positive functional. Indeed, Lemma 4.9 shows

$$\sum_{\widetilde{\sigma}^\alpha \in D(\Delta)_0} \varphi_\Delta(z(\widetilde{\sigma}^\alpha)) = \varphi_\Delta(1),$$

which proves the theorem. Q.E.D.

Remark. To obtain the induction-reduction graph between $A-B$ and $B-B$ sectors, it suffices to know the number

$$\dim(\iota\widetilde{\sigma}^\alpha, (\rho_\xi \otimes id^{op})\iota) = \dim((\sigma \otimes id^{op})\iota, (\rho_\xi \otimes id^{op})\iota) = \dim(\sigma, \rho_\xi).$$

Note that this number can be deduced from the tube algebra: it is the rank of the subalgebra $z(\widetilde{\sigma}^\alpha)\mathcal{A}_\xi \subset$ Tube Δ. Therefore, we can obtain the dual principal graph of the Longo-Rehren inclusion once we know the structure of the tube algebra.

5 S and T Matrices

In this section, we show that our definition of S and T in Section 3 is relevant; more precisely, we prove that our S and T coincide with the ones obtained from the braiding of $D(\Delta)$ through Rehren's construction in [10][30].

First, we show that S is well-defined, i.e. S preserves the center of Tube Δ.

Lemma 5.1 *Let S_0 be as in Section 3 and $z(\widetilde{\sigma}^\alpha)$ be a minimal central projection of* Tube Δ. *Then,*

$$\begin{aligned} S_0(z(\widetilde{\sigma}_\alpha)) &= \frac{d(\sigma)}{\lambda} \sum_{\xi,\eta,i} (\xi\eta | W_\sigma(\eta)_i^* \mathcal{E}_\sigma^\alpha(\xi)^* \rho_\xi(W_\sigma(\eta)_i) | \eta\xi) \\ &= \frac{d(\sigma)}{\lambda} \sum_{\xi,\eta,i} (\xi\eta | \mathcal{E}_\sigma^\alpha(\xi)^*_{(\eta,i),(\eta,i)} | \eta\xi). \end{aligned}$$

Proof. Using the definition of $z(\widetilde{\sigma}_\alpha)$ and S_0, we get

$$\begin{aligned} S_0(z(\widetilde{\sigma}_\alpha)) &= \frac{d(\sigma)}{\lambda} \sum_{\xi,\eta,i} \frac{d(\xi)}{d(\eta)} S_0((\eta\xi | \rho_\xi(W_\sigma(\eta)_i^*) \mathcal{E}_\sigma^\alpha(\xi) W_\sigma(\eta)_i | \xi\eta)) \\ &= \frac{d(\sigma)}{\lambda} \sum_{\xi,\eta,i} d(\xi) (\bar{\xi}\eta | R_\xi^* \rho_{\bar{\xi}}(\rho_\xi(W_\sigma(\eta)_i^*) \mathcal{E}_\sigma^\alpha(\xi) W_\sigma(\eta)_i \rho_\eta(\overline{R}_\xi)) | \eta\bar{\xi}) \\ &= \frac{d(\sigma)}{\lambda} \sum_{\xi,\eta,i} d(\xi) (\bar{\xi}\eta | W_\sigma(\eta)_i^* R_\xi^* \rho_{\bar{\xi}}(\mathcal{E}_\sigma^\alpha(\xi) \sigma(\overline{R}_\xi) W_\sigma(\eta)_i) | \eta\bar{\xi}). \end{aligned}$$

Thanks to the braiding fusion equation, we have

$$\rho_\xi(\mathcal{E}_\sigma^\alpha(\bar{\xi})) \mathcal{E}_\sigma^\alpha(\xi) \sigma(\overline{R}_\xi) = \overline{R}_\xi,$$

and so

$$d(\xi) R_\xi^* \rho_{\bar{\xi}}(\mathcal{E}_\sigma^\alpha(\xi) \sigma(\overline{R}_\xi)) = d(\xi) R_\xi^* \rho_{\bar{\xi}}(\rho_\xi(\mathcal{E}_\sigma^\alpha(\bar{\xi})^* \overline{R}_\xi) = \mathcal{E}_\sigma^\alpha(\bar{\xi})^*,$$

which shows the lemma. Q.E.D.

Proposition 5.2 *S_0 preserves the center of* Tube Δ.

Proof. Let $z(\tilde{\sigma}_\alpha)$ be a minimal central projection of Tube Δ as in the previous lemma. To prove the theorem, it suffices to show that $S_0(z(\tilde{\sigma}^\alpha))$ commutes with arbitrary $(\xi\zeta|X|\zeta\eta)$, or equivalently

$$\begin{aligned}&\sum_{\nu,i}(\xi\nu|W_\sigma(\nu)_i^*\mathcal{E}_\sigma^\alpha(\xi)^*\rho_\xi(W_\sigma(\nu)_i)|\nu\xi)(\xi\zeta|X|\zeta\eta)\\ &\quad=\sum_{\tau,i}(\xi\zeta|X|\zeta\eta)(\eta\tau|W_\sigma(\tau)_i^*\mathcal{E}_\sigma^\alpha(\eta)^*\rho_\eta(W_\sigma(\tau)_i)|\tau\eta).\end{aligned}$$

The left-hand side is

$$\begin{aligned}&\sum_{\nu,i}\sum_{\mu\prec\nu\zeta}\sum_{j=1}^{N_{\nu,\zeta}^\mu}(\xi\mu|T(^\mu_{\nu,\zeta})_j^*\rho_\nu(X)W_\sigma(\nu)_i^*\mathcal{E}_\sigma^\alpha(\xi)^*\rho_\xi(W_\sigma(\nu)_i)\rho_\xi(T(^\mu_{\nu,\zeta})_j)|\mu\eta)\\ &\quad=\sum_{\nu,i}\sum_{\mu\prec\nu\zeta}\sum_{j=1}^{N_{\nu,\zeta}^\mu}(\xi\mu|T(^\mu_{\nu,\zeta})_j^*W_\sigma(\nu)_i^*\sigma(X)\mathcal{E}_\sigma^\alpha(\xi)^*\rho_\xi(W_\sigma(\nu)_iT(^\mu_{\nu,\zeta})_j)|\mu\eta).\end{aligned}$$

The right-hand side is

$$\begin{aligned}&\sum_{\tau,i}\sum_{\mu\prec\zeta\tau}\sum_{j=1}^{N_{\zeta,\tau}^\mu}(\xi\mu|T(^\mu_{\zeta,\tau})_j^*\rho_\zeta(W_\sigma(\tau)_i^*\mathcal{E}_\sigma^\alpha(\eta)^*\rho_\eta(W_\sigma(\tau)_i))X\rho_\xi(T(^\mu_{\zeta,\tau})_j)|\mu\eta)\\ &\quad=\sum_{\tau,i}\sum_{\mu\prec\zeta\tau}\sum_{j=1}^{N_{\zeta,\tau}^\mu}(\xi\mu|T(^\mu_{\zeta,\tau})_j^*\rho_\zeta(W_\sigma(\tau)_i^*)\rho_\zeta(\mathcal{E}_\sigma^\alpha(\eta)^*)X\rho_\xi(\rho_\zeta(W_\sigma(\tau)_i)T(^\mu_{\zeta,\tau})_j)|\mu\eta)\end{aligned}$$

Using the braiding fusion equation, we get

$$\rho_\zeta(\mathcal{E}_\sigma^\alpha(\eta)^*)X=\mathcal{E}_\sigma^\alpha(\zeta)\sigma(X)\mathcal{E}_\sigma^\alpha(\xi)^*\rho_\xi(\mathcal{E}_\sigma^\alpha(\zeta)^*),$$

and so the right-hand side is

$$\sum_{\tau,i}\sum_{\mu\prec\zeta\tau}\sum_{j=1}^{N_{\zeta,\tau}^\mu}(\xi\mu|T(^\mu_{\zeta,\tau})_j^*\rho_\zeta(W_\sigma(\tau)_i^*)\mathcal{E}_\sigma^\alpha(\zeta)\sigma(X)\mathcal{E}_\sigma^\alpha(\xi)^*\rho_\xi(\mathcal{E}_\sigma^\alpha(\zeta)^*\rho_\zeta(W_\sigma(\tau)_i)T(^\mu_{\zeta,\tau})_j)|\mu\eta)$$

Since both $\{W_\sigma(\nu)_iT(^\mu_{\nu,\zeta})_j\}_{\nu,i,j}$ and $\{\mathcal{E}_\sigma^\alpha(\zeta)^*\rho_\zeta(W_\sigma(\tau)_i)T(^\mu_{\zeta,\tau})_j\}_{\tau,i,j}$ are orthonormal bases of $(\rho_\mu,\sigma\rho_\zeta)$, we get the equality. Q.E.D.

The following lemma is also useful to compute S and T in concrete examples, especially when we can manage to obtain all matrix units of Tube Δ.

Lemma 5.3 (i) *Let $S_{\tilde{\sigma}^\alpha,\tilde{\mu}^\beta}$ be the matrix coefficient of S with respect to the orthonormal basis $\{\frac{\sqrt{\lambda}}{d(\sigma)}z(\tilde{\sigma}^\alpha)\}$. Then,*

$$\frac{d(\sigma)}{\lambda}\sum_{\xi,i}d(\xi)\phi_\xi(\mathcal{E}_\mu^\beta(\eta)^*_{(\xi,i),(\xi,i)}\mathcal{E}_\sigma^\alpha(\xi)^*_{(\eta,j),(\eta,k)})=\delta_{j,k}S_{\tilde{\sigma}^\alpha,\tilde{\mu}^\beta}.$$

In particular,

$$\frac{1}{\lambda}\sum_{\xi,\eta,i,j}d(\xi)d(\eta)\phi_\xi(\mathcal{E}_\mu^\beta(\eta)^*_{(\xi,j),(\xi,j)}\mathcal{E}_\sigma^\alpha(\xi)^*_{(\eta,i),(\eta,i)})=S_{\tilde{\sigma}^\alpha,\tilde{\mu}^\beta}.$$

(ii) *Let be* $\mathbf{t} = \sum_{\widetilde{\sigma^\alpha}} \omega_{\widetilde{\sigma^\alpha}} z(\widetilde{\sigma}^\alpha)$, *where* $\mathbf{t}$ *is as in Section 3. If* ρ_ξ *is contained in* σ,

$$\phi_\xi(\mathcal{E}_\sigma^\alpha(\xi)_{(\xi,i),(\xi,j)}) = \delta_{i,j}\frac{1}{d(\xi)}\omega_{\widetilde{\sigma^\alpha}}.$$

Proof. (i) Thanks to Proposition 5.2, we have

$$z(\widetilde{\sigma}^\alpha)S_0(z(\widetilde{\mu}^\beta)) = \frac{d(\mu)}{d(\sigma)}S_{\widetilde{\sigma^\alpha},\widetilde{\mu^\beta}}z(\widetilde{\sigma}^\alpha),$$

and so

$$S_{\widetilde{\sigma^\alpha},\widetilde{\mu^\beta}}e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\zeta,j)} = \frac{d(\sigma)}{d(\mu)}e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\zeta,j)}S_0(z(\widetilde{\mu}^\beta)).$$

On the other hand, thanks to Lemma 4.7 and Lemma 5.1, the right-hand side equals to

$$\frac{d(\sigma)}{\lambda}\sum_{\tau,k,l}\phi_\tau(\mathcal{E}_\mu^\beta(\zeta)^*_{(\tau,k),(\tau,k)}\mathcal{E}_\sigma^\alpha(\tau)^*_{(\zeta,j),(\zeta,l)})e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\eta,l)},$$

which shows the statement. (ii) Using $\mathbf{t}^* = \sum_\eta(\eta\eta|1|\eta\eta)$ and Lemma 4.7, we get

$$\begin{aligned}\overline{\omega}_{\widetilde{\sigma^\alpha}}e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\zeta,j)} &= e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\zeta,j)}\mathbf{t}^* \\ &= d(\zeta)\sum_k \phi_\zeta(\mathcal{E}_\sigma^\alpha(\zeta)^*_{(\zeta,j),(\zeta,k)})e(\widetilde{\sigma}^\alpha)_{(\eta,i),(\zeta,k)},\end{aligned}$$

which proves the statement. Q.E.D.

Lemma 5.4 *Let* $\{\mathcal{E}(\widetilde{\sigma}^\alpha,\widetilde{\mu}^\beta)\}$ *be the braiding of* $D(\Delta)$ *defined in Theorem 4.6. Then, the following hold:*

(i)

$$S_{\widetilde{\sigma^\alpha},\widetilde{\mu^\beta}} = \frac{d(\sigma)d(\mu)}{\lambda}\phi_{\widetilde{\mu^\beta}}(\mathcal{E}(\widetilde{\mu}^\beta,\widetilde{\sigma}^\alpha)^*\mathcal{E}(\widetilde{\sigma}^\alpha,\widetilde{\mu}^\beta)^*).$$

(ii) $\omega_{\widetilde{\sigma^\alpha}}$ *is the statistical phase of* $\widetilde{\sigma}^\alpha$, *i.e.*

$$\omega_{\widetilde{\sigma^\alpha}} = d(\sigma)\phi_{\widetilde{\sigma^\alpha}}(\mathcal{E}(\widetilde{\sigma}^\alpha,\widetilde{\sigma}^\alpha)).$$

(iii)

$$\sum_{\widetilde{\sigma^\alpha}} d(\widetilde{\sigma}^\alpha)^2\overline{\omega}_{\widetilde{\sigma^\alpha}} = \lambda.$$

Proof. (i) Note that the standard left inverse of a reducible sector σ is given by

$$\phi_\sigma(x) = \sum_{\xi,i}\frac{d(\xi)}{d(\sigma)}\phi_\xi(W_\sigma(\xi)_i^* x W_\sigma(\xi)_i).$$

First, we claim

$$\phi_{\widetilde{\mu^\beta}}(\gamma(x\otimes 1)) = \gamma(\phi_\mu(x)\otimes 1).$$

Indeed, thanks to Theorem 4.6, we know that two isometries

$$\gamma(R_\sigma \otimes 1) \in (id, \widetilde{\bar{\mu}}^{\bar{\beta}}\widetilde{\mu}^{\beta}), \quad \gamma(\bar{R}_\sigma \otimes 1) \in (id, \widetilde{\mu}^{\beta}\widetilde{\bar{\mu}}^{\bar{\beta}})$$

satisfies (2.1). Thus, we get

$$\begin{aligned}\phi_{\widetilde{\mu}^{\beta}}(\gamma(x \otimes 1)) &= \gamma(R_\mu^* \otimes 1)\widetilde{\bar{\mu}}^{\bar{\beta}}(\gamma(x \otimes 1))\gamma(R_\mu \otimes 1) = \gamma(R_\mu^* \bar{\mu}(x) R_\mu \otimes 1) \\ &= \gamma(\phi_\mu(x) \otimes 1).\end{aligned}$$

Using the claim and the definition of the braiding, we get

$$\begin{aligned}\phi_{\widetilde{\mu}^{\beta}}(\mathcal{E}(\widetilde{\mu}^{\beta}, \widetilde{\sigma}^{\alpha})^* \mathcal{E}(\widetilde{\sigma}^{\alpha}, \widetilde{\mu}^{\beta})^*) &= \phi_{\widetilde{\mu}^{\beta}}(\gamma(\mathcal{E}_\mu^\beta(\sigma)^* \mathcal{E}_\sigma^\alpha(\mu)^*) \otimes 1) \\ &= \gamma(\phi_\mu(\mathcal{E}_\mu^\beta(\sigma)^* \mathcal{E}_\sigma^\alpha(\mu)^*) \otimes 1),\end{aligned}$$

which is a scalar. Thus, it equals to

$$\begin{aligned}\phi_\mu(\mathcal{E}_\mu^\beta(\sigma)^* \mathcal{E}_\sigma^\alpha(\mu)^*) &= \sum_{\xi,i} \frac{d(\xi)}{d(\mu)} \phi_\xi(W_\mu(\xi)_i^* \mathcal{E}_\mu^\beta(\sigma)^* \mathcal{E}_\sigma^\alpha(\mu)^* W_\mu(\xi)_i) \\ &= \sum_{\xi,\eta,i,j} \frac{d(\xi)}{d(\mu)} \phi_\xi(W_\mu(\xi)_i^* \mu(W_\sigma(\eta)_j) \mathcal{E}_\mu^\beta(\eta)^* W_\sigma(\eta)_j^* \sigma(W_\mu(\xi)_i) \mathcal{E}_\sigma^\alpha(\xi)^*) \\ &= \sum_{\xi,\eta,i,j} \frac{d(\xi)}{d(\mu)} W_\sigma(\eta)_j \phi_\xi(W_\mu(\xi)_i^* \mathcal{E}_\mu^\beta(\eta)^* \rho_\eta(W_\mu(\xi)_i) W_\sigma(\eta)_j^* \mathcal{E}_\sigma^\alpha(\xi)^*) \\ &= \sum_{\xi,\eta,\zeta,i,j,k} \frac{d(\xi)}{d(\mu)} W_\sigma(\eta)_j \phi_\xi(\mathcal{E}_\mu^\beta(\eta)^*_{(\xi,i),(\xi,i)} \mathcal{E}_\sigma^\alpha(\xi)^*_{(\eta,j),(\zeta,k)}) W_\sigma(\zeta)_k^*.\end{aligned}$$

Since $\phi_\xi(\mathcal{E}_\mu^\beta(\eta)^*_{(\xi,i),(\xi,i)} \mathcal{E}_\sigma^\alpha(\xi)^*_{(\eta,j),(\zeta,k)}) \in (\rho_\zeta, \rho_\eta)$, this term survives only when $\eta = \zeta$. Thus we get the following using Lemma 5.3:

$$\begin{aligned}\phi_\mu(\mathcal{E}_\mu^\beta(\sigma)^* \mathcal{E}_\sigma^\alpha(\mu)^*) &= \sum_{\xi,\eta,i,j,k} \frac{d(\xi)}{d(\mu)} \phi_\xi(\mathcal{E}_\mu^\beta(\eta)^*_{(\xi,i),(\xi,i)} \mathcal{E}_\sigma^\alpha(\xi)^*_{(\eta,j),(\eta,k)}) W_\sigma(\eta)_j W_\sigma(\eta)_k^*, \\ &= \frac{\lambda}{d(\sigma)d(\mu)} S_{\widetilde{\sigma}^{\alpha}, \widetilde{\mu}^{\beta}} \sum_{\eta,j} W_\sigma(\eta)_j W_\sigma(\eta)_j^* \\ &= \frac{\lambda}{d(\sigma)d(\mu)} S_{\widetilde{\sigma}^{\alpha}, \widetilde{\mu}^{\beta}}.\end{aligned}$$

(ii) In a similar way as above, we have $\phi_{\widetilde{\sigma}^{\alpha}}(\mathcal{E}(\widetilde{\sigma}^{\alpha}, \widetilde{\sigma}^{\alpha})) = \gamma(\phi_\sigma(\mathcal{E}_\sigma^\alpha(\sigma)) \otimes 1)$ and

$$\begin{aligned}\phi_\sigma(\mathcal{E}_\sigma^\alpha(\sigma)) &= \sum_{\xi,i} \frac{d(\xi)}{d(\sigma)} \phi_\xi(W_\sigma(\xi)_i^* \mathcal{E}_\sigma^\alpha(\sigma) W_\sigma(\xi)_i) \\ &= \sum_{\xi,i} \frac{d(\xi)}{d(\sigma)} \phi_\xi(\mathcal{E}_\sigma^\alpha(\xi) W_\sigma(\xi)_i) W_\sigma(\xi)_i^* \\ &= \sum_{\xi,\eta,i,j} \frac{d(\xi)}{d(\sigma)} \phi_\xi(\mathcal{E}_\sigma^\alpha(\xi)_{(\xi,i),(\eta,j)}) W_\sigma(\eta)_j W_\sigma(\xi)_i^* \\ &= \sum_{\xi,i} \frac{\omega_{\widetilde{\sigma}^{\alpha}}}{d(\sigma)} W_\sigma(\xi)_i W_\sigma(\xi)_i^* \\ &= \frac{\omega_{\widetilde{\sigma}^{\alpha}}}{d(\sigma)}.\end{aligned}$$

(iii) We compute $\varphi_\Delta(\mathbf{t}^*)$ in two ways. Using $\mathbf{t}^* = \sum_\xi (\xi\xi|1|\xi\xi)$, we get $\varphi_\Delta(\mathbf{t}^*) = 1$. On the other hand

$$\varphi_\Delta(\mathbf{t}^*) = \sum_{\widetilde{\sigma}^\alpha} \overline{\omega_{\widetilde{\sigma}^\alpha}} \varphi_\Delta(z(\widetilde{\sigma}^\alpha)) = \sum_{\widetilde{\sigma}^\alpha} \overline{\omega_{\widetilde{\sigma}^\alpha}} \frac{d(\sigma)^2}{\lambda}.$$

Q.E.D.

Thanks to Rehren's argument in [10][30], the above implies the following:

Theorem 5.5 *Let S and T be as above. Then, S and T satisfy the following:*

(i) $S_{i,j} = S_{j,i} = \overline{S_{i,\bar{j}}}$.

(ii) *Modular group relation:*

$$(ST)^3 = S^2 = C, \quad CT = TC,$$

where $C_{i,j} := \delta_{i,\bar{j}}$ is the conjugation matrix.

(iii) *Verlinde formula:*

$$N_{i,j}^k = \sum_l \frac{S_{i,l} S_{j,l} S_{\bar{k},l}}{S_{e,l}}.$$

6 Relation to α-Induction

Though Theorem 4.6 assures that there are enough endomorphisms with half braidings for a given system Δ, one might still wonder why it occurs or how one can obtain these endomorphisms and half braidings. Here, we give the one corresponding to the restriction $\gamma|_B$ of γ to B explicitly, which is necessary in order to obtain the α-inductions [2] of $B-B$ sectors with respect to the braiding we defined in Section 4.

We start with the following technical lemma:

Lemma 6.1 *Let E_Δ be the conditional expectation onto B. Then,*

$$E_\Delta(V_\xi(\bar{R}_\xi \bar{R}_\xi^* \otimes 1)V_\xi^*) = E_\Delta(V_\xi(1 \otimes j(\bar{R}_\xi \bar{R}_\xi^*))V_\xi^*) = \frac{1}{\lambda}.$$

Proof. Since E_Δ is given by $W^*\gamma(\cdot)W$, we get

$$\begin{aligned}
E_\Delta(V_\xi(\bar{R}_\xi \bar{R}_\xi^* \otimes 1)V_\xi^*) &= W^*\gamma(V_\xi(\bar{R}_\xi \bar{R}_\xi^* \otimes 1)V_\xi^*)W \\
&= \sum_{\eta,\zeta,\zeta'} \frac{d(\xi)d(\eta)}{\lambda\sqrt{d(\zeta)d(\zeta')}} V_{\zeta'} T_{\eta,\xi}^{\zeta'\,*} (\rho_\eta(\bar{R}_\xi \bar{R}_\xi^*) \otimes 1) T_{\eta,\xi}^{\zeta} V_\zeta^* \\
&= \sum_{\eta,\zeta,i} \frac{d(\xi)d(\eta)}{\lambda d(\zeta)} V_\zeta (T(^{\zeta}_{\eta,\xi})_i^* \rho_\eta(\bar{R}_\xi \bar{R}_\xi^*) T(^{\zeta}_{\eta,\xi})_i \otimes 1) V_\zeta^* \\
&= \sum_{\eta,\zeta,i} \frac{1}{\lambda} V_\zeta (T(^{\eta}_{\zeta,\bar{\xi}})_i T(^{\eta}_{\zeta,\bar{\xi}})_i^* \otimes 1) V_\zeta^* \\
&= \frac{1}{\lambda},
\end{aligned}$$

where we use the Frobenius reciprocity. The second equation can be proved in the same way. Q.E.D.

We set σ_0 to be the direct sum of $\rho_\xi\rho_{\bar\xi}$, more precisely,

$$\sigma_0(x) := \sum_\xi W_\xi\rho_\xi\rho_{\bar\xi}(x)W_\xi^*,$$

where $\{W_\xi\}_\xi \subset M$ are isometries satisfying $\sum_\xi W_\xi W_\xi^* = 1$. We set

$$\mathcal{E}^0_{\sigma_0}(\xi) = \sum_{\eta,\zeta,i} \rho_\xi(W_\zeta)T(\tbinom{\eta}{\xi,\zeta}_i)\rho_\eta(T(\tbinom{\bar\zeta}{\bar\eta\xi}_i^*)W_\eta^*.$$

Direct computation shows that $\{\mathcal{E}^0_{\sigma_0}(\xi)\}_\xi$ is a half braiding of σ_0 with respect to Δ. We denote by $\{\mathcal{E}(\rho,\pi)\}$ the braiding of $D(\Delta)$ given by Theorem 4.6.

Proposition 6.2 *Let σ_0 and $\mathcal{E}^0_{\sigma_0}$ be as above. Then,*

(i) *Let $X_0 := \sqrt{\lambda}\sum_\xi(W_\xi \otimes j(\bar R_\xi^*))V_\xi^*\gamma(V_{\bar\xi}^*)$ and $X := \gamma(X_0)W$. Then, X is a unitary in $(\gamma|_B, \tilde\sigma^0)$. In consequence, we have $[\tilde\sigma^0] = [\gamma|_B]$.*

(ii) *$\mathcal{E}(\tilde\mu^\beta, \gamma|_B) = \gamma(U^\beta(\mu))$.*

Proof. (i) First we show that X is an isometry. Indeed, using Lemma 6.1, we get

$$X^*X = E_\Delta(X_0^*X_0) = \lambda\sum_\xi \gamma(V_{\bar\xi})E_\Delta(V_\xi^*(1\otimes j(\bar R_\xi\bar R_\xi^*))V_\xi)\gamma(V_{\bar\xi}^*) = \sum_\xi \gamma(V_{\bar\xi}V_{\bar\xi}^*) = 1.$$

Let $E_{\gamma(A)}$ be the conditional expectation from B to $\gamma(A)$. Then,

$$E_{\gamma(A)}(XX^*) = \gamma(X_0)E_{\gamma(A)}(WW^*)\gamma(X_0^*) = \frac{1}{\lambda}\gamma(X_0X_0^*) = \sum_\xi W_\xi W_\xi^* \otimes 1 = 1,$$

which shows that X is a unitary. Since $B = \gamma(A)W$, to prove $X \in (\gamma|_B, \tilde\sigma^0)$, it suffices to show the following two

$$X \in (\gamma^2, \gamma(\sigma_0 \otimes id^{op})), \quad X\gamma(W) = \gamma(U^0(\sigma_0)^*)WX,$$

which is equivalent to

$$X_0 \in (\gamma^2, \sigma_0 \otimes id^{op}), \quad X_0\gamma(W) = U^0(\sigma_0)^*\gamma(X_0)W.$$

The first condition obviously holds. Straightforward computation using the Frobenius reciprocity shows that the both sides of the second equation above coincides with

$$\sum_{\xi,\eta,\zeta,i}(\sigma_0\otimes id)(V_\eta)(W_\xi\rho_\xi(T(\tbinom{\zeta}{\bar\xi\eta}_i) \otimes j(T(\tbinom{\eta}{\xi,\zeta}_i^*))\hat\rho_\xi(V_\zeta^*)V_\xi^*.$$

(ii) Since $\gamma|_B = Ad(X^*)\tilde{\sigma}_0^0$, we have

$$\begin{aligned}
\mathcal{E}(\tilde{\mu}^\beta, \gamma|_B) &= X^*\gamma(\mathcal{E}_\mu^\beta(\sigma_0)\otimes 1)\tilde{\mu}^\beta(X) = W^*\gamma(X_0^*(\mathcal{E}_\mu^\beta(\sigma_0)\otimes 1)(\mu\otimes id)(X_0)U^\beta(\mu)^*)W \\
&= \sum_\xi E_\Delta(X_0^*(W_\xi\otimes 1)(\mathcal{E}_\mu^\beta(\rho_\xi\rho_{\bar{\xi}})\otimes 1)(\mu\otimes id)((W_\xi^*\otimes 1)X_0)U^\beta(\mu)^*) \\
&= \lambda\sum_\xi E_\Delta(\gamma(V_{\bar{\xi}})V_\xi(\mathcal{E}_\mu^\beta(\rho_\xi\rho_{\bar{\xi}})\otimes j(\bar{R}_\xi\bar{R}_\xi^*))(\mu\otimes id)(V_\xi^*\gamma(V_{\bar{\xi}}^*))U^\beta(\mu)^*) \\
&= \lambda\sum_\xi \gamma(V_{\bar{\xi}})E_\Delta(V_\xi(\mathcal{E}_\mu^\beta(\rho_\xi\rho_{\bar{\xi}})\otimes j(\bar{R}_\xi\bar{R}_\xi^*))(\mu\otimes id)(V_\xi^*)U^\beta(\mu)^*)\gamma((\mu\otimes id)(V_{\bar{\xi}}^*)) \\
&= \lambda\sum_\xi \gamma(V_{\bar{\xi}})E_\Delta(V_\xi(\mathcal{E}_\mu^\beta(\rho_\xi\rho_{\bar{\xi}})\mathcal{E}_\mu^\beta(\xi)^*\otimes j(\bar{R}_\xi\bar{R}_\xi^*))V_\xi^*)\gamma((\mu\otimes id)(V_{\bar{\xi}}^*)) \\
&= \lambda\sum_\xi \gamma(V_{\bar{\xi}})E_\Delta(V_\xi(\rho_\xi(\mathcal{E}_\mu^\beta(\bar{\xi}))\otimes j(\bar{R}_\xi\bar{R}_\xi^*))V_\xi^*)\gamma((\mu\otimes id)(V_{\bar{\xi}}^*)) \\
&= \lambda\sum_\xi \gamma(V_{\bar{\xi}}(\mathcal{E}_\mu^\beta(\bar{\xi})\otimes 1))E_\Delta(V_\xi(1\otimes j(\bar{R}_\xi\bar{R}_\xi^*))V_\xi^*)\gamma((\mu\otimes id)(V_{\bar{\xi}}^*)) \\
&= \sum_\xi \gamma(V_{\bar{\xi}}(\mathcal{E}_\mu^\beta(\bar{\xi})\otimes 1)(\mu\otimes id)(V_{\bar{\xi}}^*)) \\
&= \gamma(U^\beta(\mu)),
\end{aligned}$$

which proves the statement. Q.E.D

We use the notation $\mathcal{E}^+(\rho,\pi) := \mathcal{E}(\rho,\pi)$, $\mathcal{E}^-(\rho,\pi) := \mathcal{E}(\pi,\rho)^*$. As in [2], for a sector $\rho \in D(\Delta)$ we define the α-induction $\alpha_\rho^\pm \in End(A)$ of ρ by

$$\gamma^{-1}\cdot Ad(\mathcal{E}^\pm(\rho,\gamma|_B))\cdot\rho\cdot\gamma.$$

The above result shows

$$\begin{aligned}
\gamma^{-1}\cdot Ad(\mathcal{E}(\tilde{\mu}^\beta,\gamma|_B))\cdot\tilde{\mu}^\beta\cdot\gamma &= \gamma^{-1}\cdot Ad(\gamma(U^\beta(\mu)))\cdot\gamma\cdot(\mu\otimes id^{op}) \\
&= Ad(U^\beta(\mu)))\cdot(\mu\otimes id^{op}).
\end{aligned}$$

Thus, we obtain

Corollary 6.3 $\alpha^+_{\tilde{\mu}^\beta} = Ad(U^\beta(\mu)))\cdot(\mu\otimes id^{op})$.

We proceed to α_ρ^-. We choose $\{W_\sigma(\xi)_i\}_i$ and $\{W_{\bar{\sigma}}(\bar{\xi})_i\}_i$ satisfying

$$R_\sigma = \sum\sqrt{\frac{d(\xi)}{d(\sigma)}}W_{\bar{\sigma}}(\bar{\xi})_i\rho_{\bar{\xi}}(W_\sigma(\xi))_i R_\xi, \tag{6.1}$$

$$\bar{R}_\sigma = \sum\sqrt{\frac{d(\xi)}{d(\sigma)}}W_\sigma(\xi)_i\rho_\xi(W_{\bar{\sigma}}(\bar{\xi}))_i\bar{R}_\xi. \tag{6.2}$$

Such a choice always exists because we may define R_σ and $\bar{R}_\sigma$ by these equalities. (Note that the pair $(R_\sigma,\bar{R}_\sigma)$ is not unique, even up to scalar multiple, when σ is reducible.)

Proposition 6.4 *Let $\tilde{\sigma}^{\alpha} \in D(\Delta)$. Then, $\tilde{\sigma}^{\alpha}$ and $\widetilde{\bar{\sigma}^{op}}^{\bar{\alpha}}$ are equivalent with a unitary intertwiner $Y(\sigma) \in (\tilde{\sigma}^{\alpha}, \widetilde{\bar{\sigma}^{op}}^{\bar{\alpha}})$ given by*

$$Y(\sigma) = \sum_{\xi,\eta,\zeta,i,k} V_{\xi}(T(\begin{smallmatrix}\xi\\ \zeta,\bar{\eta}\end{smallmatrix})_i^* \rho_{\zeta}(W_{\sigma}(\bar{\eta})_k^*) \otimes \rho_{\xi}^{op}(j(W_{\bar{\sigma}}(\eta)_k)) j(T(\begin{smallmatrix}\zeta\\ \xi,\eta\end{smallmatrix})_i)) V_{\zeta}^*,$$

where $T(\begin{smallmatrix}\xi\\ \zeta,\bar{\eta}\end{smallmatrix})_i$ and $T(\begin{smallmatrix}\zeta\\ \xi,\eta\end{smallmatrix})_i$ are related through the Frobenius reciprocity.

Proof. It is obvious that $Y(\sigma)$ is a unitary operator in

$$(\tilde{\sigma}^{\alpha} \cdot \gamma, \widetilde{\bar{\sigma}^{op}}^{\bar{\alpha}} \cdot \gamma) = (\gamma \cdot (\sigma \otimes id^{op}), \gamma \cdot (id \otimes \bar{\sigma}^{op})).$$

We set

$$Y(\sigma)_0 = \sqrt{\lambda} \sum_{\xi,k} (\bar{R}_{\xi}^* \rho_{\xi}(W_{\sigma}(\bar{\xi})_k^*) \otimes j(W_{\bar{\sigma}}(\xi)_k)) V_{\xi}^* \in (\gamma \cdot (\sigma \otimes id^{op}), id \otimes \bar{\sigma}^{op}).$$

Then, direct computation using the Frobenius reciprocity yields

$$Y(\sigma) = \gamma(Y(\sigma)_0) W \in B.$$

Since we have $B = \gamma(A)W$, what remains is to show $Y(\sigma)\tilde{\sigma}^{\alpha}(W) = \widetilde{\bar{\sigma}^{op}}^{\bar{\alpha}}(W) Y(\sigma)$. The left-hand side equals to

$$\gamma(Y(\sigma)_0) W \gamma(U^{\alpha}(\sigma)^*) W = \gamma(Y(\sigma)_0 \gamma(U^{\alpha}(\sigma)^*) W) W,$$

while the right-hand side is $\gamma(U^{\bar{\alpha}}(\bar{\sigma}^{op})^* Y(\sigma)) W$. Thus, to prove the statement, it suffices to show

$$U^{\bar{\alpha}}(\bar{\sigma}^{op}) Y(\sigma)_0 \gamma(U^{\alpha}(\sigma)^*) W = Y(\sigma).$$

Using $\mathcal{E}_{\sigma}(\eta)^* = d(\sigma)\sigma(\rho_{\eta}(R_{\sigma}^*)\mathcal{E}_{\bar{\sigma}}^{\bar{\alpha}}(\eta))\bar{R}_{\sigma}$ (See Theorem 4.6) and the fact that $Y(\sigma)_0$ belongs to $(\gamma \cdot (\sigma \otimes id^{op}), id \otimes \bar{\sigma}^{op})$, we get

$$Y(\sigma)_0 \gamma(U^{\alpha}(\sigma)^*) = d(\sigma) \sum_{\eta} (id \otimes \bar{\sigma}^{op})(V_{\eta})(\rho_{\eta}(R_{\sigma}^*)\mathcal{E}_{\bar{\sigma}}^{\bar{\alpha}}(\eta)^* \otimes 1) Y(\sigma)_0 \gamma((\bar{R}_{\sigma} \otimes 1) V_{\eta}^*).$$

Thus, $U^{\bar{\alpha}}(\bar{\sigma}^{op}) Y(\sigma)_0 \gamma(U^{\alpha}(\sigma)^*) W$ equals to

$$\sum_{\xi,\eta\zeta i,k} d(\sigma) \sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}} V_{\eta}(\rho_{\eta}(R_{\sigma}^*)\mathcal{E}_{\bar{\sigma}}^{\bar{\alpha}}(\eta)^* \otimes j(\mathcal{E}_{\bar{\sigma}}^{\bar{\alpha}}(\eta)^*)) Y(\sigma)_0 V_{\xi}(\rho_{\xi}(\bar{R}_{\sigma}) \otimes 1) T_{\xi,\eta}^{\zeta} V_{\zeta}^*.$$

Using the normalization (6.1),(6.2) of $\{W_{\sigma}(\bar{\xi})_k\}$ and $\{W_{\bar{\sigma}}(\xi)_k\}$ stated above, we have

$$Y(\sigma)_0 V_{\xi}(\rho_{\xi}(\bar{R}_{\sigma}) \otimes 1) = \sqrt{\frac{\lambda}{d(\sigma)d(\xi)}} \sum_k W_{\bar{\sigma}}(\xi)_k \otimes j(W_{\bar{\sigma}}(\xi)_k),$$

and so $U^{\bar{\alpha}}(\bar{\sigma}^{op}) Y(\sigma)_0 \gamma(U^{\alpha}(\sigma)^*) W$ equals to

$$\sum_{\xi,\eta,\zeta,i,k} \sqrt{\frac{d(\eta)}{d(\sigma)d(\zeta)}} V_{\eta}(\rho_{\eta}(R_{\sigma}^*) \otimes 1)(\mathcal{E}_{\bar{\sigma}}^{\bar{\alpha}}(\eta)^* W_{\bar{\sigma}}(\xi)_k T(\begin{smallmatrix}\zeta\\ \xi,\eta\end{smallmatrix})_i \otimes j(\mathcal{E}_{\bar{\sigma}}^{\bar{\alpha}}(\eta)^* W_{\bar{\sigma}}(\xi)_k T(\begin{smallmatrix}\zeta\\ \xi,\eta\end{smallmatrix})_i)) V_{\zeta}^*.$$

Since $\{\mathcal{E}^{\bar\alpha}_{\bar\sigma}(\eta)^* W_{\bar\sigma}(\xi)_k T(\binom{\zeta}{\xi,\eta}_i\}_{\xi,i,k}$ is an orthonormal basis of $(\rho_\zeta, \rho_\eta\bar\sigma)$, we can replace it with $\{\rho_\eta(W_{\bar\sigma}(\xi)_k)T(\binom{\zeta}{\eta,\xi}_i\}_{\xi,i,k}$ in the above formula. Thus, $U^{\bar\alpha}(\bar\sigma^{op})Y(\sigma)_0\gamma(U^\alpha(\sigma)^*)W$ equals to

$$\begin{aligned}
&\sum_{\xi,\eta,\zeta,i,k}\sqrt{\frac{d(\eta)}{d(\sigma)d(\zeta)}}V_\eta(\rho_\eta(R^*_\sigma W_{\bar\sigma}(\xi)_k)T(\tbinom{\zeta}{\eta,\xi}_i\otimes j(\rho_\eta(W_{\bar\sigma}(\xi)_k)T(\tbinom{\zeta}{\eta,\xi}_i))V^*_\zeta\\
&= \sum_{\xi,\eta,\zeta,i,k}\sqrt{\frac{d(\xi)d(\eta)}{d(\zeta)}}V_\eta(\rho_\eta(\bar R^*_\xi\rho_\xi(W_\sigma(\bar\xi)^*_k))T(\tbinom{\zeta}{\eta,\xi}_i\otimes j(\rho_\eta(W_{\bar\sigma}(\xi)_k)T(\tbinom{\zeta}{\eta,\xi}_i))V^*_\zeta\\
&= \sum_{\xi,\eta,\zeta,i,k}\sqrt{\frac{d(\xi)d(\eta)}{d(\zeta)}}V_\eta(\rho_\eta(\bar R^*_\xi)T(\tbinom{\zeta}{\eta,\xi}_i\rho_\zeta(W_\sigma(\bar\xi)^*_k)\otimes j(\rho_\eta(W_{\bar\sigma}(\xi)_k)T(\tbinom{\zeta}{\eta,\xi}_i))V^*_\zeta\\
&= \sum_{\xi,\eta,\zeta,i,k}V_\eta(T(\tbinom{\eta}{\zeta,\bar\xi}^*_i\rho_\zeta(W_\sigma(\bar\xi)^*_k)\otimes j(\rho_\eta(W_{\bar\sigma}(\xi)_k)T(\tbinom{\zeta}{\eta,\xi}_i))V^*_\zeta\\
&= Y(\sigma).
\end{aligned}$$

Q.E.D.

We set $\mathcal{E}^{op}(\widetilde{\bar\sigma^{op}}^{\bar\alpha},\widetilde{\bar\mu^{op}}^{\bar\beta}) = \gamma(1\otimes j(\mathcal{E}^{\bar\alpha}_{\bar\sigma}(\bar\mu)))$. Then, clearly $\mathcal{E}^{op}$ can extend in a natural way to a braiding of $D(\Delta)$.

Proposition 6.5 *With the above notation, we have $\mathcal{E}^- = \mathcal{E}^{op}$.*

Proof. Since $\tilde\sigma^\alpha = Ad(Y(\sigma)^*)\cdot\widetilde{\bar\sigma^{op}}^{\bar\alpha}$ and $\tilde\mu^\beta = Ad(Y(\mu)^*)\cdot\widetilde{\bar\mu^{op}}^{\bar\beta}$, we have

$$\mathcal{E}^{op}(\tilde\sigma^\alpha,\tilde\mu^\beta) = \tilde\mu^\beta(Y(\sigma)^*)Y(\mu)^*\gamma(1\otimes j(\mathcal{E}^{\bar\alpha}_{\bar\sigma}(\bar\mu)))\widetilde{\bar\sigma^{op}}^{\bar\alpha}(Y(\mu))Y(\sigma).$$

First, we show $\gamma(1\otimes j(\mathcal{E}^{\bar\alpha}_{\bar\sigma}(\bar\mu)))\widetilde{\bar\sigma^{op}}^{\bar\alpha}(Y(\mu)) = Y(\mu)$. Indeed,

$$\begin{aligned}
\gamma(1\otimes j(\mathcal{E}^{\bar\alpha}_{\bar\sigma}(\bar\mu)))\widetilde{\bar\sigma^{op}}^{\bar\alpha}(Y(\mu)) &= \gamma(1\otimes j(\mathcal{E}^{\bar\alpha}_{\bar\sigma}(\bar\mu)))\widetilde{\bar\sigma^{op}}^{\bar\alpha}(\gamma(Y(\mu)_0)W)\\
&= \gamma((1\otimes j(\mathcal{E}^{\bar\alpha}_{\bar\sigma}(\bar\mu))(id\otimes\bar\sigma^{op})(Y(\mu)_0)U^{\bar\alpha}(\bar\sigma^{op})^*)W\\
&= \sum_{\xi,i}\sqrt{\lambda}\gamma((\bar R^*_\xi\rho_\xi(W_\mu(\bar\xi)^*_i)\otimes j(\mathcal{E}^{\bar\alpha}_{\bar\sigma}(\bar\mu)\bar\sigma(W_{\bar\mu}(\xi)_i)\mathcal{E}^{\bar\alpha}_{\bar\sigma}(\bar\xi)^*))V^*_\xi)W\\
&= \sum_{\xi,i}\sqrt{\lambda}\gamma((\bar R^*_\xi\rho_\xi(W_\mu(\bar\xi)^*_i)\otimes j(W_{\bar\mu}(\xi)_i)V^*_\xi)W\\
&= Y(\mu).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{E}^{op}(\tilde\sigma^\alpha,\tilde\mu^\beta) &= \tilde\mu^\beta(Y(\sigma)^*)Y(\sigma) = \tilde\mu^\beta(W^*\gamma(Y(\sigma)^*_0))\gamma(Y(\sigma)_0)W\\
&= E_\Delta(U^\beta(\mu)(\mu\otimes id)(Y(\sigma)^*_0))Y(\sigma)_0)\\
&= \lambda\sum_{\xi,i}E_\Delta(U^\beta(\mu)(\mu\otimes id)(V_\xi)(\mu(\rho_\xi(W_\sigma(\bar\xi)_i)\bar R_\xi)\bar R^*_\xi\rho_\xi(W_\sigma(\bar\sigma)^*_i)\otimes 1)V^*_\xi)\\
&= \lambda\sum_{\xi,i}\gamma(\mu(W_\sigma(\bar\xi)_i)\otimes 1)E_\Delta(V_\xi(\mathcal{E}^\beta_\mu(\xi)\mu(\bar R_\xi)\bar R^*_\xi\otimes 1)V^*_\xi)\gamma(W_\sigma(\bar\xi)^*_i\otimes 1)
\end{aligned}$$

Using the braiding fusion equation, we have $\mathcal{E}^{\beta}_{\mu}(\xi)\mu(\bar{R}_{\xi}) = \rho_{\xi}(\mathcal{E}^{\beta}_{\mu}(\bar{\xi})^*)\bar{R}_{\xi}$. Thus,

$$\begin{aligned}
\mathcal{E}^{op}(\widetilde{\sigma}^{\alpha}, \widetilde{\mu}^{\beta}) &= \lambda \sum_{\xi,i} \gamma(\mu(W_{\sigma}(\bar{\xi})_i) \otimes 1) E_{\Delta}(V_{\xi}(\rho_{\xi}(\mathcal{E}^{\beta}_{\mu}(\bar{\xi})^*)\bar{R}_{\xi}\bar{R}^*_{\xi} \otimes 1)V^*_{\xi})\gamma(W_{\sigma}(\bar{\xi})^*_i \otimes 1) \\
&= \lambda \sum_{\xi,i} \gamma(\mu(W_{\sigma}(\bar{\xi})_i)\mathcal{E}^{\beta}_{\mu}(\bar{\xi})^* \otimes 1) E_{\Delta}(V_{\xi}(\bar{R}_{\xi}\bar{R}^*_{\xi} \otimes 1)V^*_{\xi})\gamma(W_{\sigma}(\bar{\xi})^*_i \otimes 1) \\
&= \sum_{\xi,i} \gamma(\mu(W_{\sigma}(\bar{\xi})_i)\mathcal{E}^{\beta}_{\mu}(\bar{\xi})^* W_{\sigma}(\bar{\xi})^*_i \otimes 1) \\
&= \gamma(\mathcal{E}^{\beta}_{\mu}(\sigma)^* \otimes 1).
\end{aligned}$$

This proves the statement. Q.E.D.

With the above result, it is easy to show the following:

Corollary 6.6 $\alpha^{-}_{\widetilde{\sigma}^{\beta}} = Ad(Y(\sigma)^* U^{\bar{\beta}}(\bar{\sigma}^{op})) \cdot (id \otimes \bar{\sigma}^{op})$.

7 When $\{\rho_{\xi}\}_{\xi \in \Delta_0}$ Has a Braiding

In this section, we describe the Longo-Rehren inclusions when the system Δ has a braiding. Part of the contents in this section has already been discussed by several authors [8][20][28]. However, the Galois correspondence established in subsection 2-4 allows us to perform more detailed analysis.

Throughout this section, we assume that the system Δ has a braiding $\{\mathcal{E}(\xi,\eta)\}$. We set

$$\mathcal{E}^{+}_{\xi}(\eta) := \mathcal{E}^{+}(\xi,\eta) = \mathcal{E}(\xi,\eta),$$
$$\mathcal{E}^{-}_{\xi}(\eta) := \mathcal{E}^{-}(\xi,\eta) = \mathcal{E}(\eta,\xi)^*.$$

We use the notation $(\sigma \widetilde{\otimes} \mu^{op})^{+-}$ for $\widetilde{\sigma}^{+}\widetilde{\mu^{op}}^{-}$. We denote by ω_{ξ} the statistical phase $d(\xi)\phi_{\xi}(\mathcal{E}(\xi,\xi))$.

Though we have seen in Section 4 that in general half braidings, other than ordinary braidings, are required in order to obtain all the sectors in $D(\Delta)$, one might think that there should be some way to do so just using the given braiding in this situation with extra symmetry. Indeed, it is a natural question whether the system $\{(\rho_{\xi} \widetilde{\otimes} \rho_{\eta}{}^{op})^{+-}\}_{\xi,\eta \in \Delta_0}$ gives all the irreducibles in $D(\Delta)$. Here we need to consider $\widetilde{\rho^{op}_{\eta}}^{-}$ rather than $\widetilde{\rho^{op}_{\eta}}^{+}$ because we have $[\widetilde{\rho_{\xi}}^{+}] = [\widetilde{\rho^{op}_{\bar{\xi}}}^{+}]$ thanks to Proposition 6.4. It turns out that this is the case if and only if the braiding of Δ is non-degenerate [8][20][28]. A sector ρ_{ξ} is said to be *degenerate* if the monodromy operator $m(\xi,\eta) := \mathcal{E}(\eta,\xi)\mathcal{E}(\xi,\eta)$ equals to 1 for all $\eta \in \Delta_0$. Δ is said to be *non-degenerate* if ρ_e is the only degenerate sector. We denote by Δ^d_0 the set of all ξ with degenerate ρ_{ξ}.

Before starting the proof of the above statement, we consider a more general situation as in [8] to treat the degenerate case such as the systems of even bimodules for Hecke algebra subfactors. We assume that there exists a larger system $\hat{\Delta} \supset \Delta$ with a braiding $\{\mathcal{E}(\xi,\eta)\}$ whose restriction to Δ is the given one. Even when neither $\xi \in \hat{\Delta}_0$ nor $\eta \in \hat{\Delta}_0$ belongs to Δ_0, $\rho_{\xi}\rho_{\bar{\eta}}$ may be decomposed into sectors in Δ. In such a case, we set

$$\mathcal{E}^{+-}_{\xi,\bar{\eta}}(\zeta) = \mathcal{E}^{+}_{\xi}(\zeta)\rho_{\xi}(\mathcal{E}^{-}_{\bar{\eta}}(\zeta)),$$

which is a half braiding of $\rho_{\xi,\bar{\eta}} := \rho_\xi \rho_{\bar{\eta}}$. Note that when $\xi, \eta \in \Delta_0$, we have $[\widetilde{\rho_{\xi,\bar{\eta}}}^{+-}] = [(\rho_\xi \widetilde{\otimes} \rho_\eta{}^{op})^{+-}]$ thanks to Proposition 6.4. More generally, when σ and μ are direct sums of sectors in $\hat{\Delta}$ (we do not necessarily assume that $\sigma\bar{\mu}$ is decomposed into sectors in Δ), we can define $\widetilde{\sigma\bar{\mu}}^{+-}$ and $(\sigma \widetilde{\otimes} \mu^{op})^{+-}$ in the same way as above though they may not belong to $D(\Delta)$. Following [28], we define the "permutant" Δ' of Δ in $\hat{\Delta}$ by

$$\Delta' = \{\rho_\xi \in \hat{\Delta}; m(\xi, \eta) = 1,\ \forall \eta \in \Delta_0\}.$$

It is easy to show that Δ' is a closed subsystem.

Theorem 7.1 *In the above situation, the following hold:*

(i) *If $\xi_1, \xi_2, \eta_1, \eta_2 \in \hat{\Delta}$,*

$$\begin{aligned}
&(\widetilde{\rho_{\xi_1,\bar{\eta_1}}}^{+-}, \widetilde{\rho_{\xi_2,\bar{\eta_2}}}^{+-}) \\
&\quad = \{\gamma(X \otimes 1); X \in (\rho_{\xi_1,\bar{\eta_1}}, \rho_{\xi_2,\bar{\eta_2}}),\ X\rho_{\xi_1}(m(\bar{\eta_1}, \zeta)) = \rho_{\xi_2}(m(\bar{\eta_2}, \zeta))X,\ \forall \zeta \in \Delta_0\} \\
&\quad = \{\gamma(X \otimes 1); X = \sum_{\nu \in \Delta_0'} \sum_{i=1}^{N_{\xi_1,\nu}^{\xi_2}} T_i^{(\xi_2}{}_{\xi_1,\nu)*} \rho_{\xi_1}(X(\nu)_i), \quad X(\nu)_i \in (\rho_{\bar{\eta_1}}, \rho_\nu \rho_{\bar{\eta_2}})\}.
\end{aligned}$$

In the above, the expression

$$X = \sum_{\nu \in \Delta_0'} \sum_{i=1}^{N_{\xi_1,\nu}^{\xi_2}} T_i^{(\xi_2}{}_{\xi_1,\nu)*} \rho_{\xi_1}(X(\nu)_i)$$

is unique.

(ii) *Let $\sigma_1, \sigma_2, \mu_1, \mu_2$ be finite direct sums of sectors in $\hat{\Delta}$. Then,*

$$((\sigma_1 \widetilde{\otimes} \mu_1{}^{op})^{+-}, (\sigma_2 \widetilde{\otimes} \mu_2{}^{op})^{+-}) = \{\sum_{\xi \in \Delta_0^d} \gamma(Y_\xi V_\xi^*)W; Y_\xi \in (\rho_\xi \sigma_1 \otimes \rho_\xi^{op} \mu_1^{op}, \sigma_2 \otimes \mu_2^{op})\}.$$

In particular, when Δ is non-degenerate, we have

$$((\sigma_1 \widetilde{\otimes} \mu_1{}^{op})^{+-}, (\sigma_2 \widetilde{\otimes} \mu_2{}^{op})^{+-}) = \{\gamma(Y); Y \in ((\sigma_1 \otimes \mu_1^{op}), (\sigma_2 \otimes \mu_2^{op}))\}.$$

Proof. (i) In the same way as in the proof of Theorem 4.6, we can show

$$\begin{aligned}
&(\widetilde{\rho_{\xi_1,\bar{\eta_1}}}^{+-}, \widetilde{\rho_{\xi_2,\bar{\eta_2}}}^{+-}) \\
&\quad = \{\gamma(X \otimes 1); X \in (\rho_{\xi_1,\bar{\eta_1}}, \rho_{\xi_2,\bar{\eta_2}}),\ \mathcal{E}_{\xi_2,\bar{\eta_2}}^{+-}(\zeta)X = \rho_\zeta(X)\mathcal{E}_{\xi_1,\bar{\eta_1}}^{+-}(\zeta),\ \forall \zeta \in \Delta_0\}.
\end{aligned}$$

The braiding fusion equation implies that the condition $\mathcal{E}_{\xi_2,\bar{\eta_2}}^{+-}(\zeta)X = \rho_\zeta(X)\mathcal{E}_{\xi_1,\bar{\eta_1}}^{+-}$ is equivalent to

$$X\rho_{\xi_1}(m(\bar{\eta_1}, \zeta)) = \rho_{\xi_2}(m(\bar{\eta_2}, \zeta))X, \qquad (*)$$

which shows the first equality. We claim that every $X \in (\rho_{\xi_1,\bar{\eta_1}}, \rho_{\xi_2,\bar{\eta_2}})$ can be uniquely expressed as

$$\sum_{\nu\in\hat{\Delta}_0}\sum_{i=1}^{N_{\xi_1,\nu}^{\xi_2}} T(\tbinom{\xi_2}{\xi_1,\nu})_i^*\rho_{\xi_1}(X(\nu)_i), \quad X(\nu)_i \in (\rho_{\bar{\eta_1}}, \rho_\nu\rho_{\bar{\eta_2}}).$$

Indeed, we set $X(\nu)_i = \frac{d(\xi_1)d(\nu)}{d(\xi_2)}\phi_{\xi_1}(T(\tbinom{\xi_2}{\xi_1,\nu})_i X) \in (\rho_{\bar{\eta_1}}, \rho_\nu\rho_{\bar{\eta_2}})$. Using the Frobenius reciprocity, we get

$$\begin{aligned}
\sum_{\nu\in\hat{\Delta}_0}\sum_{i=1}^{N_{\xi_1,\nu}^{\xi_2}} T(\tbinom{\xi_2}{\xi_1,\nu})_i^*\rho_{\xi_1}(X(\nu)_i) &= \sum_{\nu\in\hat{\Delta}_0}\sum_{i=1}^{N_{\xi_1,\nu}^{\xi_2}} \frac{d(\xi_1)d(\nu)}{d(\xi_2)} T(\tbinom{\xi_2}{\xi_1,\nu})_i^*\rho_{\xi_1}(\phi_{\xi_1}(T(\tbinom{\xi_2}{\xi_1,\nu})_i X)) \\
&= \sum_{\nu\in\hat{\Delta}_0}\sum_{i=1}^{N_{\xi_1,\nu}^{\xi_2}} d(\xi_1)^2 \bar{R}_{\xi_1}^*\rho_{\xi_1}(T(\tbinom{\nu}{\bar{\xi_1},\xi_2})_i\phi_{\xi_1}(\rho_{\xi_1}(T(\tbinom{\nu}{\bar{\xi_1},\xi_2})_i^*)\bar{R}_{\xi_1}X)) \\
&= \sum_{\nu\in\hat{\Delta}_0}\sum_{i=1}^{N_{\xi_1,\nu}^{\xi_2}} d(\xi_1)^2 \bar{R}_{\xi_1}^*\rho_{\xi_1}(T(\tbinom{\nu}{\bar{\xi_1},\xi_2})_i T(\tbinom{\nu}{\bar{\xi_1},\xi_2})_i^*\phi_{\xi_1}(\bar{R}_{\xi_1}X)) \\
&= d(\xi_1)^2 \bar{R}_{\xi_1}^*\rho_{\xi_1}(\phi_{\xi_1}(\bar{R}_{\xi_1}X)) \\
&= d(\xi_1)\bar{R}_{\xi_1}^*\rho_{\xi_1}(\rho_{\bar{\xi_1}}(X)R_{\xi_1}) \\
&= X,
\end{aligned}$$

which shows the claim. Since the above expression is unique, the condition $(*)$ is equivalent to

$$\rho_\nu(m(\bar{\eta}_2,\zeta))X(\nu)_i = X(\nu)_i m(\bar{\eta}_1,\zeta), \quad \forall\zeta\in\Delta_0,\ \forall\nu\in\hat{\Delta}_0.$$

The braiding fusion equation implies that this is equivalent to

$$m(\zeta,\nu)\rho_\zeta(X(\nu)_i) = \rho_\zeta(X(\nu)_i), \quad \forall\zeta\in\Delta_0,\ \forall\nu\in\hat{\Delta}_0.$$

We show that if $X(\nu)_i$ is not zero, $m(\zeta,\nu) = 1$ for all $\zeta\in\Delta_0$. Indeed, if it is the case, we have $\phi_\zeta(m(\zeta,\nu))X(\nu)_i = X(\nu)_i$, which implies $\phi_\zeta(m(\zeta,\nu)) = 1$ because it is a scalar. Thus, we get

$$\phi_\zeta((1-m(\zeta,\nu))^*)(1-m(\zeta,\nu))) = 2-\phi_\zeta(m(\zeta,\nu))-\phi_\zeta(m(\zeta,\nu)^*) = 0,$$

and so $m(\zeta,\nu) = 1$. This proves the statement.

(ii) By simple computation, we can see that $\gamma(Y)W \in ((\sigma_1 \widetilde{\otimes} {\mu_1}^{op})^{+-}, (\sigma_2 \widetilde{\otimes} {\mu_2}^{op})^{+-})$ if and only if $Y \in (\gamma(\sigma_1\otimes\mu_1^{op}), \sigma_2\otimes\mu_2^{op})$ and the following:

$$Y\gamma(U_1^*)W = U_2^*\gamma(Y)W, \qquad (**)$$

where $U_i = \sum_\xi V_\xi(\mathcal{E}_{\sigma_i}^+(\xi)\otimes j(\mathcal{E}_{\mu_i}^-(\xi)))(\sigma_i\otimes\mu_i^{op})(V_\xi^*)$. Let

$$Y_\xi = YV_\xi \in (\rho_\xi\sigma_1\otimes\rho_\xi^{op}\mu_1^{op}, \sigma_2\otimes\mu_2^{op}).$$

Then, the left-hand side of $(**)$ is

$$\begin{aligned}
Y\gamma(U_1^*)W &= \sum_{\xi,\eta,\zeta\in\Delta_0}\sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}}Y_\xi\hat{\rho}_\xi(U_1^*V_\eta)T^\zeta_{\xi,\eta}V_\zeta^* \\
&= \sum_{\xi,\eta,\zeta\in\Delta_0}\sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}}Y_\xi\hat{\rho}_\xi((\sigma_1\otimes\mu_1^{op})(V_\eta)(\mathcal{E}^+(\sigma_1,\eta)^*\otimes j(\mathcal{E}^-(\mu_1,\eta)^*))T^\zeta_{\xi,\eta}V_\zeta^* \\
&= \sum_{\xi,\eta,\zeta\in\Delta_0}\sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}}(\sigma_2\otimes\mu_2^{op})(V_\eta)Y_\xi\hat{\rho}_\xi(\mathcal{E}^+(\sigma_1,\eta)^*\otimes j(\mathcal{E}^-(\mu_1,\eta)^*))T^\zeta_{\xi,\eta}V_\zeta^*.
\end{aligned}$$

On the other hand, the right-hand side is

$$\begin{aligned}
U_2^*\gamma(Y)W &= \sum_{\xi,\eta,\zeta\in\Delta_0}\sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}}(\sigma_2\otimes\mu_2^{op})(V_\xi)(\mathcal{E}^+(\sigma_2,\xi)^*\otimes j(\mathcal{E}^-(\mu_2,\xi)^*))\hat{\rho}_\xi(Y_\eta)T^\zeta_{\xi,\eta}V_\zeta^* \\
&= \sum_{\xi,\eta,\zeta\in\Delta_0}\sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}}(\sigma_2\otimes\mu_2^{op})(V_\xi)Y_\eta\hat{\rho}_\eta(\mathcal{E}^+(\sigma_1,\xi)^*\otimes j(\mathcal{E}^-(\mu_1,\xi)^*)) \\
&\times\ (\mathcal{E}^+(\eta,\xi)^*\otimes j(\mathcal{E}^-(\eta,\xi)^*))T^\zeta_{\xi,\eta}V_\zeta^* \\
&= \sum_{\xi,\eta,\zeta\in\Delta_0}\sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}}(\sigma_2\otimes\mu_2^{op})(V_\xi)Y_\eta\hat{\rho}_\eta(\mathcal{E}^+(\sigma_1,\xi)^*\otimes j(\mathcal{E}^-(\mu_1,\xi)^*)) \\
&\times\ (m(\eta,\xi)^*\otimes 1)T^\zeta_{\eta,\xi}V_\zeta^*
\end{aligned}$$

where we use the braiding fusion equation in the second equality. Thus, $(**)$ is equivalent to the following for $\forall\eta,\zeta\in\Delta_0$.

$$\begin{aligned}
&\sum_{\xi\in\Delta_0}\sqrt{d(\xi)}Y_\xi\hat{\rho}_\xi(\mathcal{E}^+(\sigma_1,\eta)^*\otimes j(\mathcal{E}^-(\mu_1,\eta)^*))T^\zeta_{\xi,\eta} \\
&\quad= \sum_{\xi\in\Delta_0}\sqrt{d(\xi)}Y_\xi\hat{\rho}_\xi(\mathcal{E}^+(\sigma_1,\eta)^*\otimes j(\mathcal{E}^-(\mu_1,\eta)^*))(m(\xi,\eta)^*\otimes 1)T^\zeta_{\xi,\eta}.
\end{aligned}$$

This shows that if ξ is a degenerate sector, there is no further extra condition for Y_ξ. Now, we assume the above equality holds. Multiplying the both sides by $\hat{\rho}_\zeta(\bar{R}_{\sigma_1}\otimes j(\bar{R}_{\mu_1}))$ from right, we get

$$\begin{aligned}
&\sum_{\xi\in\Delta_0}\sqrt{d(\xi)}Y_\xi\hat{\rho}_\xi(\mathcal{E}^+(\sigma_1,\eta)^*\otimes j(\mathcal{E}^-(\mu_1,\eta)^*)\hat{\rho}_\eta(\bar{R}_{\sigma_1}\otimes j(\bar{R}_{\mu_1})))T^\zeta_{\xi,\eta} \\
&\quad= \sum_{\xi\in\Delta_0}\sqrt{d(\xi)}Y_\xi\hat{\rho}_\xi(\mathcal{E}^+(\sigma_1,\eta)^*\otimes j(\mathcal{E}^-(\mu_1,\eta)^*)\hat{\rho}_\eta(\bar{R}_{\sigma_1}\otimes j(\bar{R}_{\mu_1})))(m(\xi,\eta)^*\otimes 1)T^\zeta_{\xi,\eta}.
\end{aligned}$$

Using the braiding fusion equation, we have

$$(\mathcal{E}^+(\sigma_1,\eta)^*\otimes j(\mathcal{E}^-(\mu_1,\eta)^*))\hat{\rho}_\eta(\bar{R}_{\sigma_1}\otimes j(\bar{R}_{\mu_1})) = (\bar{R}_{\sigma_1}\otimes j(\bar{R}_{\mu_1}))(\mathcal{E}^+(\bar{\sigma}_1,\eta)\otimes j(\mathcal{E}^-(\bar{\mu}_1,\eta))),$$

and so

$$\begin{aligned}&\sum_{\xi\in\Delta_0}\sqrt{d(\xi)}Y'_\xi(\mathcal{E}^+(\bar\sigma_1,\eta)\otimes j(\mathcal{E}^-(\bar\mu_1,\eta)))T^\zeta_{\xi,\eta}\\ =\ &\sum_{\xi\in\Delta_0}\sqrt{d(\xi)}Y'_\xi(\mathcal{E}^+(\bar\sigma_1,\eta)\otimes j(\mathcal{E}^-(\bar\mu_1,\eta)))(m(\xi,\eta)^*\otimes 1)T^\zeta_{\xi,\eta}.\end{aligned}$$

where

$$Y'_\xi = Y_\xi\hat\rho_\xi(\bar R_{\sigma_1}\otimes j(\bar R_{\mu_1}))\in(\hat\rho_\xi,\sigma_2\bar\sigma_1\otimes\mu_2^{op}\bar\mu_1{}^{op}).$$

Note that Y'_ξ is a multiple of an isometry and $\{Y'_\xi\}_\xi$ are mutually orthogonal. Moreover, since we have

$$Y_\xi = d(\sigma_1)d(\mu_1)(\sigma_2\otimes\mu_2^{op})(R_{\sigma_1^*}\otimes R^*_{\mu_1})Y'_\xi,$$

Y_ξ is not zero if and only if Y'_ξ is not zero. Therefore, if Y_ξ is not zero, we have $T^\zeta_{\xi,\eta} = (m(\xi,\eta)^*\otimes 1)T^\zeta_{\xi,\eta}$ for all $\eta,\zeta\in\Delta_0$, or equivalently $T(^\zeta_{\xi,\eta})_i = m(\xi,\eta)^*T(^\zeta_{\xi,\eta})_i$ for all $i = 1,2,\cdots N^\zeta_{\xi,\eta}$, $\eta,\zeta\in\Delta_0$. Thus, $m(\xi,\eta)=1$ for all $\eta\in\Delta_0$, which proves the statement. Q.E.D.

The above theorem implies the following [8][20][28]:

Corollary 7.2 *If the braiding of Δ is non-degenerate, $\{(\rho_\xi\widetilde{\otimes}\rho_\eta{}^{op})^{+-}\}_{\xi,\eta\in\Delta_0}$ gives the set of irreducibles in $D(\Delta)$. $D(\Delta)$ is isomorphic, as a category, to the direct product of the category generated by Δ and its opposite. The irreducible decomposition of the restriction $\gamma|_B$ of the canonical endomorphism γ to B is given by*

$$[\gamma|_B] = \bigoplus_{\xi\in\Delta_0}[(\rho_\xi\widetilde{\otimes}\rho_\xi{}^{op})^{+-}].$$

Proof. (ii) of the above theorem shows that $\{(\rho_\xi\widetilde{\otimes}\rho_\eta{}^{op})^{+-}\}_{\xi,\eta\in\Delta_0}$ are not mutually equivalent irreducible sectors in $D(\Delta)$. Since the global index of $A\supset B$ is λ^2, there is no other irreducible sector in $D(\Delta)$. Thanks to the Frobenius reciprocity, we have

$$\begin{aligned}\dim((\rho_\xi\widetilde{\otimes}\rho_\eta{}^{op})^{+-},\gamma|_B) &= \dim(\iota,\iota\cdot(\rho_\xi\widetilde{\otimes}\rho_\eta{}^{op})^{+-})\\ &= \dim(\iota,(\rho_\xi\otimes\rho_\eta^{op})\cdot\iota)\\ &= \dim(\gamma,(\rho_\xi\otimes\rho_\eta^{op}))\\ &= \delta_{\xi,\eta},\end{aligned}$$

which proves the last statement. Q.E.D.

It is easy to show that in the above situation, the principal graph of $A\supset B$ and that of the dual inclusion are the same. For a more detailed description of the dual inclusion, we have the following:

Proposition 7.3 *We assume that the braiding of Δ is non-degenerate. We set $V'_\xi := \gamma(V'_{\xi,0})W$ where*

$$V'_{\xi,0} := \sum_{\xi,\eta,\zeta}\sqrt{\frac{d(\xi)d(\zeta)}{d(\eta)}}V_\eta T^\eta_{\zeta,\xi}{}^*(m(\zeta,\xi)^*\otimes 1)V^*_\zeta.$$

Then, V'_ξ is an isometry in $((\rho_\xi \widetilde{\otimes} {\rho_\eta}^{op})^{+-}, \gamma|_B)$. The Q-system $(\gamma|_B, W, \gamma(V))$ for the dual inclusion $B \supset \gamma(A)$ can be expressed as

$$\gamma|_B(x) = \sum_{\xi \in \Delta_0} V'_\xi (\rho_\xi \widetilde{\otimes} {\rho_\xi}^{op})^{+-}(x) {V'_\xi}^*,$$

$$W = V'_e,$$

$$\gamma(V) = \sum_{\xi,\eta,\zeta} \sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}} \gamma(V'_\eta) V'_\xi \gamma((m(\xi,\eta)\otimes 1) T^\zeta_{\xi,\eta}) {V'_\zeta}^*.$$

Proof. From the previous corollary, we know $\dim((\rho_\xi \widetilde{\otimes} {\rho_\xi}^{op})^{+-}, \gamma|_B) = 1$. As usual we can see that $\gamma(V'_{\xi,0})W \in ((\rho_\xi \widetilde{\otimes} {\rho_\xi}^{op})^{+-}, \gamma|_B)$ if and only if the following two conditions hold:

$$V'_{\xi,0} \in (\gamma \cdot \hat{\rho}_\xi, \gamma), \quad V'_{\xi,0}\gamma(U^*)W = W V'_{\xi,0},$$

where

$$U = \sum_\eta V_\eta (\mathcal{E}^+(\xi,\eta) \otimes j(\mathcal{E}^-(\xi,\eta))) \hat{\rho}_\xi(V_\eta^*).$$

We assume that these hold and show that $V'_{\xi,0}$ is proportional to the one given in the statement. Direct computation shows that when the first condition holds, the second one $V'_{\xi,0}\gamma(U^*)W = W V'_{\xi,0}$ is equivalent to

$$\sum_\mu \sqrt{\frac{d(\mu)}{d(\zeta)}} V_\nu^* V'_{\xi,0} V_\mu \hat{\rho}_\mu(\mathcal{E}^+(\xi,\eta)^* \otimes j(\mathcal{E}^-(\xi,\eta)^*)) T^\zeta_{\mu,\eta} = \sum_\tau \sqrt{\frac{d(\nu)}{d(\tau)}} T^\tau_{\nu,\eta} V_\tau^* V'_{\xi,0} V_\zeta,$$

for all $\eta, \zeta, \nu \in \Delta_0$. We set $\nu = e$. Then, the right-hand side is

$$\frac{1}{\sqrt{d(\eta)}} V_\eta^* V'_{\xi,0} V_\zeta.$$

Note that $V_e^* V'_{\xi,0} V_\mu \in (\hat{\rho}_\mu \hat{\rho}_\xi, id)$ in the left-hand side survives only when $\mu = \bar{\xi}$, and it is proportional to $R_\xi^* \otimes j(R_\xi^*)$ if $\mu = \bar{\xi}$. Thus, there exists a complex number c such that

$$V_\eta^* V'_{\xi,0} V_\zeta = c \sqrt{\frac{d(\xi)d(\eta)}{d(\zeta)}} (R_\xi^* \otimes j(R_\xi^*)) \hat{\rho}_{\bar{\xi}}(\mathcal{E}^+(\xi,\eta)^* \otimes j(\mathcal{E}^-(\xi,\eta)^*)) T^\zeta_{\bar{\xi},\eta}.$$

Using the braiding fusion equation and the Frobenius reciprocity, this equals to

$$c\overline{\omega_\xi}^2 \sqrt{\frac{d(\zeta)}{d(\xi)d(\eta)}} {T^\eta_{\xi,\zeta}}^* (\mathcal{E}^+(\xi,\zeta)^* \otimes j(\mathcal{E}^-(\xi,\zeta)^*)) = c\overline{\omega_\xi}^2 \sqrt{\frac{d(\zeta)}{d(\eta)d(\xi)}} {T^\eta_{\zeta,\xi}}^* (m(\zeta,\xi)^* \otimes 1).$$

Thus, we get

$$V'_{\xi,0} = c\overline{\omega_\xi}^2 \sum_{\eta,\zeta} \sqrt{\frac{d(\zeta)}{d(\xi)d(\eta)}} V_\eta {T^\eta_{\zeta,\xi}}^* (m(\zeta,\xi)^* \otimes 1) V_\zeta^*.$$

To see that $|c| = d(\xi)$ is the correct normalization, it suffices to show $E_\gamma(V'_\xi V'^*_\xi) = d(\xi)^2/\lambda$, where E_γ is the conditional expectation from B to $\gamma(A)$. Indeed, it is equal to

$$\begin{aligned}\gamma(V'_{\xi,0})E_\gamma(WW^*)\gamma(V'^*_{\xi,0}) &= \frac{1}{\lambda}\gamma(V'_{\xi,0}V'^*_{\xi,0}) = \frac{|c|^2}{\lambda}\sum_{\eta,\zeta}\frac{d(\zeta)}{d(\xi)d(\eta)}N^\eta_{\zeta,\xi}V_\eta V_\eta^* \\ &= \frac{|c|^2}{\lambda}\sum_\eta V_\eta V_\eta^* = \frac{|c|^2}{\lambda}.\end{aligned}$$

We set $c = d(\xi)\omega_\xi^2$. Then, $V'_e = W$ is obvious. To show the last statement, we compute the quantity

$$V'^*_\xi\gamma(V'^*_\eta)\gamma(V)V'_\zeta = W^*\gamma(V'^*_{\xi,0}W^*VV'_{\eta,0}V'_{\zeta,0})W = \frac{1}{\sqrt{\lambda}}E_\Delta(V'^*_{\xi,0}V'_{\eta,0}V'_{\zeta,0}).$$

Note that this belongs to

$$((\rho_\zeta \widetilde{\otimes} {\rho_\zeta}^{op})^{+-}, (\rho_\xi \widetilde{\otimes} {\rho_\xi}^{op})^{+-}(\rho_\eta \widetilde{\otimes} {\rho_\eta}^{op})^{+-}) = \gamma((\hat\rho_\zeta, \hat\rho_\xi\hat\rho_\eta)).$$

Thus, there exists $X \in (\hat\rho_\zeta, \hat\rho_\xi\hat\rho_\eta)$ such that $E_\Delta(V'^*_{\xi,0}V'_{\eta,0}V'_{\zeta,0}) = \gamma(X)$. Since $V^*E_\Delta(\cdot)V$ is the left inverse of γ, we get

$$X = V^*E_\Delta(V'^*_{\xi,0}V'_{\eta,0}V'_{\zeta,0})V = \frac{1}{\lambda}\sum_\nu d(\nu)^2(\phi_\nu\otimes\phi_\nu^{op})(V_\nu^*V'^*_{\xi,0}V'_{\eta,0}V'_{\zeta,0}V_\nu).$$

Using the definition of $V'_{\xi,0}$ and the braiding fusion equation, we get

$$\begin{aligned}&V_\nu^*V'^*_{\xi,0}V'_{\eta,0}V'_{\zeta,0}V_\nu \\ &= \sum_{\mu,\tau}\frac{d(\nu)\sqrt{d(\xi)d(\eta)d(\zeta)}}{d(\mu)}(m(\nu,\xi)\otimes 1)T^\tau_{\nu,\xi}(m(\tau,\eta)\otimes 1)T^\mu_{\tau,\eta}{T^\mu_{\nu,\zeta}}^*(m(\nu,\zeta)^*\otimes 1) \\ &= \sum_{\mu,\tau}\frac{d(\nu)\sqrt{d(\xi)d(\eta)d(\zeta)}}{d(\mu)}(m(\nu,\xi)\otimes 1)(m(\nu\xi,\eta)\otimes 1)T^\tau_{\nu,\xi}T^\mu_{\tau,\eta}{T^\mu_{\nu,\zeta}}^*(m(\nu,\zeta)^*\otimes 1).\end{aligned}$$

In the above, we can replace $\sum_\tau T^\tau_{\nu,\xi}T^\mu_{\tau,\eta}$ with $\sum_\tau \hat\rho_\nu(T^\tau_{\xi,\eta})T^\mu_{\nu,\tau}$, and we get

$$\begin{aligned}&V_\nu^*V'^*_{\xi,0}V'_{\eta,0}V'_{\zeta,0}V_\nu \\ &= \sum_{\mu,\tau}\frac{d(\nu)\sqrt{d(\xi)d(\eta)d(\zeta)}}{d(\mu)}(m(\nu,\xi)\otimes 1)(m(\nu\xi,\eta)\otimes 1)\hat\rho_\nu(T^\tau_{\xi,\eta})T^\mu_{\nu,\tau}{T^\mu_{\nu,\zeta}}^*(m(\nu,\zeta)^*\otimes 1).\end{aligned}$$

Using the braiding fusion equation several times (draw a diagram), we get

$$\begin{aligned}&(\phi_\nu\otimes\phi_\nu^{op})(V_\nu^*V'^*_{\xi,0}V'_{\eta,0}V'_{\zeta,0}V_\nu) \\ &= \sum_{\mu,\tau}\frac{d(\nu)\sqrt{d(\xi)d(\eta)d(\zeta)}}{d(\mu)}(m(\xi,\eta)\otimes 1)T^\tau_{\xi,\eta}(\phi_\nu\otimes\phi_\nu^{op})(T^\mu_{\nu,\tau}{T^\mu_{\nu,\zeta}}^*).\end{aligned}$$

Note that $(\phi_\nu \otimes \phi_\nu^{op})(T^\mu_{\nu,\tau} T^{\mu}_{\nu,\zeta}{}^*) \in (\hat{\rho}_\zeta, \hat{\rho}_\tau)$ is not zero only when $\tau = \zeta$, and that it is a scalar if $\tau = \zeta$. Since we have

$$\phi_\nu(T(^{\mu}_{\nu,\zeta})_i T(^{\mu}_{\nu,\zeta})^*_j) = \delta_{i,j} \frac{d(\mu)}{d(\nu)d(\zeta)},$$

the right-hand side equals to

$$\sum_\mu \frac{d(\mu)}{d(\nu)d(\zeta)} \sqrt{\frac{d(\xi)d(\eta)}{d(\zeta)}} N^\mu_{\nu,\zeta}(m(\xi,\eta) \otimes 1) T^\zeta_{\xi,\eta} = \sqrt{\frac{d(\xi)d(\eta)}{d(\zeta)}} (m(\xi,\eta) \otimes 1) T^\zeta_{\xi,\eta}.$$

Therefore, we get

$$X = \sqrt{\frac{d(\xi)d(\eta)}{d(\zeta)}} (m(\xi,\eta) \otimes 1) T^\zeta_{\xi,\eta},$$

and so

$$\gamma(V) = \sum_{\xi,\eta,\zeta} \sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}} \gamma(V'_\eta) V'_\xi \gamma((m(\xi,\eta) \otimes 1) T^\zeta_{\xi,\eta}) V'^*_\zeta.$$

Q.E.D.

Remark. We say that Two Q-systems (γ, V_1, W_1) and (γ, V_2, W_2) are *equivalent* if there exists a unitary $u \in (\gamma, \gamma)$ satisfying

$$V_2 = uV_1, \quad W_2 = u\gamma(u) W_1 u^*.$$

This equivalence relation precisely corresponds to inner conjugacy of the corresponding subfactors [17]. In particular, if γ is decomposed into automorphisms, the obstruction to this equivalence precisely corresponds to the 2 cohomology obstruction $H^2(G, \mathbf{T})$ for the group actions to cocycle conjugacy. When $\{\mathcal{E}(\xi,\eta)\}$ is a, not necessarily non-degenerate, braiding of Δ, we define $W_\mathcal{E}$ by

$$W_\mathcal{E} = \sum_{\xi,\eta,\zeta} \sqrt{\frac{d(\xi)d(\eta)}{\lambda d(\zeta)}} \gamma(V_\eta) V_\xi (m(\xi,\eta) \otimes 1) T^\zeta_{\xi,\eta} V^*_\zeta.$$

Then, direct computation shows that $(\gamma, V, W_\mathcal{E})$ is a Q-system too. We call the corresponding inclusion *$\mathcal{E}$-twisted Longo-Rehren inclusion of* Δ . The above proposition shows that if the braiding is non-degenerate, $B \supset \gamma(A)$ is isomorphic to the $\mathcal{E}$-twisted Longo-Rehren inclusion of Δ. It is easy to show that (γ, V, W) and $(\gamma, V, W_\mathcal{E})$ are equivalent only if $m(\xi,\eta)$ is a scalar for all ξ, η. Therefore, a priori we do not know if the Longo-Rehren inclusion is self-dual even when it possesses a non-degenerate braiding. However, many arguments for the Longo-Rehren inclusions work for the $\mathcal{E}$-twisted ones as well such as the Galois correspondence discussed in subsection 2-4.

In what follows, we assume that Δ is a closed subsystem of $\hat{\Delta}$ with a non-degenerate braiding $\{\mathcal{E}(\xi,\eta)\}$, whose restriction to Δ is not necessarily non-degenerate. We denote by $B_{\hat{\Delta}}$ the Longo-Rehren subfactor for $\hat{\Delta}$, and by $\hat{\imath}$ the inclusion map from $B_{\hat{\Delta}}$ to A. Let $\hat{\gamma} : A \longrightarrow B_{\hat{\Delta}}$ be the canonical endomorphism given by

$$\hat{\gamma}(x) = \sum_{\xi \in \hat{\Delta}_0} \hat{V}_\xi \hat{\rho}_\xi(x) \hat{V}^*_\xi.$$

We set

$$\hat{W} = \sum_{\xi,\eta,\zeta\in\hat{\Delta}_0} \sqrt{\frac{d(\xi)d(\eta)}{\lambda_{\hat{\Delta}} d(\zeta)}} \hat{V}_\xi \hat{\rho}_\xi(\hat{V}_\eta) T^\zeta_{\xi,\eta} \hat{V}^*_\zeta,$$

where $\lambda_{\hat{\Delta}}$ is the global index of $\hat{\Delta}$. The global index of Δ will be denoted by λ_Δ. For a finite direct sum σ of sectors in $\hat{\Delta}$ and a half braiding $\{\mathcal{E}^\alpha_\sigma(\xi)\}_{\xi\in\hat{\Delta}_0}$ of σ, we set

$$\hat{U}^\alpha(\sigma) := \sum_{\xi\in\hat{\Delta}_0} \hat{V}_\xi(\mathcal{E}^\alpha_\sigma(\xi)\otimes 1)(\sigma\otimes id^{op})(\hat{V}^*_\xi).$$

For $B_{\hat{\Delta}} - B_{\hat{\Delta}}$ sectors, we use the notation $\hat{\sigma}^\alpha$ instead of $\tilde{\sigma}^\alpha$, (though $\hat{\rho}_\xi$ still means $\rho_\xi \otimes \rho^{op}_\xi$). Note that $\{\widehat{\rho_{\xi,\bar{\eta}}}^{+-}\}_{\xi,\eta\in\hat{\Delta}_0}$ is a complete system of representatives of the irreducibles in $D(\hat{\Delta})$ thanks to Proposition 6.4 and Corollary 7.2. To avoid possible confusion, we denote by B_Δ, instead of B, the Longo-Rehren subfactor for Δ. For the Longo-Rehren inclusion $A \supset B_\Delta$ we use the same notation as before such as ι, γ, V, W. Using the Galois correspondence, we regard B_Δ as an intermediate subfactor of $A \supset B_{\hat{\Delta}}$. More precisely, we set $P_\Delta = \sum_{\xi\in\Delta_0} V_\xi V^*_\xi$. Then, the conditional expectation E_Δ from A to B_Δ is given by

$$E_\Delta(x) = \frac{\lambda_{\hat{\Delta}}}{\lambda_\Delta} \hat{W}^* \hat{\gamma}(x) P_\Delta \hat{W}.$$

We fix an isometry $R_\Delta \in B_\Delta$ satisfying $R_\Delta R^*_\Delta = P_\Delta$. Then, we have $V_\xi = R^*_\Delta \hat{V}_\xi$ for $\xi \in \Delta$ (See Subsection 2-4).

In [28] Ocneanu claims, without any hint of the proof, that the "bipermutant" theorem $(\Delta')' = \Delta$ holds. Thanks to the Galois correspondence, we can show the following stronger statement:

Theorem 7.4 *Let $\iota_1 : B_{\hat{\Delta}} \hookrightarrow B_\Delta$ be the inclusion map. We denote by $\lambda_{\Delta'}$ the global index of the "permutant" Δ' of Δ. Then,*

(i) $[\bar{\iota}_1\iota_1] = \bigoplus_{\xi\in\Delta'_0} [\widehat{\rho_{\xi,\bar{\xi}}}^{+-}]$.

(ii) $\lambda_\Delta \lambda_{\Delta'} = \lambda_{\hat{\Delta}}$. *In consequence,* $(\Delta')' = \Delta$ *holds.*

Proof. (i) Thanks to Corollary 7.2, we know that $[\bar{\iota}_1\iota_1]$ is contained in

$$[\hat{\gamma}|_{B_\Delta}] = \bigoplus_{\xi\in\hat{\Delta}_0} [\widehat{\rho_{\xi,\bar{\xi}}}^{+-}].$$

The Frobenius reciprocity implies $\dim(\widehat{\rho_{\xi,\bar{\xi}}}^{+-}, \bar{\iota}_1\iota_1) = \dim(\iota_1, \iota_1\widehat{\rho_{\xi,\bar{\xi}}}^{+-})$. Since $\widehat{\rho_{\xi,\bar{\xi}}}^{+-}$ is the restriction of $Ad(U^{+-}(\rho_{\xi,\bar{\xi}}))\cdot(\rho_{\xi,\bar{\xi}}\otimes id^{op})$ to $B_{\hat{\Delta}}$, we have

$$(\iota_1, \iota_1\widehat{\rho_{\xi,\bar{\xi}}}^{+-}) \subset (\hat{\iota}, \hat{\iota}\widehat{\rho_{\xi,\bar{\xi}}}^{+-}) = \mathbf{C}U^{+-}(\rho_{\xi,\bar{\xi}})\bar{R}_\xi.$$

Thus $\widehat{\rho_{\xi,\bar{\xi}}}^{+-}$ is contained in $\bar{\iota}_1\iota_1$ if and only if

$$E_\Delta(U^{+-}(\rho_{\xi,\bar{\xi}})\bar{R}_\xi) = U^{+-}(\rho_{\xi,\bar{\xi}})\bar{R}_\xi.$$

Using the braiding fusion equation, we get

$$\begin{aligned} U^{+-}(\rho_{\xi,\bar{\xi}})\bar{R}_\xi &= \sum_{\eta\in\hat{\Delta}_0} \hat{V}_\eta(\mathcal{E}^{+-}_{\xi,\bar{\xi}}(\eta)\otimes 1)(\rho_\xi\rho_{\bar{\xi}}\otimes id^{op})(\hat{V}_\eta^*)(\bar{R}_\xi\otimes 1) \\ &= \sum_{\eta\in\hat{\Delta}_0} \hat{V}_\eta(\mathcal{E}^{+-}_{\xi,\bar{\xi}}(\eta)\otimes 1)(\bar{R}_\xi\otimes 1)\hat{V}_\eta^* \\ &= \sum_{\eta\in\hat{\Delta}_0} \hat{V}_\eta(m(\eta,\xi)\rho_\eta(\bar{R}_\xi)\otimes 1)\hat{V}_\eta^*. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} E_\Delta(U^{+-}(\rho_{\xi,\bar{\xi}})\bar{R}_\xi) &= \sum_{\mu\in\Delta_0}\sum_{\eta,\zeta,\nu\in\hat{\Delta}_0} \frac{d(\mu)d(\eta)}{\lambda_\Delta\sqrt{d(\zeta)d(\nu)}}\hat{V}_\nu T^{\nu}_{\mu,\eta}{}^*\hat{\rho}_\mu(m(\eta,\xi)\rho_\eta(\bar{R}_\xi)\otimes 1)T^{\zeta}_{\mu,\eta}\hat{V}_\zeta^* \\ &= \sum_{\mu\in\Delta_0}\sum_{\eta,\zeta\in\hat{\Delta}_0}\sum_{i=1}^{N^\zeta_{\mu,\eta}} \frac{d(\mu)d(\eta)}{\lambda_\Delta d(\zeta)}\hat{V}_\zeta(T^{\zeta}_{\mu,\eta})^*_i\rho_\mu(m(\eta,\xi))T^{\zeta}_{\mu,\eta})_i\rho_\zeta(\bar{R}_\xi)\otimes 1)\hat{V}_\zeta^*. \end{aligned}$$

This means that $\widehat{\rho_{\xi,\bar{\xi}}}^{+-}$ is contained in $\bar{\iota}_1\iota_1$ if and only if the following holds for all $\zeta\in\hat{\Delta}_0$:

$$m(\zeta,\xi)\rho_\zeta(\bar{R}_\xi) = \sum_{\mu\in\Delta_0}\sum_{\eta\in\hat{\Delta}_0}\sum_{i=1}^{N^\zeta_{\mu,\eta}} \frac{d(\mu)d(\eta)}{\lambda_\Delta d(\zeta)}T^{\zeta}_{\mu,\eta})^*_i\rho_\mu(m(\eta,\xi))T^{\zeta}_{\mu,\eta})_i\rho_\zeta(\bar{R}_\xi).$$

Multiplying the both sides by $\rho_\zeta(\rho_\xi(R_\xi^*))$ from left, we see that this is equivalent to

$$m(\zeta,\xi) = \sum_{\mu\in\Delta_0}\sum_{\eta\in\hat{\Delta}_0}\sum_{i=1}^{N^\zeta_{\mu,\eta}} \frac{d(\mu)d(\eta)}{\lambda_\Delta d(\zeta)}T^{\zeta}_{\mu,\eta})^*_i\rho_\mu(m(\eta,\xi))T^{\zeta}_{\mu,\eta})_i. \qquad (*)$$

Thanks to the braiding fusion equation, this is also equivalent to

$$1 = \sum_{\mu\in\Delta_0}\sum_{\eta\in\hat{\Delta}_0}\sum_{i=1}^{N^\zeta_{\mu,\eta}} \frac{d(\mu)d(\eta)}{\lambda_\Delta d(\zeta)}T^{\zeta}_{\mu,\eta})^*_i\rho_\mu(\mathcal{E}(\eta,\xi)^*)m(\mu,\xi)^*\rho_\mu(\mathcal{E}(\eta,\xi))T^{\zeta}_{\mu,\eta})_i. \qquad (*')$$

Assume that $\widehat{\rho_{\xi,\bar{\xi}}}^{+-}$ is contained in $\bar{\iota}_1\iota_1$ and set $\zeta = e$ in $(*)$. Then, we get

$$1 = \sum_{\mu\in\Delta_0}\frac{d(\mu)^2}{\lambda_\Delta}\bar{R}^*_\mu\rho_\mu(m(\bar{\mu},\xi))\bar{R}_\mu = \sum_{\mu\in\Delta_0}\frac{d(\mu)^2}{\lambda_\Delta}\phi_{\bar{\mu}}(m(\bar{\mu},\xi)).$$

Since $\sum_{\mu\in\Delta_0} d(\mu)^2 = \lambda_\Delta$ and $|\phi_{\bar{\mu}}(m(\bar{\mu},\xi))| \le 1$, we get $\phi_{\bar{\mu}}(m(\bar{\mu},\xi)) = 1$ for all $\mu\in\Delta_0$. Thus, we get

$$\phi_{\bar{\mu}}((1-m(\bar{\mu},\xi))^*(1-m(\bar{\mu},\xi))) = 1$$

and so $m(\bar{\mu},\xi) = 1$ for all $\mu\in\Delta_0$. On the other hand, if $\xi\in\Delta_0'$, the right-hand side of $(*')$ is

$$\sum_{\mu\in\Delta_0}\sum_{\eta\in\hat{\Delta}_0}\sum_{i=1}^{N^\zeta_{\mu,\eta}} \frac{d(\mu)d(\eta)}{\lambda_\Delta d(\zeta)} = \sum_{\mu\in\Delta_0}\sum_{\eta\in\hat{\Delta}_0}\frac{d(\mu)d(\eta)}{\lambda_\Delta d(\zeta)}N^\zeta_{\mu,\eta} = \sum_{\mu\in\Delta_0}\frac{d(\mu)^2}{\lambda_\Delta} = 1,$$

which proves the statement. (ii) Since λ_Δ, $\lambda_{\Delta'}$, and $\lambda_{\hat{\Delta}}$ are the indices of $A \supset B_\Delta$, $B_\Delta \supset B_{\hat{\Delta}}$, and $A \supset B_{\hat{\Delta}}$ respectively, we have $\lambda_\Delta \lambda_{\Delta'} = \lambda_{\hat{\Delta}}$. Applying this equality to Δ', we get $\lambda_\Delta = \lambda_{(\Delta')'}$, which proves the statement because $(\Delta')' \supset \Delta$ obviously holds. Q.E.D.

Remark. Applying the Galois correspondence to the $\mathcal{E}$-twisted Longo-Rehren inclusion, we can show that the above theorem implies that $B_\Delta \supset B_{\hat{\Delta}}$ is isomorphic to the dual inclusion of the $\mathcal{E}$-twisted Longo-Rehren inclusion of Δ'. If Δ' is contained in Δ, or in other word, $\hat{\Delta}$ is a minimal non-degenerate extension of Δ in the sense of [28], then Δ' is completely degenerate, i.e. $m(\xi, \eta) = 1$ for all $\xi, \eta \in \Delta'$. Thus, $B_\Delta \supset B_{\hat{\Delta}}$ is actually isomorphic to the dual inclusion of the Longo-Rehren inclusion of Δ'. Therefore, the structure of the inclusion $B_\Delta \supset B_{\hat{\Delta}}$ itself depends only on Δ in this case, while at least a priory, the relationship between two inclusions $A \supset B_\Delta$ and $B_\Delta \supset B_{\hat{\Delta}}$ depends on the choice of the embedding $\Delta \subset \hat{\Delta}$. In fact, Ocneanu claims in [28] that the minimal non-degenerate extension $\Delta \subset \hat{\Delta}$ is "essentially unique" (whatever it means). It would be plausible if we could characterize the $B_\Delta - B_\Delta$ sectors associated with $B_\Delta \supset B_{\hat{\Delta}}$ only in terms of the inclusion $A \supset B_\Delta$.

We proceed to show how to construct the sectors in $D(\Delta)$ in the non-degenerate case. For braidings of $D(\hat{\Delta})$, we consider only the one given by Theorem 4.6, and we use the same notation $\mathcal{E}$. We take a canonical endomorphism $\gamma_1 : B_\Delta \longrightarrow B_{\hat{\Delta}}$ satisfying $\gamma_1 \cdot \gamma = \hat{\gamma}$. Then, since $\hat{\gamma}|_{B_{\hat{\Delta}}} = \bar{\iota}_1 \bar{\iota} \iota \iota_1$ and $\gamma_1|_{B_{\hat{\Delta}}} = \bar{\iota}_1 \iota_1$, we have

$$\gamma_1|_{B_{\hat{\Delta}}} = \gamma_1(W^*)\hat{\gamma}|_{B_{\hat{\Delta}}}(\cdot)\gamma_1(W).$$

For $\rho \in D(\hat{\Delta})$, we define the α-inductions of ρ to B_Δ and A by

$$\alpha_\rho^\Delta(x) = \gamma_1^{-1} \cdot Ad(\mathcal{E}(\rho, \gamma_1|_{B_{\hat{\Delta}}})) \cdot \rho \cdot \gamma_1, \quad x \in B_\Delta.$$

$$\alpha_\rho(x) = \hat{\gamma}^{-1} \cdot Ad(\mathcal{E}(\rho, \hat{\gamma}|_{B_{\hat{\Delta}}})) \cdot \rho \cdot \hat{\gamma}, \quad x \in A.$$

We begin with the following well-known lemma:

Lemma 7.5 *Let $P \supset Q$ be a unital inclusion of C^*-algebras with a faithful conditional expectation E. If X is an isometry in P such that $E(X)$ is also an isometry, then $X \in Q$.*

Proof. Compute $E((X - E(X))^*(X - E(X)))$. Q.E.D.

Lemma 7.6 *Let σ be a finite direct sum of sectors in $\hat{\Delta}$ with a half braiding $\{\mathcal{E}_\sigma^\beta(\xi)\}_{\xi \in \hat{\Delta}_0}$. Then, we have $\hat{U}^\beta(\sigma)U^\beta(\sigma)^* \in B_\Delta$ and*

$$\alpha_{\hat{\sigma}^\beta}^\Delta = Ad(\hat{U}^\beta(\sigma)U^\beta(\sigma)^*) \cdot \tilde{\sigma}^\beta.$$

Proof. Since $\gamma_1(W) \in (\gamma_1|_{B_{\hat{\Delta}}}, \hat{\gamma}|_{B_{\hat{\Delta}}})$, we have

$$\gamma_1(W)\mathcal{E}(\hat{\sigma}^\beta, \gamma_1|_{B_{\hat{\Delta}}}) = \mathcal{E}(\hat{\sigma}^\beta, \hat{\gamma}|_{B_{\hat{\Delta}}})\hat{\sigma}^\beta(\gamma_1(W)),$$

and in particular,

$$\mathcal{E}(\hat{\sigma}^\beta, \gamma_1|_{B_{\hat{\Delta}}}) = \gamma_1(W^*)\mathcal{E}(\hat{\sigma}^\beta, \hat{\gamma}|_{B_{\hat{\Delta}}})\hat{\sigma}^\beta(\gamma_1(W)).$$

Thus, for $x \in B_\Delta$ we have,

$$\begin{aligned}
\alpha_\rho^\Delta(x) &= \gamma_1^{-1} \cdot Ad(\mathcal{E}(\hat{\sigma}^\beta, \gamma_1|_{B_{\hat{\Delta}}})) \cdot \hat{\sigma}^\beta \cdot \gamma_1(x) \\
&= \gamma_1^{-1}(\gamma_1(W^*)\mathcal{E}(\hat{\sigma}^\beta, \hat{\gamma}|_{B_{\hat{\Delta}}})\hat{\sigma}^\beta(\gamma_1(Wx))\mathcal{E}(\hat{\sigma}^\beta, \gamma_1|_{B_{\hat{\Delta}}})^*) \\
&= \gamma_1^{-1}(\gamma_1(W^*)\mathcal{E}(\hat{\sigma}^\beta, \hat{\gamma}|_{B_{\hat{\Delta}}})\hat{\sigma}^\beta(\gamma_1(\gamma(xW)))\mathcal{E}(\hat{\sigma}^\beta, \gamma_1|_{B_{\hat{\Delta}}})^*) \\
&= \gamma_1^{-1}(\gamma_1(W^*)\mathcal{E}(\hat{\sigma}^\beta, \hat{\gamma}|_{B_{\hat{\Delta}}})\hat{\sigma}^\beta(\hat{\gamma}(x))\mathcal{E}(\hat{\sigma}^\beta, \hat{\gamma}_1|_{B_{\hat{\Delta}}})^*)\gamma_1(W)) \\
&= \gamma_1^{-1}(\gamma_1(W^*)\hat{\gamma}(\alpha_{\hat{\sigma}^\beta}(x))\gamma_1(W))) \\
&= W^*\gamma(\alpha_{\hat{\sigma}^\beta}(x))W \\
&= E_\Delta(\alpha_{\hat{\sigma}^\beta}(x)).
\end{aligned}$$

Since $\alpha_{\hat{\sigma}^\beta}^\Delta$ is a $*$-homomorphism, we obtain $E_\Delta(\alpha_{\hat{\sigma}^\beta}(x)) = \alpha_{\hat{\sigma}^\beta}(x)$ for every isometry $x \in B_\Delta$ thanks to Lemma 7.5, and so it holds for every $x \in B_\Delta$. Thus, Corollary 6.3 implies

$$\alpha_{\hat{\sigma}^\beta}^\Delta = Ad(\hat{U}^\beta(\sigma)) \cdot (\sigma \otimes id^{op})|_{B_\Delta}.$$

Using $\hat{U}^\beta(\sigma)(\sigma \otimes id^{op})(R_\Delta) = R_\Delta U^\beta(\sigma)$, we get

$$\alpha_{\hat{\sigma}^\beta}^\Delta(R_\Delta) = R_\Delta U^\beta(\sigma)\hat{U}^\beta(\sigma)^* \in B_\Delta$$

and $U^\beta(\sigma)\hat{U}^\beta(\sigma)^* \in B_\Delta$. Therefore, we obtain

$$\alpha_{\hat{\sigma}^\beta}^\Delta = Ad(\hat{U}^\beta(\sigma)U^\beta(\sigma)^*) \cdot Ad(U^\beta(\sigma)) \cdot (\sigma \otimes id^{op})|_{B_\Delta} = Ad(\hat{U}^\beta(\sigma)U^\beta(\sigma)^*) \cdot \tilde{\sigma}^\beta.$$

Q.E.D.

Theorem 7.7 *Let Δ be a closed subsystem of $\hat{\Delta}$. We assume that $\hat{\Delta}$ has a non-degenerate braiding $\{\mathcal{E}(\xi,\eta)\}_{\xi,\eta\in\hat{\Delta}_0}$, which is not necessarily non-degenerate on Δ. Then, every irreducible sector in $D(\Delta)$ is contained in $\widehat{\rho_{\xi,\bar{\eta}}}^{+-}$ for some $\xi, \eta \in \hat{\Delta}_0$.*

Proof. Let $\rho \in D(\Delta)$ be an irreducible sector. We claim that $\bar{\iota}_1\rho\iota_1$ contains $\widehat{\rho_{\xi,\bar{\eta}}}^{+-}$ for some $\xi, \eta \in \hat{\Delta}$. Indeed, note that $\hat{\iota}\bar{\iota}_1\rho\iota_1 = \iota\iota_1\bar{\iota}_1\rho\iota_1$ contains $\iota\rho\iota_1$ that is decomposed into sectors in $\{(\rho_\xi \otimes id^{op})\hat{\iota}\}_{\xi\in\hat{\Delta}_0}$ from the definition $D(\Delta)$. Thus, there exists an irreducible $B_{\hat{\Delta}} - B_{\hat{\Delta}}$ sector ρ' contained in $\bar{\iota}_1\rho\iota_1$ such that $\hat{\iota}\rho'$ contains $(\rho_\xi \otimes id^{op})\hat{\iota}$ for some $\xi \in \hat{\Delta}_0$. The Frobenius reciprocity implies that ρ' is contained in $\bar{\hat{\iota}}(\rho_\xi \otimes id^{op})\hat{\iota} \in D(\hat{\Delta})$, which shows the claim. We assume that $\widehat{\rho_{\xi,\bar{\eta}}}^{+-}$ is contained in $\bar{\iota}_1\rho\iota_1$. Thanks to the Frobenius reciprocity, we get

$$\dim(\alpha^\Delta_{\widehat{\rho_{\xi,\bar{\eta}}}^{+-}}\gamma_1, \rho) = \dim(\alpha^\Delta_{\widehat{\rho_{\xi,\bar{\eta}}}^{+-}}\iota_1, \rho\iota_1) = \dim(\iota_1\widehat{\rho_{\xi,\bar{\eta}}}^{+-}, \rho\iota_1) = \dim(\widehat{\rho_{\xi,\bar{\eta}}}^{+-}, \bar{\iota}_1\rho\iota_1) \neq 0.$$

Thus the above lemma shows that ρ is contained in $\widehat{\rho_{\xi,\bar{\eta}}}^{+-}\gamma_1$. Let π be an irreducible component of γ_1 such that $\widehat{\rho_{\xi,\bar{\eta}}}^{+-}\pi$ contains ρ. Then, $\iota\widehat{\rho_{\xi,\bar{\eta}}}^{+-}\pi = (\rho_\xi\rho_{\bar{\eta}} \otimes id^{op})\iota\pi$ contains $(\rho_\zeta \otimes id^{op})\iota$ for some $\zeta \in \Delta_0$. The Frobenius reciprocity implies that $\iota\pi$ contains $(\rho_\nu \otimes id^{op})\iota$ for some $\nu \in \hat{\Delta}_0$. On the other hand, we have

$$\dim(\iota\gamma_1, (\rho_\nu \otimes id^{op})\iota) = \dim(\hat{\iota}, (\rho_\nu \otimes id^{op})\hat{\iota}) = \dim(\hat{\gamma}, (\rho_\nu \otimes id^{op})) = \delta_{\nu,e}.$$

Since γ_1 contains id_{B_Δ} with multiplicity one, we conclude $\pi = id_{B_\Delta}$. Thus $\widetilde{\rho_{\xi,\bar{\eta}}}^{+-}$ contains ρ. Q.E.D.

Before closing this section, we apply the above results to the Hecke algebra subfactors [34]. In [8][28], a precise description of the asymptotic inclusions of the Hecke algebra subfactors is obtained in the cases of $SU(2)$ and $SU(3)$, which is indeed our motive example of the above theorem. We assume that Δ is isomorphic to the set of irreducible even bimodules of the $SU(n)_k$ Hecke algebra subfactor where k is the level. As in [8] we embed Δ into the full system of irreducibles $\hat{\Delta}$ associated with $SU(n)_k$ WZW model, or equivalently the set of positive energy irreducible projective representations of the loop group $LSU(n)$ [33]. $\hat{\Delta}$ has a natural $\mathbf{Z}_n$-grading and Δ is the set of grade 0 elements. For $\xi \in \hat{\Delta}_0$, $gr(\xi)$ denotes the grading of ξ. We set

$$\hat{\Delta}_{0,i} = \{\xi \in \hat{\Delta}_0; gr(\xi) = i\}.$$

Although $\hat{\Delta}$ is always non-degenerate, it is not always the case for Δ. As in [8], we assume that k is a multiple of n for simplicity. Then, there is a sector $\rho_\sigma \in \Delta$ with $d(\sigma) = 1$ such that ρ_σ^N is an inner automorphism and

$$\{id, \rho_\sigma, \rho_\sigma^2, \cdots \rho_\sigma^{n-1}\}$$

is precisely the set of degenerate sectors (Lemma 3.5 of [8]). We use the notation $\rho_{\sigma^i} = \rho_\sigma^i$, $i = 1, 2, \cdots n-1$. There exists a unique sector $\rho_f \in \Delta$ such that ρ_f is self-conjugate and $[\rho_\sigma \rho_f] = [\rho_f]$. ρ_f is the unique irreducible sector in $\hat{\Delta}$ such that $\rho_f \overline{\rho}_f$ contains ρ_{σ^i} for some $i = 1, 2, \cdots n-1$.

It is easy to see that $\widetilde{\rho_{\xi,\bar{\eta}}}^{+-}$ contains a sector in $D(\Delta)$ only if ξ and η have the same grading. On the other hand, if ξ and η in $\hat{\Delta}_0$ have the same grading, $\rho_{\xi,\bar{\eta}}$ is decomposed into sectors in Δ and $\widetilde{\rho_{\xi,\bar{\eta}}}^{+-} \in D(\Delta)$. Thanks to Theorem 7.1, we know that $\widetilde{\rho_{\xi,\bar{\eta}}}^{+-}$ is reducible if and only if $(\xi, \eta) = (f, f)$, and that $\widetilde{\rho_{\xi_1,\bar{\eta}_1}}^{+-}$ and $\widetilde{\rho_{\xi_2,\bar{\eta}_2}}^{+-}$ are equivalent if and only $(\xi_1, \eta_1) = (\sigma^i \xi_2, \sigma^i \eta_2)$ for some i. Thus, we introduce an equivalence relation in $\hat{\Delta}_{0,i} \times \hat{\Delta}_{0,i}$ for $i \neq gr(f)$ and $\hat{\Delta}_{0,i} \times \hat{\Delta}_{0,i} \setminus \{(f, f)\}$ for $i = gr(f)$ by setting $(\xi_1, \eta_1) \sim (\xi_2, \eta_2)$ if $(\xi_1, \eta_1) = (\sigma^j \xi_2, \sigma^j \eta_2)$ for some j. We denote by J_i, $i = 0, 1, \cdots n-1$ the quotients by this equivalence relation, and take a complete system of representatives $\{(\xi_s, \eta_s)\}_{s \in J_i}$. We set $J = \bigcup J_i$. Theorem 7.1 shows that if $(\xi, \eta) \neq (f, f)$ there is no non-zero intertwiner between $\widetilde{\rho_{\xi,\bar{\eta}}}^{+-}$ and $\widetilde{\rho_{f,\bar{f}}}^{+-}$. Therefore, Theorem 7.7 implies that the set of irreducibles of $D(\Delta)$ consists of the irreducible subsectors of $\widetilde{\rho_{f,\bar{f}}}^{+-}$ and $\bigcup_{s \in J} \{\widetilde{\rho_{\xi_s,\bar{\eta}_s}}^{+-}\}$.

The following lemma (Assumption 3.14 of [8]) was proved in [8] for $SU(2)$ and $SU(3)$ case.

Lemma 7.8 *$\widetilde{\rho_{f,\bar{f}}}^{+-}$ is decomposed into mutually not equivalent sectors with equal dimensions.*

Proof. Since $[\rho_\sigma \rho_f] = [\rho_f]$, we may and do assume $\rho_\sigma \rho_f = \rho_f$ by taking an appropriate representative ρ_σ. In this case, $\rho_\sigma^n = id$ holds because it is an inner automorphism implemented by a unitary in $(\rho_f, \rho_f) = \mathbf{C}$. Then, we have $(\rho_f, \rho_f \rho_{\sigma^i}) = \mathbf{C}\mathcal{E}(\sigma^i, f)$ and $(\rho_f, \rho_{\sigma^i} \rho_f) = \mathbf{C}1$. Thus, Theorem 7.1 implies

$$(\widetilde{\rho_{f,\bar{f}}}^{+-}, \widetilde{\rho_{f,\bar{f}}}^{+-}) = \{\sum_i c_i \gamma(\mathcal{E}(f, \sigma^i) \otimes 1); c_i \in \mathbf{C}\}.$$

The braiding fusion equation and the condition $\rho_\sigma\rho_f = \rho_f$ imply $\rho_\sigma(\mathcal{E}(f,\sigma)) = \overline{\omega_\sigma}\mathcal{E}(f,\sigma)$, where $\mathcal{E}(\sigma,\sigma) = \omega_\sigma$. Therefore, we get

$$\mathcal{E}(f,\sigma^i) = \rho_\sigma^{i-1}(\mathcal{E}(f,\sigma))\rho_\sigma^{i-2}(\mathcal{E}(f,\sigma))\cdots\mathcal{E}(f,\sigma) = \omega_\sigma^{-\frac{i(i-1)}{2}}\mathcal{E}(f,\sigma)^i.$$

In particular, we have $\mathcal{E}(f,\sigma)^n = \omega_\sigma^{\frac{n(n-1)}{2}}$ that is a scalar. Note that $\phi_f(\mathcal{E}(f,\sigma^i)) = 0$ for $i \neq 0$ because it belongs to $\in (\rho_\sigma^i, id)$. Let τ_f be the restriction of $\phi_{\widetilde{\rho_{f,\bar{f}}}^{+-}}$ to $(\widetilde{\rho_{f,\bar{f}}}^{+-}, \widetilde{\rho_{f,\bar{f}}}^{+-})$, which is the canonical trace. Then, the above argument shows that The pair $((\widetilde{\rho_{f,\bar{f}}}^{+-}, \widetilde{\rho_{f,\bar{f}}}^{+-}), \tau_f)$ is isomorphic to $(\mathbf{CZ}_n, \tau)$ where τ is the natural trace of the group algebra of $\mathbf{Z}_n$. This means that $(\widetilde{\rho_{f,\bar{f}}}^{+-}, \widetilde{\rho_{f,\bar{f}}}^{+-}) \cong \mathbf{C}^n$ and the trace evaluation of each minimal projection is equal, which proves the statement. Q.E.D.

Let $\gamma(E_0\otimes 1), \gamma(E_1\otimes 1), \cdots \gamma(E_{n-1}\otimes 1)$ be minimal projections of $(\widetilde{\rho_{f,\bar{f}}}^{+-}, \widetilde{\rho_{f,\bar{f}}}^{+-})$. We take isometries $X_i \in M$ satisfying $X_iX_i^* = E_i$, and set

$$\pi_i(x) = \gamma(X_i^*\otimes 1)\widetilde{\rho_{f,\bar{f}}}^{+-}(x)\gamma(X_i\otimes 1).$$

Then, the above argument shows the following for general $SU(n)$ case:

Corollary 7.9 $\bigcup_{s\in J}\{\widetilde{\rho_{\xi_s,\bar{\eta}_s}}^{+-}\} \cup \{\pi_0, \pi_1, \cdots \pi_{n-1}\}$ *is the set of irreducibles of* $D(\Delta)$.

Remark.

(i) To obtain the principal graph, we need to compute the irreducible decomposition of $X_i^*\rho_f^2(\cdot)X_i$.

(ii) The braiding of $D(\Delta)$ is given as follows:

$$\begin{aligned}
\mathcal{E}(\widetilde{\rho_{\xi_1,\bar{\eta_1}}}^{+-}, \widetilde{\rho_{\xi_2,\bar{\eta_2}}}^{+-}) &= \gamma((\rho_{\xi_2}(\mathcal{E}(\xi_1,\bar{\eta_2}))\mathcal{E}(\xi_1,\xi_2)\rho_{\xi_1}\rho_{\xi_2}(\mathcal{E}(\bar{\eta_2},\bar{\eta_1})^*)\rho_{\xi_1}(\mathcal{E}(\xi_2,\bar{\eta_2})^*))\otimes 1),\\
\mathcal{E}(\widetilde{\rho_{\xi,\bar{\eta}}}^{+-}, \pi_i) &= \gamma(X_i^*\otimes 1)\mathcal{E}(\widetilde{\rho_{\xi,\bar{\eta}}}^{+-}, \widetilde{\rho_{f,\bar{f}}}^{+-})\gamma(\rho_{\xi,\bar{\eta}}(X_i)\otimes 1),\\
\mathcal{E}(\pi_i, \widetilde{\rho_{\xi,\bar{\eta}}}^{+-}) &= \gamma(\rho_{\xi,\bar{\eta}}(X_i^*)\otimes 1)\mathcal{E}(\widetilde{\rho_{f,\bar{f}}}^{+-}, \widetilde{\rho_{\xi,\bar{\eta}}}^{+-})\gamma(X_i\otimes 1),\\
\mathcal{E}(\pi_i,\pi_j) &= \gamma(X_j^*\rho_{f,f}(X_i^*)\otimes 1)\mathcal{E}(\widetilde{\rho_{f,\bar{f}}}^{+-}, \widetilde{\rho_{f,\bar{f}}}^{+-})\gamma(\rho_{f,f}(X_j)X_i\otimes 1).
\end{aligned}$$

Thus, using Lemma 5.4 we can obtain explicit formulae for the S and T matrices in terms of the braiding of Δ and $\{E_i\}_{i=0}^{n-1}$. Actually we can evaluate them except for S_{π_i,π_j}:

$$\omega_{\widetilde{\rho_{\xi,\bar{\eta}}}^{+-}} = \omega_\xi\overline{\omega_\eta}, \qquad \omega_{\pi_i} = 1,$$

$$S_{\widetilde{\rho_{\xi_1,\bar{\eta_1}}}^{+-}, \widetilde{\rho_{\xi_2,\bar{\eta_2}}}^{+-}} = \frac{S_{\xi_1,\xi_2}\overline{S_{\eta_1,\eta_2}}}{\lambda|S_{e,e}|^2} = nS_{\xi_1,\xi_2}\overline{S_{\eta_1,\eta_2}},$$

$$S_{\widetilde{\rho_{\xi,\bar{\eta}}}^{+-},\pi_i} = \frac{S_{\xi,f}\overline{S_{\eta,f}}}{n\lambda|S_{e,e}|^2} = S_{\xi,f}\overline{S_{\eta,f}},$$

$$S_{\pi_i,\pi_j} = \frac{d(f)^4}{\lambda}\phi^3(\mathcal{E}\rho(\mathcal{E})\phi(\mathcal{E}^*\rho(\mathcal{E}^*)E_j\rho^2(E_i)\rho(\mathcal{E})\mathcal{E}^*)\rho(\mathcal{E})\mathcal{E}^*),$$

where $\phi = \phi_f$, $\rho = \rho_f$, $\mathcal{E} = \mathcal{E}(f,f)$. We leave the details.

(iii) Thanks to Theorem 7.4, we know that B_Δ is the crosse product of $B_{\hat{\Delta}}$ by the $\mathbf{Z}_n$ action arising from $\{\widehat{\rho_{\sigma,\bar{\sigma}}}^{+-}\}$. Thus, $D(\Delta)$ may also be obtained by applying "treatment of orbifold " to $D(\hat{\Delta})$ as in [3][27]. We give a more precise description of $B_\Delta \supset B_{\hat{\Delta}}$. As before we may and do assume $\rho_\sigma^n = id$ and $\rho_{\bar{\sigma}} = \rho_\sigma^{-1}$. Since $\theta := \widehat{\rho_{\sigma,\bar{\sigma}}}^{+-}$ is the restriction of

$$Ad(U^{+-}(\rho_{\sigma,\bar{\sigma}})) \cdot (\rho_{\sigma,\bar{\sigma}} \otimes id^{op})$$

and $\rho_{\sigma,\bar{\sigma}}$ is trivial now, θ is implemented by $U := U^{+-}(\rho_{\sigma,\bar{\sigma}})$ on $B_{\hat{\Delta}}$. Note that since $\rho_\xi \rho_\sigma$ is irreducible, the monodromy operator $m(\xi,\sigma)$ is a scalar, and in fact we have $m(\xi,\sigma) = e^{\frac{2\pi\sqrt{-1}}{n} gr(\xi)}$ (See [21]). The braiding fusion equation implies

$$\mathcal{E}^{+-}_{\rho_{\sigma,\bar{\sigma}}}(\xi) = \mathcal{E}(\sigma,\xi)\rho_\sigma(\mathcal{E}(\xi,\bar{\sigma})^*) = \mathcal{E}(\sigma,\xi)\mathcal{E}(\xi,\sigma) = m(\xi,\sigma).$$

Thus, B_Δ is generated by $B_{\hat{\Delta}}$ and the implementing unitary

$$U = \sum_{\xi \in \hat{\Delta}_0} e^{\frac{2\pi\sqrt{-1}}{n} gr(\xi)} \hat{V}_\xi \hat{V}_\xi^*.$$

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