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(iv) If K is 3-manifold for self dual singularity, and write

$$\lambda_2(K) = \frac{a}{b}, (a,b) = 1, b > 0.$$

Then there exists a prime number p and $m \in \mathbf{N}$ such that

$$b = p^m.$$

Remark 12. *By a change of the orientation of K, we have Table 4'and similar statements holds for the invariants.*

Table4'. $\lambda_2(-K)$ for Arnold's exceptional singularities

Type	Dolgachev numbers	Gabrielov numbers	H	E	$\lambda_0(-K)$	$\lambda_1(-K)$	$\lambda_2(-K)$
Q_{10}	$2,3,9$	$3,3,4$	3	$\frac{1}{18}$	$\frac{1}{3}$	$-\frac{11}{18}$	$\frac{2473}{648}$
Q_{11}	$2,4,7$	$3,3,5$	6	$\frac{3}{28}$	$\frac{1}{6}$	$-\frac{5}{36}$	$\frac{2785}{5184}$
Q_{12}	$3,3,6$	$3,3,6$	9	$\frac{1}{6}$	$\frac{1}{9}$	$-\frac{2}{27}$	$\frac{5}{27}$
S_{11}	$2,5,6$	$3,4,4$	8	$\frac{2}{15}$	$\frac{1}{8}$	$-\frac{9}{128}$	$\frac{261}{1024}$
S_{12}	$3,4,5$	$3,4,5$	13	$\frac{13}{60}$	$\frac{1}{13}$	$-\frac{6}{169}$	$\frac{170}{2197}$
U_{12}	$4,4,4$	$4,4,4$	16	$\frac{1}{4}$	$\frac{1}{16}$	$-\frac{3}{128}$	$\frac{49}{1024}$
Z_{11}	$2,3,8$	$2,4,5$	2	$\frac{1}{24}$	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{695}{64}$
Z_{12}	$2,4,6$	$2,4,6$	4	$\frac{1}{12}$	$\frac{1}{4}$	$-\frac{3}{8}$	$\frac{95}{64}$
Z_{13}	$3,3,5$	$2,4,7$	6	$\frac{2}{15}$	$\frac{1}{6}$	$-\frac{7}{36}$	$\frac{2593}{5184}$
W_{12}	$2,5,5$	$2,5,5$	5	$\frac{1}{10}$	$\frac{1}{5}$	$-\frac{6}{25}$	$\frac{104}{125}$
W_{13}	$3,4,4$	$2,5,6$	8	$\frac{1}{6}$	$\frac{1}{8}$	$-\frac{15}{28}$	$\frac{261}{1024}$
K_{12}	$2,3,7$	$2,3,7$	1	$\frac{1}{42}$	1	-6	69
K_{13}	$2,4,5$	$2,3,8$	2	$\frac{1}{20}$	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{551}{64}$
K_{14}	$3,3,4$	$2,3,9$	3	$\frac{1}{12}$	$\frac{1}{3}$	$-\frac{13}{18}$	$\frac{1693}{648}$

Proposition 11 is an appearance of the Arnold's strange duality in the invariants. The comprehensive understanding of this matter is expected.

Acknowledgement.

Table 4. $\lambda_2(K)$ for Arnold's exceptional singularities

Type	Dolgachev numbers	Gabrielov numbers	H	E	$\lambda_0(K)$	$\lambda_1(K)$	$\lambda_2(K)$	Dual type
Q_{10}	$2,3,9$	$3,3,4$	3	$-\frac{1}{18}$	$\frac{1}{3}$	$\frac{11}{18}$	$\frac{2077}{648}$	K_{14}
Q_{11}	$2,4,7$	$3,3,5$	6	$-\frac{3}{28}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{2065}{5184}$	Z_{13}
Q_{12}	$3,3,6$	$3,3,6$	9	$-\frac{1}{6}$	$\frac{1}{9}$	$\frac{2}{27}$	$\frac{1}{9}$	Q_{12}
S_{11}	$2,5,6$	$3,4,4$	8	$-\frac{2}{15}$	$\frac{1}{8}$	$\frac{9}{128}$	$\frac{189}{1024}$	W_{13}
S_{12}	$3,4,5$	$3,4,5$	13	$-\frac{13}{60}$	$\frac{1}{13}$	$\frac{6}{169}$	$\frac{92}{2197}$	S_{12}
U_{12}	$4,4,4$	$4,4,4$	16	$-\frac{1}{4}$	$\frac{1}{16}$	$\frac{3}{128}$	$\frac{25}{1024}$	U_{12}
Z_{11}	$2,3,8$	$2,4,5$	2	$-\frac{1}{24}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{599}{64}$	K_{13}
Z_{12}	$2,4,6$	$2,4,6$	4	$-\frac{1}{12}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{71}{64}$	Z_{12}
Z_{13}	$3,3,5$	$2,4,7$	6	$-\frac{2}{15}$	$\frac{1}{6}$	$\frac{7}{36}$	$\frac{1585}{5184}$	Q_{11}
W_{12}	$2,5,5$	$2,5,5$	5	$-\frac{1}{10}$	$\frac{1}{5}$	$\frac{6}{25}$	$\frac{74}{125}$	W_{12}
W_{13}	$3,4,4$	$2,5,6$	8	$-\frac{1}{6}$	$\frac{1}{8}$	$\frac{15}{128}$	$\frac{141}{1024}$	S_{11}
K_{12}	$2,3,7$	$2,3,7$	1	$-\frac{1}{42}$	1	6	63	K_{12}
K_{13}	$2,4,5$	$2,3,8$	2	$-\frac{1}{20}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{455}{64}$	Z_{11}
K_{14}	$3,3,4$	$2,3,9$	3	$-\frac{1}{12}$	$\frac{1}{3}$	$\frac{13}{18}$	$\frac{1225}{648}$	Q_{10}

Now we have the following proposition from Table 4.

Proposition 11. *Let K and K' be the 3-manifolds for dual singularities.*

(i) $\lambda_0(K) = \lambda_0(K')$.

(ii) The Casson-Walker invariants have the duality

$$\lambda_1(K) + \lambda_1(K') = \frac{12}{H^2}.$$

(iii) If we write

$$\lambda_2(K) = \frac{a}{b}, \lambda_2(K') = \frac{a'}{b'}, (a,b) = (a',b') = 1, b > 0, b' > 0.$$

Then

$$b = b'.$$

Theorem 10 (Sato [13]). *Let $M = M(Oo0|b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ be a Seifert fibered rational homology sphere over a 2-sphere. The second Ohtsuki invariant for M is given as follows:*

$$\lambda_2(M) = \frac{3}{H}\left\{ -\left(\frac{1}{12E}\right)^2 \left(\sum_{j=1}^{n}\left(\frac{1}{\alpha_j}\right)^4 - n + 2\right) + \frac{5}{2}X^2 + (Y-1)X + \frac{1}{6}(Y^2 - 2Y) + \frac{1}{9}\right\},$$

where H denotes the order of $H_1(M; \mathbf{Z})$, $E = H\varepsilon(E)/(\alpha_1 \ldots \alpha_s) = b + \sum_{j=1}^{n}(\beta_j/\alpha_j)$ and $\varepsilon(E) = E/|E|$. Also X and Y are given by

$$X = \frac{1}{12E}\left\{\sum_{j=1}^{n}\left(\frac{1}{\alpha_j}\right)^2 - n + 2\right\},$$

$$Y = \frac{1}{4}\left\{2 + E - 3\varepsilon(E) - 12\sum_{j=1}^{n} s(\beta_j, \alpha_j)\right\},$$

where s(,) is the Dedekind sum.

By Lescop [4] and Murakami [7],

$$\lambda_1(M) = \frac{6}{H}\left\{-\frac{\varepsilon(E)}{8} + \frac{1}{24E}\left(E^2 + \sum_{j=1}^{n}\left(\frac{1}{\alpha_j}\right)^2 - n + 2\right) - \frac{1}{2}\sum_{j=1}^{n} s(\beta_j, \alpha_j)\right\}$$

(see also Sato [13].)
Therefore we obtain the table 4.

where $\mathbf{Z}_r$ *is the ring of* r*-adic integers and* i *is the natural inclusion and* p *is the projection, that is,* $\underline{\lambda_n}$ *is a reduction of* λ_n *modulo* r*.*

Definition 8 (Ohtsuki [8]). *A topological invariant* $\tau(M)$ *for any oriented rational homology 3-sphere* M *is given by*

$$\tau(M) = \sum_{n=0}^{\infty} \lambda_n (t-1)^n,$$

which is formal power series in $t-1$ *with coefficients in* $\mathbf{Q}$*.*

Proposition 9 (Ohtsuki [8]). $\tau(M)$ *satisfies the following properties;*

$$\tau(S^3) = 1,$$

$$\tau(M_1 \# M_2) = \tau(M_1)\tau(M_2),$$

$$\tau(-M)(t) = \tau(M)(t^{-1}),$$

where $M_1 \# M_2$ *is the connected sum of* M_1 *and* M_2*, and* $-M$ *is* M *with the opposite orientation.*

By H.Murakami's results [7], we have

$$\lambda_0(M) = \frac{1}{|H_1(M;\mathbf{Z})|},$$

$$\lambda_1(M) = \frac{6}{|H_1(M;\mathbf{Z})|}\lambda_{CW}(M),$$

where $\lambda_{CW}(M)$ is Casson-Walker invariant [15]. Here we use Casson's normalization, which is half of Walker's one.

The formulas of $\lambda_{CW}(M)$ for Seifert manifolds M are given in [4] and the formulas of $\lambda_2(M)$ for Seifert manifolds M are given in [13].

4 Invariants of an oriented closed 3-manifold

Witten [16] introduced a topological invariant of a closed oriented 3-manifold M, which is called the quantum G -invariant $Z_k(M;G)$, where k is an integer and G is a compact Lie group. The invariant was given in term of the Feyman path integral.

Reshetikhin and Turaev [11] gave a mathematical definition of the quantum $SU(2)$ -invariant ${\tau_r}^{SU(2)}(M)$ for closed oriented 3-manifolds and positive integer $r = k + 2$. Sato [13] gave a formula for the second Ohtsuki invariant of a Seifert fibered rational sphere over a 2-sphere.

For an odd integer r not less than 3, Kirby and Melvin [3] defined the topological invariant $\tau_r^{SO(3)}(M)$ called quantum $SO(3)$ -invariant and showed that $\tau_r^{SU(2)}(M)$ splits into the product of $\tau_3^{SU(2)}(M)$ and $\tau_r^{SO(3)}(M)$. The values of $\tau_3^{SU(2)}(M)$ are well known [3].

Let r be an odd prime. Let M a $\mathbf{Z}/r\mathbf{Z}$-homology 3-sphere ,i.e.

$$H_*(M;\mathbf{Z}/r\mathbf{Z}) \cong H_*(S^3;\mathbf{Z}/r\mathbf{Z}).$$

Theorem 6 (Murakami [7]). *Let r be an odd prime and M a $\mathbf{Z}/r\mathbf{Z}$ -homology 3-sphere. Then*

$$\tau_r^{SO(3)}(M) \in \mathbf{Z}[q],$$

where $q = \exp(2\pi\sqrt{-1}/r)$.

Let $\mathbf{Z}_{(m)}$ be $\mathbf{Z}[1/2, 1/3, \ldots, 1/m]$. We mean the order of $H_1(M;\mathbf{Z})$ by $|H_1(M;\mathbf{Z})|$.

Theorem 7 (Ohtsuki [8]). *Let M be a rational homology 3-sphere. Then there exist series of topological invariants of M, $\lambda_n = \lambda_n(M) \in \mathbf{Z}_{(\max\{|H_1(M;\mathbf{Z})|, 2n+1\})}$ for $n = 0, 1, 2, \ldots$ such that for any odd prime r which is greater than $|H_1(M;\mathbf{Z})|$ the following formula holds;*

$$\begin{aligned}{\tau_r}^{SO(3)}(M) = \left(\frac{|H_1(M;\mathbf{Z})|}{r}\right) \{&\underline{\lambda_0} + \underline{\lambda_1}(q-1) + \underline{\lambda_2}(q-1)^2 + \cdots \\ &+ \underline{\lambda_{(r-3)/2}}(q-1)^{(r-3)/2}\} + u(q-1)^{(r-1)/2},\end{aligned}$$

where $u \in \mathbf{Z}[q]$, $(\frac{x}{r})$ is the Legendre symbol. Here $\underline{\lambda_n}$ is the image of λ_n by the mapping;

$$\mathbf{Z}_{(\max\{|H_1(M;\mathbf{Z})|, 2n+1\})} \xrightarrow{i} \mathbf{Z}_r \xrightarrow{p} \mathbf{Z}/r\mathbf{Z} \xrightarrow{=} \{0, 1, \ldots, r-1\},$$

Then

$$H_i(K;\mathbf{Z}) \cong \begin{cases} \mathbf{Z} & \textit{for } i = 0, 3, \\ \mathbf{Z}/H\mathbf{Z} & \textit{for } i = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. K is a fiber space whose fiber is homeomorphic to S^1 and its base space is homeomorphic to S^2. Also K is orientable, smooth and without boundary [5]. Therefore we obtain $H_3(K;\mathbf{Z}) \cong \mathbf{Z}$.

Since S^1 is connected and S^2 is simply connected, there is the exact sequence of a fiber space. By Gysin exact sequence, we have

$$\longrightarrow H^1(S^2;\mathbf{Z}) \longrightarrow H^1(K;\mathbf{Z}) \longrightarrow H^0(S^2;\mathbf{Z})$$

$$\longrightarrow H^2(S^2;\mathbf{Z}) \longrightarrow H^2(K;\mathbf{Z}) \longrightarrow H^1(S^2;\mathbf{Z})$$

$$\longrightarrow H^3(S^2;\mathbf{Z}) \longrightarrow H^3(K;\mathbf{Z}) \longrightarrow H^2(S^2;\mathbf{Z}) \longrightarrow 0.$$

Hence we obtain

$$H^3(K;\mathbf{Z}) \cong \mathbf{Z},$$

$$H^1(K;\mathbf{Z}) \cong 0,$$

$$H^2(K;\mathbf{Z}) \cong \mathbf{Z}/m\mathbf{Z} (m \in \mathbf{N}).$$

By Poincaré duality,

$$H_0(K;\mathbf{Z}) \cong \mathbf{Z},$$

$$H_2(K;\mathbf{Z}) \cong 0,$$

$$H_1(K;\mathbf{Z}) \cong \mathbf{Z}/m\mathbf{Z}.$$

$H_1(K;\mathbf{Z}) \cong \mathbf{Z}/H\mathbf{Z}$ because $\#H_1(K;\mathbf{Z}) = H$.

This completes the proof of Lemma 4. □

Now we have the following corollary by the universal coefficient theorem.

Corollary 5. *The Seifert manifold K is a rational homology sphere.*

Table 3. E, H of Seifert manifolds for Arnold's exceptional singularities

Type	Dolgachev numbers	Gabrielov numbers	H	E
Q_{10}	$2,3,9$	$3,3,4$	3	$-\frac{1}{18}$
Q_{11}	$2,4,7$	$3,3,5$	6	$-\frac{3}{28}$
Q_{12}	$3,3,6$	$3,3,6$	9	$-\frac{1}{6}$
S_{11}	$2,5,6$	$3,4,4$	8	$-\frac{2}{15}$
S_{12}	$3,4,5$	$3,4,5$	13	$-\frac{13}{60}$
U_{12}	$4,4,4$	$4,4,4$	16	$-\frac{1}{4}$
Z_{11}	$2,3,8$	$2,4,5$	2	$-\frac{1}{24}$
Z_{12}	$2,4,6$	$2,4,6$	4	$-\frac{1}{12}$
Z_{13}	$3,3,5$	$2,4,7$	6	$-\frac{2}{15}$
W_{12}	$2,5,5$	$2,5,5$	5	$-\frac{1}{10}$
W_{13}	$3,4,4$	$2,5,6$	8	$-\frac{1}{6}$
K_{12}	$2,3,7$	$2,3,7$	1	$-\frac{1}{42}$
K_{13}	$2,4,5$	$2,3,8$	2	$-\frac{1}{20}$
K_{14}	$3,3,4$	$2,3,9$	3	$-\frac{1}{12}$

There is so-called strange duality among Arnold's 14 exceptional singularities. This duality is given by Dolgachev numbers and Gabrielov numbers for the singularities. Dolgachev numbers are self-intersection numbers for irreducible components of resolutions of the exceptional singularities. Gabrielov numbers are numbers determined by the homology lattice of Milnor fibers.

For example, looking at the table 3, Q_{10} has Dolgachev numbers $(2,3,9)$ and Gabrielov numbers $(3,3,4)$. On the other hand,K_{14} has Dolgachev numbers $(3,3,4)$ and Gabrielov numbers $(2,3,9)$.This is the duality. We can find that the orders H for dual singularities coincide. For exceptional singularities, we can find that α_j coincide with Gabrielov numbers.

Lemma 4. *Let K be the Seifert manifold for Arnold's 14 exceptional singularities and $H = \#H_1(K; \mathbf{Z})$.*

Table 2. Seifert invariants of 3-manifolds for exceptional singularities

Type	Invariants
Q_{10}	$(Oo0\|-1;(2,1),(3,1),(9,1))$
Q_{11}	$(Oo0\|-1;(2,1),(4,1),(7,1))$
Q_{12}	$(Oo0\|-1;(3,1),(3,1),(6,1))$
S_{11}	$(Oo0\|-1;(2,1),(5,1),(6,1))$
S_{12}	$(Oo0\|-1;(3,1),(4,1),(5,1))$
U_{12}	$(Oo0\|-1;(4,1),(4,1),(4,1))$
Z_{11}	$(Oo0\|-1;(2,1),(3,1),(8,1))$
Z_{12}	$(Oo0\|-1;(2,1),(4,1),(6,1))$
Z_{13}	$(Oo0\|-1;(3,1),(3,1),(5,1))$
W_{12}	$(Oo0\|-1;(2,1),(5,1),(5,1))$
W_{13}	$(Oo0\|-1;(3,1),(4,1),(4,1))$
K_{12}	$(Oo0\|-1;(2,1),(3,1),(7,1))$
K_{13}	$(Oo0\|-1;(2,1),(4,1),(5,1))$
K_{14}	$(Oo0\|-1;(3,1),(3,1),(4,1))$

The rational Euler number E of Seifert fibration for Seifert manifold $K=(Oog|b;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ is given by

$$E=b+\sum_{j=1}^{n}\frac{\beta_j}{\alpha_j}.$$

The rank of $H_1(K;\mathbf{Z})$ as a module coincides with $2g$ [9]. Since $g=0$ for exceptional singularities, $H_1(K;\mathbf{Z})$ is isomorphic to a finite abelian group. The number H of elements of $H_1(K;\mathbf{Z})$ is given by [4]

$$H=(\alpha_1\ldots\alpha_n)|E|.$$

We call H the order of $H_1(K;\mathbf{Z})$. We obtain the table 3 (see also [12]).

has a closed neighborhood, consisting of a union of fibers, which is homeomorphic to a fibered solid torus via a fiber-preserving homeomorphism. We call such a neighborhood of a fiber a fibered-neighborhood.

Any oriented Seifert fibered 3-manifold M with an orientable base space can be presented by the system of parameters [14] :

$$(Oog|b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)).$$

Here O means that M is orientable and og means that its base space is an oientable closed surface with g handles. The integer b and the coprime integers (α_j, β_j) are given for a regular fiber and singular fibers.

Assume now that V is an algebraic surface with an isolated singularity given as the locus of zeros of some polynomials in $\mathbf{C}^{n+1}$ and it is invariant under a good $\mathbf{C}^*$ action. If V has an isolated singularity and a good $\mathbf{C}^*$-action. Then K is a Seifert manifold [9] .

Now if f is a normal form in the Table 1. Variety $V = f^{-1}(0) \subseteq \mathbf{C}^3$ has the point $(0, 0, 0)$ as an isolated singularity [1] and has a natural good $\mathbf{C}^*$ -action. Hence for an exceptional singularity, K is a Seifert manifold.

The Seifert invariants of K is given in [10] (see also [9]). Thus we have Seifert invariants of K for Arnold's 14 exceptional singularities which is listed in Table 2.

Let M be an oriented Seifert manifold with invariants

$$(Oog|b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)).$$

A change of orientation gives the Seifert invariants for $-M$

$$-M = (Oog| - b - n; (\alpha_1, \alpha_1 - \beta_1), \ldots, (\alpha_n, \alpha_n - \beta_n)).$$

Table1. Normal forms and weights of Arnold's exceptional singularities

Type	Normal form	Weights (q_0, q_1, q_2)	d
Q_{10}	$x^2z + y^3 + z^4$	$(9, 8, 6)$	24
Q_{11}	$x^2z + y^3 + yz^3$	$(7, 6, 4)$	18
Q_{12}	$x^2z + y^3 + z^5$	$(6, 5, 3)$	15
S_{11}	$x^2z + yz^2 + y^4$	$(5, 4, 6)$	16
S_{12}	$x^2z + yz^2 + xy^3$	$(4, 3, 5)$	13
U_{12}	$x^3 + y^3 + z^4$	$(4, 4, 3)$	12
Z_{11}	$x^3y + y^5 + z^2$	$(8, 6, 15)$	30
Z_{12}	$x^3y + xy^4 + z^2$	$(6, 4, 11)$	22
Z_{13}	$x^3y + y^6 + z^2$	$(5, 3, 9)$	18
W_{12}	$x^4 + y^5 + z^2$	$(5, 4, 10)$	20
W_{13}	$x^4 + xy^4 + z^2$	$(4, 3, 8)$	16
K_{12}	$x^3 + y^7 + z^2$	$(14, 6, 21)$	42
K_{13}	$x^3 + xy^5 + z^2$	$(10, 4, 15)$	30
K_{14}	$x^3 + y^8 + z^2$	$(8, 3, 12)$	24

3 Seifert manifold K

Definition 2 ([2]). *Let (μ, ν) be a pair of relatively prime integers. Let D^2 be the unit disc in $\mathbf{R}^2$ defined in polar coordinates as $D^2 = \{(r, \theta) : 0 \leq r \leq 1\}$. A fibered solid torus of type (μ, ν) is the quotient of the cylinder $D^2 \times I$ via the identification*

$$((r, \theta), 1) = ((r, \theta + 2\pi\nu/\mu), 0).$$

The fibers are the images of the arcs $\{x\} \times I$. Thus the core(or former cylinder axis) arises form the identification of $\{0\} \times I$ and meets the meridianal disc $D^2 \times 0$ once. Every other fiber meets $D^2 \times 0$ exactly μ times. Up to fiber-preserving homeomorphism we may assume that $\mu > 0$ and $0 < \nu < \mu/2$. The integer μ is called the index. If $\mu > 1$, we call the core a singular fiber; otherwise, we call the core a regular fiber.

Definition 3 ([2]). *A Seifert fibered 3-manifold, M, is a 3-manifold which is a union of pairwise disjoint simple closed curves(fibers) such that each fiber*

The purposes of this paper is to find a duality for these invariants comming from the strange duality of Arnold's 14 exceptional singularities. We can find a duality of the Casson-Walker invariants and some kind of duality of the secand Ohtsuki invarinats.

2 Arnold's 14 exceptional singularities

Definition 1. *A polynomial $h(z_0, \ldots, z_n)$ is called weighted homogeneous if there exist non-zero integers $q_0, \ldots, q_n$ and a positive integer d so that*

$$h(t^{q_0} z_0, \ldots, t^{q_n} z_n) = t^d h(z_0, \ldots, z_n).$$

We call the integers $q_0, \ldots, q_n$ weights.

Let V be the variety defined by weighted homogeneous polynomials $h_1, \ldots, h_r$ with weight $(q_0, \ldots, q_n)$. Then there is a natural $\mathbf{C}^*$ action

$$\sigma(t, (z_0, \ldots, z_n)) = (t^{q_0} z_0, \ldots, t^{q_n} z_n) \quad t \in \mathbf{C}^*.$$

We call this action good if it is effective and $q_i > 0$ for all i.

A normal form f of Arnold's 14 exceptional singularities aregiven in Table 1 [1]. They are weighted homologeous polynomials. Hence variety $V = f^{-1}(0) \subseteq \mathbf{C}^3$ has a natural $\mathbf{C}^*$ -action

$$\sigma(t, (z_0, z_1, z_2)) = (t^{q_0} z_0, t^{q_1} z_1, t^{q_2} z_2).$$

Since the action is effective and $q_i > 0$ for all i, it is good action.

Let $f(z_0, z_1, z_2)$ be a polynomial in 3 complex variables and let $V \subseteq \mathbf{C}^*$ be the set of zeros of f. Suppose V has a good $\mathbf{C}^*$ action. If V has an isolated singularity, then the intersection $K = V \cap S_\varepsilon$ of V with a small sphere S_ε centered at the singular point of V is a smooth 3-manifold with S^1 -action [5].

The Casson-Walker invariants and Arnold's 14 exceptional singularities

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Abstract

We calculate the Casson-Walker invariants and second Ohtsuki invariants of 3-manifolds determined by Arnold's 14 exeptional singularities which have so-called strange duality. We study the duality in the invariants of 3-manifolds for these singularities.

1 Introduction

There is so-called strange duality among Arnold's 14 exceptional singularities [1]. This is the duality between Dolgachev numbers and Gabrielov numbers for the singularities. Dolgachev numbers are self-intersection numbers for irreducible components of resolutions of the exceptional singularities. Gabrielov numbers are numbers determined by the homology lattice of Milnor fibers.

Since an exceptional singularity is an isolated singularity of a hypersurface, the intersection of the hypersurface with a sufficiently small 5-sphere centered at the isolated singularity is a smooth,orientable closed 3-manifold [5].

Let $f(z_0, z_1, z_2)$ be a polynomial in 3 complex variables and let $V \subseteq \mathbf{C}^*$ be the set of zeros of f. Suppose V has a good $\mathbf{C}^*$ action (see §2). If V has an isolated singularity, then the intersection $K = V \cap S_\varepsilon$ of V with a small sphere S_ε centered at the singular point of V is a smooth 3-manifold with S^1 -action [5].

In this paper, we calculate the Casson-Walker invariants and the second Ohtsuki invariant of oriented closed 3-manifolds obtained as above for Arnold's 14 exceptional singularities.