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$$
\begin{aligned}
-\sigma_{(10)(26)} &: (a_i; s; x : y : z) \longmapsto \left(-a_1, -a_0, -a_2\,; s;\ xz : -(yz - x^2 - sxz) : z^2\right), \\
w_1 &: (a_i; s; x : y : z) \longmapsto (-a_1, a_2 + a_1, a_0 + a_1\,; s;\ x : y - a_1 z : z), \\
w_2 &= \sigma_{(15)(24)} \circ w_1 \circ \sigma_{(15)(24)}, \quad w_0 = \sigma_{(10)(26)} \circ w_2 \circ \sigma_{(10)(26)}.
\end{aligned}
$$

Add 10. $E_7^{(1)}$-surface:
$\mathrm{Aut}(E_7^{(1)}) \ltimes W(A_1^{(1)}) \simeq \widetilde{W}(A_1^{(1)})$-symmetry

$$
\begin{aligned}
\sigma_{(15)(24)(60)} &: (a_1, a_0\,; t;\ x : y : z) \longmapsto \left(a_0, a_1\,; s;\ x(y^2 z - szx^2 - x^3) : -y(y^2 z - szx^2 - x^3) : zx^3\right), \\
w_1 &: (a_1, a_0; s; x : y : z) \longmapsto (-a_1, a_0 + 2a_1; s; x : y - a_1 z : z), \\
w_0 &= \sigma_{(15)(24)(60)} \circ w_1 \circ \sigma_{(15)(24)(60)}.
\end{aligned}
$$

Acknowledgement. The author expresses his sincere gratitude to Professor Masatoshi Noumi, Professor Kazuo Okamoto, Professor Masahiko Saito, Professor Kyouichi Takano, Professor Hiroshi Umemura and Professor Yasuhiko Yamada for many helpful advices and constant encouragement. The ideas in this paper have matured under the stimulus of their important and interesting results. The author is most grateful to Professor Michio Jimbo and Professor Jun-ichi Matsuzawa, who spent much time to read this paper and gave many useful advices. This work is partially supported by the Research Fellowships of the Japan Society for the Promotion of Science.

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$$w_3 \;:\; (a_i;s;x:y:z)\longmapsto \left(\begin{matrix} a_1 & a_2+a_3 & -a_3 \\ & a_4 & a_0 \end{matrix};s;\; x:y-a_3z:z\right),$$
$$w_4 \;:\; (a_i;s;x:y:z)\longmapsto \left(\begin{matrix} a_1 & a_2+a_4 & a_3 \\ & -a_4 & a_0 \end{matrix};s;\; x:y-a_4x:z\right),$$
$$w_0 \;:\; (a_i;s;x:y:z)\longmapsto \left(\begin{matrix} a_1 & a_2+a_0 & a_3 \\ a_4 & & -a_0 \end{matrix};s;\; \begin{matrix} x(sx+(1-s)z):y(sx+(1-s)z)-a_0xz \\ :z(sx+(1-s)z) \end{matrix}\right).$$

Add 5. $D_5^{(1)}$-surface:
$\mathrm{Aut}(D_5^{(1)})\ltimes W(A_3^{(1)})\simeq \widetilde{W}(A_3^{(1)})$-symmetry

$$\begin{aligned}
\sigma_{(14)(23)(50)} \;&:\; (a_1,a_2,a_3,a_0\,;s;\; x:y:z)\longmapsto \\
&\qquad (a_2,a_1,a_0,a_3\,;-s;\; yz(z-x):-y(y(z-x)-sxz):z(y(z-x)-sxz)),\\
\sigma_{(45)} \;&:\; (a_i;s;x:y:z)\longmapsto (a_3,a_2,a_1,a_0\,;-s;\; z:y:x),\\
w_1 \;&:\; (a_i;s;x:y:z)\longmapsto (-a_1,a_2+a_1,a_3,a_0+a_1\,;s;\; x:y-a_1x:z),\\
w_3 \;&:\; (a_i;s;x:y:z)\longmapsto (a_1,a_2+a_3,-a_3,a_0+a_3\,;s;\; x:y-a_3z:z),\\
w_2 \;&=\; \sigma_{(14)(23)(50)}\circ w_1\circ\sigma_{(14)(23)(50)},\quad w_0=\sigma_{(14)(23)(50)}\circ w_3\circ\sigma_{(14)(23)(50)}.
\end{aligned}$$

Add 6. $D_6^{(1)}$-surface:
$\mathrm{Aut}(D_6^{(1)})\ltimes W((2A_1)^{(1)})$-symmetry

$$\begin{aligned}
b_0: \;&=\; a_0+a_1-b_1,\\
\sigma_{(16)(24)(50)} \;&:\; \left(\begin{matrix} a_1 & a_0 \\ b_1 & b_0 \end{matrix};s;\; x:y:z\right)\longmapsto\\
&\qquad \left(\begin{matrix} a_0 & a_1 \\ b_1 & b_0 \end{matrix};s;\; \begin{matrix} x(y(y-b_1x)(z-x)-szx^2):-(y-b_1x)(y(y-b_1x)(z-x)-szx^2): \\ :xy(y-b_1x)(z-x) \end{matrix}\right),\\
\sigma_{(56)} \;&:\; (a_1,a_0,b_1,b_0\,;s;\; x:y:z)\longmapsto (b_1,b_0,a_1,a_0\,;-s;\; z:y:x),\\
w_1 \;&:\; (a_1,a_0,b_1,b_0\,;s;\; x:y:z)\longmapsto (-a_1,a_0+2a_1,b_1,b_0\,;s;\; x:y-a_1z:z),\\
w_1' \;&:\; (a_1,a_0,b_1,b_0\,;s;\; x:y:z)\longmapsto (a_1,a_0,-b_1,b_0+2b_1\,;s;\; x:y-b_1x:z),\\
w_0 \;&=\; \sigma_{(16)(24)(50)}\circ w_1\circ\sigma_{(16)(24)(50)},\\
w_0' \;&=\; \sigma_{(56)}\circ\sigma_{(16)(24)(50)}\circ\sigma_{(56)}\circ w_1'\circ\sigma_{(56)}\circ\sigma_{(16)(24)(50)}\circ\sigma_{(56)}.
\end{aligned}$$

Add 7. $D_7^{(1)}$-surface:
$\widetilde{W}(A_1^{(1)})$-symmetry

$$\begin{aligned}
\sigma\,(=\sigma_{(67)}) \;&:\; (a_1,a_0;s;x:y:z)\mapsto (-a_1,a_0+2a_1\,;-s;\; x:y-a_1x-z:-z),\\
\sigma'\,(=\sigma_{(16)(25)(34)(70)}) \;&:\; (a_1,a_0;s;x:y:z)\mapsto (a_0,\,a_1\,;-s;\; -xyz:zy^2:sx^3).
\end{aligned}$$

Add 8. $D_8^{(1)}$-surface:
$\mathfrak{S}_2$-symmetry

$$\sigma_{(17)(26)(35)(80)}:(\lambda;s;x:y:z)\mapsto \left(\lambda;\; s\,;\; zx:(y-\lambda x/2)\,z:sx^2\right).$$

Add 9. $E_6^{(1)}$-surface:
$\mathrm{Aut}(E_6^{(1)})\ltimes W(A_2^{(1)})\simeq \widetilde{W}(A_2^{(1)})$-symmetry

$$-\sigma_{(15)(24)} \;:\; (a_1,a_2,a_0\,;s;\; x:y:z)\longmapsto (-a_2,-a_1,-a_0\,;s;\; yz:-xy:-zx),$$

Mul 7. $A_6^{(1)}$-surface:
$(W(A_1^{(1)}) \rtimes \{\sigma^n \,|\, n \in \mathbb{Z}\}) \rtimes \mathfrak{S}_2$ -symmetry

$$\begin{aligned}
\sigma\,(=\sigma_{(1642053)}) \quad &: \quad (a_1,\, a_0,\, b\,;\, x:y:z) \longmapsto (a_0,\, a_1,\, -a_1 b\,;\, z(z-x):bx(z-x):yz)\\
-\sigma_{(15)(24)(60)} \quad &: \quad (a_1,\, a_0,\, b\,;\, x:y:z) \longmapsto\\
&\qquad \left(1/a_0,\, 1/a_1,\, b\,;\, z(x-z)(x+y-z):y((z-bx)(x-z)+yz):bx(x-z)^2\right),\\
w_1 \quad &: \quad (a_1,\, a_0,\, b\,;\, x:y:z) \longmapsto (1/a_1,\, a_1^2 a_0,\, ba_1\,;\, x(y-a_1 z):y(y-z):a_1 z(y-z)),\\
w_0 \quad &= \quad \sigma_{(15)(24)(60)} \circ w_1 \circ \sigma_{(15)(24)(60)}.
\end{aligned}$$

Mul 8. $A_7^{(1)}$-surface:
$\{\sigma^n \,|\, n \in \mathbb{Z}\} \rtimes \mathfrak{S}_2$-symmetry

$$\begin{aligned}
\sigma\,(=\sigma_{(14725036)}) \quad &: \quad (a_1,\, a_0\,;\, x:y:z) \mapsto (a_1^2 a_0,\, 1/a_1\,;\, yz:a_1 z(x-z):a_1 x(x-z)),\\
-\sigma_{(14)(23)(50)(67)} \quad &: \quad (a_1,\, a_0\,;\, x:y:z) \mapsto (1/a_0,\, 1/a_1\,;\, zx:yz:xy).
\end{aligned}$$

Mul 9. $A_7^{(1)\prime}$-surface:
$\mathfrak{D}_8 \ltimes W(A_1^{(1)})$ -symmetry

$$\begin{aligned}
\sigma_{(1357)(2460)} \quad &: \quad (a_1,\, b,\, c\,;\, x:y:z) \longmapsto \left(bc,\, a_1/b,\, b\,;\, xy:byz:x^2\right),\\
-\sigma_{(13)(40)(57)} \quad &: \quad (a_1,\, b,\, c\,;\, x:y:z) \longmapsto (1/a_1,\, 1/c,\, 1/b\,;\, x:z:y),\\
w_1 \quad &: \quad (a_1,\, b,\, c\,;\, x:y:z) \longmapsto (1/a_1,\, a_1 b,\, a_1 c\,;\, x(y-a_1 z):y(z-y):a_1 z(z-y)),\\
w_0 \quad &= \quad \sigma_{(13)(40)(57)} \circ w_1 \circ \sigma_{(13)(40)(57)}.
\end{aligned}$$

Mul 10. $A_8^{(1)}$-surface:
$\mathfrak{D}_6$ -symmetry

$$\begin{aligned}
\sigma_{(147)(258)(360)} \quad &: \quad (b,\, c\,;\, x:y:z) \mapsto (c^2,\, b/c\,;\, cy:z:cx),\\
-\sigma_{(15)(24)(60)(78)} \quad &: \quad (b,\, c\,;\, x:y:z) \mapsto (1/c,\, 1/b\,;\, zx:yz:xy).
\end{aligned}$$

Add 4. $D_4^{(1)}$-surface:
$\mathrm{Aut}(D_4^{(1)}) \ltimes W(D_4^{(1)}) \simeq \widetilde{W}(D_4^{(1)})$-symmetry

$$\begin{aligned}
\sigma_{(40)} \quad &: \quad \left(\begin{matrix} a_1 & a_2 & a_3 \\ a_4 & & a_0 \end{matrix}\,;s;\, x:y:z\right) \longmapsto \left(\begin{matrix} a_1 & a_2 & a_0 \\ a_4 & & a_3 \end{matrix}\,;\frac{s}{s-1};\, xz:y(sx+(1-s)z):z(sx+(1-s)z)\right),\\
\sigma_{(14)} \quad &: \quad (a_i;s;x:y:z) \longmapsto \left(\begin{matrix} a_1 & a_2 & a_4 \\ a_3 & & a_0 \end{matrix}\,;\frac{1}{s};\, x:-y-a_2 x:x-z\right),\\
\sigma_{(34)} \quad &: \quad (a_i;s;x:y:z) \longmapsto \left(\begin{matrix} a_3 & a_2 & a_1 \\ a_4 & & a_0 \end{matrix}\,;1-s;\, z:y:x\right),\\
w_1 \quad &: \quad (a_i;s;x:y:z) \longmapsto \left(\begin{matrix} -a_1 & a_2+a_1 & a_3 \\ & a_4 & a_0 \end{matrix}\,;s;\, x:y:z\right),\\
w_2 \quad &: \quad (a_i;s;x:y:z) \longmapsto \left(\begin{matrix} a_1+a_2 & -a_2 & a_3+a_2 \\ a_4+a_2 & & a_0+a_2 \end{matrix}\,;s;\, x(y+a_2 z):(y+a_2 z)(a_2 x+y):z(a_2 x+y)\right),
\end{aligned}$$

$$
\begin{aligned}
w_3 &: (a_i; x:y:z) \longmapsto \\
&\qquad \left(\begin{matrix} a_8a_1a_4 & 1/a_4 & -a_2a_6 & -a_3a_6 \\ 1/a_6 & -a_5a_1 & 1/a_1 & -a_7a_4 \end{matrix}; \begin{matrix} x(-a_6x+y+a_6a_1z) : y(a_4a_6x-a_4y+z) : \\ : z(x+a_1a_4y-a_1z) \end{matrix}\right), \\
w_0 &: (a_i; x:y:z) \longmapsto \left(\begin{matrix} 1/a_1 & 1/a_8 & a_2 & a_3 \\ a_4a_1 & a_5a_1 & a_6a_8 & a_7a_8 \end{matrix}; x(x-a_1z) : y(a_8x-z) : z(x-a_1z)\right), \\
w_1 &: (a_2,a_3;\, x:y:z) \longmapsto (a_3,a_2; x:y:z), \qquad w_2 : (a_1,a_2) \longmapsto (a_2,a_1), \\
w_4 &: (a_4,a_5) \longmapsto (a_5,a_4), \qquad w_5 : (a_6,a_7) \longmapsto (a_7,a_6).
\end{aligned}
$$

Mul 5. $A_4^{(1)}$-surface:
$\mathrm{Aut}(A_4^{(1)}) \ltimes W(A_4^{(1)}) \simeq \widetilde{W}(A_4^{(1)})$-symmetry

$$
\begin{aligned}
\sigma_{(12340)} &: \left(\begin{matrix} a_1 & a_2 & a_3 \\ a_4 & a_0 \end{matrix}; x:y:z\right) \longmapsto \\
&\qquad \left(\begin{matrix} a_3 & a_4 & a_0 \\ a_1 & a_2 \end{matrix}; a_4xy(z-x) : a_2z(x+y-z)(x+a_4y-z) : x(x-z)^2\right), \\
-\sigma_{(13)(40)} &: (a_i; x:y:z) \longmapsto \\
&\qquad \left(\begin{matrix} 1/a_1 & 1/a_0 & 1/a_4 \\ 1/a_3 & 1/a_2 \end{matrix}; \begin{matrix} a_2z(x-z)(x+y-z) : \\ : y((z-x)(a_0x+a_2z)-a_2yz) : a_0x(x-z)^2 \end{matrix}\right), \\
w_3 &: (a_i; x:y:z) \longmapsto \\
&\qquad \left(\begin{matrix} a_1 & a_2a_3 & 1/a_3 \\ a_3a_4 & a_0 \end{matrix}; a_3x(a_3x+y-a_3z) : y(a_3x+y-z) : a_3z(x+y-z)\right), \\
w_1 &: (a_i; x:y:z) \longmapsto \left(\begin{matrix} 1/a_1 & a_1a_2 & a_3 \\ a_4 & a_1a_0 \end{matrix}; x:y:z\right), \\
w_2 &: (a_i; x:y:z) \longmapsto \left(\begin{matrix} a_1a_2 & 1/a_2 & a_2a_3 \\ a_4 & a_0 \end{matrix}; x:a_2y:a_2z\right), \\
w_4 &: (a_i; x:y:z) \longmapsto \left(\begin{matrix} a_1 & a_2 & a_3a_4 \\ 1/a_4 & a_4a_0 \end{matrix}; x:a_4y:z\right), \qquad w_0 = \sigma^3_{(12340)} \circ w_1 \circ \sigma^2_{(12340)}.
\end{aligned}
$$

Mul 6. $A_5^{(1)}$-surface:
$\mathrm{Aut}(A_5^{(1)}) \ltimes W((A_2+A_1)^{(1)})$ -symmetry

$$
\begin{aligned}
\sigma_{(123450)} &: \left(\begin{matrix} a_1 & a_2 & a_0 \\ b_1 & b_0 \end{matrix}; x:y:z\right) \longmapsto \\
&\qquad \left(\begin{matrix} a_2 & a_0 & a_1 \\ b_0 & b_1 \end{matrix}; b_1xy(z-x) : -a_2yz(x+y-z) : b_1x(x-z)^2\right), \\
-\sigma_{(12)(30)(45)} &: (a_i,b_j; x:y:z) \longmapsto \left(\begin{matrix} 1/a_2 & 1/a_1 & 1/a_0 \\ 1/b_0 & 1/b_1 \end{matrix}; x(z-x) : z(x+y-z) : yx\right), \\
w_2 &: (a_i,b_j; x:y:z) \longmapsto \\
&\qquad \left(\begin{matrix} a_1a_2 & 1/a_2 & a_2a_0 \\ b_1 & b_0 \end{matrix}; x(a_2x+y-a_2z) : -y(a_2x+y-z) : z(x+y-z)\right), \\
w_1 &: (a_i,b_j; x:y:z) \longmapsto \left(\begin{matrix} 1/a_1 & a_1a_2 & a_1a_0 \\ b_1 & b_0 \end{matrix}; x:a_1y:a_1z\right), \\
w_0 &= \sigma^4_{(123450)} \circ w_1 \circ \sigma^2_{(123450)}, \\
w_1' &: (a_i,b_j; x:y:z) \longmapsto \left(\begin{matrix} a_1 & a_2 & a_0 \\ 1/b_1 & b_1^2b_0 \end{matrix}; x(y+a_2x) : -b_1y(y+a_2x) : z(a_2x-b_1y)\right), \\
w_0' &= \sigma^3_{(123450)} \circ w_1' \circ \sigma^3_{(123450)}.
\end{aligned}
$$

where

$$(\overline{x}:\overline{y}:\overline{z})=\bar{l}_{\theta_1,\theta_2,\theta_3}(x:y:z)\begin{pmatrix} p_1(\theta_1,\theta_2,\theta_3)\wp(\frac{\theta_1-2\theta_2-2\theta_3}{3}) & p_1(\theta_1,\theta_2,\theta_3)\wp'(\frac{\theta_1-2\theta_2-2\theta_3}{3}) & p_1(\theta_1,\theta_2,\theta_3)\\ p_2(\theta_1,\theta_2,\theta_3)\wp(\frac{-2\theta_1+\theta_2-2\theta_3}{3}) & p_2(\theta_1,\theta_2,\theta_3)\wp'(\frac{-2\theta_1+\theta_2-2\theta_3}{3}) & p_2(\theta_1,\theta_2,\theta_3)\\ p_3(\theta_1,\theta_2,\theta_3)\wp(\frac{-2\theta_1-2\theta_2+\theta_3}{3}) & p_3(\theta_1,\theta_2,\theta_3)\wp'(\frac{-2\theta_1-2\theta_2+\theta_3}{3}) & p_3(\theta_1,\theta_2,\theta_3)\end{pmatrix},$$

$$\bar{l}_{\alpha,\beta,\gamma}(x:y:z)=\begin{pmatrix} l_{\gamma,\alpha}(x:y:z)\cdot l_{\alpha,\beta}(x:y:z) : l_{\alpha,\beta}(x:y:z)\cdot l_{\beta,\gamma}(x:y:z) :\\ : l_{\beta,\gamma}(x:y:z)\cdot l_{\gamma,\alpha}(x:y:z)\end{pmatrix},$$

$$l_{\alpha,\beta}(x:y:z)=\det\begin{pmatrix} x & y & z\\ \wp(\alpha) & \wp'(\alpha) & 1\\ \wp(\beta) & \wp'(\beta) & 1\end{pmatrix},$$

$$\begin{pmatrix} p_1(\alpha,\beta,\gamma)\\ p_2(\alpha,\beta,\gamma)\\ p_3(\alpha,\beta,\gamma)\end{pmatrix}=M\left(\frac{\alpha-2\beta-2\gamma}{3},\frac{-2\alpha+\beta-2\gamma}{3},\frac{-2\alpha-2\beta+\gamma}{3}\right)\begin{pmatrix} \wp\left(\frac{\alpha+\beta+\gamma}{3}\right)(\wp(\beta)-\wp(\gamma))\\ \wp'\left(\frac{\alpha+\beta+\gamma}{3}\right)(\wp(\gamma)-\wp(\alpha))\\ \wp(\alpha)-\wp(\beta)\end{pmatrix},$$

$$M(\alpha,\beta,\gamma)=\begin{pmatrix} \wp'(\alpha)-\wp'(\beta) & \wp'(\beta)-\wp'(\gamma) & \wp'(\gamma)-\wp'(\alpha)\\ \wp(\beta)-\wp(\gamma) & \wp(\gamma)-\wp(\alpha) & \wp(\alpha)-\wp(\beta)\\ \wp'(\beta)\wp(\alpha)-\wp'(\gamma)\wp(\beta) & \wp'(\gamma)\wp(\beta)-\wp'(\alpha)\wp(\gamma) & \wp'(\alpha)\wp(\gamma)-\wp'(\beta)\wp(\alpha)\end{pmatrix},$$

$$\begin{aligned} w_i &: (\theta_i,\theta_{i+1};\tau;x:y:z)\longmapsto(\theta_{i+1},\theta_i;\tau;x:y:z),\quad \text{for } i=1,\dots,7,\\ w_0 &: (\theta_8,\theta_9)\longmapsto(\theta_9,\theta_8).\end{aligned}$$

Mul 3. $A_2^{(1)}$-surface:
$\mathrm{Aut}(A_2^{(1)})\ltimes W(E_6^{(1)})\simeq\widetilde{W}(E_6^{(1)})$-symmetry

$$-\sigma_{(12)} : \begin{pmatrix} a_1 & a_2 & a_3\\ a_4 & a_5 & a_9\\ a_6 & a_7 & a_8\end{pmatrix};\ x:y:z\Big)\longmapsto\begin{pmatrix} 1/a_6 & 1/a_7 & 1/a_8\\ 1/a_4 & 1/a_5 & 1/a_9\\ 1/a_1 & 1/a_2 & 1/a_3\end{pmatrix};\ y:x:z\Big),$$

$$-\sigma_{(10)} : (a_i;x:y:z)\longmapsto\begin{pmatrix} 1/a_1 & 1/a_2 & 1/a_3\\ 1/a_6 & 1/a_7 & 1/a_8\\ 1/a_4 & 1/a_5 & 1/a_9\end{pmatrix};\ x:z:y\Big),$$

$$w_3 : (a_i;x:y:z)\longmapsto\begin{pmatrix} 1/a_4 & -a_2a_6 & -a_3a_6\\ 1/a_6 & -a_5a_1 & -a_9a_1\\ 1/a_1 & -a_7a_4 & -a_8a_4\end{pmatrix};\ \begin{matrix} x(-a_6x+y+a_6a_1z):y(a_4a_6x-a_4y+z):\\ :z(x+a_1a_4y-a_1z)\end{matrix}\Big),$$

$$w_1 : (a_2,a_3;\,x:y:z)\longmapsto(a_3,a_2;x:y:z),\qquad w_2 :\ (a_1,a_2)\longmapsto(a_2,a_1),$$

$$w_4 : (a_6,a_7)\longmapsto(a_7,a_6),\qquad w_5 :\ (a_7,a_8)\longmapsto(a_8,a_7),$$

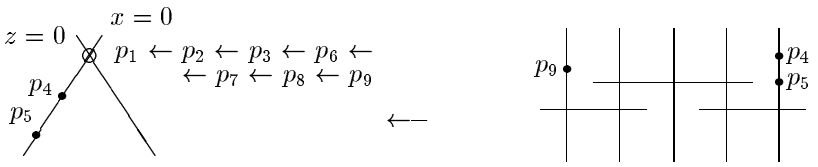
$$w_6 : (a_4,a_5)\longmapsto(a_5,a_4),\qquad w_0 :\ (a_5,a_9)\longmapsto(a_9,a_5).$$

Mul 4. $A_3^{(1)}$-surface:
$\mathrm{Aut}(A_3^{(1)})\ltimes W(D_5^{(1)})\simeq\widetilde{W}(D_5^{(1)})$-symmetry

$$-\sigma_{(12)(30)} : \begin{pmatrix} a_8 & a_1 & a_2 & a_3\\ a_4 & a_5 & a_6 & a_7\end{pmatrix};x:y:z\Big)\longmapsto$$
$$\begin{pmatrix} -1/a_5a_1 & -a_4a_1 & 1/a_6 & 1/a_7\\ 1/a_4 & 1/a_8a_1a_4 & 1/a_2 & 1/a_3\end{pmatrix};\ x(x-a_1z):z(x+a_1a_4y-a_1z):xy\Big),$$

$$-\sigma_{(13)} : (a_i;x:y:z)\longmapsto\begin{pmatrix} 1/a_8 & 1/a_1 & 1/a_2 & 1/a_3\\ 1/a_6 & 1/a_7 & 1/a_4 & 1/a_5\end{pmatrix};\ z:y:x\Big),$$

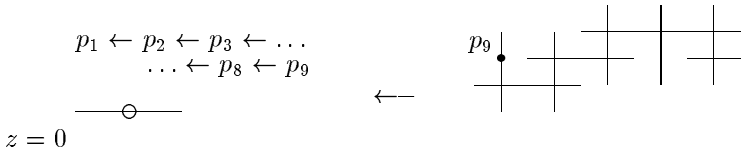
Add 10. $E_7^{(1)}$-surface:



$$p_1 : (0:1:0) \quad \leftarrow p_2 : \left(\frac{x}{y}, \frac{z}{x}\right) = (0,0) \leftarrow p_3 : \left(\frac{x}{y}, \frac{yz}{x^2}\right) = (0,0) \leftarrow$$
$$\leftarrow p_6 : \left(\frac{x}{y}, \frac{y^2 z}{x^3}\right) = (0,1) \leftarrow p_7 : \left(\frac{x}{y}, \frac{y(y^2z - x^3)}{x^4}\right) = (0,0) \leftarrow$$
$$\leftarrow p_8 : \left(\frac{x}{y}, \frac{y^2(y^2z - x^3)}{x^5}\right) = (0,-s) \leftarrow p_9 : \left(\frac{x}{y}, \frac{y(y^2(y^2z - x^3) + sx^5)}{x^6}\right) = (0,-a_0),$$
$$p_4 : (0:0:1), \qquad p_5 : (0:a_1:1).$$

$$(a_1,\ a_0\,;\, s\,;\, x:y:z) \sim (\mu^3 a_1,\ \mu^3 a_0\,;\, \mu^2 s\,;\, x:\mu y:\mu^2 z),$$
$$\chi(\alpha_1) = a_1/\lambda, \quad \chi(\alpha_0) = a_0/\lambda, \quad \text{where } \lambda = a_1 + a_0.$$

Add 11. $E_8^{(1)}$-surface:



$$p_1 : (0:1:0) \quad \leftarrow p_2 : \left(\frac{x}{y}, \frac{z}{x}\right) = (0,0) \leftarrow p_3 : \left(\frac{x}{y}, \frac{yz}{x^2}\right) = (0,0) \leftarrow$$
$$\leftarrow p_4 : \left(\frac{x}{y}, \frac{y^2 z}{x^3}\right) = (0,1) \leftarrow p_5 : \left(\frac{x}{y}, \frac{y(y^2z - x^3)}{x^4}\right) = (0,0) \leftarrow$$
$$\leftarrow p_6 : \left(\frac{x}{y}, \frac{y^2(y^2z - x^3)}{x^5}\right) = (0,0) \leftarrow p_7 : \left(\frac{x}{y}, \frac{y^3(y^2z - x^3)}{x^6}\right) = (0,0) \leftarrow$$
$$\leftarrow p_8 : \left(\frac{x}{y}, \frac{y^4(y^2z - x^3)}{x^7}\right) = (0,s) \leftarrow p_9 : \left(\frac{x}{y}, \frac{y(y^4(y^2z - x^3) - sx^7)}{x^7}\right) = (0,\lambda).$$

$$(\lambda\,;\, s\,;\, x:y:z) \sim (\mu^2\lambda\,;\, \mu s\,;\, \mu^2 x:\mu^3 y:z), \quad \chi(\delta) = \lambda/\lambda = 1.$$

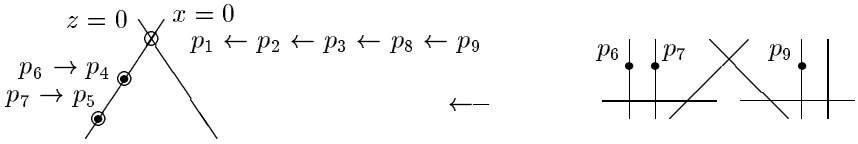
C Cremona action

It is useful to calculate generators of the Cremona action for the purpose of a construction of the discrete Painlevé equations. We give some calculations here. Some surfaces are omitted.

Ell 1. $A_0^{(1)}$-surface:
$W(E_8^{(1)})$-symmetry

$$w_8 : \left(\begin{array}{ccc} \theta_1 & \theta_2 & \theta_3 \\ \theta_4 & \theta_5 & \theta_6 \\ \theta_7 & \theta_8 & \theta_9 \end{array} ;\tau\,;\, x:y:z \right) \longmapsto \left(\begin{array}{ccc} \frac{\theta_1 - 2\theta_2 - 2\theta_3}{3} & \frac{-2\theta_1 + \theta_2 - 2\theta_3}{3} & \frac{-2\theta_1 - 2\theta_2 + \theta_3}{3} \\ \theta_4 + \frac{\theta_1+\theta_2+\theta_3}{3} & \theta_5 + \frac{\theta_1+\theta_2+\theta_3}{3} & \theta_6 + \frac{\theta_1+\theta_2+\theta_3}{3} \\ \theta_7 + \frac{\theta_1+\theta_2+\theta_3}{3} & \theta_8 + \frac{\theta_1+\theta_2+\theta_3}{3} & \theta_9 + \frac{\theta_1+\theta_2+\theta_3}{3} \end{array} ;\tau\,;\, \overline{x}:\overline{y}:\overline{z} \right),$$

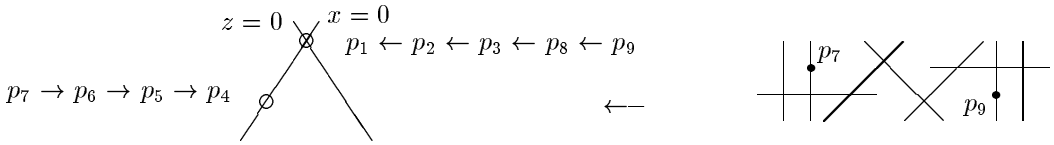
Add 7. $D_7^{(1)}$-surface:



$$p_1:\ (0:1:0)\quad \leftarrow p_2:\ \left(\frac{x}{y},\frac{z}{x}\right)=(0,0)\leftarrow p_3:\ \left(\frac{x}{y},\frac{yz}{x^2}\right)=(0,0)\leftarrow$$
$$\leftarrow p_8:\ \left(\frac{x}{y},\frac{y^2z}{x^3}\right)=(0,s)\leftarrow p_9:\ \left(\frac{x}{y},\frac{y(y^2z-sx^3)}{x^4}\right)=(0,-a_0s),$$
$$p_4:\ (0:0:1)\quad \leftarrow p_6:\ \left(\frac{x}{z},\frac{y}{x}\right)=(0,0),\quad p_5:\ (0:1:1)\leftarrow p_7:\ \left(\frac{x}{z},\frac{y-z}{x}\right)=(0,a_1).$$

$$(a_1,\,a_0\,;\,s\,;\,x:y:z)\sim(\mu a_1,\,\mu a_0\,;\,\mu^3 s\,;\,x:\mu y:\mu z),$$
$$\chi(\alpha_1)=\frac{a_1}{\lambda},\quad \chi(\alpha_0)=\frac{a_0}{\lambda},\qquad \text{where } \lambda=a_1+a_0.$$

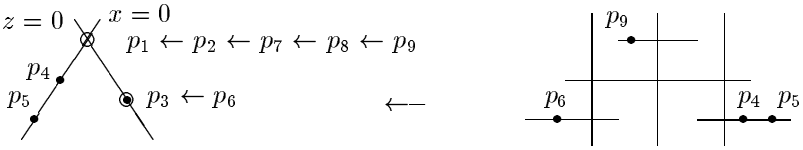
Add 8. $D_8^{(1)}$-surface:



$$p_1:\ (0:1:0)\quad \leftarrow p_2:\ \left(\frac{x}{y},\frac{z}{x}\right)=(0,0)\leftarrow p_3:\ \left(\frac{x}{y},\frac{yz}{x^2}\right)=(0,0)\leftarrow$$
$$\leftarrow p_8:\ \left(\frac{x}{y},\frac{y^2z}{x^3}\right)=(0,s)\leftarrow p_9:\ \left(\frac{x}{y},\frac{y(y^2z-sx^3)}{x^4}\right)=(0,-\lambda s),$$
$$p_4:\ (0:0:1)\quad \leftarrow p_5:\ \left(\frac{y}{z},\frac{x}{y}\right)=(0,0)\leftarrow p_6:\ \left(\frac{y}{z},\frac{zx}{y^2}\right)=(0,1)\leftarrow p_7:\ \left(\frac{y}{z},\frac{z(zx-y^2)}{y^3}\right)=(0,0).$$

$$(\lambda;\,s\,;\,x:y:z)\sim(\mu\lambda;\,\mu^4 s\,;\,\mu^{-1}x:y:\mu z)$$
$$\chi(\delta)=\lambda/\lambda=1.$$

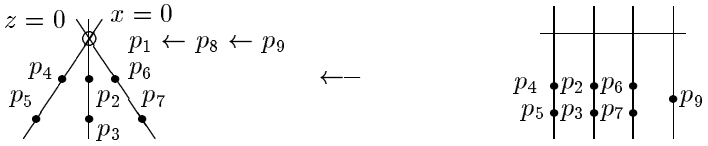
Add 9. $E_6^{(1)}$-surface:



$$p_1:\ (0:1:0)\quad \leftarrow p_2:\ \left(\frac{x}{y},\frac{z}{x}\right)=(0,0)\leftarrow p_7:\ \left(\frac{x}{y},\frac{yz}{x^2}\right)=(0,1)\leftarrow$$
$$\leftarrow p_8:\ \left(\frac{x}{y},\frac{y(yz-x^2)}{x^3}\right)=(0,s)\leftarrow p_9:\ \left(\frac{x}{y},\frac{y(y(yz-x^2)-sx^3)}{x^4}\right)=(0,-a_0+s^2),$$
$$p_4:\ (0:0:1),\qquad p_5:\ (0:a_1:1),\quad p_3:\ (1:0:0)\leftarrow p_6:\ \left(\frac{z}{x},\frac{y}{z}\right)=(0,-a_2).$$

$$(a_i\,;\,s\,;\,x:y:z)\sim(\mu^2 a_i\,;\,\mu s\,;\,x:\mu y:\mu^{-1}z),$$
$$\chi(\alpha_1)=a_1/\lambda,\quad \chi(\alpha_2)=a_2/\lambda,\quad \chi(\alpha_0)=a_0/\lambda,\quad \text{where } \lambda=\textstyle\sum_{i=0}^{2}a_i.$$

Add 4. $D_4^{(1)}$-surface:



$$p_1:\ (0:1:0) \leftarrow p_8:\ \left(\frac{x}{y},\frac{z}{x}\right)=\left(0,\frac{s}{s-1}\right) \leftarrow p_9:\ \left(\frac{x}{y},\frac{y(sx+(1-s)y)}{x^2}\right)=\left(0,\frac{s}{s-1}a_0\right),$$

$$p_2:\ (1:-a_2:1),\qquad p_3:\ (1:-a_2-a_1:1),$$

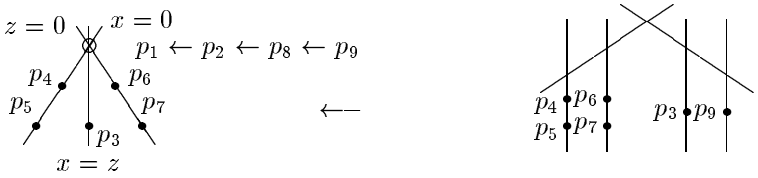
$$p_4:\ (0:0:1),\quad p_5:\ (0:a_3:1),\quad p_6:\ (1:0:0),\quad p_7:\ (1:a_4:0).$$

$$(a_i\,;\,s\,;\,x:y:z)\sim(\mu a_i\,;\,s\,;\,x:\mu y:z),$$

$$\chi(\alpha_1)=a_1/\lambda,\quad \chi(\alpha_2)=a_2/\lambda,\quad \chi(\alpha_3)=a_3/\lambda,\quad \chi(\alpha_4)=a_4/\lambda,\quad \chi(\alpha_0)=a_0/\lambda,$$

where $\lambda=\sum_{i=0}^{4}a_i$.

Add 5. $D_5^{(1)}$-surface:



$$p_1:\ (0:1:0) \leftarrow p_2:\ \left(\frac{x}{y},\frac{z}{x}\right)=(0,1) \leftarrow p_8:\ \left(\frac{x}{y},\frac{y(z-x)}{x^2}\right)=(0,s) \leftarrow$$

$$\leftarrow p_9:\ \left(\frac{x}{y},\frac{y(y(z-x)-tx^2)}{x^3}\right)=(0,s(s-a_0)),$$

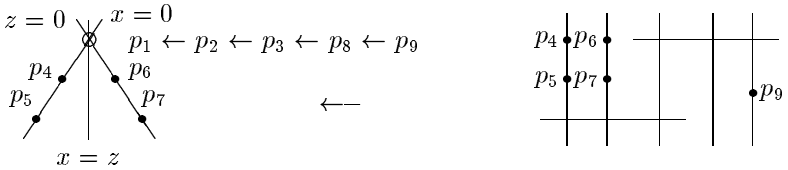
$$p_3:\ (1:-a_2:1),\qquad p_4:\ (0:0:1),\qquad p_5:\ (0:a_1:1),$$

$$p_6:\ (1:0:0),\qquad p_7:\ (1:a_3:0).$$

$$(a_i\,;\,s\,;\,x:y:z)\sim(\mu a_i\,;\,\mu s\,;\,x:\mu y:z),$$

$$\chi(\alpha_1)=a_1/\lambda,\quad \chi(\alpha_2)=a_2/\lambda,\quad \chi(\alpha_3)=a_3/\lambda,\quad \chi(\alpha_0)=a_0/\lambda,\quad \text{where } \lambda=\sum_{i=0}^{3}a_i.$$

Add 6. $D_6^{(1)}$-surface:



$$p_1:\ (0:1:0) \leftarrow p_2:\ \left(\frac{x}{y},\frac{z}{x}\right)=(0,1) \leftarrow p_3:\ \left(\frac{x}{y},\frac{y(z-x)}{x^2}\right)=(0,0) \leftarrow$$

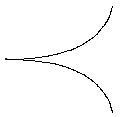
$$\leftarrow p_8:\ \left(\frac{x}{y},\frac{y^2(z-x)}{x^3}\right)=(0,s) \leftarrow p_9:\ \left(\frac{x}{y},\frac{y(y^2(z-x)-sx^3)}{x^4}\right)=(0,s(b_1-a_0)),$$

$$p_4:\ (0:0:1),\qquad p_5:\ (0:a_1:1),\quad p_6:\ (1:0:0),\quad p_7:\ (1:b_1:0).$$

$$\begin{pmatrix}a_1 & a_0\\ b_1 & b_0\end{pmatrix}\,;\,s\,;\,x:y:z\Big)\sim\begin{pmatrix}\mu a_1 & \mu a_0\\ \mu b_1 & \mu b_0\end{pmatrix}\,;\,\mu^2 s\,;\,x:\mu y:z\Big),$$

$$\chi(\alpha_1)=a_1/\lambda,\quad \chi(\alpha_0)=a_0/\lambda,\quad \chi(\alpha_1')=b_1/\lambda,\quad \chi(\alpha_0')=(a_1+a_0-b_1)/\lambda,\quad \text{where } \lambda=a_1+a_0.$$

Add 1. $A_0^{(1)**}$-surface:

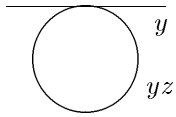


$$y^2z = 4x^3$$

$$p_i : \ (a_i : -2 : a_i^3) \quad (i = 1, \dots, 9),$$

$$(a_i\,;\, x : y : z) \sim (\mu a_i\,;\, \mu x : y : \mu^3 z),$$
$$\chi(\alpha_i) = (a_{i+1} - a_i)/\lambda \quad (i = 1, \dots, 7), \qquad \chi(\alpha_8) = (a_1 + a_2 + a_3)/\lambda, \quad \chi(\alpha_0) = (a_9 - a_8)/\lambda,$$
$$\text{where } \lambda = \textstyle\sum_{i=1}^{9} a_i.$$

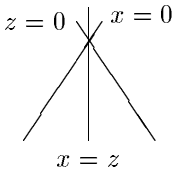
Add 2. $A_1^{(1)*}$-surface:



$$p_1 : \ (1 : 0 : a_1), \qquad p_2 : \ (1 : 0 : a_2), \qquad p_3 : \ (1 : 0 : a_3),$$
$$p_4 : \ (-a_4 : 1 : a_4^2), \qquad p_5 : \ (-a_5 : 1 : a_5^2), \qquad p_6 : \ (-a_6 : 1 :: a_6^2),$$
$$p_7 : \ (-a_7 : 1 : a_7^2), \qquad p_8 : \ (-a_8 : 1 : a_8^2), \qquad p_9 : \ (-a_9 : 1 : a_9^2).$$

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} ; x : y : z \sim \begin{pmatrix} \mu a_1 + 2\nu & \mu a_2 + 2\nu & \mu a_3 + 2\nu \\ \mu a_4 - \nu & \mu a_5 - \nu & \mu a_6 - \nu \\ \mu a_7 - \nu & \mu a_8 - \nu & \mu a_9 - \nu \end{pmatrix} ; \mu x + \nu y : y : 2\mu\nu x + \nu^2 y + \mu^2 z \Bigg),$$
$$\chi(\alpha_1) = (a_3 - a_2)/\lambda, \quad \chi(\alpha_2) = (a_2 - a_1)/\lambda, \quad \chi(\alpha_3) = (a_1 + a_4 + a_5)/\lambda,$$
$$\chi(\alpha_4) = (a_6 - a_5)/\lambda, \quad \chi(\alpha_5) = (a_7 - a_6)/\lambda, \quad \chi(\alpha_6) = (a_8 - a_7)/\lambda,$$
$$\chi(\alpha_7) = (a_5 - a_4)/\lambda, \quad \chi(\alpha_0) = (a_9 - a_8)/\lambda, \quad \text{where } \lambda = \textstyle\sum_{i=1}^{9} a_i.$$

Add 3. $A_2^{(1)*}$-surface:

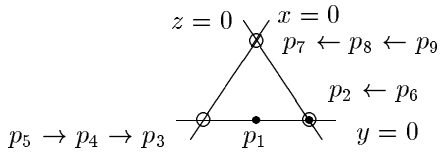


$$p_1 : \ (1 : -a_1 : 1), \qquad p_2 : \ (1 : -a_2 : 1), \qquad p_3 : \ (1 : -a_3 : 1),$$
$$p_4 : \ (0 : a_4 : 1), \qquad p_5 : \ (0 : a_5 : 1), \qquad p_9 : \ (0 : a_9 : 1),$$
$$p_6 : \ (1 : a_6 : 0), \qquad p_7 : \ (1 : a_7 : 0), \qquad p_8 : \ (1 : a_8 : 0).$$

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_9 \\ a_6 & a_7 & a_8 \end{pmatrix} ; x : y : z \sim \begin{pmatrix} \nu a_1 - \mu - \eta & \nu a_2 - \mu - \eta & \nu a_3 - \mu - \eta \\ \nu a_4 + \eta & \nu a_5 + \eta & \nu a_9 + \eta \\ \nu a_6 + \mu & \nu a_7 + \mu & \nu a_8 + \mu \end{pmatrix} ; x : \mu x + \nu y + \eta z : z \Bigg),$$
$$\chi(\alpha_1) = (a_3 - a_2)/\lambda, \quad \chi(\alpha_2) = (a_2 - a_1)/\lambda, \quad \chi(\alpha_3) = (a_1 + a_4 + a_6)/\lambda, \quad \chi(\alpha_4) = (a_7 - a_6)/\lambda,$$
$$\chi(\alpha_5) = (a_8 - a_7)/\lambda, \quad \chi(\alpha_6) = (a_5 - a_4)/\lambda, \quad \chi(\alpha_0) = (a_9 - a_5)/\lambda, \quad \text{where } \lambda = \textstyle\sum_{i=1}^{9} a_i.$$

$$e^{2\pi\sqrt{-1}\chi(\alpha_1)} = a_1, \quad e^{2\pi\sqrt{-1}\chi(\alpha_0)} = a_0, \quad e^{2\pi\sqrt{-1}\chi(\alpha_1')} = a_1 a_0^2/b^2, \quad e^{2\pi\sqrt{-1}\chi(\alpha_0')} = b^2/a_0.$$

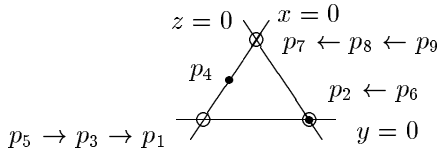
Mul 8. $A_7^{(1)}$-surface:



$$p_1 : (1:0:1), \quad p_2 : (1:0:0) \leftarrow p_6 : \left(\frac{z}{x}, \frac{y}{z}\right) = (0,1),$$
$$p_3 : (0:0:1) \leftarrow p_4 : \left(\frac{y}{z}, \frac{x}{y}\right) = (0,0) \leftarrow p_5 : \left(\frac{y}{z}, \frac{zx}{y^2}\right) = (0,a_0),$$
$$p_7 : (0:1:0) \leftarrow p_8 : \left(\frac{x}{y}, \frac{z}{x}\right) = (0,0) \leftarrow p_9 : \left(\frac{x}{y}, \frac{yz}{x^2}\right) = (0,a_1).$$

$$e^{2\pi\sqrt{-1}\chi(\alpha_1)} = a_1, \quad e^{2\pi\sqrt{-1}\chi(\alpha_0)} = a_0.$$

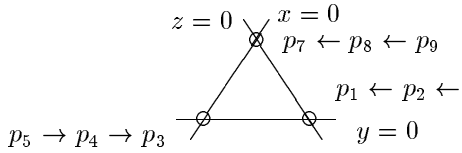
Mul 9. $A_7^{(1)\prime}$-surface:



$$p_1 : (0:0:1) \leftarrow p_3 : \left(\frac{x}{z}, \frac{y}{x}\right) = (0,0) \leftarrow p_5 : \left(\frac{y}{x}, \frac{x^2}{yz}\right) = (0,b)$$
$$p_2 : (1:0:0) \leftarrow p_6 : \left(\frac{z}{x}, \frac{y}{z}\right) = (0,a_1), \qquad p_4 : (0:1:1),$$
$$p_7 : (0:1:0) \leftarrow p_8 : \left(\frac{x}{y}, \frac{z}{x}\right) = (0,0) \leftarrow p_9 : \left(\frac{x}{y}, \frac{yz}{x^2}\right) = (0,c).$$

$$(a_1,\, b,\, c\,;\, x:y:z) \sim \left(a_1,\, \mu^2 b,\, \tfrac{c}{\mu^2}\,;\, \mu x:y:z\right),$$
$$e^{2\pi\sqrt{-1}\chi(\alpha_1)} = a_1, \quad e^{2\pi\sqrt{-1}\chi(\alpha_0)} = bc.$$

Mul 10. $A_8^{(1)}$-surface:



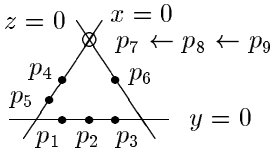
$$p_1 : (1:0:0) \leftarrow p_2 : \left(\frac{z}{x}, \frac{y}{z}\right) = (0,0) \leftarrow p_6 : \left(\frac{z}{x}, \frac{xy}{z^2}\right) = (0,1),$$
$$p_3 : (0:0:1) \leftarrow p_4 : \left(\frac{y}{z}, \frac{x}{y}\right) = (0,0) \leftarrow p_5 : \left(\frac{y}{z}, \frac{zx}{y^2}\right) = (0,b),$$
$$p_7 : (0:1:0) \leftarrow p_8 : \left(\frac{x}{y}, \frac{z}{x}\right) = (0,0) \leftarrow p_9 : \left(\frac{x}{y}, \frac{yz}{x^2}\right) = (0,c).$$

$$(b,\, c\,;\, x:y:z) \sim (\mu^3 b,\, \mu^{-3} c\,;\, \mu x:\mu^{-1} y:z),$$
$$e^{2\pi\sqrt{-1}\chi(\delta)} = bc.$$

$$\begin{pmatrix} a_8 & a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 & a_7 \end{pmatrix}; x:y:z\Big) \sim \left(\begin{matrix} \frac{1}{\mu}a_8 & \mu a_1 & \mu a_2 & \mu a_3 \\ \frac{1}{\nu}a_4 & \frac{1}{\nu}a_5 & \frac{\nu}{\mu}a_6 & \frac{\nu}{\mu}a_7 \end{matrix}; \mu x:\nu y:z \right),$$

$$e^{2\pi\sqrt{-1}\chi(\alpha_1)} = a_3/a_2, \quad e^{2\pi\sqrt{-1}\chi(\alpha_2)} = a_2/a_1, \quad e^{2\pi\sqrt{-1}\chi(\alpha_3)} = -a_1a_4a_6,$$
$$e^{2\pi\sqrt{-1}\chi(\alpha_4)} = a_7/a_6, \quad e^{2\pi\sqrt{-1}\chi(\alpha_5)} = a_5/a_4, \quad e^{2\pi\sqrt{-1}\chi(\alpha_0)} = a_1a_8.$$

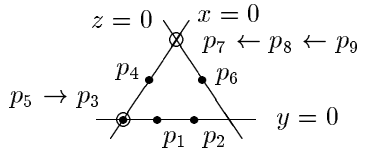
Mul 5. $A_4^{(1)}$-surface:



$$p_1 : (1:0:1), \qquad p_2 : (a_2:0:1), \qquad p_3 : (a_1a_2:0:1),$$
$$p_4 : (0:1:1), \qquad p_5 : (0:1:a_4), \qquad p_6 : (1:-a_3:0),$$
$$p_7 : (0:1:0) \leftarrow p_8 : \left(\frac{x}{y}, \frac{z}{x}\right) = (0,0) \leftarrow p_9 : \left(\frac{x}{y}, \frac{yz}{x^2}\right) = (0, a_0/a_2).$$

$$e^{2\pi\sqrt{-1}\chi(\alpha_1)} = a_1, \quad e^{2\pi\sqrt{-1}\chi(\alpha_2)} = a_2, \quad e^{2\pi\sqrt{-1}\chi(\alpha_3)} = a_3,$$
$$e^{2\pi\sqrt{-1}\chi(\alpha_4)} = a_4, \quad e^{2\pi\sqrt{-1}\chi(\alpha_0)} = a_0.$$

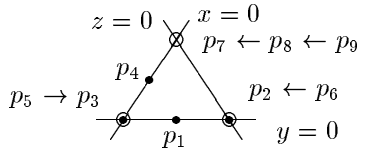
Mul 6. $A_5^{(1)}$-surface:



$$p_1 : (1:0:1), \quad p_2 : (a_1:0:1), \quad p_4 : (0:1:1), \quad p_6 : (1:-a_2:0),$$
$$p_3 : (0:0:1) \leftarrow p_5 : \left(\frac{y}{z}, \frac{x}{y}\right) = (0, b_1/a_2),$$
$$p_7 : (0:1:0) \leftarrow p_8 : \left(\frac{x}{y}, \frac{z}{x}\right) = (0,0) \leftarrow p_9 : \left(\frac{x}{y}, \frac{yz}{x^2}\right) = (0, b_0/a_1).$$

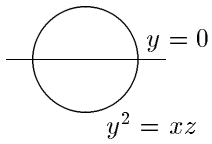
$$e^{2\pi\sqrt{-1}\chi(\alpha_1)} = a_1, \quad e^{2\pi\sqrt{-1}\chi(\alpha_2)} = a_2, \quad e^{2\pi\sqrt{-1}\chi(\alpha_0)} = b_1b_0/a_1a_2,$$
$$e^{2\pi\sqrt{-1}\chi(\alpha_1')} = b_1, \quad e^{2\pi\sqrt{-1}\chi(\alpha_0')} = b_0.$$

Mul 7. $A_6^{(1)}$-surface:



$$p_1 : (1:0:1), \qquad p_4 : (0:1:1),$$
$$p_2 : (1:0:0) \leftarrow p_6 : \left(\frac{z}{x}, \frac{y}{z}\right) = (0, a_1),$$
$$p_3 : (0:0:1) \leftarrow p_5 : \left(\frac{y}{z}, \frac{x}{y}\right) = (0, a_0/b),$$
$$p_7 : (0:1:0) \leftarrow p_8 : \left(\frac{x}{y}, \frac{z}{x}\right) = (0,0) \leftarrow p_9 : \left(\frac{x}{y}, \frac{yz}{x^2}\right) = (0, -b).$$

Mul 2. $A_1^{(1)}$-surface:

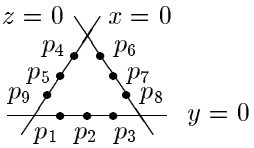


$$p_1 : (a_1 : 0 : 1), \quad p_2 : (a_2 : 0 : 1), \quad p_3 : (a_3 : 0 : 1),$$
$$p_4 : (a_4^2 : a_4 : 1), \quad p_5 : (a_5^2 : a_5 : 1), \quad p_6 : (a_6^2 : a_6 : 1),$$
$$p_7 : (a_7^2 : a_7 : 1), \quad p_8 : (a_8^2 : a_8 : 1), \quad p_9 : (a_9^2 : a_9 : 1).$$

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} ; x : y : z \Bigg) \sim \begin{pmatrix} \mu^2 a_1 & \mu^2 a_2 & \mu^2 a_3 \\ \mu a_4 & \mu a_5 & \mu a_6 \\ \mu a_7 & \mu a_8 & \mu a_9 \end{pmatrix} ; \mu^2 x : \mu y : z \Bigg),$$

$$e^{2\pi\sqrt{-1}\chi(\alpha_1)} = a_3/a_2, \quad e^{2\pi\sqrt{-1}\chi(\alpha_2)} = a_2/a_1, \quad e^{2\pi\sqrt{-1}\chi(\alpha_3)} = -a_1/a_4a_5, \quad e^{2\pi\sqrt{-1}\chi(\alpha_4)} = a_5/a_6,$$
$$e^{2\pi\sqrt{-1}\chi(\alpha_5)} = a_6/a_7, \quad e^{2\pi\sqrt{-1}\chi(\alpha_6)} = a_7/a_8, \quad e^{2\pi\sqrt{-1}\chi(\alpha_7)} = a_4/a_5, \quad e^{2\pi\sqrt{-1}\chi(\alpha_0)} = a_8/a_9.$$

Mul 3. $A_2^{(1)}$-surface:

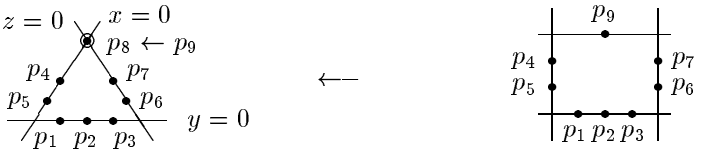


$$p_1 : (a_1 : 0 : 1), \quad p_2 : (a_2 : 0 : 1), \quad p_3 : (a_3 : 0 : 1),$$
$$p_4 : (0 : 1 : a_4), \quad p_5 : (0 : 1 : a_5), \quad p_9 : (0 : 1 : a_9),$$
$$p_6 : (1 : a_6 : 0), \quad p_7 : (1 : a_7 : 0), \quad p_8 : (1 : a_8 : 0).$$

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_9 \\ a_6 & a_7 & a_8 \end{pmatrix} ; x : y : z \Bigg) \sim \begin{pmatrix} \mu a_1 & \mu a_2 & \mu a_3 \\ \frac{1}{\nu} a_4 & \frac{1}{\nu} a_5 & \frac{1}{\nu} a_9 \\ \frac{\nu}{\mu} a_6 & \frac{\nu}{\mu} a_7 & \frac{\nu}{\mu} a_8 \end{pmatrix} ; \mu x : \nu y : z \Bigg),$$

$$e^{2\pi\sqrt{-1}\chi(\alpha_1)} = a_3/a_2, \quad e^{2\pi\sqrt{-1}\chi(\alpha_2)} = a_2/a_1, \quad e^{2\pi\sqrt{-1}\chi(\alpha_3)} = -a_1a_4a_6,$$
$$e^{2\pi\sqrt{-1}\chi(\alpha_4)} = a_7/a_6, \quad e^{2\pi\sqrt{-1}\chi(\alpha_5)} = a_8/a_7, \quad e^{2\pi\sqrt{-1}\chi(\alpha_6)} = a_5/a_4, \quad e^{2\pi\sqrt{-1}\chi(\alpha_0)} = a_9/a_5.$$

Mul 4. $A_3^{(1)}$-surface:



$$p_1 : (a_1 : 0 : 1), \quad p_2 : (a_2 : 0 : 1), \quad p_3 : (a_3 : 0 : 1),$$
$$p_4 : (0 : 1 : a_4), \quad p_5 : (0 : 1 : a_5), \quad p_6 : (1 : a_6 : 0), \quad p_7 : (1 : a_7 : 0),$$
$$p_8 : (0 : 1 : 0) \leftarrow p_9 : \left(\frac{x}{y}, \frac{z}{x}\right) = (0, a_8).$$

Add 10. $E_7^{(1)}$-surface:

$$\begin{aligned}
\Pi(E_7^{(1)}) \quad &: \quad \mathcal{D}_0 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_5, \quad \mathcal{D}_1 = \mathcal{E}_1 - \mathcal{E}_2, \quad \mathcal{D}_2 = \mathcal{E}_2 - \mathcal{E}_3, \\
&\quad\ \mathcal{D}_3 = \mathcal{E}_3 - \mathcal{E}_6, \quad \mathcal{D}_4 = \mathcal{E}_6 - \mathcal{E}_7, \quad \mathcal{D}_5 = \mathcal{E}_7 - \mathcal{E}_8, \\
&\quad\ \mathcal{D}_6 = \mathcal{E}_8 - \mathcal{E}_9, \quad \mathcal{D}_7 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \\
&\quad\ \delta = 2\mathcal{D}_1 + 3\mathcal{D}_2 + 4\mathcal{D}_3 + 3\mathcal{D}_4 + 2\mathcal{D}_5 + \mathcal{D}_6 + 2\mathcal{D}_7 + \mathcal{D}_0, \\
E_7^{(1)\perp} \quad &= \quad A_1^{(1)} \\
\Pi(A_1^{(1)}) \quad &: \quad \alpha_1 = \mathcal{E}_4 - \mathcal{E}_5, \quad \alpha_0 = 3\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - 2\mathcal{E}_4 - \mathcal{E}_6 - \cdots - \mathcal{E}_9, \\
&\quad\ \delta = \alpha_1 + \alpha_0.
\end{aligned}$$

Add 11. $E_8^{(1)}$-surface:

$$\begin{aligned}
\Pi(E_8^{(1)}) \quad &: \quad \mathcal{D}_i = \mathcal{E}_i - \mathcal{E}_{i+1} (\text{for } i = 1, \ldots, 7), \quad \mathcal{D}_8 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \\
&\quad\ \mathcal{D}_0 = \mathcal{E}_8 - \mathcal{E}_9, \\
&\quad\ \delta = 2\mathcal{D}_1 + 4\mathcal{D}_2 + 6\mathcal{D}_3 + 5\mathcal{D}_4 + 4\mathcal{D}_5 + 3\mathcal{D}_6 + 2\mathcal{D}_7 + \mathcal{D}_0 + 3\mathcal{D}_8, \\
E_8^{(1)\perp} \quad &= \quad A_0^{(1)} \\
Q(A_0^{(1)}) \quad &= \quad \mathbb{Z}\delta.
\end{aligned}$$

B Realization of generalized Halphen surfaces with $\dim |-\mathcal{K}_X| = 0$

We have a blowing down morphism : $X \to \mathbb{P}^2$ which gives the basis $(\mathcal{E}_i)_{i=0}^9$ of $\mathrm{Pic}(X)$ in the relation of Appendix A with D_i's, for a generalized Halphen surface X with $\dim |-\mathcal{K}_X| = 0$. We normalize $\mathbb{P}^2$ by PGL(3) and get the following list. Any generalized Halphen surface is obtained by blowing up $\mathbb{P}^2$ with the centers at p_i's. See § 5, **3** and **4**.

Ell 1. $A_0^{(1)}$-surface:

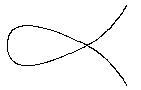
D: a smooth elliptic curve;

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3,$$
$$p_i : (\wp(\theta_i) : \wp'(\theta_i) : 1) = (a_i : a_i' : 1) \quad (i = 1, \ldots, 9),$$

$$\chi(\alpha_i) = \theta_{i+1} - \theta_i \quad (i = 1, \ldots, 7), \qquad \chi(\alpha_8) = \theta_1 + \theta_2 + \theta_3, \quad \chi(\alpha_0) = \theta_9 - \theta_8.$$

Mul 1. $A_0^{(1)*}$-surface:

D: a rational curve with a node;



$$y^2 z = 4(x + \tfrac{1}{3}z)^2 (x - \tfrac{2}{3}z)$$

$$p_i : \left(\frac{1}{\sin^2 \theta_i} - \frac{1}{3} : \frac{-2\cos\theta_i}{\sin^3 \theta_i} : 1 \right) = (a_i : a_i' : 1) \quad (i = 1, \ldots, 9).$$

$$\chi(\alpha_i) = \theta_{i+1} - \theta_i \quad (i = 1, \ldots, 7), \qquad \chi(\alpha_8) = \theta_1 + \theta_2 + \theta_3, \quad \chi(\alpha_0) = \theta_9 - \theta_8.$$

$$
\begin{aligned}
& \mathcal{D}_4 = \mathcal{E}_1 - \mathcal{E}_2, \quad \mathcal{D}_5 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_5, \quad \mathcal{D}_6 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_6 - \mathcal{E}_7, \\
& \mathcal{D}_0 = \mathcal{E}_8 - \mathcal{E}_9, \\
& \delta = \mathcal{D}_1 + 2\mathcal{D}_2 + 2\mathcal{D}_3 + 2\mathcal{D}_4 + \mathcal{D}_5 + \mathcal{D}_6 + \mathcal{D}_0, \\
D_6^{(1)\perp} &= (2A_1)^{(1)} \\
\Pi((2A_1)^{(1)}) &: \alpha_1 = \mathcal{E}_4 - \mathcal{E}_5, \quad \alpha_0 = 3\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - 2\mathcal{E}_4 - \mathcal{E}_6 - \cdots - \mathcal{E}_9, \\
& \alpha_1' = \mathcal{E}_6 - \mathcal{E}_7, \quad \alpha_0' = 3\mathcal{E}_0 - \mathcal{E}_1 - \cdots - \mathcal{E}_5 - 2\mathcal{E}_6 - \mathcal{E}_8 - \mathcal{E}_9, \\
& \delta = \alpha_1 + \alpha_0 = \alpha_1' + \alpha_0'.
\end{aligned}
$$

Add 7. $D_7^{(1)}$-surface:

$$
\begin{aligned}
\Pi(D_7^{(1)}) &: \mathcal{D}_1 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \quad \mathcal{D}_2 = \mathcal{E}_3 - \mathcal{E}_8, \quad \mathcal{D}_3 = \mathcal{E}_2 - \mathcal{E}_3, \\
& \mathcal{D}_4 = \mathcal{E}_1 - \mathcal{E}_2, \quad \mathcal{D}_5 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_5, \quad \mathcal{D}_6 = \mathcal{E}_4 - \mathcal{E}_6, \quad \mathcal{D}_7 = \mathcal{E}_5 - \mathcal{E}_7, \\
& \mathcal{D}_0 = \mathcal{E}_8 - \mathcal{E}_9, \\
& \delta = \mathcal{D}_1 + 2\mathcal{D}_2 + 2\mathcal{D}_3 + 2\mathcal{D}_4 + 2\mathcal{D}_5 + \mathcal{D}_6 + \mathcal{D}_7 + \mathcal{D}_0, \\
D_7^{(1)\perp} &= A_{1,|\alpha|^2=4}^{(1)} \\
\Pi(A_{1,|\alpha|^2=4}^{(1)}) &: \alpha_1 = \mathcal{E}_4 - \mathcal{E}_5 + \mathcal{E}_6 - \mathcal{E}_7, \quad \alpha_0 = 3\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - 2\mathcal{E}_4 - 2\mathcal{E}_6 - \mathcal{E}_8 - \mathcal{E}_9, \\
& \delta = \alpha_1 + \alpha_0.
\end{aligned}
$$

Add 8. $D_8^{(1)}$-surface:

$$
\begin{aligned}
\Pi(D_8^{(1)}) &: \mathcal{D}_1 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \quad \mathcal{D}_2 = \mathcal{E}_3 - \mathcal{E}_8, \quad \mathcal{D}_3 = \mathcal{E}_2 - \mathcal{E}_3, \\
& \mathcal{D}_4 = \mathcal{E}_1 - \mathcal{E}_2, \quad \mathcal{D}_5 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_5, \quad \mathcal{D}_6 = \mathcal{E}_5 - \mathcal{E}_6, \quad \mathcal{D}_7 = \mathcal{E}_4 - \mathcal{E}_5, \\
& \mathcal{D}_8 = \mathcal{E}_6 - \mathcal{E}_7, \quad \mathcal{D}_0 = \mathcal{E}_8 - \mathcal{E}_9, \\
& \delta = \mathcal{D}_1 + 2\mathcal{D}_2 + 2\mathcal{D}_3 + 2\mathcal{D}_4 + 2\mathcal{D}_5 + 2\mathcal{D}_6 + \mathcal{D}_7 + \mathcal{D}_8 + \mathcal{D}_0, \\
D_8^{(1)\perp} &= A_0^{(1)} \\
Q(A_0^{(1)}) &= \mathbb{Z}\delta.
\end{aligned}
$$

Add 9. $E_6^{(1)}$-surface:

$$
\begin{aligned}
\Pi(E_6^{(1)}) &: \mathcal{D}_1 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_5, \quad \mathcal{D}_2 = \mathcal{E}_1 - \mathcal{E}_2, \quad \mathcal{D}_3 = \mathcal{E}_2 - \mathcal{E}_7, \\
& \mathcal{D}_4 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \quad \mathcal{D}_5 = \mathcal{E}_3 - \mathcal{E}_6, \\
& \mathcal{D}_6 = \mathcal{E}_7 - \mathcal{E}_8, \quad \mathcal{D}_0 = \mathcal{E}_8 - \mathcal{E}_9, \\
& \delta = \mathcal{D}_1 + 2\mathcal{D}_2 + 3\mathcal{D}_3 + 2\mathcal{D}_4 + \mathcal{D}_5 + 2\mathcal{D}_6 + \mathcal{D}_0, \\
E_6^{(1)\perp} &= A_2^{(1)} \\
\Pi(A_2^{(1)}) &: \alpha_1 = \mathcal{E}_4 - \mathcal{E}_5, \quad \alpha_2 = \mathcal{E}_0 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_6, \\
& \alpha_0 = 2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8 - \mathcal{E}_9, \\
& \delta = \alpha_1 + \alpha_2 + \alpha_0.
\end{aligned}
$$

$$\begin{aligned}\Pi(A_1^{(1)}) \quad &: \quad \alpha_1 = \mathcal{E}_0 - \mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_6, \quad \alpha_0 = 2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_3 - \mathcal{E}_5 - \mathcal{E}_7 - \mathcal{E}_8 - \mathcal{E}_9,\\ &\quad \delta = \alpha_1 + \alpha_0.\end{aligned}$$

Mul 10. $A_8^{(1)}$-surface:

$$\begin{aligned}\Pi(A_8^{(1)}) \quad &: \quad \mathcal{D}_1 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_7 - \mathcal{E}_8, \quad \mathcal{D}_2 = \mathcal{E}_1 - \mathcal{E}_2, \quad \mathcal{D}_3 = \mathcal{E}_2 - \mathcal{E}_6,\\ &\quad \mathcal{D}_4 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \quad \mathcal{D}_5 = \mathcal{E}_3 - \mathcal{E}_4, \quad \mathcal{D}_6 = \mathcal{E}_4 - \mathcal{E}_5,\\ &\quad \mathcal{D}_7 = \mathcal{E}_0 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_7, \quad \mathcal{D}_8 = \mathcal{E}_7 - \mathcal{E}_8, \quad \mathcal{D}_0 = \mathcal{E}_8 - \mathcal{E}_9,\\ &\quad \delta = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4 + \mathcal{D}_5 + \mathcal{D}_6 + \mathcal{D}_7 + \mathcal{D}_8 + \mathcal{D}_0,\\ A_8^{(1)\perp} \quad &= \quad A_0^{(1)}\\ Q(A_0^{(1)}) \quad &= \quad \mathbb{Z}\delta.\end{aligned}$$

Add 1. $A_0^{(1)**}$-surface:

The same with **Ell 1** and **Mul 1**.

Add 2. $A_1^{(1)*}$-surface:

The same with **Mul 2**.

Add 3. $A_2^{(1)*}$-surface:

The same with **Mul 3**.

Add 4. $D_4^{(1)}$-surface:

$$\begin{aligned}\Pi(D_4^{(1)}) \quad &: \quad \mathcal{D}_1 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \quad \mathcal{D}_2 = \mathcal{E}_1 - \mathcal{E}_8, \quad \mathcal{D}_3 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_5,\\ &\quad \mathcal{D}_4 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_6 - \mathcal{E}_7, \quad \mathcal{D}_0 = \mathcal{E}_8 - \mathcal{E}_9,\\ &\quad \delta = \mathcal{D}_1 + 2\mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4 + \mathcal{D}_0,\\ D_4^{(1)\perp} \quad &= \quad D_4^{(1)}\\ \Pi(D_4^{(1)}) \quad &: \quad \alpha_1 = \mathcal{E}_2 - \mathcal{E}_3, \quad \alpha_2 = \mathcal{E}_0 - \mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_6, \quad \alpha_3 = \mathcal{E}_4 - \mathcal{E}_5,\\ &\quad \alpha_4 = \mathcal{E}_6 - \mathcal{E}_7, \quad \alpha_0 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_8 - \mathcal{E}_9,\\ &\quad \delta = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 + \alpha_0.\end{aligned}$$

Add 5. $D_5^{(1)}$-surface:

$$\begin{aligned}\Pi(D_5^{(1)}) \quad &: \quad \mathcal{D}_1 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \quad \mathcal{D}_2 = \mathcal{E}_2 - \mathcal{E}_8, \quad \mathcal{D}_3 = \mathcal{E}_1 - \mathcal{E}_2,\\ &\quad \mathcal{D}_4 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_5, \quad \mathcal{D}_5 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_6 - \mathcal{E}_7, \quad \mathcal{D}_0 = \mathcal{E}_8 - \mathcal{E}_9,\\ &\quad \delta = \mathcal{D}_1 + 2\mathcal{D}_2 + 2\mathcal{D}_3 + \mathcal{D}_4 + \mathcal{D}_5 + \mathcal{D}_0,\\ D_5^{(1)\perp} \quad &= \quad A_3^{(1)}\\ \Pi(A_3^{(1)}) \quad &: \quad \alpha_1 = \mathcal{E}_4 - \mathcal{E}_5, \quad \alpha_2 = \mathcal{E}_0 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_6,\\ &\quad \alpha_3 = \mathcal{E}_6 - \mathcal{E}_7, \quad \alpha_0 = 2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_6 - \mathcal{E}_8 - \mathcal{E}_9,\\ &\quad \delta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_0.\end{aligned}$$

Add 6. $D_6^{(1)}$-surface:

$$\Pi(D_6^{(1)}) \quad : \quad \mathcal{D}_1 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \quad \mathcal{D}_2 = \mathcal{E}_3 - \mathcal{E}_8, \quad \mathcal{D}_3 = \mathcal{E}_2 - \mathcal{E}_3,$$

Mul 6. $A_5^{(1)}$-surface:

$$\begin{aligned}
\Pi(A_5^{(1)}) \quad &: \quad \mathcal{D}_1 = \mathcal{E}_0 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8, \quad \mathcal{D}_2 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \quad \mathcal{D}_3 = \mathcal{E}_3 - \mathcal{E}_5,\\
&\quad \mathcal{D}_4 = \mathcal{E}_0 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_7, \quad \mathcal{D}_5 = \mathcal{E}_7 - \mathcal{E}_8, \quad \mathcal{D}_0 = \mathcal{E}_8 - \mathcal{E}_9,\\
&\quad \delta = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4 + \mathcal{D}_5 + \mathcal{D}_0,\\
A_5^{(1)\perp} \quad &= \quad (A_2 + A_1)^{(1)}\\
\Pi((A_2 + A_1)^{(1)}) \quad &: \quad \alpha_1 = \mathcal{E}_1 - \mathcal{E}_2, \quad \alpha_2 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_6,\\
&\quad \alpha_0 = 2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_3 - \mathcal{E}_5 - \mathcal{E}_7 - \mathcal{E}_8 - \mathcal{E}_9,\\
&\quad \alpha_1' = \mathcal{E}_0 - \mathcal{E}_3 - \mathcal{E}_5 - \mathcal{E}_6, \quad \alpha_0' = 2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8 - \mathcal{E}_9,\\
&\quad \delta = \alpha_1 + \alpha_0 = \alpha_1' + \alpha_0'.
\end{aligned}$$

Mul 7. $A_6^{(1)}$-surface:

$$\begin{aligned}
\Pi(A_6^{(1)}) \quad &: \quad \mathcal{D}_1 = \mathcal{E}_0 - \mathcal{E}_2 - \mathcal{E}_7 - \mathcal{E}_8, \quad \mathcal{D}_2 = \mathcal{E}_2 - \mathcal{E}_6, \quad \mathcal{D}_3 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3,\\
&\quad \mathcal{D}_4 = \mathcal{E}_3 - \mathcal{E}_5, \quad \mathcal{D}_5 = \mathcal{E}_0 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_7, \quad \mathcal{D}_6 = \mathcal{E}_7 - \mathcal{E}_8,\\
&\quad \mathcal{D}_0 = \mathcal{E}_8 - \mathcal{E}_9,\\
&\quad \delta = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4 + \mathcal{D}_5 + \mathcal{D}_6 + \mathcal{D}_0,\\
A_6^{(1)\perp} \quad &= \quad (A_1 + A_{1,|\alpha|^2=14})^{(1)}\\
\Pi((A_1 + A_{1,|\alpha|^2=14})^{(1)}) \quad &: \quad \alpha_1 = \mathcal{E}_0 - \mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_6, \quad \alpha_0 = 2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_3 - \mathcal{E}_5 - \mathcal{E}_7 - \mathcal{E}_8 - \mathcal{E}_9,\\
&\quad \alpha_1' = \mathcal{E}_0 + 2\mathcal{E}_1 - \mathcal{E}_2 - 2\mathcal{E}_3 + \mathcal{E}_4 - 2\mathcal{E}_5 - \mathcal{E}_6,\\
&\quad \alpha_0' = 2\mathcal{E}_0 - 3\mathcal{E}_1 + \mathcal{E}_3 - 2\mathcal{E}_4 + \mathcal{E}_5 - \mathcal{E}_7 - \mathcal{E}_8 - \mathcal{E}_9,\\
&\quad \delta = \alpha_1 + \alpha_0 = \alpha_1' + \alpha_0'.
\end{aligned}$$

Mul 8. $A_7^{(1)}$-surface:

$$\begin{aligned}
\Pi(A_7^{(1)}) \quad &: \quad \mathcal{D}_1 = \mathcal{E}_0 - \mathcal{E}_2 - \mathcal{E}_7 - \mathcal{E}_8, \quad \mathcal{D}_2 = \mathcal{E}_2 - \mathcal{E}_6, \quad \mathcal{D}_3 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3,\\
&\quad \mathcal{D}_4 = \mathcal{E}_3 - \mathcal{E}_4, \quad \mathcal{D}_5 = \mathcal{E}_4 - \mathcal{E}_5, \quad \mathcal{D}_6 = \mathcal{E}_0 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_7,\\
&\quad \mathcal{D}_7 = \mathcal{E}_7 - \mathcal{E}_8, \quad \mathcal{D}_0 = \mathcal{E}_8 - \mathcal{E}_9,\\
&\quad \delta = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4 + \mathcal{D}_5 + \mathcal{D}_6 + \mathcal{D}_7 + \mathcal{D}_0,\\
A_7^{(1)\perp} \quad &= \quad A_{1,|\alpha|^2=8}^{(1)}\\
\Pi(A_{1,|\alpha|^2=8}^{(1)}) \quad &: \quad \alpha_1 = \mathcal{E}_0 - 2\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8 - \mathcal{E}_9, \quad \alpha_0 = 2\mathcal{E}_0 + \mathcal{E}_1 - 2\mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - 2\mathcal{E}_6,\\
&\quad \delta = \alpha_1 + \alpha_0.
\end{aligned}$$

Mul 9. $A_7^{(1)\prime}$-surface:

$$\begin{aligned}
\Pi(A_7^{(1)\prime}) \quad &: \quad \mathcal{D}_1 = \mathcal{E}_0 - \mathcal{E}_2 - \mathcal{E}_7 - \mathcal{E}_8, \quad \mathcal{D}_2 = \mathcal{E}_2 - \mathcal{E}_6, \quad \mathcal{D}_3 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3,\\
&\quad \mathcal{D}_4 = \mathcal{E}_3 - \mathcal{E}_5, \quad \mathcal{D}_5 = \mathcal{E}_1 - \mathcal{E}_3, \quad \mathcal{D}_6 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_7,\\
&\quad \mathcal{D}_7 = \mathcal{E}_7 - \mathcal{E}_8, \quad \mathcal{D}_0 = \mathcal{E}_8 - \mathcal{E}_9,\\
&\quad \delta = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4 + \mathcal{D}_5 + \mathcal{D}_6 + \mathcal{D}_7 + \mathcal{D}_0,\\
A_7^{(1)\prime\perp} \quad &= \quad A_1^{(1)}
\end{aligned}$$

Mul 2. $A_1^{(1)}$-surface:

$$\begin{aligned}
\Pi(A_1^{(1)}) \quad &: \quad \mathcal{D}_1 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \quad \mathcal{D}_0 = 2\mathcal{E}_0 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8 - \mathcal{E}_9, \\
&\quad \delta = \mathcal{D}_1 + \mathcal{D}_0, \\
A_1^{(1)\perp} \quad &= \quad E_7^{(1)} \\
\Pi(E_7^{(1)}) \quad &: \quad \alpha_1 = \mathcal{E}_2 - \mathcal{E}_3, \quad \alpha_2 = \mathcal{E}_1 - \mathcal{E}_2, \quad \alpha_3 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_5, \quad \alpha_4 = \mathcal{E}_5 - \mathcal{E}_6, \\
&\quad \alpha_5 = \mathcal{E}_6 - \mathcal{E}_7, \quad \alpha_6 = \mathcal{E}_7 - \mathcal{E}_8, \quad \alpha_7 = \mathcal{E}_4 - \mathcal{E}_5, \quad \alpha_0 = \mathcal{E}_8 - \mathcal{E}_9, \\
&\quad \delta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_0 + 2\alpha_7.
\end{aligned}$$

Mul 3. $A_2^{(1)}$-surface:

$$\begin{aligned}
\Pi(A_2^{(1)}) \quad &: \quad \mathcal{D}_1 = \mathcal{E}_0 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8, \quad \mathcal{D}_2 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \\
&\quad \mathcal{D}_0 = \mathcal{E}_0 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_9, \\
&\quad \delta = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_0, \\
A_2^{(1)\perp} \quad &= \quad E_6^{(1)} \\
\Pi(E_6^{(1)}) \quad &: \quad \alpha_1 = \mathcal{E}_2 - \mathcal{E}_3, \quad \alpha_2 = \mathcal{E}_1 - \mathcal{E}_2, \quad \alpha_3 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_6, \\
&\quad \alpha_4 = \mathcal{E}_6 - \mathcal{E}_7, \quad \alpha_5 = \mathcal{E}_7 - \mathcal{E}_8, \quad \alpha_6 = \mathcal{E}_4 - \mathcal{E}_5, \quad \alpha_0 = \mathcal{E}_5 - \mathcal{E}_9, \\
&\quad \delta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 + \alpha_0.
\end{aligned}$$

Mul 4. $A_3^{(1)}$-surface:

$$\begin{aligned}
\Pi(A_3^{(1)}) \quad &: \quad \mathcal{D}_1 = \mathcal{E}_0 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8, \quad \mathcal{D}_2 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \\
&\quad \mathcal{D}_3 = \mathcal{E}_0 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_8, \quad \mathcal{D}_0 = \mathcal{E}_8 - \mathcal{E}_9, \\
&\quad \delta = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_0, \\
A_3^{(1)\perp} \quad &= \quad D_5^{(1)} \\
\Pi(D_5^{(1)}) \quad &: \quad \alpha_1 = \mathcal{E}_2 - \mathcal{E}_3, \quad \alpha_2 = \mathcal{E}_1 - \mathcal{E}_2, \quad \alpha_3 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_6, \\
&\quad \alpha_4 = \mathcal{E}_6 - \mathcal{E}_7, \quad \alpha_5 = \mathcal{E}_4 - \mathcal{E}_5, \quad \alpha_0 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_8 - \mathcal{E}_9, \\
&\quad \delta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_0.
\end{aligned}$$

Mul 5. $A_4^{(1)}$-surface:

$$\begin{aligned}
\Pi(A_4^{(1)}) \quad &: \quad \mathcal{D}_1 = \mathcal{E}_0 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8, \quad \mathcal{D}_2 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \\
&\quad \mathcal{D}_3 = \mathcal{E}_0 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_7, \quad \mathcal{D}_4 = \mathcal{E}_7 - \mathcal{E}_8, \quad \mathcal{D}_0 = \mathcal{E}_8 - \mathcal{E}_9, \\
&\quad \delta = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4 + \mathcal{D}_0, \\
A_4^{(1)\perp} \quad &= \quad A_4^{(1)} \\
\Pi(A_4^{(1)}) \quad &: \quad \alpha_1 = \mathcal{E}_2 - \mathcal{E}_3, \quad \alpha_2 = \mathcal{E}_1 - \mathcal{E}_2, \quad \alpha_3 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_6, \\
&\quad \alpha_4 = \mathcal{E}_4 - \mathcal{E}_5, \quad \alpha_0 = 2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8 - \mathcal{E}_9, \\
&\quad \delta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_0.
\end{aligned}$$

$$
\begin{aligned}
(\overline{x}:\overline{y}:\overline{z}) &= P_{\left(\theta_4-\frac{2}{3}\lambda_{456}+\frac{\lambda}{3},\theta_5-\frac{2}{3}\lambda_{456}+\frac{\lambda}{3},\theta_6-\frac{2}{3}\lambda_{456}+\frac{\lambda}{3}\right)}\circ P_{\left(\theta_7+\frac{2}{3}\lambda_{123}+\frac{\lambda_{456}}{3},\theta_8+\frac{2}{3}\lambda_{123}+\frac{\lambda_{456}}{3},\theta_9+\frac{2}{3}\lambda_{123}+\frac{\lambda_{456}}{3},\right)}\circ \\
&\circ P_{\left(\theta_4+\frac{\lambda_{123}}{3},\theta_5+\frac{\lambda_{123}}{3},\theta_6+\frac{\lambda_{123}}{3}\right)}\circ P_{(\theta_1,\theta_2,\theta_3)}(x:y:z), \\
&\lambda=\lambda_{123}+\lambda_{456}+\lambda_{789}=\sum_{i=1}^{9}\theta_i,\quad \lambda_{ijk}=\theta_i+\theta_j+\theta_k,
\end{aligned}
$$

where

$$
P_{(\alpha,\beta,\gamma)}(x:y:z)=\bar{l}_{\alpha,\beta,\gamma}(x:y:z)\begin{pmatrix} p_1(\alpha,\beta,\gamma)\wp(\frac{\alpha-2\beta-2\gamma}{3}) & p_1(\alpha,\beta,\gamma)\wp'(\frac{\alpha-2\beta-2\gamma}{3}) & p_1(\alpha,\beta,\gamma)\\ p_2(\alpha,\beta,\gamma)\wp(\frac{-2\alpha+\beta-2\gamma}{3}) & p_2(\alpha,\beta,\gamma)\wp'(\frac{-2\alpha+\beta-2\gamma}{3}) & p_2(\alpha,\beta,\gamma)\\ p_3(\alpha,\beta,\gamma)\wp(\frac{-2\alpha-2\beta+\gamma}{3}) & p_3(\alpha,\beta,\gamma)\wp'(\frac{-2\alpha-2\beta+\gamma}{3}) & p_3(\alpha,\beta,\gamma)\end{pmatrix},
$$

$$
\bar{l}_{\alpha,\beta,\gamma}(x:y:z)=\begin{pmatrix} l_{\gamma,\alpha}(x:y:z)\cdot l_{\alpha,\beta}(x:y:z) : l_{\alpha,\beta}(x:y:z)\cdot l_{\beta,\gamma}(x:y:z) : \\ : l_{\beta,\gamma}(x:y:z)\cdot l_{\gamma,\alpha}(x:y:z)\end{pmatrix},
$$

$$
l_{\alpha,\beta}(x:y:z)=\det\begin{pmatrix} x & y & z\\ \wp(\alpha) & \wp'(\alpha) & 1\\ \wp(\beta) & \wp'(\beta) & 1\end{pmatrix},
$$

$$
\begin{pmatrix} p_1(\alpha,\beta,\gamma)\\ p_2(\alpha,\beta,\gamma)\\ p_3(\alpha,\beta,\gamma)\end{pmatrix}=M\left(\frac{\alpha-2\beta-2\gamma}{3},\frac{-2\alpha+\beta-2\gamma}{3},\frac{-2\alpha-2\beta+\gamma}{3}\right)\begin{pmatrix} \wp\left(\frac{\alpha+\beta+\gamma}{3}\right)(\wp(\beta)-\wp(\gamma))\\ \wp'\left(\frac{\alpha+\beta+\gamma}{3}\right)(\wp(\gamma)-\wp(\alpha))\\ \wp(\alpha)-\wp(\beta)\end{pmatrix},
$$

$$
M(\alpha,\beta,\gamma)=\begin{pmatrix} \wp'(\alpha)-\wp'(\beta) & \wp'(\beta)-\wp'(\gamma) & \wp'(\gamma)-\wp'(\alpha)\\ \wp(\beta)-\wp(\gamma) & \wp(\gamma)-\wp(\alpha) & \wp(\alpha)-\wp(\beta)\\ \wp'(\beta)\wp(\alpha)-\wp'(\gamma)\wp(\beta) & \wp'(\gamma)\wp(\beta)-\wp'(\alpha)\wp(\gamma) & \wp'(\alpha)\wp(\gamma)-\wp'(\beta)\wp(\alpha)\end{pmatrix}.
$$

A Correspondence between standard basis of Λ_{10} and canonical root bases

Here we list the correspondence between the standard basis and the canonical root bases for the Picard group $\mathrm{Pic}(X)$ of the generalized Halphen surface X with $\dim|-\mathcal{K}_X|=0$. Now we denote by $(\mathcal{E}_i)_{i=0}^9$ the standard basis which defines the blowing-down structure, and denote by $\mathcal{D}_i=[D_i]$ the class of the irreducible component of the unique anti-canonical divisor. We take a root basis $\Pi(R)$ as for the following α_i's. There are different correspondences, but we can take the blowing-down structue as in the list, for X and $D=\sum m_iD_i$. See § 4, **4**.

Ell 1. $A_0^{(1)}$-surface:

$$
\begin{aligned}
Q(A_0^{(1)}) &= \mathbb{Z}\delta,\\
A_0^{(1)\perp} &= E_8^{(1)}\\
\Pi(E_8^{(1)}) &: \alpha_i=\mathcal{E}_i-\mathcal{E}_{i+1}(\text{for } i=1,\dots,7),\quad \alpha_8=\mathcal{E}_0-\mathcal{E}_1-\mathcal{E}_2-\mathcal{E}_3,\quad \alpha_0=\mathcal{E}_8-\mathcal{E}_9,\\
&\quad \delta=2\alpha_1+4\alpha_2+6\alpha_3+5\alpha_4+4\alpha_5+3\alpha_6+2\alpha_7+\alpha_0+3\alpha_8.
\end{aligned}
$$

Mul 1. $A_0^{(1)*}$-surface:

The same with **Ell 1**.

On $E_6^{(1)}$-surface (in $\widetilde{W}(A_2^{(1)})$-symmetry)

$$\begin{aligned} d\text{-}P_{II} &= w_2 \circ w_1 \circ \sigma_{(15)(24)} \circ \sigma_{(10)(26)} : \\ &\quad (a_i; t; f, g) \mapsto (\overline{a}_i; \overline{t}; \overline{f}, \overline{g}) = \left(a_1, a_2 - \lambda, a_0 + \lambda; s; s - f - g + \frac{a_0}{g}, s - g - \overline{f} - \frac{a_2 - \lambda}{\overline{f}} \right), \\ &\quad \lambda = a_1 + a_2 + a_0, \end{aligned}$$

where $f = -y/x$ and $g = s - \dfrac{x}{z} + \dfrac{y}{x}$.

On $E_7^{(1)}$-surface (in $\widetilde{W}(A_1^{(1)})$-symmetry)

$$\begin{aligned} \text{alt.}d\text{-}P_I &= w_1 \circ \sigma_{(15)(24)(60)} : \\ &\quad (a_i; s; f, g) \mapsto (\overline{a}_i; \overline{s}; \overline{f}, \overline{g}) = \left(a_1 - \lambda, a_0 + \lambda; s; -f - \frac{a_0}{\overline{g}}, s + f^2 - g \right), \\ &\quad \lambda = a_1 + a_0, \end{aligned}$$

where $f = y/x$ and $g = x/z$.

In this paper we have presented a new approach for both of discrete and differential Painlevé equations. Now we can summarize how to get the differential Painlevé equations. In order to obtain the Painlevé equation, we have constructed

- X : a generalized Halphen surface with $\dim |-\mathcal{K}_X| = 0$,
- $\mathcal{X}$: a family of surfaces X,
- $\mathrm{Cr}(X)$: the group of the Cremona isometries (an action on $\mathrm{Pic}(X)$),
- the Cremona action : an action of $\mathrm{Cr}(X)$ on $\mathcal{X}$,
- discrete Painlevé equations as a Cremona action,

and the Painlevé differential equations are obtained as a limit of discrete Painlevé equations.

4. We close this section with the introduction of the most generic discrete equation. Although it has no differential equation as a limit directly, all discrete equations which appear in this article can be regarded as its degeneration.

We often use terminology such as "q-difference equation" when we call a discrete equation which have coefficients to change multiplicatively with respect to the independent variables. The next equation have coefficients which change in compliance with the addition formula for the elliptic functions. If we call this kind of discrete equations elliptic-difference eqations, then the next is the elliptic-difference Painlevé equation.

On $A_0^{(1)}$-surface (in $W(E_8^{(1)})$-symmetry)

$$\begin{aligned} \text{ell.}P &= w_3 \circ w_2 \circ w_4 \circ w_3 \circ w_1 \circ w_2 \circ w_5 \circ w_4 \circ w_3 \circ w_6 \circ w_5 \circ w_4 \circ w_8 \circ w_3 \circ w_2 \circ w_1 \circ w_7 \circ w_6 \circ w_5 \circ w_4 \circ \\ &\quad \circ w_3 \circ w_2 \circ w_0 \circ w_7 \circ w_6 \circ w_5 \circ w_4 \circ w_3 \circ w_8 \circ w_3 \circ w_4 \circ w_5 \circ w_6 \circ w_7 \circ w_0 \circ w_2 \circ w_3 \circ w_4 \circ w_5 \circ \\ &\quad \circ w_6 \circ w_7 \circ w_1 \circ w_2 \circ w_3 \circ w_8 \circ w_4 \circ w_5 \circ w_6 \circ w_3 \circ w_4 \circ w_5 \circ w_2 \circ w_1 \circ w_3 \circ w_4 \circ w_2 \circ w_3 \circ w_8 : \\ &\quad \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 & \\ \theta_4 & \theta_5 & \theta_6; & x : y : z \\ \theta_7 & \theta_8 & \theta_9 & \end{pmatrix} \mapsto \begin{pmatrix} \theta_1 + \frac{2}{3}\lambda & \theta_2 + \frac{2}{3}\lambda & \theta_3 + \frac{2}{3}\lambda & \\ \theta_4 - \frac{1}{3}\lambda & \theta_5 - \frac{1}{3}\lambda & \theta_6 - \frac{1}{3}\lambda; & \overline{x} : \overline{y} : \overline{z} \\ \theta_7 - \frac{1}{3}\lambda & \theta_8 - \frac{1}{3}\lambda & \theta_9 - \frac{1}{3}\lambda & \end{pmatrix}, \end{aligned}$$

On $D_4^{(1)}$-surface (in $\widetilde{W}(D_4^{(1)})$-symmetry)

$$
\begin{aligned}
d\text{-}P_V \quad = \quad & \sigma_{(14)} \circ w_2 \circ w_3 \circ w_4 \circ \sigma_{(40)} \circ \sigma_{(34)} \circ \sigma_{(40)} \circ w_2 \circ w_1 \circ w_0 : \\
& \left(\begin{matrix} a_1 & a_2 & a_3 \\ a_4 & a_0 \end{matrix} ; s; f, g \right) \mapsto \left(\begin{matrix} a_1 + \lambda & a_2 - \lambda & a_3 \\ a_4 & a_0 + \lambda \end{matrix} ; s; \overline{f}, \overline{g} \right), \\
& \overline{f} + f = a_3 + \frac{a_1}{g+1} + \frac{a_0}{sg+1}, \\
& \overline{g} g = \frac{(\overline{f} - \lambda + a_2)(\overline{f} - \lambda + a_2 + a_4)}{s\overline{f}(\overline{f} - a_3)}, \\
& \lambda = a_1 + a_2 + a_3 + a_4 + a_0,
\end{aligned}
$$

where $f = y/x$ and $g = (x - z)/z$.

On $D_5^{(1)}$-surface (in $\widetilde{W}(A_3^{(1)})$-symmetry)

$$
\begin{aligned}
d\text{-}P_{IV} \quad = \quad & w_3 \circ w_2 \circ w_1 \circ \sigma_{(45)} \circ \sigma_{(14)(50)} : \\
& \left(\begin{matrix} a_1 & a_2 \\ a_3 & a_0 \end{matrix} ; s; f, g \right) \mapsto \left(\begin{matrix} a_1 & a_2 \\ a_3 - \lambda & a_0 + \lambda \end{matrix} ; s; \overline{f}, \overline{g} \right), \\
& \overline{f} f = \frac{s\overline{g}}{(\overline{g} - a_3 + \lambda)(\overline{g} + a_0 + \lambda)}, \\
& \overline{g} + g = \frac{s}{f} + \frac{a_1 + a_0}{1 - f} - \lambda + a_3 - a_0, \\
& \lambda = a_1 + a_2 + a_3 + a_0,
\end{aligned}
$$

where $f = \dfrac{(x - z)(y(z - x) - szx)}{x((y + a_0 z)(z - x) - szx)}$ and $g = \dfrac{y(z - x) - szx}{z(z - x)}$.

$$
\begin{aligned}
d\text{-}P_{III} \quad = \quad & (w_3 \circ w_2 \circ w_1 \circ \sigma_{(45)} \circ \sigma_{(14)(50)}) \circ (w_2 \circ w_1 \circ w_0 \circ \sigma_{(45)} \circ \sigma_{(14)(50)}) \\
= \quad & w_2 \circ w_3 \circ w_1 \circ w_2 \circ (\sigma_{(45)} \circ \sigma_{(14)(50)})^2 : \\
& \left(\begin{matrix} a_1 & a_2 \\ a_3 & a_0 \end{matrix} ; s; f, g \right) \mapsto \left(\begin{matrix} a_1 & a_2 + \lambda \\ a_3 & a_0 - \lambda \end{matrix} ; s; \overline{f}, \overline{g} \right), \\
& \overline{f} + f = 1 + \frac{a_0 - \lambda}{s - \overline{g}} + \frac{a + 2 - \lambda}{\overline{g}}, \\
& \overline{g} + g = s - \frac{a_1 + a_0}{1 - f} + \frac{a_3 + a_0}{f}, \\
& \lambda = a_1 + a_2 + a_3 + a_0,
\end{aligned}
$$

where $f = \dfrac{x((y + a_0 z)(z - x) - szx)}{(x - z)(y(z - x) - szx)}$ and $g = \dfrac{y(z - x) - szx}{z(z - x)}$.

On $D_6^{(1)}$-surface (in $\widetilde{W}((2A_1)^{(1)})$-symmetry)

$$
\begin{aligned}
\text{alt. } d\text{-}P_{II} \quad = \quad & w_1' \circ \sigma_{(56)} \circ \sigma_{(16)(24)(50)} \circ \sigma_{(56)} : \\
& \left(\begin{matrix} a_1 & a_0 \\ b_1 & b_0 \end{matrix} ; s; f, g \right) \mapsto \left(\begin{matrix} a_1 & a_0 \\ b_1 - \lambda & b_0 + \lambda \end{matrix} ; s; \overline{f}, \overline{g} \right), \\
& \overline{f} f = \frac{g(g - b_1)}{s}, \\
& \overline{g} + g = \frac{b_1}{1 - \overline{f}}, \\
& \lambda = a_1 + a_0 = b_1 + b_0,
\end{aligned}
$$

where $f = x/(x - z)$ and $g = y/z$.

On $A_3^{(1)}$-surface (in $\widetilde{W}(D_5^{(1)})$-symmetry);

$$\begin{aligned}
q\text{-}P_{VI} &= \sigma_{(20)} \circ w_2 \circ w_1 \circ w_0 \circ w_2 \circ \sigma_{(13)} \circ w_3 \circ w_5 \circ w_4 \circ w_3 : \\
&\quad \left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix}; f, g\right) \mapsto \left(\begin{matrix} qb_1 & qb_2 & b_3 & b_4 \\ qb_5 & qb_6 & b_7 & b_8 \end{matrix}; \overline{f}, \overline{g}\right), \\
&\quad \frac{f\overline{f}}{b_7 b_8} = \frac{\overline{g} - qb_1}{\overline{g} - b_3}\frac{\overline{g} - qb_2}{\overline{g} - b_4}, \\
&\quad \frac{g\overline{g}}{b_3 b_4} = \frac{f - b_5}{f - b_7}\frac{f - b_6}{f - b_8}, \\
&\quad q = \frac{b_3 b_4 b_5 b_6}{b_1 b_2 b_7 b_8},
\end{aligned}$$

where

$$\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix}; f, g\right) = \left(\begin{matrix} 1/a_2 & 1/a_3 & 1/a_1 & a_8 \\ a_6 & a_7 & -1/a_4 a_1 & -1/a_5 a_1 \end{matrix}; \frac{y}{x - a_1 z}, \frac{z}{x}\right).$$

On $A_4^{(1)}$-surface (in $\widetilde{W}(A_4^{(1)})$-symmetry);

$$\begin{aligned}
q\text{-}P_{V} &= w_4 \circ w_3 \circ w_2 \circ w_1 \circ \sigma_{(12340)}^2 : \\
&\quad \left(\begin{matrix} a_1 & a_2 & a_3 \\ a_4 & a_0 \end{matrix}; f, g\right) \mapsto \left(\begin{matrix} a_1 & a_2 & a_3 \\ a_4/q & qa_0 \end{matrix}; \overline{f}, \overline{g}\right), \\
&\quad \overline{f} f = \frac{a_1 a_2}{a_3}\frac{1 - g}{(1 + a_2 g)(1 - a_1 a_2 g)}, \\
&\quad \overline{g} g = \frac{a_4 a_0}{a_2}\frac{(1 - a_3\overline{f})(1 - f)}{\overline{f}(a_4 - q\overline{f})},
\end{aligned}$$

where $f = (z - x)/y$ and $g = z(x + y - z)/x(z - x)$.

On $A_5^{(1)}$-surface (in $\widetilde{W}((A_2 + A_1)^{(1)})$-symmetry);

$$\begin{aligned}
q\text{-}P_{IV} &= w_1' \circ \sigma_{(123450)}^3 : \\
&\quad \left(\begin{matrix} a_1 & a_2 & a_0 \\ b_1 & b_0 \end{matrix}; f, g\right) \mapsto \left(\begin{matrix} a_1 & a_2 & a_0 \\ b_1/q & qb_0 \end{matrix}; \overline{f}, \overline{g}\right), \\
&\quad \overline{f} f = -(a_2 b_0 - \frac{a_0 b_1}{g})(1 - g), \\
&\quad a_1 b_0 \overline{g} g = -\frac{\overline{f} - a_2 b_0}{\overline{f} + a_2}, \\
&\quad q = a_1 a_2 a_0 = b_1 b_0,
\end{aligned}$$

where $f = y/x$ and $g = x/z$.

$$\begin{aligned}
q\text{-}P_{III} &= w_2 \circ w_1 \circ \sigma_{(123450)}^2 : \\
&\quad \left(\begin{matrix} a_1 & a_2 & a_0 \\ b_1 & b_0 \end{matrix}; f, g\right) \mapsto \left(\begin{matrix} a_1 & a_2/q & qa_0 \\ & b_1 & b_0 \end{matrix}; \overline{f}, \overline{g}\right), \\
&\quad g f \overline{f} = -a_1 \frac{g - b_1}{a_0 g - b_1}, \\
&\quad g \overline{f} \overline{g} = b_1 \frac{\overline{f} g + b_1}{a_2 a_0 \overline{f} g + b_1},
\end{aligned}$$

where $f = (z - x)/y$ and $g = yz/x(z - x)$.

This is well known as one of the difference second Painlevé equations $d\text{-}P_{II}$.

In order to degenerate the $E_6^{(1)}$-surface to the $E_7^{(1)}$-surface we have to make p_3 infinitely near p_1. We take a new coordinate of $E_6^{(1)}$-surface as $(\tilde{x} : \tilde{y} : \tilde{z}) = (-x : (x+y)/\epsilon : \epsilon^2 z)$. In this coordinate a line $y = \tilde{x} + \epsilon\tilde{y} = 0$, which passes through p_5 and p_3, get joined with $x = -\tilde{x} = 0$. We have

$$
\begin{aligned}
p_1 : (0:1:0) \quad &\leftarrow \quad p_2 : \left(\frac{\tilde{x}}{\tilde{y}}, \frac{\tilde{z}}{\tilde{x}}\right) = (0,0) \leftarrow p_3 : \left(\frac{\tilde{x}}{\tilde{y}}, \frac{\tilde{y}\tilde{z}}{\tilde{x}^2}\right) = (-\epsilon, 0) \leftarrow \\
&\leftarrow \quad p_6 : \left(\widetilde{\left(\frac{\tilde{x}}{\tilde{y}}\right)}, \widetilde{\left(\frac{\tilde{y}}{\tilde{x}}\right)}\frac{\tilde{y}\tilde{z}}{\tilde{x}^2}\right) = \left(0, -\frac{1}{a_2}\right) \leftarrow \\
&\leftarrow \quad p_7 : \left(\widetilde{\left(\frac{\tilde{x}}{\tilde{y}}\right)}, \widetilde{\left(\frac{\tilde{y}}{\tilde{x}}\right)}\left(\widetilde{\left(\frac{\tilde{y}}{\tilde{x}}\right)}\frac{\tilde{y}\tilde{z}}{\tilde{x}^2} + \frac{1}{a_2}\right)\right) = \left(\epsilon, \frac{1}{\epsilon}\left(\frac{1}{a_2} + 1\right)\right) \leftarrow \\
&\leftarrow \quad p_8 : \left(\frac{\tilde{x}}{\tilde{y}}, \frac{\tilde{y}}{\tilde{x}}\left(\widetilde{\left(\frac{\tilde{y}}{\tilde{x}}\right)}\left(\widetilde{\left(\frac{\tilde{y}}{\tilde{x}}\right)}\frac{\tilde{y}\tilde{z}}{\tilde{x}^2} + \frac{1}{a_2}\right) - \frac{1}{\epsilon}\left(\frac{1}{a_2} + 1\right)\right)\right) = \left(0, -\frac{1}{\epsilon^2}\left(3 + \frac{1}{a_2} - s\right)\right) \leftarrow \\
&\leftarrow \quad p_9 : \left(\frac{\tilde{x}}{\tilde{y}}, \frac{\tilde{y}}{\tilde{x}}\left(\frac{\tilde{y}}{\tilde{x}}\left(\widetilde{\left(\frac{\tilde{y}}{\tilde{x}}\right)}\left(\widetilde{\left(\frac{\tilde{y}}{\tilde{x}}\right)}\frac{\tilde{y}\tilde{z}}{\tilde{x}^2} + \frac{1}{a_2}\right) - \frac{1}{\epsilon}\left(\frac{1}{a_2} + 1\right)\right) + \frac{1}{\epsilon^2}\left(3 + \frac{1}{a_2} - s\right)\right)\right) \\
&\qquad = \left(0, \frac{1}{\epsilon^3}\left(6 - 4s + s^2 + \frac{1}{a_2} - a_0\right)\right), \\
p_4 : (0:0:1), \qquad & p_5 : (0 : a_1 : \epsilon^3),
\end{aligned}
$$

where $\widetilde{\left(\frac{\tilde{x}}{\tilde{y}}\right)} = \frac{\tilde{x}}{\tilde{y}} + \epsilon$ and $\widetilde{\left(\frac{\tilde{y}}{\tilde{x}}\right)} = \widetilde{\left(\frac{\tilde{x}}{\tilde{y}}\right)}^{-1} = \left(\frac{\tilde{x}}{\tilde{y}} + \epsilon\right)^{-1}$.

When we take

$$
s = 2, a_0 = 1 + \tilde{s}\epsilon^2 + \beta_0\epsilon^3, a_2 = -1 - \tilde{s}\epsilon^2 + \beta_2\epsilon^3, a_1 = \beta_1\epsilon^3 \ (\tilde{a}_0 = \beta_0 + \beta_2, \tilde{a}_1 = \beta_1),
$$

we have the $E_7^{(1)}$-surface as a limit $\epsilon \to 0$.

If we take $\tilde{f} = \frac{\tilde{y}}{\tilde{x}} - \frac{1}{2}\epsilon\frac{\tilde{x}}{\tilde{z}}$ $(\to \tilde{y}/\tilde{x})$ and $\tilde{g} = \frac{\tilde{x}}{\tilde{z}} - \epsilon\frac{\beta_0 - \beta_2}{4}$ $(\to \tilde{x}/\tilde{z})$ then the difference equation $d\text{-}P_{II}$ goes to the differential equation P_{II} as a continuous limit:

$$
\begin{aligned}
\tilde{f}' &= \tilde{f}^2 - 2\tilde{g} - \tilde{s}, \\
\tilde{g}' &= \beta_1 - 2\tilde{f}\tilde{g}, \\
\tilde{s}' &= \beta_1 + \beta_2 + \beta_0.
\end{aligned}
$$

At the end we have got the second Painlevé differential equation only from surface theory.

Remark. This calculation is a formal one. We assume $\tilde{f}, \tilde{g}$ and $\tilde{s}$ to be functions of a formal parameter t. We take $\overline{\tilde{f}} = T(\tilde{f}) = \tilde{f} + \epsilon\frac{d\tilde{f}}{dt} + \frac{\epsilon^2}{2}\frac{d^2\tilde{f}}{dt^2} + \cdots$, and so on. □

The commutativity between T and $W(A_1^{(1)})$ in $\widetilde{W}(A_2^{(1)})$ is the reason why the Bäcklund transformations occur;

$$
T \circ w_1 = w_1 \circ T, \quad T \circ w_0 \circ w_2 \circ w_0 = w_0 \circ w_2 \circ w_0 \circ T,
$$

where $W(A_2^{(1)}) \supset W(A_1^{(1)}) = \langle w_1, w_0 \circ w_2 \circ w_0 \rangle$ and $\langle w_1, w_0 \circ w_2 \circ w_0 \rangle$ goes to the action of $\mathrm{Cr}(X(E_7^{(1)}), D)$ on $\mathcal{X}(E_7^{(1)})$ as a limit. The fact that $(\alpha_0 - \alpha_2) \perp (\mathbb{Z}\alpha_1 + \mathbb{Z}(\alpha_2 + \alpha_0))$ causes this commutativity.

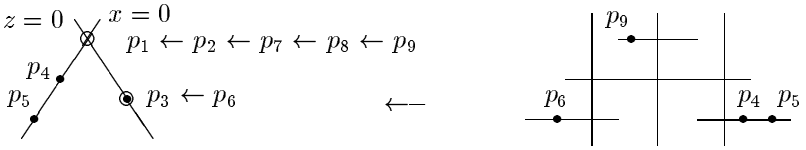
3. These settings we have seen in the case of the $E_7^{(1)}$-surface and P_{II} are applicable to all the Painlevé equations. We write down some discrete Painlevé equations.

2. The space of initial conditions $X(R)$ is constructed and the Bäcklund transformatins occurs from the Cremona isometries. We have achieved here from theory of rational surfaces without the differential equation. Then, can we obtain the differential equation itself from surface theory?

In the case of the $E_7^{(1)}$-surface, the $\widetilde{W}(A_1^{(1)})$-symmetry is an action with respect to a_1 and a_0, but we have another parameter s. This s is not a parameter arising from the period mapping with respect to $H_2(X - D_{red}, \mathbb{Z})$. In the theory of Painlevé differential equations the continuous action from differential equation is a deformation of X with respect to s, but this is a trivial deformation of $X - D_{red}$ because the Painlevé property ensures that $\mathcal{X}_{(a_1,a_0,s_0)} - D_{red} \to \mathcal{X}_{(a_1,a_0,s_0+s)} - D_{red}$ is an isomorphism as complex analytic spaces.

Anyway we want to construct this deformation. In the following, the second Painlevé diffential equation is obtained not directly but as a limit of discete action on further generic rational surfaces.

The $E_6^{(1)}$-surface and the $D_6^{(1)}$-surface have the $E_7^{(1)}$-surface as its degeneration. We choose the $E_6^{(1)}$-surface among them. This surface is constructed in a similar way to the case $R = E_7^{(1)}$ above;



$$p_1 : (0:1:0) \quad \leftarrow p_2 : \left(\frac{x}{y}, \frac{z}{x}\right) = (0,0) \leftarrow p_7 : \left(\frac{x}{y}, \frac{yz}{x^2}\right) = (0,1) \leftarrow$$

$$\leftarrow p_8 : \left(\frac{x}{y}, \frac{y(yz-x^2)}{x^3}\right) = (0,s) \leftarrow p_9 : \left(\frac{x}{y}, \frac{y(y(yz-x^2)-sx^3)}{x^4}\right) = (0, -a_0 + s^2),$$

$$p_4 : (0:0:1), \qquad p_5 : (0:a_1:1), \quad p_3 : (1:0:0) \leftarrow p_6 : \left(\frac{z}{x}, \frac{y}{z}\right) = (0, -a_2).$$

Now $Q(E_6^{(1)})^{\perp} = Q(A_2^{(1)})$ and we can write the action of $\mathrm{Aut}(E_6^{(1)}) \ltimes W(A_2^{(1)}) \simeq \widetilde{W}(A_2^{(1)})$ as;

$$\begin{aligned}
-\sigma_{(15)(24)} &: (a_1, a_2, a_0\,; s;\, x:y:z) \longmapsto (-a_2, -a_1, -a_0\,; s;\, yz : -xy : -zx)\,,\\
-\sigma_{(10)(26)} &: (a_i; s; x:y:z) \longmapsto \left(-a_1, -a_0, -a_2\,; s;\, xz : -(yz - x^2 - sxz) : z^2\right),\\
w_1 &: (a_i; s; x:y:z) \longmapsto (-a_1, a_2 + a_1, a_0 + a_1\,; s;\, x : y - a_1 z : z)\,,\\
w_2 &= \sigma_{(15)(24)} \circ w_1 \circ \sigma_{(15)(24)}, \quad w_0 = \sigma_{(10)(26)} \circ w_2 \circ \sigma_{(10)(26)}.
\end{aligned}$$

Let us take $f = -y/x$ and $g = s - \dfrac{x}{z} + \dfrac{y}{x}$ as generators of $K(\mathcal{X})$ with a_1, a_2, a_0 and s. Then we have

$$\begin{aligned}
w_1 &: (a_1, a_2, a_0; s; f, g) \longmapsto \left(-a_1, a_2 + a_1, a_0 + a_1; s; f + \frac{a_1}{s - f - g}, g - \frac{a_1}{s - f - g}\right),\\
w_2 &: (a_1, a_2, a_0; s; f, g) \longmapsto \left(a_1 + a_2, -a_2, a_0 + a_2; s; f, g + \frac{a_2}{f}\right),\\
w_0 &: (a_1, a_2, a_0; s; f, g) \longmapsto \left(a_1 + a_0, a_2 + a_0, -a_0; s; f - \frac{a_0}{g}, g\right),\\
-\sigma_{(15)(24)} &: (a_1, a_2, a_0; s; f, g) \longmapsto (-a_2, -a_1, -a_0; s; s - f - g, g),\\
-\sigma_{(10)(26)} &: (a_1, a_2, a_0; s; f, g) \longmapsto (-a_1, -a_0, -a_2; s; g, f).
\end{aligned}$$

We have a translation T in $\widetilde{W}(A_2^{(1)})$:

$$\begin{aligned}
T &= w_2 \circ w_1 \circ \sigma_{(15)(24)} \circ \sigma_{(10)(26)} :\\
&(a_i; s; f, g) \mapsto (\overline{a}_i; \overline{s}; \overline{f}, \overline{g}) = \left(a_1, a_2 - \lambda, a_0 + \lambda; s; s - f - g + \frac{a_0}{g}, s - g - \overline{f} - \frac{a_2 - \lambda}{\overline{f}}\right).
\end{aligned}$$

Remark. Usually P_{III} is written in the for

$$\frac{d^2f}{dt^2} = \frac{1}{f}\left(\frac{df}{dt}\right)^2 - \frac{1}{t}\frac{df}{dt} + \frac{1}{t}(\alpha f^2 + \beta) + \gamma f^3 + \frac{\delta}{f}. \tag{24}$$

It is equivalent to

$$\frac{d^2f}{dt^2} = \frac{1}{f}\left(\frac{df}{dt}\right)^2 - \frac{1}{t}\frac{df}{dt} + \frac{f^2}{4t^2}(\gamma f + \alpha) + \frac{\beta}{4t} + \frac{\delta}{4f}, \tag{25}$$

by replacing in the latter equation, t by t^2 and f by tf.

If $\gamma\delta \neq 0$, then we can normalize $\gamma = -\delta = 4$ without loss of generality. In this case we get $P_{III}^{D_6^{(1)}}$ (16)–(18).

The parameters (δ, β) and $(-\alpha, -\gamma)$ are symmetric; if one of γ and δ equals zero (not both), then we have $P_{III}^{D_7^{(1)}}$ (19)–(21). When $\gamma = \delta = 0$, we have $P_{III}^{D_8^{(1)}}$ (22)–(23).

Moreover P_V

$$\frac{d^2f}{dt^2} = \left(\frac{1}{2f} + \frac{1}{1-f}\right)\left(\frac{df}{dt}\right)^2 - \frac{1}{t}\frac{df}{dt} + \frac{(f-1)^2}{t^2}\left(\alpha f + \frac{\beta}{f}\right) + \frac{\gamma}{t}f + \delta\frac{f(f+1)}{f-1} \tag{26}$$

is also reduced to $P_{III}^{D_6^{(1)}}$ (16)–(18) when $\delta = 0$. If $\delta \neq 0$, then we can normalize it as (13)–(15). In detail, see [20]. □

$$P_{IV}: \quad \frac{df}{dt} = f(2g - f - s) - a_1, \tag{27}$$

$$\frac{dg}{dt} = g(2f - g + s) + a_2, \tag{28}$$

$$\frac{ds}{dt} = 2\lambda = 2(a_1 + a_2 + a_0), \tag{29}$$

where $f = x/z$ and $g = y/x$.

$$P_{II}: \quad \frac{df}{dt} = f^2 - s - 2g, \tag{30}$$

$$\frac{dg}{dt} = a_1 - 2fg, \tag{31}$$

$$\frac{ds}{dt} = \lambda = a_1 + a_0, \tag{32}$$

where $f = y/x$ and $g = x/z$.

$$P_I: \quad \frac{df}{dt} = g, \tag{33}$$

$$\frac{dg}{dt} = 6f^2 + 2s, \tag{34}$$

$$\frac{ds}{dt} = \lambda, \tag{35}$$

where $f = x/z$ and $g = 2y/z$.

Remark. If $\lambda = 0$, then the surface X is a Halphen surface of index one. Each Painlevé equation is reduced to the equation which is satisfied by an elliptic function in this limit. If $\lambda \neq 0$, then we can normalize each Painlevé equation as $\lambda = 1$ without loss of generality. □

the type of surface	$D_4^{(1)}$	$D_5^{(1)}$	$D_6^{(1)}$	$D_7^{(1)}$	$D_8^{(1)}$
Painlevé equation	P_{VI}	P_V	$P_{III}^{D_6^{(1)}}$	$P_{III}^{D_7^{(1)}}$	$P_{III}^{D_8^{(1)}}$
Cremona action	$\widetilde{W}(D_4^{(1)})$	$\widetilde{W}(A_3^{(1)})$	$\mathrm{Aut}(D_6^{(1)}) \ltimes W((2A_1)^{(1)})$	$\widetilde{W}(A_1^{(1)})$	$\mathfrak{S}_2$

$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$
P_{IV}	P_{II}	P_I
$\widetilde{W}(A_2^{(1)})$	$\widetilde{W}(A_1^{(1)})$	-

Table 6. $X(R)$, $\mathrm{Cr}(X)$ and Painlevé equations

Each Painlevé equation is expressed by elements of $K(\mathcal{X})$;

$$P_{VI}: \quad \frac{df}{dt} = \frac{1}{s(s-1)}\left[2f(f-1)(f-s)g + (a_1+2a_2)f(f-1) + a_3(s-1)f + a_4 s(f-1)\right], \tag{10}$$

$$\frac{dg}{dt} = -\frac{1}{s(s-1)}[\{3f^2 - 2(s+1)f + s\}g^2 + \\ +\{(a_1+2a_2)(2f-1) + a_3(s-1) + a_4 s\}g + a_2(a_1+a_2)], \tag{11}$$

$$\frac{ds}{dt} = \lambda = a_1 + a_2 + a_3 + a_4 + a_0, \tag{12}$$

where $f = z/(z-x)$ and $g = y(z-x)/xz$.

$$P_V: \quad \frac{df}{dt} = \frac{1}{s}[(2g+s)f(f-1) - (a_1+a_3)f + a_1], \tag{13}$$

$$\frac{dg}{dt} = -\frac{1}{s}[(2f-1)g(g+s) - (a_1+a_3)g + a_2 s], \tag{14}$$

$$\frac{ds}{dt} = \lambda = a_1 + a_2 + a_3 + a_0, \tag{15}$$

where $f = x/(x-z)$ and $g = y(x-z)/xz$.

$$P_{III}^{D_6^{(1)}}: \quad \frac{df}{dt} = \frac{4}{s}[2f^2 g - (f^2 + (a_1+b_1)f - \frac{s}{4})], \tag{16}$$

$$\frac{dg}{dt} = -\frac{4}{s}[2fg^2 - (2f + (a_1+b_1)g + a_1)], \tag{17}$$

$$\frac{ds}{dt} = 4\lambda = 4(a_1+a_0) = 4(b_1+b_0), \tag{18}$$

where $f = y(z-x)/xz$ and $g = x/(z-x)$.

$$P_{III}^{D_7^{(1)}}: \quad \frac{df}{dt} = \frac{2}{s}[2f^2 g - a_1 f + \frac{s}{2}], \tag{19}$$

$$\frac{dg}{dt} = -\frac{2}{s}[2fg^2 - a_1 g + \frac{1}{2}], \tag{20}$$

$$\frac{ds}{dt} = 2\lambda = 2(a_1+a_0), \tag{21}$$

where $f = yz/x^2$ and $g = x/z$.

$$P_{III}^{D_8^{(1)}}: \quad \frac{d^2 f}{dt^2} = \frac{1}{f}\left(\frac{df}{dt}\right)^2 - \frac{2\lambda}{s}\frac{df}{dt} - \frac{4}{s^2}f^2 - \frac{4}{s}, \tag{22}$$

$$\frac{ds}{dt} = 2\lambda, \tag{23}$$

where $f = z/x$.

$$\uparrow \rho_3$$
$$\left(\frac{x}{y}, \frac{y^2z}{x^3}\right) \cup \left(\frac{x^3}{y^2z}, \frac{yz}{x^2}\right)$$

$$\mathbb{F}_2 = \left(\frac{y}{x}, \frac{z}{x}\right) \cup \left(\frac{y}{x}, \frac{x}{z}\right) \cup \left(\frac{x}{y}, \frac{y^2z}{x^3}\right) \cup \left(\frac{x}{y}, \frac{x^3}{y^2z}\right)$$

$$\uparrow \rho_6$$
$$\left(\frac{x}{y}, \frac{y(y^2z-x^3)}{x^4}\right) \cup \left(\frac{x^4}{y(y^2z-x^3)}, \frac{y^2z}{x^3} - 1\right)$$

$$\mathbb{F}_1 = \left(\frac{y}{x}, \frac{z}{x}\right) \cup \left(\frac{y}{x}, \frac{x}{z}\right) \cup \left(\frac{x}{y}, \frac{y(y^2z-x^3)}{x^4}\right) \cup \left(\frac{x}{y}, \frac{x^4}{y(y^2z-x^3)}\right)$$

$$\sim \left(\frac{y}{x}, \frac{y^2z-x^3}{x^2z}\right) \cup \left(\frac{y}{x}, \frac{x^2z}{y^2z-x^3}\right) \cup \left(\frac{x}{y}, \frac{y^2z-x^3}{xyz}\right) \cup \left(\frac{x}{y}, \frac{xyz}{y^2z-x^3}\right)$$

$$\uparrow \rho_7$$
$$\left(\frac{x}{y}, \frac{y^2z-x^3}{x^2z}\right) \cup \left(\frac{x^2z}{y^2z-x^3}, \frac{y^2z-x^3}{xyz}\right)$$

$$\mathbb{P}^1 \times \mathbb{P}^1 = \left(\frac{y}{x}, \frac{y^2z-x^3}{x^2z}\right) \cup \left(\frac{y}{x}, \frac{x^2z}{y^2z-x^3}\right) \cup \left(\frac{x}{y}, \frac{y^2z-x^3}{x^2z}\right) \cup \left(\frac{x}{y}, \frac{x^2z}{y^2z-x^3}\right)$$

$$\sim \left(\frac{y}{x}, \frac{y^2z-x^3+sx^2z}{x^2z}\right) \cup \left(\frac{y}{x}, \frac{x^2z}{y^2z-x^3+sx^2z}\right) \cup \left(\frac{x}{y}, \frac{y^2z-x^3+sx^2z}{x^2z}\right) \cup \left(\frac{x}{y}, \frac{x^2z}{y^2z-x^3+sx^2z}\right)$$

$$\uparrow \rho_8$$
$$\left(\frac{x}{y}, \frac{y(y^2z-x^3+sx^2z)}{x^3z}\right) \cup \left(\frac{x^3z}{y(y^2z-x^3+sx^2z)}, \frac{y^2z-x^3+sx^2z}{x^2z}\right)$$

$$\to \mathbb{P}^2 = (x(y^2z-x^3+sx^2z) : y(y^2z-x^3+sx^2z) : x^3z)$$

Thus we have

$$\sigma_{(15)(24)(60)} : (a_1, a_0\,; s;\; x : y : z) \longmapsto \left(a_0, a_1\,; s;\; x(y^2z + szx^2 - x^3) : -y(y^2z + szx^2 - x^3) : zx^3\right).$$

Taking $f = y/x$ and $g = x/z$ as generators of $K(\mathcal{X})$ with a_1, a_0, s, we have a simpler decription of w_1 and $\sigma_{(15)(24)(60)}$;

$$\begin{aligned} w_1 \quad &: \quad (a_1, a_0; s; f, g) \longmapsto \left(-a_1, a_0 + 2a_1; s; f - \frac{a_1}{g}, g\right), \\ \sigma_{(15)(24)(60)} \quad &: \quad (a_1, a_0; s; f, g) \longmapsto (a_0, a_1; s; -f, f^2 + s - g). \end{aligned}$$

They are isomorphisms of fibers : $\mathcal{X}_{(a_1,a_0,s)} \to \mathcal{X}_{(\overline{a_1},\overline{a_0},\overline{s})}$, where $\overline{\xi}$ is the image of ξ by $\sigma_{(15)(24)(60)}$ or w_1. This is because $\mathcal{X}_{(a_1,a_0,s)}$ and $\mathcal{X}_{(\overline{a_1},\overline{a_0},\overline{s})}$ are the same surfaces and we have changed only blowing-down structures.

When $a_1 = a_0$, $\sigma_{(15)(24)(60)}$ is an automorphism different from the identity.

7 Painlevé equations

1. In the case of an R-surface in the list of Table 4, $X(R)$ is isomorphic to the space of initial conditions of P_J for $J = I, II, III, IV, V, VI$ [19]. Furthermore the action of $\mathrm{Cr}(X)$ coincides with the Bäcklund transformations of P_J [20].

Here we explain the calculations in the case of the $E_7^{(1)}$-surfaces as an example. It has $\mathrm{Aut}(E_7^{(1)}) \ltimes W(A_1^{(1)}) \simeq \widetilde{W}(A_1^{(1)})$ symmetry. Let $\sigma_{(15)(24)(60)} \in \mathrm{Aut}(E_7^{(1)}) \simeq \mathrm{Aut}(A_1^{(1)})$ and $w_1, w_0 \in W(A_1^{(1)})$ be the generators. We have only to get the descriptions of w_1 and $\sigma_{(15)(24)(60)}$ because $w_0 = \sigma_{(15)(24)(60)} \circ w_1 \circ \sigma_{(15)(24)(60)}$.

The reflection w_1 is easy. It is obtained from permuting the divisor classes $\mathcal{E}_4$ and $\mathcal{E}_5$. This means permuting distinct points p_4 and p_5 and it means permuting the order of blowings-down. We have

$$w_1 : (a_1, a_0; s; x : y : z) \longmapsto (-a_1, a_0 + 2a_1; s; x : y - a_1 z : z).$$

The Cremona isometry $\sigma_{(15)(24)(60)} \in \mathrm{Cr}(X(E_7^{(1)}))$ is described as follows;

$$\begin{aligned}
\overline{\mathcal{E}}_9 &= \mathcal{E}_5, \quad \overline{\mathcal{E}}_8 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4, \quad \overline{\mathcal{E}}_7 = \mathcal{E}_0 - \mathcal{E}_2 - \mathcal{E}_4, \quad \overline{\mathcal{E}}_6 = \mathcal{E}_0 - \mathcal{E}_3 - \mathcal{E}_4, \\
\overline{\mathcal{E}}_3 &= \mathcal{E}_0 - \mathcal{E}_4 - \mathcal{E}_6, \quad \overline{\mathcal{E}}_2 = \mathcal{E}_0 - \mathcal{E}_4 - \mathcal{E}_7, \quad \overline{\mathcal{E}}_1 = \mathcal{E}_0 - \mathcal{E}_4 - \mathcal{E}_8, \quad \overline{\mathcal{E}}_5 = \mathcal{E}_9, \\
\overline{\mathcal{E}}_4 &= 3\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - 2\mathcal{E}_4 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8, \quad \overline{\mathcal{E}}_0 = 4\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - 3\mathcal{E}_4 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8.
\end{aligned}$$

In order to obtain the action of $\sigma_{(15)(24)(60)}$ on $\mathcal{X}$, we have to blow up $\mathbb{P}^2$ successively and get the surface X with the blowing-down structure $\rho : X \to \mathbb{P}^2$. Then we blow down the obtained surface to $\mathbb{P}^2$ with another blowing-down structure $\overline{\rho} : X \to \mathbb{P}^2$. But we take these operation alternately for saving descriptions and we can omit some operations. For example $\overline{\mathcal{E}}_9 = \mathcal{E}_5$ means that we have to blow up the point p_5 and blow down the exceptional curve, and we can omit these operations because these two operations are canceled out.

First, we construct the surface obtained by blowing up $\mathbb{P}^2$ with the centers at p_1 and p_4. We have $\overline{\mathcal{E}}_9 = \mathcal{E}_5$, so we can regard that $\overline{\mathcal{E}}_9$ has been already blown down. Next $\overline{\mathcal{E}}_8 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4$ is to be blown down. This is the proper transform of the line which passes through p_1 and p_4, namely the line $(x = 0)$. Blowing it down, we get $\mathbb{P}^1 \times \mathbb{P}^1$. We abbreviate $\mathbb{P}^2 = \mathrm{Spec}\,\mathbb{C}[\frac{x}{z}, \frac{y}{z}] \cup \mathrm{Spec}\,\mathbb{C}[\frac{y}{x}, \frac{z}{x}] \cup \mathrm{Spec}\,\mathbb{C}[\frac{z}{y}, \frac{x}{y}]$ as $\mathbb{P}^2 = (\frac{x}{z}, \frac{y}{z}) \cup (\frac{y}{x}, \frac{z}{x}) \cup (\frac{z}{y}, \frac{x}{y})$. This process is described as follows;

$$\begin{array}{ccc}
\mathbb{P}^2 = \left(\frac{x}{z}, \frac{y}{z}\right) \cup \left(\frac{y}{x}, \frac{z}{x}\right) \cup \left(\frac{z}{y}, \frac{x}{y}\right) & & \\
\uparrow \rho_4 & & \uparrow \rho_1 \\
\left(\frac{x}{z}, \frac{y}{z}\right) \cup \left(\frac{x}{y}, \frac{y}{z}\right) & \left(\frac{z}{y}, \frac{x}{z}\right) & \cup \left(\frac{z}{x}, \frac{x}{y}\right) \\
\mathbb{P}^1 \times \mathbb{P}^1 = \left(\frac{y}{x}, \frac{z}{x}\right) \cup \left(\frac{y}{x}, \frac{x}{z}\right) & \cup \left(\frac{x}{y}, \frac{z}{x}\right) & \cup \left(\frac{x}{y}, \frac{x}{z}\right)
\end{array}$$

We regard each coordinate as an element of the rational function field $K(X) \simeq K(\mathbb{P}^2)$.

Proceeding with these operations, we obtain the next description;

$$\begin{array}{ccc}
\mathbb{P}^2 = \left(\frac{x}{z}, \frac{y}{z}\right) \cup \left(\frac{y}{x}, \frac{z}{x}\right) \cup \left(\frac{z}{y}, \frac{x}{y}\right) & & \\
\uparrow \rho_4 & & \uparrow \rho_1 \\
\left(\frac{x}{z}, \frac{y}{z}\right) \cup \left(\frac{x}{y}, \frac{y}{z}\right) & \left(\frac{z}{y}, \frac{x}{z}\right) & \cup \left(\frac{z}{x}, \frac{x}{y}\right) \\
\mathbb{P}^1 \times \mathbb{P}^1 = \left(\frac{y}{x}, \frac{z}{x}\right) \cup \left(\frac{y}{x}, \frac{x}{z}\right) & \cup \left(\frac{x}{y}, \frac{z}{x}\right) & \cup \left(\frac{x}{y}, \frac{x}{z}\right) \\
& \uparrow \rho_2 & \\
& \left(\frac{x}{y}, \frac{yz}{x^2}\right) \cup \left(\frac{x^2}{yz}, \frac{z}{x}\right) & \\
\mathbb{F}_1 = \left(\frac{y}{x}, \frac{z}{x}\right) \cup \left(\frac{y}{x}, \frac{x}{z}\right) & \cup \left(\frac{x}{y}, \frac{yz}{x^2}\right) & \cup \left(\frac{x}{y}, \frac{x^2}{yz}\right)
\end{array}$$

Now $w \in \mathrm{Cr}(X,D)$ and $\sigma \circ w^{-1}$ fixes the simple roots α_i's and $\mathcal{D}_i$'s. Let $(\mathcal{E}_i)$ and $(\alpha_j, \mathcal{D}_k)$ be the standard basis and the simple roots in Appendix A. If we set $(\mathcal{E}'_i) = (\sigma \circ w^{-1}(\mathcal{E}_i))$ then $(\mathcal{E}_i)$ and $(\mathcal{E}'_i)$ have same intersections with $(\alpha_j, \mathcal{D}_k)$. $(\mathcal{E}_i - \mathcal{E}'_i)\cdot\alpha_j = (\mathcal{E}_i - \mathcal{E}'_i)\cdot\mathcal{D}_k = 0$ so $\mathcal{E}'_i = \mathcal{E}_i + n_i\delta$. But $|\mathcal{E}'_0|^2 = |\mathcal{E}_0|^2 + 2n_0(\mathcal{E}_0|\delta) = -1$ so $n_0 = 0$ and $\mathcal{E}_0 = \mathcal{E}'_0$. Similarly $\mathcal{E}_i = \mathcal{E}'_i$ for $i = 1, \ldots, 9$. Hence $\sigma \circ w^{-1}$ trivially acts on $\mathrm{Pic}(X)$.

b) There is a homomorphism $1 \to \mathrm{Cr}(X)/\mathrm{Cr}(X,D) \to \mathrm{Aut}(R)$ so we have only to construct a homomorphism $\mathrm{Aut}(R)_{\Delta^{nod}} \to \mathrm{Cr}(X)$. It can be shown in each case respectively (We will construct it as the Cremona action in Appendix C). We will see the $E_7^{(1)}$-surface as an example. We assume that the generator σ of $\mathrm{Aut}(E_7^{(1)})$ belongs to $\mathrm{Aut}(E_7^{(1)})_{\Delta^{nod}}$. We want the action of $\sigma : (\mathcal{E}_i) \mapsto (\overline{\mathcal{E}}_i)$ on $\mathrm{Pic}(X)$. Then $\overline{\mathcal{E}}_9$ is an irreducible exceptional class which intersects with $\sigma(\mathcal{D}_6) = \mathcal{D}_0$. We take $\overline{\mathcal{E}}_9 = \mathcal{E}_5$ and it determines the action of σ on $\mathrm{Pic}(X)$. In order to show this we look at $\overline{\mathcal{E}}_i$ for all i.

$$
\begin{aligned}
\overline{\mathcal{E}}_9 &= \mathcal{E}_5, \quad \overline{\mathcal{E}}_8 = \overline{\mathcal{E}}_9 + \sigma(\mathcal{D}_6) = \mathcal{E}_5 + \mathcal{D}_0 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4, \quad \overline{\mathcal{E}}_7 = \mathcal{E}_0 - \mathcal{E}_2 - \mathcal{E}_4, \\
\overline{\mathcal{E}}_6 &= \mathcal{E}_0 - \mathcal{E}_3 - \mathcal{E}_4, \quad \overline{\mathcal{E}}_3 = \mathcal{E}_0 - \mathcal{E}_4 - \mathcal{E}_6, \quad \overline{\mathcal{E}}_2 = \mathcal{E}_0 - \mathcal{E}_4 - \mathcal{E}_7, \quad \overline{\mathcal{E}}_1 = \mathcal{E}_0 - \mathcal{E}_4 - \mathcal{E}_8, \quad \overline{\mathcal{E}}_5 = \mathcal{E}_9, \\
\overline{\mathcal{E}}_4 &= \alpha_0 + \mathcal{E}_9 = 3\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - 2\mathcal{E}_4 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8, \\
\overline{\mathcal{E}}_0 &= \sigma(\mathcal{D}_7 + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3) = 4\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - 3\mathcal{E}_4 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8.
\end{aligned}
$$

Now σ is a Cremona isometry because $\sigma \in W(E_8^{(1)}) = \mathrm{Aut}(Q(E_8^{(1)}))/\{\pm 1\}$ and $\sigma(\mathrm{Comp}(D)) = \mathrm{Comp}(D)$. It can be verified that $\mathrm{Aut}(R) \ltimes W(R^\perp) \simeq \widetilde{W}(R^\perp)$ in each case respectively.

d) According to the discussion above, $\mathrm{Cr}(X,D) \subset W(A^{(1)}_{1,|\alpha|^2=8})$, but w_{α_0} and w_{α_1} are not elements of $\mathrm{Aut}(\mathrm{Pic}(X))$ but elements of $\mathrm{Aut}(\mathrm{Pic}(X) \otimes \mathbb{Q})$. An element of $W(A^{(1)}_{1,|\alpha|^2=8})$ is expressed as $(w_{\alpha_0} \circ w_{\alpha_1})^{\pm n}$ or $((w_{\alpha_0} \circ w_{\alpha_1})^n \circ w_{\alpha_0})^{\pm}$. The smallest element which is in $\mathrm{Aut}(\mathrm{Pic}(X))$ is $(w_{\alpha_0} \circ w_{\alpha_1})^4$. It also induces an element of $\mathrm{Aut}(Q(E_8^{(1)}))$, so belongs to $\mathrm{Cr}(X,D)$. We have $\sigma^8 = (w_{\alpha_0} \circ w_{\alpha_1})^4$ by a simple calculation.

Let σ' be a reflection in $\mathfrak{D}_{16}$ such that $\sigma'^2 = 1$. Then σ' can be constructed as an element of $\mathrm{Aut}(\mathrm{Pic}(X))$ (See Appendix C). Here σ is an element of the inverse image of a rotation in $\mathrm{Aut}(A_7^{(1)}) = \mathfrak{D}_{16}$. For any $\sigma'' \in \mathrm{Cr}(X)$, $\sigma'' \circ \sigma^k \circ \sigma'^i$ is an element of $\mathrm{Cr}(X,D)$ for some $0 \le k \le 7$ and $i = 0$ or 1. So the second statement is concluded. Similarly, c) and e) can be proved.

f) For $R = A_7^{(1)\prime}$, $\mathfrak{D}_8 \subset \mathrm{Aut}(A_7^{(1)}) = \mathfrak{D}_{16}$. We can construct $\sigma \in \mathfrak{D}_8$ as an automorphism of $\mathrm{Pic}(X)$ (See Appendix C). We have to show that the elements of $\mathfrak{D}_{16} \setminus \mathfrak{D}_8$ do not induce an automorphism of $\mathrm{Pic}(X)$. This is obvious because $\mathcal{D}_2, \mathcal{D}_4, \mathcal{D}_6$ and $\mathcal{D}_0$ intersect an irreucible exceptional divisor respectively and the other $\mathcal{D}_i$'s do not. The case of $R = A_8^{(1)}$ and $D_8^{(1)}$ can be treated similarly. □

2. Next we calculate a concrete expression of Cremona actions of $\mathrm{Cr}(X)$ as a change of canonical coordinates. Let $\mathcal{X}$ be the family of R-surfaces which we constructed in the previous section. We describe these actions by using corresponding actions on $K(\mathcal{X})$. They are automorphisms of $K(\mathcal{X})$ and birational actions on $\mathcal{X}$.

These actions are obtained from the actions on blowing-down structures. So, in order to obtain these, it is enough to blow up the projective plane and blow it down with another blowing-down structure. In particular it is just a replacement of parametrizations of the same surface so the restriction of the action to each fiber is an isomorphism between R-surfaces.

We have the list of results of calculations in Appendix C. Let us explain the meaning of the symbols. For instance, for the $E_6^{(1)}$-surface the symbol $\sigma_{(10)(26)}$ corresponds to the permutation $(10)(26) = \left(\begin{smallmatrix}1234560\\0634521\end{smallmatrix}\right) \in \mathfrak{S}_7$ where we view $\mathrm{Aut}(E_6^{(1)}) \hookrightarrow \mathfrak{S}_7$. Now $(10)(26) \in \mathfrak{S}_7$ means that the vertices $\mathcal{D}_1$ and $\mathcal{D}_0$, $\mathcal{D}_2$ and $\mathcal{D}_6$ are interchanged. We have an isomorphism $\mathrm{Aut}(E_6^{(1)}) \to \mathrm{Aut}(A_2^{(1)})$ and $\sigma_{(10)(26)}$ interchanges α_2 and α_0. Notice this interchange is denoted by $\sigma_{(10)(26)}$ and not by $\sigma_{(20)}$. The minus means that $-\sigma$ transforms $\lambda = a_1 + a_2 + a_0$ to $-\lambda$.

6 Cremona action

1. In order to study the discrete and differential Painlevé equations we need to construct the elements of $\mathrm{Cr}(X)$. The main result of this section is the next theorem. Looijenga calculate $\mathrm{Cr}(X(R), D)$ for $R = A_0^{(1)*}, A_1^{(1)}, A_2^{(1)}, A_3^{(1)}$ and $A_4^{(1)}$ [15]. This theorem is an extension.

Here $\mathrm{Cr}(X, D)$ is the group of the Cremona isometries which leaves each class $\mathcal{D}_i$ fixed. We use the notation $G_S = \{g \in G \,|\, g(S) = S\} \subset G$. For α with $(\alpha^2) = -2$, $w_\alpha \in W(R^\perp)$ acts on not only $Q(R^\perp)$ but on $\mathrm{Pic}(X)$ by $w_\alpha(\mathcal{F}) = \mathcal{F} + (\alpha \cdot \mathcal{F})\,\alpha$.

Theorem 26 *a)* $\mathrm{Cr}(X(R), D) = W(R^\perp)_{\Delta^{nod}}$ *for* $R \neq A_6^{(1)}, A_7^{(1)}$ *or* $D_7^{(1)}$.

b) For $R \neq A_6^{(1)}, A_7^{(1)}, A_7^{(1)\prime}, A_8^{(1)}, D_7^{(1)}$ *or* $D_8^{(1)}$, $\sigma \in \mathrm{Aut}(R)$ *defines an action on* $\mathrm{Pic}(X)$ *and* $\mathrm{Cr}(X(R)) = (\mathrm{Aut}(R) \ltimes W(R^\perp))_{\Delta^{nod}} \simeq (\widetilde{W}(R^\perp))_{\Delta^{nod}}$.

c) $\mathrm{Cr}(X(A_6^{(1)}), D) = (W(A_1^{(1)}) \times \{\sigma^{7n} \,|\, n \in \mathbb{Z}\})_{\Delta^{nod}}$, $\mathrm{Cr}(X(A_6^{(1)})) = ((W(A_1^{(1)}) \rtimes \{\sigma^n \,|\, n \in \mathbb{Z}\}) \rtimes \mathfrak{S}_2)_{\Delta^{nod}}$, *where*

$$\begin{aligned}
\sigma \;:\; & (\mathcal{E}_0, \dots, \mathcal{E}_9) \mapsto (\overline{\mathcal{E}}_0, \dots, \overline{\mathcal{E}}_9);\\
& \overline{\mathcal{E}}_0 = 2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_7 - \mathcal{E}_8, \quad \overline{\mathcal{E}}_1 = \mathcal{E}_4, \quad \overline{\mathcal{E}}_2 = \mathcal{E}_3, \quad \overline{\mathcal{E}}_3 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_8, \quad \overline{\mathcal{E}}_4 = \mathcal{E}_9,\\
& \overline{\mathcal{E}}_5 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_7, \quad \overline{\mathcal{E}}_6 = \mathcal{E}_5, \quad \overline{\mathcal{E}}_7 = \mathcal{E}_0 - \mathcal{E}_7 - \mathcal{E}_8, \quad \overline{\mathcal{E}}_8 = \mathcal{E}_2, \quad \overline{\mathcal{E}}_9 = \mathcal{E}_6.
\end{aligned}$$

d) $\mathrm{Cr}(X(A_7^{(1)}), D) = \{\sigma^{8n} \,|\, n \in \mathbb{Z}\}_{\Delta^{nod}}$, $\mathrm{Cr}(X(A_7^{(1)})) = (\{\sigma^n \,|\, n \in \mathbb{Z}\} \rtimes \mathfrak{S}_2)_{\Delta^{nod}}$, *where*

$$\begin{aligned}
\sigma \;:\; & (\mathcal{E}_0, \dots, \mathcal{E}_9) \mapsto (\overline{\mathcal{E}}_0, \dots, \overline{\mathcal{E}}_9);\\
& \overline{\mathcal{E}}_0 = 2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_7 - \mathcal{E}_8, \quad \overline{\mathcal{E}}_1 = \mathcal{E}_9, \quad \overline{\mathcal{E}}_2 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_8, \quad \overline{\mathcal{E}}_3 = \mathcal{E}_0 - \mathcal{E}_7 - \mathcal{E}_8,\\
& \overline{\mathcal{E}}_4 = \mathcal{E}_2, \quad \overline{\mathcal{E}}_5 = \mathcal{E}_6, \quad \overline{\mathcal{E}}_6 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_7, \quad \overline{\mathcal{E}}_7 = \mathcal{E}_3, \quad \overline{\mathcal{E}}_8 = \mathcal{E}_4, \quad \overline{\mathcal{E}}_9 = \mathcal{E}_5.
\end{aligned}$$

e) $\mathrm{Cr}(X(D_7^{(1)}), D) = \{(\sigma \circ \sigma')^{4n} \,|\, n \in \mathbb{Z}\}_{\Delta^{nod}}$, $\mathrm{Cr}(X(D_7^{(1)})) = \langle \sigma, \sigma' \rangle_{\Delta^{nod}} \simeq \widetilde{W}(A_1^{(1)})_{\Delta^{nod}}$, *where*

$$\begin{aligned}
\sigma \;:\; & \mathcal{E}_4 \leftrightarrow \mathcal{E}_5, \quad \mathcal{E}_6 \leftrightarrow \mathcal{E}_7,\\
\sigma' \;:\; & (\mathcal{E}_0, \dots, \mathcal{E}_9) \mapsto (\overline{\mathcal{E}}_0, \dots, \overline{\mathcal{E}}_9);\\
& \overline{\mathcal{E}}_0 = 3\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - 2\mathcal{E}_4 - \mathcal{E}_6, \quad \overline{\mathcal{E}}_1 = \mathcal{E}_0 - \mathcal{E}_3 - \mathcal{E}_4, \quad \overline{\mathcal{E}}_2 = \mathcal{E}_0 - \mathcal{E}_2 - \mathcal{E}_4,\\
& \overline{\mathcal{E}}_3 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4, \quad \overline{\mathcal{E}}_4 = 2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_6, \quad \overline{\mathcal{E}}_5 = \mathcal{E}_8,\\
& \overline{\mathcal{E}}_6 = \mathcal{E}_0 - \mathcal{E}_4 - \mathcal{E}_6, \quad \overline{\mathcal{E}}_7 = \mathcal{E}_9, \quad \overline{\mathcal{E}}_8 = \mathcal{E}_5, \quad \overline{\mathcal{E}}_9 = \mathcal{E}_7.
\end{aligned}$$

f) $\mathrm{Cr}(X(A_7^{(1)\prime})) = (\mathfrak{D}_8 \ltimes W(A_1^{(1)}))_{\Delta^{nod}}$, $\mathrm{Cr}(X(A_8^{(1)})) = (\mathfrak{D}_6)_{\Delta^{nod}}$, $\mathrm{Cr}(X(D_8^{(1)})) = (\mathfrak{S}_2)_{\Delta^{nod}}$, *where* $\mathfrak{D}_{2n}$ *is the dihedral group of order* $2n$.

Proof. a) Let $w \in W(R^\perp)$ be such that $w(\Delta^{nod}) = \Delta^{nod}$ ($R \neq A_6^{(1)}, A_7^{(1)}$ or $D_7^{(1)}$). Then w leaves $\mathrm{EX} = \{\mathcal{F} \in \mathrm{Pic}(X) | (\mathcal{F}^2) = -1, \mathcal{F} \cdot \mathcal{K}_X = -1\}$ invariant. Furthermore NEFF is also invariant from Proposition 20 because w leaves the blowing-down structures invariant from Proposition 18. Proposition 5 says that $\mathrm{Pic}^+(X)$ remains through the action of w.

Conversely let $\sigma \in \mathrm{Cr}(X, D)$. Then σ induces an automorphism of $Q(R^\perp)$. Therefore there exists a $w \in \mathrm{Aut}(Q(R^\perp)) = \pm\mathrm{Aut}(R^\perp) \ltimes W(R^\perp)$ (Proposition 13) induced by σ. However an automorphism of the Dynkin diagram of $R^\perp$ moves $\mathcal{D}_j$ except the identity. It is because each Dynkin automorphism of $R^\perp$ moves a terminal vertex α_i which intersects an exceptional curve $\mathcal{F}$ such that $\mathcal{F} \cdot \alpha_i = \mathcal{F} \cdot \mathcal{D}_j = 1$ for some j and $\mathcal{F} \cdot \alpha_k = \mathcal{F} \cdot \mathcal{D}_l = 0$ for $k \neq i, l \neq j$. So $w \in W(R^\perp)$ and of course $w(\Delta^{nod}) = \Delta^{nod}$.

If ϕ : $\mathrm{Pic}(X) \to \mathrm{Pic}(\widetilde{X})$ *is an isometry such that*

$$a) \quad \phi(\mathcal{D}_i) = \widetilde{\mathcal{D}}_i, \qquad b) \quad \phi(\Delta^{nod}) = \widetilde{\Delta}^{nod},$$

then for the standard basis $(\mathcal{E}_i)_{i=0}^9$ *which defines a blowing-down structure* ρ, $(\phi(\mathcal{E}_i))_{i=0}^9$ *defines a blowing-down structure.*

Furthermore, denote the extra-parmeter defined for (X, ρ) *(resp.* $(\widetilde{X}, \tilde{\rho})$*) by* ζ *(resp.* $\tilde{\zeta}$*) for* R *lised in Table* 5. *If* ϕ *and* (X, ρ), $(\widetilde{X}, \tilde{\rho})$ *satisfies*

$$c) \ \tau = \tilde{\tau} \in \mathcal{H} \quad \text{when } R = A_0^{(1)}, \quad \tau \text{ is not a fixed point of an element of } \mathrm{PSL}(2, \mathbb{Z}),$$
$$d) \ \phi^*(\tilde{\chi}) = \chi, \qquad e) \ \zeta = \tilde{\zeta} \in B \quad \text{when } R \text{ is listed in Table } 5,$$

then ϕ *is induced by an isomorphism* $\Phi : \widetilde{X} \to X$, *and this* Φ *is unique when* $R \neq A_7^{(1)\prime}, A_8^{(1)}$.

Proof. The former part follows from Proposition 18. To apply this proposition, we take a basis $(\widetilde{\mathcal{E}}_i)$ of $\mathrm{Pic}(\widetilde{X})$ satisfying the correspondence with $\widetilde{\mathcal{D}}_i$ in Appendix A which defines a blowing-down structure. The change of basis $(\phi(\mathcal{E}_i)) \mapsto (\widetilde{\mathcal{E}}_i)$ is induced from an automorphism of $Q(E_8^{(1)})$ because $-\mathcal{K}_{\bar{X}} = \sum m_i \widetilde{\mathcal{D}}_i$ is invariant. From $\mathrm{Aut}(Q(E_8^{(1)})) = \pm W(E_8^{(1)})$ we have the conclusion.

In the latter part, the existence of an isomorphism Φ follows from the discussion similar to the case of $E_7^{(1)}$ above. (A realization of surfaces, normalized by PGL(3), is given in Appendix B.)

We prove the uniqueness for $X(R)$, where $R \neq A_7^{(1)\prime}, A_8^{(1)}$. If we have two different isomorphisms, then we have a non-trivial automorphism of the blowing-down structure of X (§ 3, **3**), which does not change the values of τ, χ and ζ. It induces an automorphism of $\mathbb{P}^2$. From Appendix B, it is clear that there is no such automorphism in the case $R \neq A_0^{(1)}$.

When $R = A_0^{(1)}$, such a transformation induces an automorphism of D: $y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3$. From the view poit of $D = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, it is a composition of a translation and a change of a basis of the lattice (a translation permutes nine inflection points $\frac{k}{3} + \frac{l\tau}{3}$ of D). A change of basis moves τ and a translation changes $\chi(\alpha_8)$, so there is no such automorphism. □

Remark. There is an ambiguity in the parametrization in the case of $A_0^{(1)}$, A_0^* and $A_1^{(1)}$. For $R = A_0^{(1)*}, A_1^{(1)}$, it is caused by the orientation of D, i.e. the choice of a basis of $H_1(D_{red}, \mathbb{Z})$. For $R = A_0^{(1)}$, it is caused by the choice of a basis of $H_1(D_{red}, \mathbb{Z})$ and the choice of the inflection point $(0 : 1 : 0)$. □

4. We have parametrized p_i for $i = 1, \ldots, 9$. From these data we could construct each R-surface as a surface obtained by blowing up the projective plane. The parameters are the gluing data of affine schemes and we have considered them as complex numbers.

If we regard the parameters as transcendental elements, then we obtain a family of $X(R)$. We illustrate it by taking the case of $E_7^{(1)}$ as an example. We regard a_1, a_0 and s as elements of $\mathbb{C}[a_1, a_0, s, 1/(a_1 + a_0)]$ and glue each affine plane which is difined over $\mathbb{C}[a_1, a_0, s, 1/(a_1 + a_0)]$. Then we get a family of $X(E_7^{(1)})$. We denote it by $\mathcal{X} = \mathcal{X}(E_7^{(1)})$.

We have the canonical morphism

$$\pi : \mathcal{X} \mapsto \mathrm{Spec}\, \mathbb{C}[a_1, a_0, s, 1/(a_1 + a_0)]$$

and a fiber $X = \mathcal{X}_{(a_1, a_0, s)} = \pi^{-1}((a_1, a_0, s) = (c, c', c''))$ is a generalized Halphen surface, where $c, c', c'' \in \mathbb{C}$ such that $c + c' \neq 0$. It is redundant for the parametrization of $X(E_8^{(1)})$, but we take this family for the purpose of a clear description of Cremona transformations.

In the next section we investigate an action of $\mathrm{Cr}(X)$ on $\mathcal{X}$. It is called the Cremona action.

It is possible that p_5 is infinitely near p_4 but in this case $p_5 : (y/z, x/y) = (0,0)$ because $\mathcal{E}_5$ intersects with D_0. In particular p_5 does not have extra-dimension of parameters so this case is simply parametrized by $a_1 = 0$. Generally, such cases are characterized by the following proposition.

Proposition 24 *Let $\alpha \in \Delta(R^\perp)$ with $(\alpha^2) = -2$. Denote by W^{nod} the subgroup of $W(E_8^{(1)})$ which is generated by the reflections with respect to the elements of Δ^{nod}. Then $\alpha \in W^{nod} \cdot \Delta^{nod}$ if and only if $\chi(\alpha) = 0$.*

Proof. If $[N] \in \Delta^{nod}$, then $N \subset X - D_{red}$ and the restriction of ω to N is holomorphic. Hence $\chi([N]) = 0$. Furthermore $\chi([N]) = 0$ for $[N] \in W^{nod} \cdot \Delta^{nod}$ because $w([N]) = [N] \bmod \mathbb{Z}\Delta^{nod}$ for $w \in W^{nod}$.

Conversely, suppose $\alpha \in \Delta(R^\perp)$ is such that $(\alpha^2) = -2$ and $\chi(\alpha) = 0$. Then $\alpha = w(\alpha_i)$ for some $\alpha_i \in \Pi(R^\perp)$ satisfying $(\alpha_i^2) = -2$. We can write $w = w' \circ w''$ with $w' \in W^{nod}$ and w'' such that $w'(\Delta^{nod}) \subset \Delta_+$. From Proposition 18, $w''(\alpha_i)$ can be expressed as $[C_0 - C_1]$, where C_0 and C_1 are exceptional curves which have no common component with D and meet D in the same component D_j. In particular $C_0 \cap C_1 = \emptyset$ or $C_0 \supset C_1$. Then $\chi([C_0 - C_1]) = \chi(w'([C_0 - C_1])) = 0$. It means that $C_0 \cap D_j = C_1 \cap D_j$ because of the concrete expression of $\mathrm{Res}_{D_j}\omega$ and Lemma 21. Therefore $[C_0 - C_1]$ is effective and $\alpha = w'([C_0 - C_1]) \in W^{nod} \cdot \Delta^{nod}$. □

Remark. This proposition is due to Looijenga [15]. This has a significance in Painlevé analysis because it gives a characterization of parameters which have the classical solutions of Riccati type. The Riccati differential equation is derived from a flow in $\mathbb{P}^1$ and this, as a special solution of the Painlevé equation, comes in sight as a flow in a nodal root. It happens from the invariance of nodal roots under the time evolution of the Painlevé equation. The fact that the Painlevé differential equation is obtained as a limit of an element of $\mathrm{Cr}(X)$ explains this invariance. □

Furthermore we have a projective transformation :

$$(a_1, a_0\,;\, s\,;\, x : y : z) \simeq (\mu^3 a_1, \mu^3 a_0\,;\, \mu^2 s\,;\, x : \mu y : \mu^2 z) \quad \text{for } \mu \in \mathbb{C}^*.$$

In the case $R = E_7^{(1)}$, notice that the value of the period mapping $\chi(\alpha_1) = a_1/\lambda \in \mathbb{C}$ does not determine the isomorphism class of the surfaces. We need an extra-paramter $\zeta = s^3/\lambda \in \mathbb{C}$. We see that the parameters $\chi(\alpha_1) = a_1/\lambda$ and s^3/λ determine the isomorphism class of the surfaces from the construction above.

The following is the list of surfaces which have such an extra-parameter $\zeta \in B$. This $\zeta \in B$ is defined through the construction in Appendix B, for the surface X in the list and the blowing-down structure. The symbol s is as in Appendix B.

the type of surface	$D_4^{(1)}$	$D_5^{(1)}$	$D_6^{(1)}$	$D_7^{(1)}$	$D_8^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$
ζ	s	s/λ	s/λ^2	s/λ^3	s/λ^4	s^2/λ	s^3/λ^2	s^2/λ
B	$\mathbb{C} \setminus \{0,1\}$	$\mathbb{C}^*$	$\mathbb{C}^*$	$\mathbb{C}^*$	$\mathbb{C}^*$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$

Table 5. Extra-parameter

Then, this parametrization is complete in some sense. Here is a theorem of Torelli type.

Theorem 25 *Let X and $\widetilde{X}$ be R-surfaces. Denote the irreducible components of the anti-canonical divisor, the set of nodal roots, the values of the period mapping with respect to the root lattice of X and $\widetilde{X}$ by the usual symbols and the tilded ones respectively. Also denote the value of the period mapping with respect to a basis of $H_1(D_{red}\,;\,\mathbb{Z})$ (resp. $H_1(\widetilde{D}_{red}\,;\,\mathbb{Z})$) by $(1,\tau)$ (resp. $(1,\tilde{\tau})$).*

Let Q be $-(p_1+p_2)$, let R be $-(p_3+p_4)$ and let S be $-(Q+R)$. Then $p_1,\dots,p_{3m},Q,R$ and S lie on the curve of degree $m+1$ which is the product of C and a line passing through Q, R and S.

We know the fact that if three points, which are among $3k$ intersecting points of a cubic curve and a curve of degree k, lie on a line, then there exists a curve of degree $k-1$ such that it intersects this cubic curve at the remaining $3(k-1)$ points. Now p_1,p_2,Q and p_3,p_4,R lie on a line respectively, so $p_5,\dots,p_{3m},S$ lie on a curve of degree $k-1$. From the assumption, we have $p_5+\cdots+p_{3m}+S=0$. From $p_1+p_2+Q=0$, $p_3+p_4+R=0$ and $Q+R+S=0$, we arrive at the conclusion. □

Remark. This proposition tells the fact that the spaces of initial conditions of the Painlevé equations have an elliptic surface as a limit. It corresponds to an elliptic function as a limit of the Painlevé transcendents. See Remark at the last part of § 7, **1**. □

3. We can realize any R-surface as a surface obtained by blowing up the projective plane. Let us normalize the surface by using projective transformations. The list is in the Appendix B. We will see closely the case of $E_7^{(1)}$-surface. First of all the images of D_0 and D_7 by ρ are lines on $\mathbb{P}^2$. We can regard they are $(x=0)$ and $(z=0)$ by a choice of a coordinate.

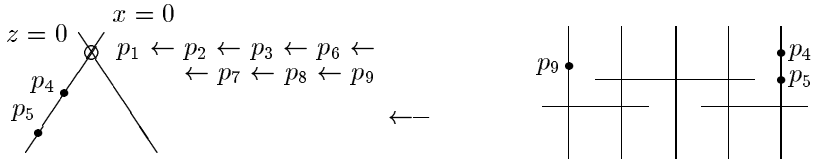
The surface obtained by blowing up the affine space $\operatorname{Spec}\mathbb{C}[x,y]$ with the center $(x,y)=(0,0)$ is

$$W=\{(x,y;\zeta_0:\zeta_1)\in\operatorname{Spec}\mathbb{C}[x,y]\times\mathbb{P}^1\mid x\zeta_0=y\zeta_1\}=\operatorname{Spec}\mathbb{C}[x,\zeta_0/\zeta_1]\cup\operatorname{Spec}\mathbb{C}[\zeta_1/\zeta_0,y].$$

In general a surface obtained by blowing-up is constructed by gluing such affine schemes. For economy of symbols we write x/y and y/x to stand for ζ_1/ζ_0 and ζ_0/ζ_1 respectively.

First, p_1 is the point which is expressed in this coordinate as $(x:y:z)=(0:1:0)$ because $\mathcal{E}_1$ intersects with D_0 and D_7. We write it as $p_1:(x:y:z)=(0:1:0)$. We can take $p_4:(0:0:1)$ and $p_5:(0:a_1:1)$ for $a_1\in\mathbb{C}$. The point p_2 is infinitely near p_1 and $p_2\in\operatorname{Spec}\mathbb{C}[\frac{x}{y},\frac{z}{y}/\frac{x}{y}]$. Because $\mathcal{E}_2$ intersects with D_7, $p_2:(x/y,z/x)=(0,0)$. Similarly $p_3:(x/y,yz/x^2)=(0,0)$. Now $p_6\in\operatorname{Spec}\mathbb{C}[\frac{x}{y},\frac{y^2z}{x^3}]$ and $p_6:(x/y,y^2z/x^3)=(0,c)$ for $c\in\mathbb{C}^*$, but by a projective transformation $(x:y:z)\mapsto(x:y:cz)$ we can set $p_6:(x/y,y^2z/x^3)=(0,1)$. Next $p_7\in\operatorname{Spec}\mathbb{C}[\frac{x}{y},\frac{y(y^2z-x^3)}{x^4}]$ and $p_7:(x/y,y(y^2z-x^3)/x^4)=(0,c')$ for $c'\in\mathbb{C}$, but by a projective transformation $(x:y:z)\mapsto(x:y+\frac{c'}{2}x:z)$ we can set $p_7:(x/y,y(y^2z-x^3)/x^4)=(0,0)$. The point p_8 is in $\operatorname{Spec}\mathbb{C}[\frac{x}{y},\frac{y^2(y^2z-x^3)}{x^5}]$ and $p_8:(x/y,y^2(y^2z-x^3)/x^5)=(0,s)$ for $s\in\mathbb{C}$. Finally, $p_9:(x/y,y(y^2(y^2z-x^3)-sx^5)/x^6)=(0,a_0)$ for $a_0\in\mathbb{C}$.

As a result, we have



$$\begin{aligned}
p_1:\ (0:1:0)\quad &\leftarrow p_2:\ \left(\frac{x}{y},\frac{z}{x}\right)=(0,0)\leftarrow p_3:\ \left(\frac{x}{y},\frac{yz}{x^2}\right)=(0,0)\leftarrow\\
&\leftarrow p_6:\ \left(\frac{x}{y},\frac{y^2z}{x^3}\right)=(0,1)\leftarrow p_7:\ \left(\frac{x}{y},\frac{y(y^2z-x^3)}{x^4}\right)=(0,0)\leftarrow\\
&\leftarrow p_8:\ \left(\frac{x}{y},\frac{y^2(y^2z-x^3)}{x^5}\right)=(0,s)\leftarrow p_9:\ \left(\frac{x}{y},\frac{y(y^2(y^2z-x^3)-sx^5)}{x^6}\right)=(0,a_0),\\
p_4:\ (0:0:1),\qquad &p_5:\ (0:a_1:1).
\end{aligned}$$

Here an arrow $p_i\leftarrow p_j$ means that p_j is infinitely near p_i.

For α_3 we regard it as $(\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4) - \mathcal{E}_6$, then the first part is represented by the line passing through p_1 and p_4 and this line intersects with D_1 at $p_{1,4} : (1 : -1/a_1a_4 : 0)$, hence

$$\chi(\alpha_3) = \frac{1}{2\pi\sqrt{-1}} \int_{p_{1,4}}^{p_6} \frac{dw}{w} = \frac{1}{2\pi\sqrt{-1}} \log(-a_1a_4a_6) \mod \mathbb{Z}.$$

Similar calculation works on in the case of the other multiplicative types.

Add 3. $\mathrm{Res}_{D_l}\omega$ has a double pole at $(0 : 1 : 0)$ and is holomorphic elsewhere. So $\mathrm{Res}_{D_l}\omega = const.dw$, where w is the coordinate on $\mathbb{P}^1$ for which $w = \infty$ coincides with $(0 : 1 : 0)$. We calculate first an α_k of the form $\mathcal{E}_i - \mathcal{E}_j$, $\chi(\alpha_k)$. If p_i and p_j are written as $(a_i : 1)$, etc. on $D_l \simeq \mathbb{P}^1$ then

$$\chi(\alpha_k) = \frac{1}{\lambda} \int_{p_i}^{p_j} dw = \frac{a_j - a_i}{\lambda},$$

where $\lambda = \chi([D])$.

For α_3 we regard it as $(\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4) - \mathcal{E}_6$, then the first part is represented by the line passing through p_1 and p_4 and this line intersects with D_1 at $p_{1,4} : (1 : -a_1 - a_4 : 0)$, hence

$$\chi(\alpha_3) = \frac{1}{\lambda} \int_{p_{1,4}}^{p_6} dw = \frac{a_1 + a_4 + a_6}{\lambda}.$$

Similar calculation works on in the case of the other additive types.

Now we characterize a Halphen surface of index one by using the period mapping, but a Halphen surface of index one allows $|-\mathcal{K}_X|$ to have many elements D's. The definiton of χ depends on the choice of ω. In this case we denote this mapping by χ_ω.

Proposition 22 *Let X be a generalized Halphen surface. Then $\chi_\omega([D]) = \chi_\omega(-\mathcal{K}_X) = 0$ if and only if X is a Halphen surface of index one.*

Proof. We know that X is a Halphen surface of index one if and only if $\dim |-\mathcal{K}_X| = 1$. An element of $|-\mathcal{K}_X|$ is a proper transform of a cubic curve passing through the nine points p_i's. If we show that $\dim |-\mathcal{K}_X| = 1$ implies $\chi_\omega([D]) = 0$, the converse can be concluded because arbitrary eight points determines the ninth point uniquely such that there exists a pencil of cubic curves passing through these nine points.

We assume $\dim |-\mathcal{K}_X| = 1$. The value of $\chi_\omega([D])$ depends on the choice of $D \in |-\mathcal{K}_X|$, but we can vary D and ω continuously. So we show it in the generic case, namely the case that D is a proper transform of an irreducible smooth cubic curve.

It means that $\theta_1 + \cdots + \theta_9 = 0 \mod \mathbb{Z} + \mathbb{Z}\tau$ from the calculus in the case of elliptic type, where $p_i = (\wp(\theta_i) : \wp'(\theta_i) : 1)$ with a certain coordinate of $\mathbb{P}^2$, and this reduces to the next well-known classical result. □

Lemma 23 *Let D be a plane cubic curve, and let C be a plane curve of degree m intersecting D at $3m$ points p_i $(i = 1, \ldots, 3m)$. Then $p_1 + \cdots + p_{3m} = 0$ with respect to the group operation on the elliptic curve D.*

Proof. Recall the group operation of an elliptic curve. We set 0 to be an inflection point. Given a point P, $-P$ is defined to be the other point where D intersects a line passing through P and 0. The operation $P + Q = R$ means that P, Q and $-R$ are intersection points of D and a line.

We show the lemma by induction. When $m = 1$, $p_1 + p_2 + p_3 = 0$ means that these three points lie on a line, so it is trivial. We assume this lemma for a curve of degree $m - 1$ and show it in the case of a curve C of degree m.

Lemma 21 *Let α_i be a simple root such that $(\alpha_i^2) = -2$ and $\alpha_i \notin \Delta^{nod}$. Then there exist irreducible exceptional curves $C_{i,0}$, $C_{i,1}$ and a D_j such that $\alpha_i = [C_{i,0} - C_{i,1}]$ and $D_j \cdot C_{i,0} = D_j \cdot C_{i,1} = 1$. Furthermore we have*

$$\chi(\alpha) = 2\pi\sqrt{-1} \int_{D_j \cap C_{i,0}}^{D_j \cap C_{i,1}} \mathrm{Res}_{D_j}\omega \quad \mod \hat{\chi}(H_1(D_{red};\mathbb{Z})).$$

Proof. The first part of this lemma is obvious from the concrete expressions of α_i given in Appendix A.

Let T be a closed tubular neighborhood of D_j in X such that $T \cap C_{i,0}$ and $T \cap C_{i,1}$ are fibres. Let l be an injective path in D_j from $C_{i,0} \cap D_j$ to $C_{i,1} \cap D_j$. We take

$$\Gamma = (C_{i,0} \setminus (C_{i,0} \cap T)) \cup \partial T|_l \cup (C_{i,1} \setminus (C_{i,1} \cap T)).$$

Then we can choose an orientation such that Γ is homologous to $C_{i,0} - C_{i,1}$.

We have $\int_{C_{i,0}\setminus(C_{i,0}\cap T)} \omega = \int_{C_{i,1}\setminus(C_{i,1}\cap T)} \omega = 0$. Therefore $\int_\Gamma \omega = \int_{\partial T|_l} \omega$. By the residue formula we have

$$\int_{\partial T|_l} \omega = 2\pi\sqrt{-1} \int_l \mathrm{Res}_{D_j}\omega. \qquad \square$$

Because $\mathrm{div}(\omega) = -D$,

$$\mathrm{div}(\mathrm{Res}_{D_j}\omega) = -\sum_{D_k \cap D_j \neq \emptyset} m_k (D_k \cap D_j),$$

where m_k is the multiplicity of D_k in D (the Kac label in the language of root systems). This divisor on D_j determines the $\mathrm{Res}_{D_j}(\omega)$ on D_j up to constant.

Let us calculate some examples.

Ell 1. We take $D = E_{(g_2,g_3)} = \{y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3\}$. Then we have

$$\mathrm{Res}_D \omega = \frac{1}{2\pi\sqrt{-1}} \frac{dx}{y},$$

the holomorphic 1-form on the elliptic curve. The isomorphism $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \simeq D$ pulls back $\frac{1}{2\pi\sqrt{-1}}\frac{dx}{y}$ to $\frac{1}{2\pi\sqrt{-1}} dw$ where w is the coordinate on $\mathbb{C}$ which is invariant with the additive action and $w = 0$ corresponds to an inflection point $(0:1:0) \in E_{(g_2,g_3)}$.

For an α_k of the form $\mathcal{E}_i - \mathcal{E}_j$, $\chi(\alpha_k)$ is easy to calculate. If p_i and p_j are written as $(\wp(\theta_i) : \wp'(\theta_i) : 1)$, $(\wp(\theta_j) : \wp'(\theta_j) : 1)$ then $\chi(\alpha_k) = \theta_j - \theta_i \mod \mathbb{Z} + \tau\mathbb{Z}$.

For α_8 we regard it as $(\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2) - \mathcal{E}_3$, then the first part is represented by the line passing through p_1 and p_2 and this line intersects with D at $(\wp(-\theta_1 - \theta_2) : \wp'(-\theta_1 - \theta_2) : 1)$. Hence we have $\chi(\alpha_8) = \theta_1 + \theta_2 + \theta_3 \mod \mathbb{Z} + \tau\mathbb{Z}$. This is the same when we view α_8 as $(\mathcal{E}_0 - \mathcal{E}_2 - \mathcal{E}_3) - \mathcal{E}_1$ or $(\mathcal{E}_0 - \mathcal{E}_3 - \mathcal{E}_1) - \mathcal{E}_2$.

Mult 3. Since D_j intersects D_{j+1} and D_{j-1} with multiplicity 1, respectively ($j \in \mathbb{Z}/3\mathbb{Z}$), we have

$$\mathrm{Res}_{D_j}\omega = \frac{1}{(2\pi\sqrt{-1})^2} \frac{dw}{w},$$

where w is a coordinate of $D_j \simeq \mathbb{P}^1$ such that D_j intersects D_{j-1} (resp. D_{j+1}) at 0 (resp. ∞) ($j \in \mathbb{Z}/3\mathbb{Z}$). For an α_k of the form $\mathcal{E}_i - \mathcal{E}_j$, $\chi(\alpha_k)$ is easy to calculate. If p_i and p_j are written as $(a_i : 1)$, etc. on $D_l \simeq \mathbb{P}^1$ then

$$\chi(\alpha_k) = \frac{1}{2\pi\sqrt{-1}} \int_{p_i}^{p_j} \frac{dw}{w} = \frac{1}{2\pi\sqrt{-1}} \log \frac{a_j}{a_i} \mod \mathbb{Z}.$$

(Cr2) σ leaves the canonical class of X fixed,

(Cr3) σ leaves the semi-group of effective classes $\mathrm{Pic}^+(X)$ invariant.

We will determine the group of the Cremona isometries $\mathrm{Cr}(X)$ for each surface $X = X(R)$ and calculate the corresponding Cremona transformations in Section 6.

5 Parametrization of isomorphism classes of surfaces

1. In the subsequent sections X is a generalized Halphen surface with $\dim |-\mathcal{K}_X| = 0$. We usually parametrize the surfaces which are obtained by blowing up the projective plane in terms of the moduli of ordered point sets on the projective plane. But we need more degenerated configurations of points including many infinitely near points and more specific configurations of irreducible components of the anti-canonical divisors.

We use the divisor D as a ruler. The nine points p_j's are put on D. We will see that these configurations of p_j's are described by the period mapping below.

Let ω be a meromorphic 2-form on X with $\mathrm{div}(\omega) = -D$. Then ω determines the period mapping $\hat{\chi} : H_2(X - D_{red}; \mathbb{Z}) \to \mathbb{C}$ which sends $\Gamma \in H_2(X - D_{red}, ; \mathbb{Z})$ to $\int_\Gamma \omega$.

We investigate $H_2(X - D_{red}; \mathbb{Z})$. The exact sequence of the homology for the pair $(X, X - D_{red})$ is the following:

$$\begin{array}{ccccccccc} \to & H_3(X;\mathbb{Z}) & \to H_3(X, X - D_{red};\mathbb{Z}) & \stackrel{\partial^*}{\to} & H_2(X - D_{red};\mathbb{Z}) & \stackrel{i^*}{\to} & H_2(X;\mathbb{Z}) & \stackrel{j^*}{\to} & H_2(X, X - D_{red};\mathbb{Z}) \to \\ & \| & & & & & & & \\ & 0 & & & & & & & \end{array}$$

We identify $H_k(X, X - D_{red}; \mathbb{Z})$ with $H^{4-k}(D_{red}; \mathbb{Z})$ by the Poincaré duality. The kernel of j^* is the root lattice $Q(R^\perp)$ because $H_2(X, X - D_{red}; \mathbb{Z}) = H^2(D_{red}; \mathbb{Z})$ is spanned by the irreducible components D_i of D.

Now, from the Poincaré duality $H^1(D_{red}; \mathbb{Z}) = H_1(D_{red}; \mathbb{Z})$ and the short exact sequence:

$$0 \to H^1(D_{red}; \mathbb{Z}) \to H_2(X - D_{red}; \mathbb{Z}) \to Q(R^\perp) \to 0,$$

we obtain the mapping

$$\chi : Q(R^\perp) \to \mathbb{C} \quad \mod \hat{\chi}(H_1(D_{red}; \mathbb{Z}))$$

through the period mapping $\hat{\chi}$.

But ω is not determined uniquely and we need a normalization. Here ω is specified as follows;

a) elliptic type: We fix a basis (γ_0, γ_1) of $H_1(D_{red}; \mathbb{Z})$ and set $\hat{\chi}(\gamma_0) = 1$ and $\tau = \hat{\chi}(\gamma_1) \in \mathcal{H} = \{v \in \mathbb{C} | \mathrm{Im}\, v > 0\}$.

b) multiplicative type: $H_1(D_{red}; \mathbb{Z})$ has a generator γ which determines an orientation of D, so we set $\hat{\chi}(\gamma) = 1$.

c) additive type: $H_2(X - D_{red}; \mathbb{Z}) = Q(R^\perp)$ so we set ω as $\chi([D]) = 1$. The fact that $\chi([D]) \neq 0$ follows from Proposition 22.

2. Next we compute $\chi(\alpha_i)$.

We can blow down the surface $X(E_7^{(1)})$ in each manner. From the first diagram we choose $(\mathcal{E}_i)_{i=0}^9$ by using the elements of $\mathrm{Comp}(D)$ as follows;

We assign $\mathcal{E}_9$ to an irreducible exceptional class which intersects $\mathcal{D}_6$;

$\mathcal{E}_8 = \mathcal{E}_9 + \mathcal{D}_6$; $\mathcal{E}_7 = \mathcal{E}_8 + \mathcal{D}_5$; $\mathcal{E}_6 = \mathcal{E}_7 + \mathcal{D}_4$;

$\mathcal{E}_5$ and $\mathcal{E}_4$ are assigned to irreducible exceptional classes which intersect $\mathcal{D}_0$;

$\mathcal{E}_3 = \mathcal{E}_6 + \mathcal{D}_3$; $\mathcal{E}_2 = \mathcal{E}_3 + \mathcal{D}_2$; $\mathcal{E}_1 = \mathcal{E}_2 + \mathcal{D}_1$.

Conversely we have the canonical root basis from $(\mathcal{E}_i)_{i=0}^9$;

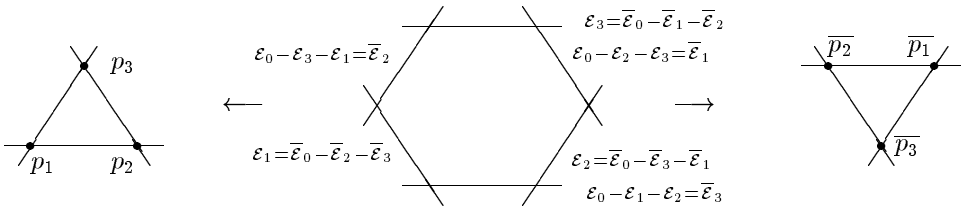
$$\mathcal{D}_0 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_5, \quad \mathcal{D}_1 = \mathcal{E}_1 - \mathcal{E}_2, \quad \mathcal{D}_2 = \mathcal{E}_2 - \mathcal{E}_3, \quad \mathcal{D}_3 = \mathcal{E}_3 - \mathcal{E}_6,$$
$$\mathcal{D}_4 = \mathcal{E}_6 - \mathcal{E}_7, \quad \mathcal{D}_5 = \mathcal{E}_7 - \mathcal{E}_8, \quad \mathcal{D}_6 = \mathcal{E}_8 - \mathcal{E}_9, \quad \mathcal{D}_7 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3.$$

We choose the correspondence between the standard basis of Λ_{10} and each canonical root basis of $Q(R)$ and $Q(R^\perp)$ as in Appendix A

5. The blowing-down structure is not determined even if we fix the span of $\mathcal{D}_i$ in $\mathrm{Pic}(X)$. For example, in the case of $X(E_7^{(1)})$, we could have chosen $\mathcal{E}_9' = 3\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - 2\mathcal{E}_4 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8$ instead of $\mathcal{E}_9$, if $\mathcal{E}_9'$ is irreducible. We could have changed the indices as $(\mathcal{E}_4', \mathcal{E}_5') = (\mathcal{E}_5, \mathcal{E}_4)$ or $(\mathcal{D}_i')_{i=0}^7 = (\mathcal{D}_6, \mathcal{D}_5, \mathcal{D}_4, \mathcal{D}_3, \mathcal{D}_2, \mathcal{D}_1, \mathcal{D}_0, \mathcal{D}_7)$.

A change of blowing-down structures causes a change of parametrers of isomorphism classes of the surface X and gives rise to a Cremona transformations. Here a Cremona transformation means an automorphism of the field $K(\mathbb{P}^2) \simeq K(X)$. We will see that such transformations are an origin of the Painlevé equations (§ 7).

Let us illustrate the correspondence between changes of blowing-down structures and Cremona transformations on a simple example. Recall the standard Cremona transformation on $\mathbb{P}^2$: $(x : y : z) \mapsto (yz : zx : xy)$,



$$Q_{p_3} Q_{p_2} Q_{p_1}(\mathbb{P}^2) = Q_{\overline{p_3}} Q_{\overline{p_2}} Q_{\overline{p_1}}(\mathbb{P}^2)$$
$$\swarrow \qquad\qquad\qquad\qquad \searrow$$
$$\mathbb{P}^2 \qquad\qquad - \to \qquad\qquad \mathbb{P}^2,$$

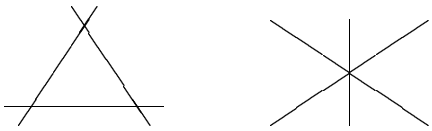
where $Q_p(X)$ is the surface obtained by blowing up X with the center p. This transformation is brought about by the change of the blowing-down structures:

$$\begin{pmatrix} \overline{\mathcal{E}}_0 \\ \overline{\mathcal{E}}_1 \\ \overline{\mathcal{E}}_2 \\ \overline{\mathcal{E}}_3 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \end{pmatrix}.$$

In order to investigate these transformations, it is enough to consider choices of blowing-down structures, i.e. choices of standard bases of $\mathrm{Pic}(X)$. We need to know these on the whole. So we prepare the terminology; an automorphism σ of $\mathrm{Pic}(X)$ is called a Cremona isometry (cf. [15],[4]) if the following properties are satisfied:

(Cr1) σ preserves the intersection form in $\mathrm{Pic}(X)$,

$R = A_2^{(1)} : D = D_0 + D_1 + D_2$ and $D_0 \cap D_1 \cap D_2 = \emptyset$; $D_i \cdot D_j = 1$ $(i \neq j)$, $R = A_2^{(1)*} : D = D_0 + D_1 + D_2$ and $D_0 \cap D_1 \cap D_2 \neq \emptyset$; $D_i \cdot D_j = 1$ $(i \neq j)$.



Remark. If the points p_i admit two distinct cubic curves $F_0(x : y : z) = 0$, $F_1(x : y : z) = 0$ passing through them, then $|-\mathcal{K}_X|$ is a pencil which consists of the proper transform of $\mu F_0 + \nu F_1 = 0$. Then the morphism $X \supset (\mu F_0 + \nu F_1 = 0) \ni x \mapsto (\mu : \nu) \in \mathbb{P}^1$ is the elliptic fibration of a Halphen surface X. □

4. We can describe the isomorphism classes of the surfaces by the ordered point sets $(p_i)_{i=1}^9$, without using detailed informations about X as a surface obtained by gluing affine spaces. That is, naïvely the parameter space of surfaces is

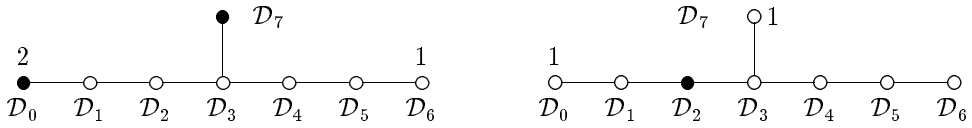
$$\mathrm{PGL}(3)\backslash\left\{\begin{pmatrix} x_1 & x_2 & \cdots & x_9 \\ y_1 & y_2 & \cdots & y_9 \\ z_1 & z_2 & \cdots & z_9 \end{pmatrix}\right\}/(\mathbb{C}^*)^9,$$

where $p_i = (x_i : y_i : z_i)$. However we need more degenerated configurations of points including many infinitely near points and more specific configurations of the anti-canonical divisor. In the next section we will closely investigate the parametrization of isomorphism classes of our surfaces, together with the blowing-down structures.

This parametrization is very convenient to describe the surfaces, but it depends on the blowing-down structures. The blowing-down structure is determined by a choice of divisor classes $(\mathcal{E}_0, \ldots, \mathcal{E}_9)$. It corresponds to the standard basis of Λ_{10}. An isomorphism $\varphi : \Lambda_{10} \to \mathrm{Pic}(X)$ defined by $\varphi(v_i) = \mathcal{E}_i$ $(i = 0, \ldots, 9)$ via the blowing-down structure on X is called a strict geometric marking of X.

If φ is a strict geometric marking, then $\varphi(\delta) = -\mathcal{K}_X$, where $\delta = \delta_9 = 3v_0 - v_1 - \cdots - v_9$. We can choose a unique root basis for $\varphi((\delta)^\perp) \simeq Q(E_8^{(1)})$ in $\mathrm{Pic}(X)$ through this marking. We call it the canonical root basis. This holds in the case of sublattice $Q(R)$ or $Q(R^\perp) \subset Q(E_8^{(1)})$.

We illustrate the setting above with the case of $X(E_7^{(1)})$ as an example. There are two types of possible blowing-down structures;



Each vertex $\mathcal{D}_i$ is a divisor class of an irreducible component D_i and a line between vertices means that these two divisor classes intersect at one point. The number associated with each vertex is the number of irreducible exceptional classes which are blown down by $\rho : X \to \mathbb{P}^2$ and intersect with corresponding D_i. Here $\bullet$ means a divisor class of type $\mathcal{E}_0 - \mathcal{E}_{i_1} - \mathcal{E}_{i_2} - \mathcal{E}_{i_3}$ $(i_1, i_2, i_3 \geq 1)$ and $\circ$ means a divisor class of type $\mathcal{E}_{j_1} - \mathcal{E}_{j_2}$ $(1 \leq j_1 < j_2)$.

realizations in the lattice $Q(E_8^{(1)})$. We use the symbol $A_7^{(1)}$ for the one which has no orthogonal real root of $E_8^{(1)}$ in its complement, and $A_7^{(1)\prime}$ for the other one. The symbol $A^1_{1,|\alpha|^2=l}$ means the root subsystem of type $A_1^{(1)}$ whose square length of roots is l, that is, it has an intersection form such that

$$-\begin{pmatrix} l & -l \\ -l & l \end{pmatrix}.$$

We use unusual symbols $(2A_1)^{(1)}$, $(A_2+A_1)^{(1)}$ and $(A_1+A_{1,|\alpha|^2=14})^{(1)}$. They mean that they have the following intersection forms.

$$\begin{aligned}
(2A_1)^{(1)} &= A_1^{(1)}+A_1 : -\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \sim A_1+A_1^{(1)} : -\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}, \\
(A_2+A_1)^{(1)} &= A_2^{(1)}+A_1 : -\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \sim A_2+A_1^{(1)} : -\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{pmatrix}, \\
(A_1+A_{1,|\alpha|^2=14})^{(1)} &= A_1^{(1)}+A_{1,|\alpha|^2=14} : -\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 14 \end{pmatrix} \sim A_1+A^{(1)}_{1,|\alpha|^2=14} : -\begin{pmatrix} 2 & 0 & 0 \\ 0 & 14 & -14 \\ 0 & -14 & 14 \end{pmatrix}.
\end{aligned}$$

The list of R can be obtained by the list of root subsystems of $E_8^{(1)}$ which are indecomposable and of affine type (see [2]). Notice that R is of affine type because the null root $-\mathcal{K}_X$ belongs to $Q(R)=\sum \mathbb{Z}\mathcal{D}_i$, and is indecomposable because of Proposition 6.

Remark. In Table 3, the surfaces corresponding to R's encapsulated in the frame are the spaces of initial conditions of the Painlevé differential equations [19]. In § 6, a subsystem $R^\perp$ plays an important role in the constructions of symmetry. A discrete Painlevé equation is a translation part of this symmetry. The operation of taking the continuous limit to the Painlevé differential equation has a corresponding arrow in Table 4 and corresponding degeneration of the surface. In detail, see § 7. □

3. The root system R in Table 3 slightly differs from the R in the classification of surfaces (Table 2). We need a more detailed study of $D=\sum m_iD_i$.

According to Proposition 2 we have a birational morphism $\rho : X \to \mathbb{P}^2$ where ρ is a composition of nine successive blowings-up of points, which are possibly infinitely near each, in $\mathbb{P}^2$;

$$X = X_9 \overset{\rho_9}{\to} X_8 \overset{\rho_8}{\to} \cdots \overset{\rho_2}{\to} X_1 \overset{\rho_1}{\to} X_0 = \mathbb{P}^2.$$

Each ρ_i is a blowing-up of a closed point $p_i \in X_{i-1}$. We denote the image of p_i in $\mathbb{P}^2$ by the same letter p_i.

In the generic case, there exists a unique cubic curve passing through the nine points p_i. Then the proper transform of this cubic curve in X is D because $-\mathcal{K}_X = 3\mathcal{E}_0-\mathcal{E}_1-\cdots-\mathcal{E}_9$. We have classified D via the span of $\mathrm{Comp}(D)$ in $\mathrm{Pic}(X)$. If D is irreducible, corresponding to $A_0^{(1)}$, D is either a smooth elliptic curve, a rational curve with a node, or a rational curve with a cusp. Similarly $A_1^{(1)}$ and $A_2^{(1)}$ have different types of divisors D. We draw distinction among these as follows;

$R=A_0^{(1)} : D$ is a smooth elliptic curve, $R=A_0^{(1)*} : D$ is a rational curve with a node, $R=A_0^{(1)**} : D$ is a rational curve with a cusp,

$R=A_1^{(1)} : D=D_0+D_1$ and D_0 intersects D_1 at two points, $R=A_1^{(1)*} : D=D_0+D_1$ and D_0 is tangent at one point; $D_0\cdot D_1=2$,

	R
elliptic type	$A_0^{(1)}(=I_0)$
multiplicative type	$A_0^{(1)*}(=I_1),\ A_1^{(1)}(=I_2),\ A_2^{(1)}(=I_3),\dots,\ A_7^{(1)},\ A_7^{(1)\prime}(=I_8), A_8^{(1)}(=I_9)$
additive type	$A_0^{(1)**}(=II),\ A_1^{(1)*}(=III),\ A_2^{(1)*}(=IV)$, $D_4^{(1)}(=I_0^*),\dots,D_8^{(1)}(=I_4^*)$, $E_6^{(1)}(=IV^*),\ E_7^{(1)}(=III^*),\ E_8^{(1)}(=II^*)$

Table 2. Classification of generalized Halphen surfaces with $\dim|-\mathcal{K}_X|=0$

The symbols in the parentheses correspond to the ones from Kodaira's elliptic singular fibers. We divide all cases with $\dim|-\mathcal{K}_X|=0$ into three classes according as $\dim H_1(D_{red},\mathbb{Z})=2,1$ or 0, where $D_{red}=\cup D_i$ for $D=\sum m_iD_i$. We call them the elliptic type, the multiplicative type and the additive type respectively. We will see later that this classification corresponds to the types of discrete equations from the affine Weyl group symmetry: what we call elliptic-difference equation, q-difference equation and usual difference equation.

We will obtain this classification, first by studying the divisor classes $\mathcal{D}_i=[D_i]\in\mathrm{Pic}(X)$, and then by a much finer investigation on the diviors D_i.

2. We have two important root subsystems R and $R^\perp$ in $Q(E_8^{(1)})=(-\mathcal{K}_X)^\perp\subset\mathrm{Pic}(X)$. (The symbol R in the classification of surfaces in Table 2 needs more detailed distinction than that of the root system R. See **3.**) The root lattice of R is the span $Q(R)=\sum\mathbb{Z}\mathcal{D}_i$ of the classes of irreducible components D_i of D, and that of $R^\perp$ is its orthogonal complement $Q(R^\perp):=Q(R)^\perp=\{v\in\mathrm{Pic}(X)\,|\,(v|\mathcal{D}_i)=0\text{ for all }i\}$. The lists of R and the corresponding $R^\perp$ are the following.

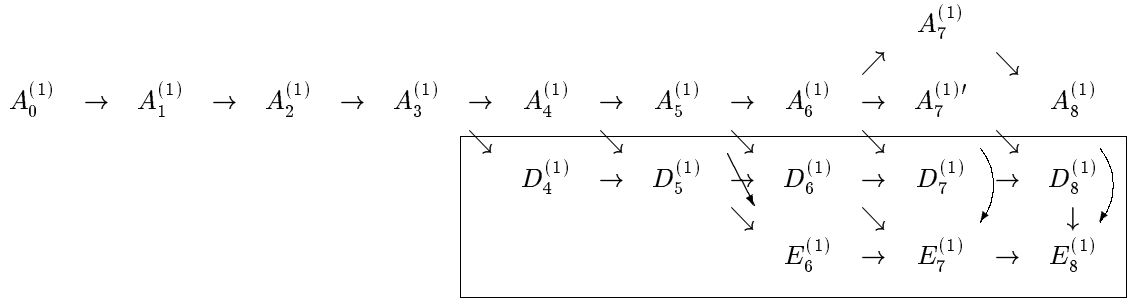


Table 3. R

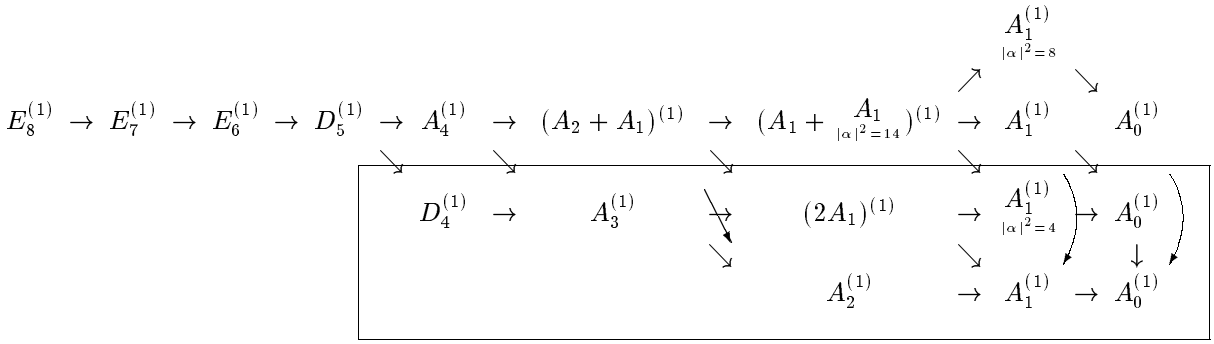


Table 4. $R^\perp$

The arrows mean inclusions ($R\to R'\Leftrightarrow R^\perp\to R'^\perp\Leftrightarrow Q(R)\subset Q(R')\Leftrightarrow Q(R^\perp)\supset Q(R'^\perp)$). Let $R=A_0^{(1)}$ stand for the lattice $Q(R)=\mathbb{Z}\mathcal{K}_X$ in the case that D itself is irreducible. Now $A_7^{(1)}$ has two types of

nodal curve, then $\mathcal{G}$ belongs to $\Delta_+(E_8^{(1)})$ by a) of Proposition 18 and $\mathcal{F}\cdot r_i \geq 0$ for all i. Thus we have only to prove it for $\mathcal{G} \in \mathrm{EX}$.

When we set $\mathcal{G} = \sum a_i\mathcal{E}_i$ then $a_0 \geq 0$. From $a_0^2 - \sum_{i=1}^9 a_i^2 = -1$, we have easily $\mathcal{F}\cdot\mathcal{G} \geq 0$ for $\mathcal{F} = \mathcal{E}_0, \mathcal{E}_0 - \mathcal{E}_1$ and $2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_1$. In the case $\mathcal{F} = -\mathcal{K}_i$ we have also $\mathcal{F}\cdot\mathcal{G} > 0$. For $\mathcal{G} \in \mathrm{Ex}(X_i)$, we have that $-\mathcal{K}_{X_i}\cdot\mathcal{G} = -\mathcal{K}_i\cdot\mathcal{G} = 1$ in $X_i = \rho_i\circ\cdots\circ\rho_9(X)$ and the total transform of a numerically effective divisor class is numerically effective.

Conversely if $\mathcal{F}$ is numerically effective then we have only to show $\mathcal{F}\cdot r_i \geq 0$ for some $(\mathcal{E}_i)_{i=0}^9$ by the above lemma. Set $\mathcal{F} = \sum_{i=0}^9 a_i\mathcal{E}_i$ in terms of some $(\mathcal{E}_i)_{i=0}^9$. If $\mathcal{F}\cdot r_i < 0$ then r_i is not effective and so $(w(\mathcal{E}_i))$ defines a blowing-down structure, where w is a reflection by a root r_i. In the case $i = 0$ this operation decreases a_0, and in the case $i > 0$, this operation decreases $\sum_{i=1}^8 \max\{-\mathcal{F}\cdot r_i, 0\} = \sum_{i=1}^8 \max\{a_i - a_{i+1}, 0\}$ with a_0 unchanged. If we do not achieve $(\mathcal{E}_i)$ such that $\mathcal{F}\cdot r_i \geq 0$ for $i = 0, \ldots, 8$ by these processes then a_0 becomes negative. But it contradicts to the fact that $\mathcal{E}_0$ is effective. □

4 Generalized Halphen surface

1. We assume X to be a rational surface such that $|-\mathcal{K}_X|$ has an effective divisor of canonical type (we denote it by D). Here, by a divisor of canonical type we mean the following.

Definition 1 *Let $D = \sum_{i\in I} m_iD_i$ be a divisor on X with irreducible components D_i. We say that D is of canonical type if*

$$\mathcal{K}_X\cdot[D_i] = D\cdot D_i = 0 \quad \textit{for all } i.$$

In this case we have $c_1^2(X) = (-\mathcal{K}_X)^2 = -\sum_{i\in I} m_i\mathcal{K}_X\cdot[D_i] = 0$.

If $|-\mathcal{K}_X|$ has a positive dimension, X is a rational elliptic surface called a Halphen surface of index one (a Halphen surface of index l is characterized by the condition that $\dim|-l\mathcal{K}_X| > 0$). It is known that the anti-canonical divisors of a Halphen surface are of canonical type [1]. So our X is a generalization of this rational elliptic surface.

Definition 2 *Let X be a smooth projective rational surface. We call X a generalized Halphen surface if X has an anti-canonical divisor of canonical type.*

Generalized Halphen surfaces are devided into two types;

- $\dim|-\mathcal{K}_X| = 1$: a Halphen surface of index one,
- $\dim|-\mathcal{K}_X| = 0$.

We classify the surfaces through the classification of anti-canonical divisors, but a Halphen surface of index one have many types of anti-canonical divisors. A Halphen surface is a well-known object [1] and the elliptic functions appear instead of the Painlevé equations. We study mainly generalized Halphen surfaces with $\dim|-\mathcal{K}_X| = 0$.

If $|-\mathcal{K}_X|$ has a unique divisor D, then X is classified according to the type R of D, where R is in the next table. We call X an R-surface and denote it by $X(R)$.

Proposition 18 *Let $\mathcal{E} = (\mathcal{E}_0, \ldots, \mathcal{E}_9)$ define a blowing-down structure.*

a) $\Delta^{nod} \cup \bigcup_{D \in |-\mathcal{K}_X|} \mathrm{Comp}(D) \subset \Delta_+(E_8^{(1)})$.

b) Let $w \in W(E_8^{(1)})$. Then $w\mathcal{E} = (w(\mathcal{E}_0), \ldots, w(\mathcal{E}_9))$ defines a blowing-down structure if and only if $w(\Delta^{nod} \cup \bigcup_{D \in |-\mathcal{K}_X|} \mathrm{Comp}(D)) \subset \Delta_+(E_8^{(1)})$.

c) Let $w_i \in W(E_8^{(1)})$ be the reflection by a simple root r_i. Then $w_i\mathcal{E}$ defines a blowing-down structure if and only if r_i is not effective.

Proof. a) If $r \in \Delta^{nod} \cup \bigcup_{D \in |-\mathcal{K}_X|} \mathrm{Comp}(D)$, then $\mathcal{E}_0 \cdot r \geq 0$. If $\mathcal{E}_0 \cdot r > 0$ then r as a sum of $r_0 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3$, $r_i = \mathcal{E}_i - \mathcal{E}_{i+1}$, $i = 1, \ldots, 8$ has r_0 with a positive coefficient, hence r is positive. If $\mathcal{E}_0 \cdot r = 0$ then r is of the form $\mathcal{E}_i - \mathcal{E}_j$. From the effectivity of r, p_j is infinitely near p_i, so $i < j$ and r is positive.

b) If $w\mathcal{E}$ defines a blowing-down structure, then w preserves the positivity of Δ^{nod} and an element of $\mathrm{Comp}(D)$ from a).

To prove the converse we show the numerical effectivity of $w(\mathcal{E}_0)$ first. But since an irreducible class $\mathcal{F}$ with $\mathcal{F} \cdot \mathcal{F} \leq -1$ is either an element of $\mathrm{Comp}(D)$, Δ^{nod} or EX, it is enough to show the effectivity of $w(\mathcal{E}_0)$ and $\mathcal{F} \cdot w(\mathcal{E}_0) \geq 0$ for $\mathcal{F} \in \bigcup_{D \in |-\mathcal{K}_X|} \mathrm{Comp}(D) \cup \Delta^{nod} \cup \mathrm{Ex}$.

If $\mathcal{F} \in \Delta^{nod} \cup \bigcup_{D \in |-\mathcal{K}_X|} \mathrm{Comp}(D)$, then $\mathcal{F} \cdot w(\mathcal{E}_0) \geq 0$ from the hypothesis. Let $\mathcal{F} \in \mathrm{EX}$. From Lemma 14 and 15 an element $\mathcal{F}' \in \mathrm{Ex}$ has the form $\mathcal{E}_9 + \mathcal{G}$ where $\mathcal{G} \in \mathbb{Z}_{\geq 0}.\Pi(E_8^{(1)})$. Taking $\mathcal{F}' = w^{-1}(\mathcal{F})$, we have $w(\mathcal{E}_0) \cdot \mathcal{F} = \mathcal{E}_0 \cdot w^{-1}(\mathcal{F}) \geq 0$. Let us show that $w(\mathcal{E}_0)$ is effective. By Lemma 15, $w(\mathcal{E}_0) \cdot \mathcal{E}_0 \geq 0$ holds, so $\dim H^2(X, w(\mathcal{E}_0)) = 0$ by Lemma 3. The Riemann-Roch inequality says $\dim H^0(X, w(\mathcal{E}_0)) \geq 3$. Therefore $w(\mathcal{E}_0)$ is numerically effective.

Lemma 16 and 17 say that $w(\mathcal{E}_0)$ induces a morphism $\varphi : X \to \mathbb{P}^2$ which is birational since $(w(\mathcal{E}_0)^2) = 1$. φ is a contraction of nine exceptional curves of the first kind which do not meet $w(\mathcal{E}_0)$. The intersection form determines these classes up to ordering, so these are $w(\mathcal{E}_i)$ $(i = 1, \ldots, 9)$. But from the hypothesis, if $w(\mathcal{E}_i) - w(\mathcal{E}_j)$ $(0 < i \neq j)$ is effective, then $i < j$. Thus we obtain the statement.

Because w_i preserves the positivity of all roots except r_i, c) is a consequence of b). The converse is clear because $w_i(r_i) = -r_i \notin \Delta_+(E_8^{(1)})$. □

Lemma 19 *Let $(\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_9)$ define a blowing-down structure, and suppose $\mathcal{F}$ is a class such that $\mathcal{F} \cdot r_i \geq 0$ $(i = 0, \ldots, 8)$, and $\mathcal{F} \cdot \mathcal{E}_i \geq 0$ $(i = 0, \ldots, 9)$. Then $\mathcal{F}$ is a nonnegative sum of the classes $\mathcal{E}_0$, $\mathcal{E}_0 - \mathcal{E}_1$, $2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2$ and $-\mathcal{K}_i$ for $i \geq 3$.*

Proof. If $\mathcal{F} \cdot \mathcal{E}_i = 0$ for every i, then $\mathcal{F} = 0$. Let k be the largest index such that $\mathcal{F} \cdot \mathcal{E}_k > 0$.

If $k = 0$, then $\mathcal{F} = a\mathcal{E}_0$, $a > 0$. If $k = 1$, then $\mathcal{F} = a\mathcal{E}_0 - b\mathcal{E}_1$ $(a \geq b > 0)$, so $\mathcal{F} = (a-b)\mathcal{E}_0 + b(\mathcal{E}_0 - \mathcal{E}_1)$. If $k = 2$, then $\mathcal{F} = a\mathcal{E}_0 - b\mathcal{E}_1 - c\mathcal{E}_2$ and $a \geq b + c > b \geq c > 0$, so $\mathcal{F} = (a-b-c)\mathcal{E}_0 + (b-c)(\mathcal{E}_0 - \mathcal{E}_1) + c(2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2)$.

If $k \geq 3$ and $\mathcal{F} = \sum_{i=0}^{9} a_i \mathcal{E}_i$, then $a_0 \geq -(a_1 + a_2 + a_3) \geq -3a_3$ and $-a_1 \geq -a_2 \geq \cdots \geq -a_k > 0 = a_{k+1} = \cdots = a_9$. Then $\mathcal{F}' = \mathcal{F} - a_k \mathcal{K}_k$ satisfies the conditions above and has a smaller k. By induction on k, $\mathcal{F}$ has the required form. □

Proposition 20 $\mathcal{F} \in \mathrm{NEFF}$ *if and only if there exists a blowing-down structure such that $\mathcal{F}$ is a sum of the classes $\mathcal{E}_0$, $\mathcal{E}_0 - \mathcal{E}_1$, $2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2$ and $-\mathcal{K}_i = -\mathcal{K}_X + \mathcal{E}_9 + \cdots + \mathcal{E}_{i+1}$ $(i \geq 3)$ with nonnegative coefficients. In particular $\mathcal{F}$ is effective.*

Proof. First we show that $\mathcal{F} = \mathcal{E}_0, \mathcal{E}_0 - \mathcal{E}_1, 2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2, -\mathcal{K}_i$ $(i \geq 3)$ are numerically effective. Since $\mathcal{F}$ is effective, it is enough to show $\mathcal{F} \cdot \mathcal{G} \geq 0$ for arbitrary irreducible divisor class $\mathcal{G}$ with $(\mathcal{G}^2) < 0$. If $\mathcal{G}$ is a

This means that $a_1 - \frac{a_0}{3} = a_2 - \frac{a_0}{3} = a_3 - \frac{a_0}{3} = 0$. Let $a_1 = a_2 = a_3 = n$ then we have $(a_i) = (3n, n, n, n, n, \ldots, n, n-1)$, because $3a_i - a_0 \leq 0$ and $\sum_{i=1}^{9}(a_i - \frac{a_0}{3}) = -1$. Now $a_0^2 - \sum_{i>0} a_i^2 = 9n^2 - (9n^2 - 2n + 1) = -1$ leads to $a_0 = 0$, and it is a contradiction.

In this case we can operate w_0 on $\mathcal{F}$ where w_0 is the reflection by a root $r_0 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3$. This operation reduces a_0 so recursive operations lead to the case of $a_0 = 0$. □

Lemma 15 *Let $\mathcal{F} \in \mathrm{Pic}(X)$ and $\mathcal{F} \cdot r_i \geq 0$ for $r_i \in \Pi(E_8^{(1)})$. Then $w(\mathcal{F}) - \mathcal{F} \in \mathbb{Z}_{\geq 0}.\Pi(E_8^{(1)})$ for $w \in W(E_8^{(1)})$.*

Proof. We take a reduced expression $w = w_{i_1} \circ \cdots \circ w_{i_t}$. For a simple reflection w_{i_t}, $w_{i_t}(\mathcal{F}) - \mathcal{F} \in \mathbb{Z}_{\geq 0}.r_{i_t}$. If i_s is the smallest number such that $w_{i_{s+1}} \circ \cdots \circ w_{i_t}(\mathcal{F}) - \mathcal{F} \notin \mathbb{Z}_{\geq 0}.\Pi$, then $r_{i_s} \cdot w_{i_{s+1}} \circ \cdots \circ w_{i_t}(\mathcal{F}) < 0$. Therefore $w_{i_t} \circ \cdots \circ w_{i_{s+1}}(r_{i_s}) \notin \Delta_+(E_8^{(1)})$ and it contradicts to the fact that this is a reduced expression ([14] § 3.11). □

Lemma 16 *If $w(\mathcal{E}_0)$ is numerically effective, then the linear system of $w(\mathcal{E}_0)$ has no base points and is effective.*

Proof. Since $w(\mathcal{E}_0)$ is numerically effective, we have $\mathcal{E}_0 \cdot w(\mathcal{E}_0) \geq 0$. From b) of Lemma 3 and the Riemann-Roch inequality, $\dim H^0(X, w(\mathcal{E}_0)) \geq 3$. Hence $w(\mathcal{E}_0)$ is effective.

Suppose $p \in X$ is a base point of $w(\mathcal{E}_0)$ and $\pi : \widetilde{X} \to X$ is the blowing-up of X at p. Then the total transform $\widetilde{\mathcal{E}}_0$ of $w(\mathcal{E}_0)$ is numerically effective and the total transform of p is a fixed component of $\widetilde{\mathcal{E}}_0$. Thus it is enough to show that $\widetilde{\mathcal{E}}_0$ has no fixed component.

Let $\mathcal{F}$ be the class of the fixed part of $\widetilde{\mathcal{E}}_0$ and write $\widetilde{\mathcal{E}}_0 = \mathcal{H} + \mathcal{F}$. Then $\mathcal{H}$ has no fixed components, so $(\mathcal{H}^2) \geq 0$.

Because $\widetilde{\mathcal{E}}_0$ is numerically effective, $\widetilde{\mathcal{E}}_0 \cdot \mathcal{H} \geq 0$. If $\widetilde{\mathcal{E}}_0 \cdot \mathcal{H} = 0$, then $(\mathcal{H}^2) = 0$. It follows that $\mathcal{H} = 0$ because $\mathcal{E}_0^{\perp} \subset \mathrm{Pic}(\widetilde{X})$ has the negative definite intersection form, and hence $\dim H^0(X, \mathcal{H}) = 1$. This contradicts to $\dim H^0(X, \mathcal{H}) = \dim H^0(X, \widetilde{\mathcal{E}}_0) \geq 3$. On the other hand, if $\widetilde{\mathcal{E}}_0 \cdot \mathcal{H} > 1$, then $\widetilde{\mathcal{E}}_0 \cdot \mathcal{F} < 0$. It is against the effectivity of $\mathcal{F}$. Thus $\widetilde{\mathcal{E}}_0 \cdot \mathcal{H} = 1$. Moreover we have $(\mathcal{H}^2) = 0$ because $(\mathcal{H}^2) > 0$ implies that $\mathcal{F} = 0$ from $\mathcal{F} \perp \mathcal{E}_0$.

The global sections of $\mathcal{H}$ induce a morphism $\varphi : \widetilde{X} \to \mathbb{P}^l$ whose image is a curve C. Since $\widetilde{X}$ is nonsingular and rational, and φ is induced by a complete linear system, we conclude that $C \simeq \mathbb{P}^1$.

Because $\widetilde{\mathcal{E}}_0 \cdot \mathcal{H} = 1$, $\mathcal{H}$ is not a nontrivial multiple of any element of $\mathrm{Pic}(\widetilde{X})$, and a general hyperplane L of $\mathbb{P}^l$ meets C at a single point. Thus C is a linear subvariety of $\mathbb{P}^l$. Since φ is induced by a linear system, this means $\mathbb{P}^l = C \simeq \mathbb{P}^1$. Therefore $H^0(\widetilde{X}, \mathcal{H}) = H^0(\mathbb{P}^1, \mathcal{O}(1))$, and $\dim H^0(\mathbb{P}^1, \mathcal{O}(1)) = 2$ leads to a contradiction. □

Lemma 17 *If $w(\mathcal{E}_0)$ is numerically effective, then* $\dim H^0(X, w(\mathcal{E}_0)) = 3$.

Proof. From the above lemma, $w(\mathcal{E}_0)$ is effective and has no base points. Since $(\mathcal{E}_0^2) > 0$, the linear system of $w(\mathcal{E}_0)$ is not composite with a pencil. Bertini's theorem says that $w(\mathcal{E}_0)$ is irreducible. The arithmetic genus of $w(\mathcal{E}_0)$ is zero. So $w(\mathcal{E}_0)$ has a divisor $C \simeq \mathbb{P}^1$. We consider the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X \otimes w(\mathcal{E}_0) \to \mathcal{O}_C \otimes w(\mathcal{E}_0) \to 0.$$

Then $\deg \mathcal{O}_C \otimes w(\mathcal{E}_0) = 1$, and from the Riemann-Roch theorem for curves, we have $\dim H^0(C, \mathcal{O}_C \otimes w(\mathcal{E}_0)) = 2$. By taking the cohomology of the above exact sequence, noting that $\dim H^0(X, \mathcal{O}_X) = 1$ and $\dim H^1(X, \mathcal{O}_X) = 0$, we have $\dim H^0(X, w(\mathcal{E}_0)) = 3$. □

5. Now we turn to the description of the Weyl groups of affine type. The Weyl group $W = W(R)$ is generated by the simple reflections w_i's, where

$$w_i(\lambda) = \lambda - \langle \alpha^\vee, \lambda \rangle \, \alpha_i \qquad \text{for } \lambda \in V^*.$$

For any real root α, w_α ($w_\alpha(\lambda) = \lambda - \langle \alpha^\vee, \lambda \rangle \, \alpha$) belongs to W. In fact if $\alpha = w(\alpha_i)$ for some $w \in W$, then $w_\alpha = w \circ w_i \circ w^{-1}$.

Proposition 11 ([14] § 6.5) *Let $\alpha \in \Delta_+^{re} = \Delta^{re} \cap \Delta_+$ be such that $\beta = \delta - a\alpha \in \Delta_+^{re}$ for some $a \in \mathbb{Z}$. Then, for $\lambda \in V^*$, we have*

$$w_\alpha \circ w_\beta(\lambda) = \lambda + \langle c, \lambda \rangle \nu(\beta^\vee) - (\langle \beta^\vee, \lambda \rangle + \frac{1}{2}|\beta^\vee|^2 \langle c, \lambda \rangle) \, \delta. \qquad \square$$

Let $\overset{\circ}{R}$ be the root system obtained from the generalized Cartan matrix of R by deleting the 0-th row and column. Denote by $\overset{\circ}{W}$ the subgroup of W generated by $w_1, \ldots, w_{l-1}$. Let $M = \nu(\mathbb{Z}(\overset{\circ}{W} \cdot \theta^\vee)) \subset \overset{\circ}{V}{}^*$, $\theta = \delta - a_0\alpha_0$, and set $T = \{t_\alpha | \alpha \in M\}$, where

$$t_\alpha(\lambda) = \lambda + \langle c, \lambda \rangle \alpha - ((\alpha|\lambda) + \frac{1}{2}|\alpha|^2 \langle c, \lambda \rangle) \, \delta \qquad \text{for } \lambda \in V^*.$$

We have $M = \overset{\circ}{Q} = Q(\overset{\circ}{R})$ if R is symmetric and of affine type.

Proposition 12 ([14] § 6.5) $W \simeq \overset{\circ}{W} \ltimes T$. $\square$

Let $\mathrm{Aut}(R)$ be the group of Dynkin automorphisms. The extension $\mathrm{Aut}(R) \ltimes W(R)$ is called the extended Weyl group and is denoted as $\widetilde{W} = \widetilde{W}(R)$.

Proposition 13 ([14] § 5.10) *a) If R is indecomposable, then the group of all automorphisms of Q preserving Δ is $\pm\widetilde{W}(R)$.*

b) If R is symmetric and of finite, affine or hyperbolic type, then the group of all automorphisms of Q preserving $(.|.)$ is $\pm\widetilde{W}(R)$. $\square$

6. In the rest of this section, suppose X is a surface obtained by nine successive blowings-up from $\mathbb{P}^2$ with the center at each point. Then $w \in W(E_8^{(1)})$ acts not only on $Q(E_8^{(1)}) \subset \mathrm{Pic}(X)$ but on $\mathrm{Pic}(X)$ by $w_{r_i}(\mathcal{F}) = \mathcal{F} + (r_i \cdot \mathcal{F}) \, r_i$, where $r_0 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3$ and $r_i = \mathcal{E}_i - \mathcal{E}_{i+1}$ $(i = 1, \ldots, 8)$. We set $\Pi(E_8^{(1)}) = \{r_i \ (i = 0, \ldots, 8)\}$.

Furthermore we assume that $\mathcal{D}_i \cdot \mathcal{K}_X = 0$ for any $\mathcal{D}_i = [D_i] \in \mathrm{Comp}(D)$ and any $D \in |-\mathcal{K}_X|$. In this case we have such $\mathcal{D}_i$'s in $\Delta^{re}(E_8^{(1)})$ because $\mathcal{D}_i \perp \mathcal{K}_X$ and $(\mathcal{D}_i^2) = -2$ (it is because we have $(\mathcal{D}_i^2) \geq -2$ from the virtual genus $\pi(D_i)$ of the irreducible curve D_i). The same argument shows that $\Delta^{nod} \subset \Delta^{re}(E_8^{(1)})$.

Lemma 14 *$W(E_8^{(1)})$ acts transitively on* EX.

Proof. Let $\mathcal{F} = \sum_{i=0}^9 a_i \mathcal{E}_i \in \mathrm{EX}$. We have $3a_0 - \sum_{i>0} a_i = 1$, $a_0^2 - \sum_{i>0} a_i^2 = -1$. From the effectivity of $\mathcal{F}$ and the numerically effectivity of $\mathcal{E}_0$, we find $a_0 \geq 0$.

If $a_0 = 0$, then there is an $\mathcal{E}_i$ $(i > 0)$ such that $\mathcal{F} = \mathcal{E}_i$ and EX is permuted by $\mathfrak{S}_9 \subset W(E_8^{(1)})$ transitively.

Let $a_0 > 0$. Acting with $\mathfrak{S}_9$ as necessary, we assume $a_1 \geq a_2 \geq \cdots \geq a_9$. We show $a_0 < a_1 + a_2 + a_3$. Suppose the contrary, then we have

$$-1 = \sum_{i>0} (a_i - \frac{a_0}{3}) \leq (a_1 + a_2 + a_3 - a_0) + 6(a_3 - \frac{a_0}{3}) \leq 2(3a_3 - a_0) \leq 0.$$

Proof. Suppose $\mathcal{F} \in \mathrm{Pic}^+(X)$ has a negative intersection number with an irreducible curve C. Then $C \in \bigcap_{F \in |\mathcal{F}|} \mathrm{Supp}(F)$ and $[C] \in \mathrm{EX}$, Δ^{nod} or $\mathrm{Comp}(D)$ for $D \in |-\mathcal{K}_X|$. We get a numerically effective class by subtracting these $[C]$ from $\mathcal{F}$. □

Proposition 6 *Let $D \in |-\mathcal{K}_X|$. Then D is connected.*

Proof. First we show 1) $\dim H^1(D, \mathcal{O}_D) = 1$ and 2) $H^1(D', \mathcal{O}_{D'}) = 0$ for a divisor D' with $0 < D' < D$.

The exact sequence $0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ leads to $H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_D) \to H^2(\mathcal{O}_X(-D)) \to H^2(\mathcal{O}_X)$. Now $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$ and $H^2(\mathcal{O}_X(-D)) \simeq H^0(\mathcal{O}_X(\mathcal{K}_X + D)) = H^0(\mathcal{O}_X) = \mathbb{C}$. Hence 1) is proved. From $\mathcal{K}_X + D' < 0$ we have $H^2(\mathcal{O}_X(-D')) \simeq H^0(\mathcal{O}_X(\mathcal{K}_X + D')) = 0$, so 2) is proved similarly.

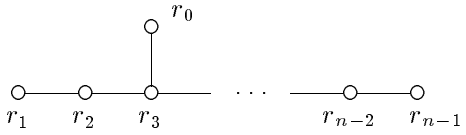
If $D = D' + D''$, $D' \cap D'' = \emptyset$, then $\mathbb{C} \simeq H^1(\mathcal{O}_D) \simeq H^1(\mathcal{O}_{D'}) \oplus H^1(\mathcal{O}_{D''}) \simeq 0$. Hence D is connected. □

4. The standard Lorentzian lattice Λ_n is a free $\mathbb{Z}$-module of rank n, $\mathbb{Z}v_0 \oplus \mathbb{Z}v_1 \oplus \cdots \mathbb{Z}v_{n-1}$, equipped with a symmetric bilinear form $(.|.)$ given by

$$|\sum_{i=0}^{n-1} a_i v_i|^2 = -a_0^2 + a_1^2 + \cdots + a_{n-1}^2,$$

where $|v|^2 = (v|v)$.

Proposition 7 *The notation being as above, let $\delta_n = 3v_0 - v_1 - \cdots - v_n$. Then $(\delta_n)^\perp = \{v \in \Lambda_{n+1} \mid (v|\delta_n) = 0\}$ is the root lattice corresponding to the generalized Cartan matrix with the following Dynkin diagram:*



A root basis can be chosen as

$$r_0 = v_0 - v_1 - v_2 - v_3, \quad r_i = v_i - v_{i+1} \ (i = 1, \ldots, n-1).$$ □

Proposition 8 *Let X be a rational surface with* $\mathrm{rank}\ \mathrm{Pic}(X) = n \geq 3$. *Then the Picard group* $\mathrm{Pic}(X)$ *with a bilinear form $(\mathcal{F}|\mathcal{G}) = -\mathcal{F} \cdot \mathcal{G}$ $(\mathcal{F}, \mathcal{G} \in \mathrm{Pic}(X))$ is isomorphic to Λ_n. Moreover there exists an isomorphism ϕ:$\Lambda_n \to \mathrm{Pic}(X)$ such that $\delta_{n-1} \mapsto -\mathcal{K}_X$.* □

We have special interest in the case $n = 9$. In this case the root system of type $E_8^{(1)}$ appears. We quote from Kac's book [14] some results about root systems. See the page 5 as for the notation R, Q, Π, etc.

Proposition 9 ([14] § 5.10) *Let R be a root system of finite, affine or hyperbolic type. Then*

a) $\Delta^{re} = \{\alpha \in Q \mid (\alpha|\alpha) = 2\}$, *when R is symmetric.*

b) $\Delta^{im} = \{\alpha \in Q \setminus \{0\} \mid (\alpha|\alpha) \leq 0\}$. □

Proposition 10 ([14] § 5.6, § 4.8) *Let R be an indecomposable root system.*

a) If R is of finite type, then the set Δ^{im} is empty.

b) If R is of affine type, then there is a unique element $\delta \in \Delta$ such that $\Delta_+^{im} = \Delta^{im} \cap \Delta_+ = \{n\delta \ (n = 1, 2, \ldots)\}$. □

If $Z = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, we take H_0 and H_1 to be the lines passing through p of bidegree $(0,1)$ and $(1,0)$, respectively. Then we can blow down the proper transform of $H_0 \cup H_1$, and the image is $\mathbb{P}^2$.

In the case $Z = \mathbb{F}_2$, p does not lie on the curve S with $(S^2) = -2$ for the reason indicated in the proof of c). Let F be the fiber of $\mathbb{F}_2$ passing through p. We can blow down the proper transform of $F \cup S$, and the image is $\mathbb{P}^2$. □

3. Suppose that X admits a rational morphism $\rho : X \to \mathbb{P}^2$, where $\rho = \rho_1 \circ \cdots \rho_n$ is the composition of blowing-down ρ_i with the center at $p_i \in X_i = \rho_i \circ \cdots \circ \rho_n(X)$. We call such a pair of a surface and a composition of blowing-down morphisms (X, ρ) a *blowing-down structure*. Two blowing-down structures (X, ρ) and (X', ρ') are isomorphic if there exist isomorphisms $\psi_i : X_i \to X'_i$ such that $\rho'_i \circ \psi_i = \psi_{i-1} \circ \rho_i$ $(i = 0, \ldots, 9)$.

We denote by $\mathcal{E}_0$ (resp. $\mathcal{E}_i$ $(i = 1, \ldots, n)$) the class of the total transform of a line in $\mathbb{P}^2$ (resp. of the closed point p_i which is the center of ρ_i). Then the Picard group of X is described as follows.

$$\begin{aligned} \mathrm{Pic}(X) &= \mathbb{Z}\mathcal{E}_0 + \mathbb{Z}\mathcal{E}_1 + \cdots + \mathbb{Z}\mathcal{E}_n, \\ (\mathcal{E}_0^2) &= 1,\ (\mathcal{E}_i^2) = -1\ (i \neq 0),\ \mathcal{E}_i \cdot \mathcal{E}_j = 0\ (i \neq j), \\ \mathcal{K}_X &= -3\mathcal{E}_0 + \mathcal{E}_1 + \cdots + \mathcal{E}_n. \end{aligned}$$

Let $(\mathcal{E}'_0, \mathcal{E}'_1, \ldots, \mathcal{E}'_n)$ be a basis of $\mathrm{Pic}(X)$ with the intersection pairing given as above. If there exists a series of blowing-down morphisms whose class of total transform of a line (resp. of an exceptional curve of the first kind) is $\mathcal{E}'_0$ (resp. $\mathcal{E}'_i$ $(i = 1, \ldots, n)$), then we say that $(\mathcal{E}'_i)_{i=0}^n$ defines a blowing-down structure.

Now $\mathcal{E}_0$ is numerically effective. It is because $\mathcal{E}_0$ is irreducible and $\mathcal{E}_0 \cdot \mathcal{E}_0 = 1$. So the next lemma is immediate. (The latter follows from a) for $\mathcal{K}_X - \mathcal{F}$ and the Serre duality.)

Lemma 3 *a) If* $\mathcal{F} \cdot \mathcal{E}_0 \leq -1$, *then* $H^0(X, \mathcal{F}) = 0$. *b) If* $\mathcal{F} \cdot \mathcal{E}_0 \geq -2$, *then* $H^2(X, \mathcal{F}) = 0$. □

Next we study the semi-group of effective classes $\mathrm{Pic}^+(X)$. Let C be an irreducible curve on X which is not a component of $D \in |-\mathcal{K}_X|$. The virtual genus of C is

$$\pi(C) = 1 + \frac{1}{2}[C] \cdot ([C] + \mathcal{K}) \leq 1 + \frac{1}{2}(C^2).$$

Hence $(C^2) \geq -2$. If $(C^2) = -2$, then $[C] \cdot \mathcal{K}_X = 0$ and $\pi(C) = 0$. It implies that C is a smooth rational curve. A smooth rational curve with self-intersection -2 is called a nodal curve. A nodal curve is either an irreducible component of $D \in |-\mathcal{K}_X|$ or disjoint from $\mathcal{K}_X$. If $(C^2) = -1$, then $C \cdot \mathcal{K}_X = -1$ and $\pi(C) = 0$. So C is an exceptional curve of the first kind and meets $-\mathcal{K}_X$ simply in a unique irreducible component.

We set the following notatiton;
NEFF : the set of numerically effective classes,
$\mathrm{EX} = \{\mathcal{F} \in \mathrm{Pic}(X) | (\mathcal{F}^2) = -1, \mathcal{F} \cdot \mathcal{K}_X = -1\}$: the set of exceptional classes,
Δ^{nod} : the set of classes of nodal curves disjoint from the irreducible components of $|-\mathcal{K}_X|$.

Lemma 4 *An exceptional class* $\mathcal{F} \in \mathrm{EX}$ *is effective.*

Proof. Let $\mathcal{G} = (\mathcal{F} \cdot \mathcal{E}_0)\mathcal{K}_X + 3\mathcal{F}$, then $\mathcal{E}_0 \cdot \mathcal{G} = 0$ and so $(\mathcal{G}^2) \leq 0$. It is equivalent to $-6(\mathcal{F} \cdot \mathcal{E}_0) - 9 \leq 0$, so $\mathcal{F} \cdot \mathcal{E}_0 \geq -\frac{9}{6} \geq -2$. By Lemma 3 we have $H^2(X, \mathcal{F}) = 0$, and $\dim H^0(X, \mathcal{F}) \geq 1 + \frac{1}{2}\mathcal{F} \cdot (\mathcal{F} - \mathcal{K}_X) = 1$ by the Riemann-Roch inequality. □

Proposition 5 $\mathrm{Pic}^+(X)$ *is generated by* $\mathrm{NEFF} \cap \mathrm{Pic}^+(X)$, EX, Δ^{nod} *and* $\mathrm{Comp}(D)$ *for* $D \in |-\mathcal{K}_X|$.

From the next section we take the position that the surfaces teach us everything. We once forget about dynamical systems and begin with characterization of surfaces.

3 Preliminaries on rational surfaces and root systems

This section is devoted to a review of known results which we will use later. Rational surfaces have been studied by many authors. Most relevant to us are the works of P. du Val [6], M. Nagata [17], Y. Manin [16], M. Demazure [3], E. Looijenga [15] and B. Harbourne [11]. For reader's convenience we give brief proofs to some of the statements. See notation and conventions at the page 5 for the definition of symbols.

1. First we recall some useful formulas which hold for a smooth, complete surface X. They are called by the name of Gauss-Bonnet, Noether, Riemann-Roch and Serre, respectively.

- $c_2(X) = \chi(X)$,
- $\chi(\mathcal{O}) = 1 - q + p_g = \frac{1}{12}(c_1^2 + c_2)$,
- $\chi(\mathcal{F}) = \frac{1}{2}\mathcal{F}\cdot(\mathcal{F} - \mathcal{K}_X) + \chi(\mathcal{O})$,
  in particular, $\dim H^0(X,\mathcal{F}) + \dim H^2(X,\mathcal{F}) \geq \frac{1}{2}\mathcal{F}\cdot(\mathcal{F} - \mathcal{K}_X) + \chi(\mathcal{O})$,
- $\dim H^i(X,\mathcal{F}) = \dim H^{2-i}(X, \mathcal{K}_X - \mathcal{F})$.

2. A smooth compact connected surface free of exceptional curves of the first kind is said to be relatively minimal. A rational relatively minimal surface is either $\mathbb{P}^2$ or one of the Hirzebruch surfaces $\mathbb{F}_l = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-l))$ $(l = 0, 2, 3, 4, \dots)$. The latter is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ having a section S with $S \cdot S = -l$.

We consider the Picard group $\mathrm{Pic}(X) = H^1(X, \mathcal{O}^*)$. The exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \stackrel{exp}{\to} \mathcal{O}^* \to 0$ leads to $H^1(X,\mathcal{O}) \to H^1(X,\mathcal{O}^*) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O})$, while the rationality implies $q = p_g = 0$. Hence $\mathrm{Pic}(X) \simeq H^2(X,\mathbb{Z})$ holds for a rational surface X.

Proposition 2 *Let X be a rational surface with $(\mathcal{K}_X^2) = 0$. Then*

a) $\mathrm{rank}\ \mathrm{Pic}(X) = 10$.

b) $|-\mathcal{K}_X| \neq \emptyset$.

c) Suppose that $|-\mathcal{K}_X|$ has a divisor $D = \sum_i m_i D_i$ such that $[D_i]\cdot\mathcal{K}_X = 0$ for any i. Let $\rho : X \to Z$ be a birational morphism of X onto a relatively minimal surface Z. Then Z is either $\mathbb{P}^2$, $\mathbb{F}_0$ $(= \mathbb{P}^1 \times \mathbb{P}^1)$ or $\mathbb{F}_2$.

d) Under the same condition for $|-\mathcal{K}_X|$, there exists a birational morphism $\rho : X \to \mathbb{P}^2$.

Proof. a) The Noether formula says that $1 = \frac{1}{12}(c_1^2 + c_2)$, where $c_2(X) = \chi(X) = \sum_{i=0}^4 (-1)^i b_i = 2 - 2b_1 + b_2$ and $b_1 = 2q = 0$. It follows from $c_1^2 = (\mathcal{K}_X^2) = 0$ that $\mathrm{rank}\ \mathrm{Pic}(X) = b_2 = 10$.

b) From the Riemann-Roch inequality we have $\dim H^0(X, 2\mathcal{K}_X) + \dim H^0(X, -\mathcal{K}_X) \geq 1$. The first term in the left hand side vanishes.

c) If $Z = \mathbb{F}_l$ $(l \geq 3)$, then Z has a nonsingular rational curve S with $(S^2) \leq -3$. For such a curve we have $S\cdot\mathcal{K}_X = 2(\pi(S) - 1) - (S^2) = -2 - (S^2) \geq 1$. Then $S \notin \mathrm{Comp}(D)$ because $S \in \mathrm{Comp}(D)$ would mean $S \cdot D = 0$. But $S \notin \mathrm{Comp}(D)$ leads to $S \cdot D \geq 0$. It is a contradiction.

d) Let $\rho : X \to Z$ be a birational morphism onto a relatively minimal model Z. Suppose that $Z \neq \mathbb{P}^2$. Then $\mathrm{rank}\ \mathrm{Pic}(Z) = 2$ and $\mathrm{rank}\ \mathrm{Pic}(X) = 10$, so that ρ can be written as a composition $\rho = \rho_1 \circ \rho_2 \circ \cdots \circ \rho_8$, where ρ_i is the blowing-down of an exceptional curve E_i $(1 \leq i \leq 8)$. Set $p = \rho_1(E_1)$ and $\rho' = \rho_2 \circ \rho_3 \circ \cdots \circ \rho_8$.

Thus we can think of X_b as the space of initial conditions of the sixth Painlevé q-difference equation.

Let us show the proposition. As a mapping $q\text{-}P_{VI}$ can be decomposed into more elementary mappings between spaces of initial conditions as follows.

$$q\text{-}P_{VI} = \sigma_2 \circ w_2 \circ w_1 \circ w_0 \circ w_2 \circ \sigma_1 \circ w_3 \circ w_5 \circ w_4 \circ w_3, \tag{9}$$

$$\begin{aligned}
\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; f, g\Big) \quad &\overset{w_3}{\mapsto} \quad \left(\begin{matrix} b_1\frac{b_7}{b_5} & b_2\frac{b_7}{b_5} & b_3 & b_4 \\ b_7 & b_6 & b_5 & b_8 \end{matrix}; f,\ g\frac{f-b_7}{f-b_5}\right) \\
&\overset{w_4}{\mapsto} \quad \left(\begin{matrix} b_1\frac{b_7}{b_5} & b_2\frac{b_7}{b_5} & b_3 & b_4 \\ b_7 & b_6 & b_8 & b_5 \end{matrix}; f,\ g\frac{f-b_7}{f-b_5}\right) \\
&\overset{w_5}{\mapsto} \quad \left(\begin{matrix} b_1\frac{b_7}{b_5} & b_2\frac{b_7}{b_5} & b_3 & b_4 \\ b_6 & b_7 & b_8 & b_5 \end{matrix}; f,\ g\frac{f-b_7}{f-b_5}\right) \\
&\overset{w_3}{\mapsto} \quad \left(\begin{matrix} b_1\frac{b_7b_8}{b_5b_6} & b_2\frac{b_7b_8}{b_5b_6} & b_3 & b_4 \\ b_8 & b_7 & b_6 & b_5 \end{matrix}; f,\ g\frac{f-b_7}{f-b_5}\frac{f-b_8}{f-b_6}\right) \\
&\overset{\sigma_1}{\mapsto} \quad \left(\begin{matrix} qb_2 & qb_1 & b_3 & b_4 \\ b_6 & b_5 & b_8 & b_7 \end{matrix}; f,\ \overline{g} = \frac{b_3b_4}{g}\frac{f-b_5}{f-b_7}\frac{b_6}{f-b_8}\right) \\
&\overset{w_2}{\mapsto} \quad \left(\begin{matrix} b_4 & qb_1 & qb_2 & b_3 \\ b_6\frac{b_4}{qb_2} & b_5\frac{b_4}{qb_2} & b_8 & b_7 \end{matrix}; f\frac{\overline{g}-b_4}{\overline{g}-qb_2},\ \overline{g}\right) \\
&\overset{w_0}{\mapsto} \quad \left(\begin{matrix} b_3 & b_4 & qb_1 & qb_2 \\ b_6\frac{b_4}{qb_2} & b_5\frac{b_4}{qb_2} & b_8 & b_7 \end{matrix}; f\frac{\overline{g}-b_4}{\overline{g}-qb_2},\ \overline{g}\right) \\
&\overset{w_1}{\mapsto} \quad \left(\begin{matrix} qb_1 & b_4 & b_3 & qb_2 \\ b_6\frac{b_4}{qb_2} & b_5\frac{b_4}{qb_2} & b_8 & b_7 \end{matrix}; f\frac{\overline{g}-b_4}{\overline{g}-qb_2},\ \overline{g}\right) \\
&\overset{w_2}{\mapsto} \quad \left(\begin{matrix} b_3 & b_4 & qb_1 & qb_2 \\ b_6\frac{b_4}{qb_2}\frac{b_3}{qb_1} & b_5\frac{b_4}{qb_2}\frac{b_3}{qb_1} & b_8 & b_7 \end{matrix}; f\frac{\overline{g}-b_4}{\overline{g}-qb_2}\frac{\overline{g}-b_3}{\overline{g}-qb_1},\overline{g}\right) \\
&\overset{\sigma_3}{\mapsto} \quad \left(\begin{matrix} qb_1 & qb_2 & b_3 & b_4 \\ qb_5 & qb_6 & b_7 & b_8 \end{matrix}; \overline{f} = \frac{b_7b_8}{f}\frac{\overline{g}-qb_1}{\overline{g}-b_3}\frac{\overline{g}-qb_2}{\overline{g}-b_4},\ \overline{g}\right).
\end{aligned}$$

We have only to show each step is an isomorphism. But this is just an interchange of blowing-down structures of the same surface. For example, w_3 is obtained by blowing-up $\mathbb{P}^1 \times \mathbb{P}^1$ at $(f, g) = (b_5, 0)$, (b_7, ∞) and blowing-down the exceptional curves of the first kind: pull-backs of $(f - b_5 = 0)$ and $(f - b_7 = 0)$.

Moreover these mappings generate the extended affine Weyl group of $D_5^{(1)}$; w_i is the reflection with respect to a simple root and σ_i comes from an automorphism of the Dynkin diagram. It naturally emerges from configurations of exceptional curves on the space of initial conditions. So we have two root systems $A_3^{(1)}$ and $D_5^{(1)}$ associated with one surface X_b. The root system $A_3^{(1)}$ arises as the pole divisors of a 2-form ω, and $D_5^{(1)}$ plays the role of the symmetry. We will see later the same situation in more general cases.

Remark. For uniformity of description, we will consider in the following sections X_b as a surface obtained by blowing up $\mathbb{P}^2$ with the centers a nine points, rather than taking $\mathbb{P}^1 \times \mathbb{P}^1$. The correspondence between the parameters in this section and those of Appendix B is

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; f, g\Big) = \left(\begin{matrix} 1/a_2 & 1/a_3 & 1/a_1 & a_8 \\ a_6 & a_7 & -1/a_4a_1 & -1/a_5a_1 \end{matrix}; \frac{y}{x - a_1z}, \frac{z}{x}\right),$$

where $(x : y : z)$ is the homogeneous coordinate of $\mathbb{P}^2$. As for the a_i, see the item '$A_3^{(1)}$-surface' in Appendix B. The Dynkin automorphisms σ_1 (resp. σ_2) is described as $-\sigma_{(13)}$ (resp. $-\sigma_{(20)}$) there. □

To simplify the notation let us rename b_1t, b_2t, b_5t, b_6t as b_1, b_2, b_5, b_6, so that $\overline{b}_i = qb_it$ for $i = 1, 2, 5, 6$. Then we have an automorphism of the field $\mathbb{C}(b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, f, g)$

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; f, g \Big) \mapsto \begin{pmatrix} qb_1 & qb_2 & b_3 & b_4 \\ qb_5 & qb_6 & b_7 & b_8 \end{pmatrix}; \overline{f}, \overline{g} \Big),$$

$$\overline{g} = \frac{b_3b_4}{g} \frac{f - b_5}{f - b_7} \frac{f - b_6}{f - b_8},$$

$$\overline{f} = \frac{b_7b_8}{f} \frac{\overline{g} - qb_1}{\overline{g} - b_3} \frac{\overline{g} - qb_2}{\overline{g} - b_4},$$

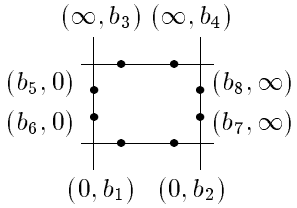
$$q = \frac{b_3b_4b_5b_6}{b_1b_2b_7b_8}.$$

Let us see how the singularity confinement works in our case of q-P_{VI}. Singularities occur for the values $f = b_7, b_8$ or $g = b_3, b_4$. We must also consider $f = b_5, b_6$ and $g = b_1, b_2$ which are the sources of the singularities for the next values of $1/f$ and $1/g$. For example, the input datum $(\underline{f}, \underline{g}) = (b_7, g_0)$ gives us $(f, g) \sim (b_8, \infty)$. In order to calculate the next value $\overline{g}$, we take the input datum $(b_7 + \epsilon, g_0)$ and take the limit $\epsilon \to 0$. Then

$$\begin{aligned} g &= \frac{b_3b_4}{\epsilon g_0} \frac{(b_7 - b_5/q)(b_7 - b_6/q)}{b_7 - b_8}, \\ f - b_8 &= \epsilon \left(\frac{g_0b_8(b_7 - b_8)(b_3 + b_4 - b_1 - b_2)}{b_3b_4(b_7 - b_5/q)(b_7 - b_6/q)} - \frac{b_8}{b_7} \right) + O(\epsilon^2) \end{aligned}$$

($O(\epsilon^2)$ means the usual symbol of Landau). We see that, even though $g = \infty$, $\overline{g}$ is finite, and moreover the input value g_0 can be recovered from it. The secret is the fact that $g(f - b_8)$ remains finite when the infinity appears in this process. So let us introduce the coordinate $h = g(f - b_8)$ and construct a surface by gluing $\mathbb{C}^2 \cup \mathbb{C}^2 = (f, 1/h) \cup (1/g, h)$ via the relation $(f - b_8)/h = 1/g$. This is nothing but the operation of blowing-up. The equation then makes sense on this surface.

Let us construct the space of initial conditions for q-P_{VI}. For this purpose let us first compactify $\mathbb{C}^2$ and take $\mathbb{P}^1 \times \mathbb{P}^1$. Writing $b = (b_1, \ldots, b_8)$, let X_b be the surface obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ with the center (b_7, ∞), (b_8, ∞), (∞, b_3), (∞, b_4), and $(b_5, 0)$, $(b_6, 0)$, $(0, b_1)$, $(0, b_2)$.



Notice that the positions of these eight points are not arbitrary. There are four pairs of points, each of which lie on a line D_i ($i = 0, \ldots, 3$). Two of the lines have bidegree $(1, 0)$ and the other two have bidegree $(0, 1)$. The meaning of these lines is that they are pole divisors of a 2-form which is invariant under the mapping q-P_{VI}. Indeed, let $\omega = \dfrac{df \wedge dg}{fg}$. Then it is easy to see that

$$\frac{df \wedge dg}{fg} = -\frac{df \wedge d\overline{g}}{f\overline{g}} = \frac{d\overline{f} \wedge d\overline{g}}{\overline{f}\,\overline{g}},$$

where we regard t as constant. The intersection matrix for these divisors D_i is the generalized Cartan matrix of type $A_3^{(1)}$ multiplied by -1. For this reason, we call X_b an $A_3^{(1)}$-surface. The merit of introducing X_b is that q-P_{VI} gives (not just a birational but) a one-to-one correspondence between X_b and $X_{\overline{b}}$.

Proposition 1 *The sixth Painlevé q-difference equation gives an isomorphism from X_b to $X_{\overline{b}}$.*

$\mathcal{F} \cdot \mathcal{G} = c_1(\mathcal{F}) \cdot c_1(\mathcal{G}) \in \mathbb{Z}$ the intersection index of divisors or divisor classes on X, and by $(\mathcal{F}^2)$ the self-intersection index. In particular, for effective classes $\mathcal{F} = [F]$ and $\mathcal{G} = [G]$ such that F and G have no common irreducible components, we have

$$\mathcal{F} \cdot \mathcal{G} = \sum_{x \in X} \dim {}_k\mathcal{O}_{X,x}/(f_x, g_x) \geq 0,$$

where $f_x = 0$ (resp. $g_x = 0$) define F (resp. G) locally near x.

The type of root system is denoted by the symbol R. In this paper, R is assumed to be symmetrizable.

- $\Pi = \Pi(R) = \{\alpha_1, \ldots \alpha_l\}$: the root basis, $\Pi^\vee = \Pi^\vee(R) = \{\alpha_1^\vee, \ldots \alpha_l^\vee\}$: the coroot basis,
- $(V, \Pi, \Pi^\vee)$: a realization of R ($\Pi \subset V^*$, $\Pi^\vee \subset V$), $\nu : V \to V^*$: the isomorphism induced from the bilinear form,
- $Q = Q(R) = \sum_{i=1}^l \mathbb{Z}\alpha_i$: the root lattice,
- $W = W(R) \subset GL(V^*)$: the Weyl group,
- $\Delta = \Delta(R)$: the set of roots, Δ_+: the set of positive roots, Δ^{re}: the set of real roots, Δ^{im}: the set of imaginary roots.

In the case of affine type, we enumerate the roots by $i = 0, \ldots, l-1$ for convenience.

2 From discrete equations to surface theory

The study of discrete Painlevé equations is currently the focus of an intensive activity. These discrete Painlevé systems are of interest, as they share many properties in common with the differential ones, including the affine Weyl group symmetry, the Lax pair, Hirota's bilinear form and the special solutions. We can add the simplicity of their algebraic and geometric structures to the reasons why we study discrete systems.

Introduction of the singularity confinement method brought forth intriguing developments [22]. Rational mappings of a discrete equation may have poles which give rise to infinities at particular input values of solutions. The singularity confinement demands that after a finite iteration of steps these infinities disappear and memory of the input data be recovered [9]. We can regard it as a discrete counterpart of Painlevé property. In this section we review briefly the singularity confinement method taking the sixth q-Painlevé equation (q-P_{VI}) as an example, and give a geometric interpretation to it in non-technical terms.

Originally q-P_{VI} was derived from the connection preserving deformation of a linear q-difference equation [13]. It is the following system of nonlinear q-difference equations

$$q\text{-}P_{VI}: \quad \frac{f(qt)f(t)}{b_7 b_8} = \frac{g(qt) - qb_1 t}{g(qt) - b_3}\frac{g(qt) - qb_2 t}{g(qt) - b_4}, \tag{7}$$

$$\frac{g(qt)g(t)}{b_3 b_4} = \frac{f(t) - b_5 t}{f(t) - b_7}\frac{f(t) - b_6 t}{f(t) - b_8}. \tag{8}$$

Here b_i's are constants subject to the constraint $qb_1b_2/b_3b_4 = b_5b_6/b_7b_8$. Provided that $|q| \neq 1$, the behavior of f and g near $t = 0$ determines the values of $f(t)$ and $g(t)$ for arbitrary $t \in \mathbb{C} \setminus \{0\}$ because the calculation of $f(t)$ and $g(t)$ can be reduced to the values of $f(q^n t)$, $g(q^n t)$ or $f(q^{-n}t)$, $g(q^{-n}t)$. Here we view (7),(8) as a chain of birational mappings $\cdots \mapsto (\underline{f}, \underline{g}) \mapsto (f, g) \mapsto (\overline{f}, \overline{g}) \mapsto \cdots$, where $\underline{f} = f(q^{-1}t)$, $\overline{f} = f(qt)$, and so forth. Each step is an automorphism of the field of rational functions $\mathbb{C}(f, g)$. To be precise, since the system is not autonomous we should consider the mapping

$$\begin{pmatrix} b_1 t & b_2 t & b_3 & b_4 \\ b_5 t & b_6 t & b_7 & b_8 \end{pmatrix}; f, g \Bigg) \mapsto \begin{pmatrix} qb_1 t & qb_2 t & b_3 & b_4 \\ qb_5 t & qb_6 t & b_7 & b_8 \end{pmatrix}; \overline{f}, \overline{g} \Bigg).$$

In Section 2 we motivate our construction on the example of a q-analog of the sixth Painlevé equation. We take an example from discrete systems because they seem to be less known as compared to the differential case, and also because they are easier to describe. This section is meant to be an introduction to our approach.

From the next Section 3, we forget about the differential/difference equations for a while and concentrate on surface theory. In Section 3 we recall known results about rational surfaces. We also mention about root systems and the Weyl groups. In Section 4, we classify the surfaces and the anti-canonical divisors which satisfy our conditions.

For us, of particular significance is the Cremona action. We introduce in Section 4 Cremona isometries as certain automorphisms of $\mathrm{Pic}(X)$, and study them in detail in Section 6. In Section 5 we parametrize our surfaces and construct a family of them. We then realize the Cremona action as birational transformations of this family in Section 6.

In Section 7 we construct discrete Painlevé equations in terms of the Cremona action. We obtain the Painlevé differential equations by degenerating surfaces. We obtain, in particular, an "elliptic" difference equation, which seems to be the most generic one (see § 7, **4**).

Notation and conventions

Throughout this paper the base field is $\mathbb{C}$. We use the following symbols.

- X: a smooth projective surface,
- $\mathcal{K} = \mathcal{K}_X$: the canonical divisor class on X,
- $b_i = \mathrm{rank}\ H^i(X, \mathbb{Z})$: the Betti number,
- $h^{p,q} = \dim H^q(X, \Omega^p)$: the Hodge number,
- $q = q(X) = h^{0,1}$: the irregularity,
- $p_g = p_g(X) = h^{0,2}$: the geometric genus,
- $c_i = c_i(X)$: the Chern classes,
- $\chi(\mathcal{F}) := \sum (-1)^i \dim H^i(X, \mathcal{F})$: the Euler characteristic of $\mathcal{F}$,
- $\pi(C) = \frac{1}{2}((C^2) + \mathcal{K}_X \cdot C) + 1$: the virtual genus of a curve C.

We have $b_i = \sum_{p+q=i} h^{p,q}$, $h^{p,q} = h^{q,p} = h^{2-p,2-q}$. They are birational invariants. In the case of a rational surface X we have $q(X) = p(X) = 0$, and more generally, $H^0(X, l\mathcal{K}_X) = 0$ for $l \geq 1$.

By $\sim$ we denote the linear equivalence of divisors. The group of classes of divisors with respect to the linear (resp. numerical) equivalence is denoted by $\mathrm{Pic}(X)$ (resp. $\mathrm{Num}(X)$). For a rational surface X we have $\mathrm{Pic}(X) = \mathrm{Num}(X)$. $\mathrm{Pic}(X)$ is also regarded as the group of invertible sheaves $H^1(X, \mathcal{O}^*)$. We write operations on invertible sheaves additively.

For a divisor D or a divisor class $\mathcal{D} = [D]$, we denote by $|D| = |\mathcal{D}|$ the set of all positive divisors C on X such that $C \sim D$. It is clear that $\dim |D| = \dim H^0(X, \mathcal{D}) - 1$. We call an element of $|-\mathcal{K}_X|$ an anti-canonical divisor.

A component of a divisor is an irreducible component of its support. We denote by $\mathrm{Comp}(D)$ the set of components of D. We use the symbol $\mathrm{Comp}(D)$ to denote the set of classes of components in $\mathrm{Pic}(X)$, when there is no fear of confusion. For a divisor $D = \sum m_i D_i$, we write $D_{red} = \sum D_i$.

Let $\mathcal{F}$ be a divisor class. We say $\mathcal{F}$ is *effective* if $|\mathcal{F}| \neq \emptyset$, *numerically effective* if $\mathcal{F} \cdot [C] \geq 0$ holds for any irreducible curve C, and *irreducible* if there exists an irreducible curve $C \in |\mathcal{F}|$. We denote by

a) there are no infinitely near points, b) no three points are collinear, c) no six points lie on a conic, and d) in the case $m = 8$, these points do not lie on a cubic with a singular point at one of the points. We consider the case of nine points in more degenerate configurations. Generalizations of the del Pezzo surface are studied, in particular in connection with the Weyl group, by P. du Val [6], M. Nagata [17], Y. Manin [16], M. Demazure [3] E. Looijenga [15] and B. Harbourne [11]. Among others, Looijenga [15] gave a detailed analysis for degenerate cases of what we call 'multiplicative type' in this paper (§ 4). We shall closely follow his method and consider all cases of elliptic, multiplicative and additive types.

In the Picard group $\mathrm{Pic}(X)$, the orthogonal complement of D (with respect to the intersection pairing) becomes a root lattice of type $E_8^{(1)}$. The latter has two interesting root subsystems of affine type, R and $R^\perp$. The root lattice of R is the span $Q(R) = \sum \mathbb{Z}[D_i]$ of the classes of irreducible components D_i of D, and that of $R^\perp$ is its orthogonal complement $Q(R^\perp)$. We classify all possible surfaces X by classifying R. The Weyl group of $R^\perp$ acts as automorphisms of $\mathrm{Pic}(X)$ preserving the above structure. We call them the Cremona isometries. (For a more precise definition, see § 4.) This action can be lifted further to a birational action of the (families of) surfaces X. We then show that the translation part of the Weyl group gives rise to discrete Painlevé equations, whereas the whole group acts as their group of symmetries (Bäcklund transformations).

We also obtain as the degenerate cases the space of initial conditions of the six Painlevé differential equations. More precisely, the latter is the noncompact surface obtained by removing the anti-canonical divisor D (the vertical leaves) from X.

The present geometric setting makes transparent algebraic properties of the Painlevé equations besides the Bäcklund transformations. It is known that there exist special solutions of Riccati type when the parameters lie on the reflection hyperplanes. From our standpoint, this condition is interpreted as that of the existence of an effective root ($\mathbb{P}^1$ with self-intersection -2) which is left invariant under the flow of the differential equation. The Painlevé equations have also the differential equation for elliptic functions as a limit. For example, if $a_0 + a_1 = 0$ in (2)–(4) then the equation can be solved by quadratures using elliptic functions. This corresponds to a degeneration of the space of initial conditions into rational elliptic surfaces.

In this paper our treatment of the Painlevé differential equations is somewhat indirect, in that they appear as a limit of the translation part of the affine symmetry. A more direct approach based on the Kodaira-Spencer deformation theory is being developed in a recent work of H. Umemura and M.-H. Saito. Their work can be viewed as a further extension of the framework of the present paper.

5. Let us mention further related problems.

A natural question that comes to mind is whether our approach is related to monodromy preserving deformations. We imagine that there may be a correspondence between some invariants of differential equations and the family $\mathcal{X}$ in this paper.

Another problem is an extention to the multi-variable cases. The Painlevé equations have Cremona transformations as their origin. We believe that a similar correspondence between birational geometry and integrable systems exists in higher dimensional situations as well, though algebraic geometry becomes more difficult. Theory of generalized del Pezzo varieties [5] may give us some hints to this idea.

A different approach to the Painlevé equations also based on the affine symmetry and discrete systems has been developed by M. Noumi and Y. Yamada [18]. They constructed representations of affine Weyl groups on the field of rational functions and found a new presentation of the Painlevé equations. This formulation makes the symmetry apparent and allows for straightforward extention to the higher dimensional case. Our approach offers a geometric characterization of various structures of the Painlevé equations, while their theory provides an effective algorithmic tool for studying them.

6. The text is organized as follows.

of P_{II},

$$y' = y^2 - 2z + s, \tag{2}$$

$$z' = a_1 - 2yz, \tag{3}$$

$$s' = a_1 + a_0, \tag{4}$$

where a_0, a_1 are parameters and the prime denotes derivation. (It is possible to normalize $a_1 + a_0$ to be 1 as is usually done. However we find it convenient to keep this redundancy for describing the symmetry.) Let $\mathbb{C}(a_0, a_1, s, y, z)$ be the differential field defined by the derivation rule (2)–(4) along with $a_0' = a_1' = 0$. Then the following maps are automorphisms of differential fields on $\mathbb{C}(a_0, a_1, s, y, z)$.

$$w_1 \ : \ a_1 \mapsto -a_1, \quad a_0 \mapsto a_0 + 2a_1, \ s \mapsto s, \quad y \mapsto y - \frac{a_1}{z}, \quad z \mapsto z, \tag{5}$$

$$\sigma \ : \ a_1 \mapsto a_0, \quad a_0 \mapsto a_1, \ s \mapsto s, \quad y \mapsto y, \quad z \mapsto y^2 + s - z. \tag{6}$$

Moreover w_1, σ generate the extended affine Weyl group of type $A_1^{(1)}$.

In the literature, such transformations of differential equations are often called Bäcklund transformations. Similar actions of affine Weyl groups are also available for the other Painlevé equations as well. We quote the result from [19, 20] in the following table. (See the page 17 for the notation $(2A_1)^{(1)}$.)

	P_I	P_{II}	P_{III}	P_{IV}	P_V	P_{VI}
vertical leaves	$E_8^{(1)}$	$E_7^{(1)}$	$D_6^{(1)}$	$E_6^{(1)}$	$D_5^{(1)}$	$D_4^{(1)}$
symmetry	-	$A_1^{(1)}$	$(2A_1)^{(1)}$	$A_2^{(1)}$	$A_3^{(1)}$	$D_4^{(1)}$

Table 1. Configuration of vertical leaves and symmetry

For comparison, we have included in the table the type of the configuration of vertical leaves mentioned above. From the list we observe that the root systems of the configurations and the symmetries are orthogonal complement to each other in a common root lattice of type $E_8^{(1)}$. We will come back to this point shortly.

In recent years, difference versions of the Painlevé equations have been studied extensively by several authors [9, 10, 13, 22]. These are discrete mappings, which in an appropriate continuous limit reduce to the Painlevé differential equations, and more significantly, are 'integrable' in certain sense. (For difference equations it is not so clear how to formulate the notion of integrability. An important step in this direction is the singularity confinement test proposed by A. Ramani and B. Grammaticos as a difference counterpart of the Painlevé property. We shall touch upon this method from our viewpoint in § 2.) It has gradually become clear that, in the discrete setting, difference equations and their symmetries should be treated on an equal footing. In fact, many of the discrete Painlevé equations found by Ramani and Grammaticos [23] turned out to be the translation part of Okamoto's affine Weyl group symmetries.

4. The result of Takano et al. suggests the possibility of a purely geometric framework which enables us to understand various features of the Painlevé equations. The present article is an attempt toward this goal. Unlike Okamoto, who started from differential equations and obtained the space of initial conditions, we will start from varieties with certain properties and end up with equations.

The main objects of our study are compact rational surfaces X which admit a unique anti-canonical divisor D of a special kind (called 'canonical type' in [1]). Such a surface X can be obtained by blowing up the projective plane $\mathbb{P}^2$ centered at nine points. Surfaces obtained by blowing up $\mathbb{P}^2$ are deeply studied from moduli theoretical viewpoint, that is, as theory of invariants of finite point sets in a projective plane [5]. A well-studied example is the del Pezzo surface which is a smooth rational surface with ample anti-canonical divisor class. This corresponds to m points in $\mathbb{P}^2$ $(1 \leq m \leq 8)$ satisfying the following conditions:

1 Introduction

1. As is well known, the Painlevé differential equations have two independent origins. The first is due to P. Painlevé and coworkers [21, 8]. In their original work, the Painlevé equations were discovered as differential equations whose solutions do not have movable singularities other than poles. This feature, now called the Painlevé property, has become a guiding principle in the theory of integrable systems. The second is the monodromy preserving deformation of linear differential equations [7, 12]. Given a system of linear differential equations, the problem is to deform it in such a way that the monodromy group stays constant. Here the Painlevé equations arise as the condition for the invariance of the monodromy, written in terms of the coefficients of the underlying linear system.

Our aim in this paper is to characterize the Painlevé equations from yet another point of view: theory of rational surfaces. Before entering into the detailed discussions, we begin by reviewing previous works which directly motivated the present study. In the following, we shall follow the standard notation $P_I, \cdots, P_{VI}$ to represent the six types of the Painlevé equations.

2. In the work [19], K. Okamoto introduced the notion of *space of initial conditions* for the Painlevé equations. The basic idea is as follows. Consider one of the Painlevé equations written in the form

$$\frac{dy}{dt} = P(t, y, z), \quad \frac{dz}{dt} = Q(t, y, z), \tag{1}$$

where P and Q are polynomials in y, z, depending rationally on t with no poles in a domain $B \subset \mathbb{C}$ (for example, $B = \mathbb{C}\backslash\{0, 1\}$ for P_{VI}). Very naïvely the space of initial conditions for (1) is $\mathbb{C}^2$, since its solutions holomorphic at a fixed base point $t_0 \in B$ are uniquely specified by initial values $(y(t_0), z(t_0))$. However we must also take into account solutions which have a pole at t_0. For that purpose we compactify $\mathbb{C}^2$. After this it can happen that many solutions pass through a common point, and we need to separate them. This requires the operation of blowing-up the space.

Starting with a trivial bundle $\mathcal{Q} = (\Sigma \times B, \pi, B)$ (where Σ is a compactification of $\mathbb{C}^2$) and repeating the above process, Okamoto constructed for each P_I–P_{VI} a fiber bundle $\mathcal{P} = (E, \pi, B)$ with the following properties: There is a foliation $\mathcal{F}$ on $\mathcal{P}$ extending the Painlevé equations on $\mathcal{Q}$ such that

a) Each leaf of $\mathcal{F}$ intersects with each fiber transversally;

b) Each path γ on B can be lifted to a leaf γ_p that runs through a given point $p \in \pi^{-1}(\gamma(0))$;

c) $\pi|_{\gamma_p} : \gamma_p \to B$ is surjective and γ_p is a covering space of B by π.

In this situation, each fiber of $\mathcal{P}$ is called the space of initial conditions. Note that any two fibers are isomorphic by (b) and (c). The space E is not compact, since at the last stage of the construction it is necessary to remove *vertical leaves* which are the pole divisor of the symplectic form of the Hamiltonian structure for the Painlevé equations. Remarkably, the configurations of irreducible components of vertical leaves are described by Dynkin diagrams [19], in a way very similar to Kodaira's classification of singular fibers of elliptic surfaces (see Table 1 below).

Subsequently, the study of the space of initial conditions was pursued further by K. Takano et al. [24]. In particular, they obtained the following uniqueness result: There exists only one Hamiltonian structure (that of the Painlevé equation) which is holomorphic on E and extends rationally to its compactification $\overline{E}$. In other words, the pair $(E, \overline{E})$ carries all information about the Painlevé equations.

3. Another aspect of the theory of Painlevé equations relevant to us is their symmetries under affine Weyl groups. This subject was also initiated and developed by Okamoto [20]. Let us illustrate it on the example

Rational surfaces associated with affine root systems and geometry of the Painlevé equations

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May 17, 1999

Abstract

We present a geometric approach to the theory of Painlevé equations based on rational surfaces. Our starting point is a compact smooth rational surface X which has a unique anti-canonical divisor D of canonical type. We classify all such surfaces X. To each X, there corresponds a root subsystems of $E_8^{(1)}$ inside the Picard lattice of X. We realize the action of the corresponding affine Weyl group as the Cremona action on a family of these surfaces. We show that the translation part of the affine Weyl group gives rise to discrete Painlevé equations, and that the above action constitutes their group of symmetries by Bäcklund transformations. The six Painlevé differential equations appear as degenerate cases of this construction. In the latter context, X is Okamoto's space of initial conditions and D is the pole divisor of the symplectic form defining the Hamiltonian structure.

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