Let us choose $x_1, x_2, x_3 \in \Gamma \subset V$ with $d(x_i, x_j) \leq 1$. Let Δ be the triangle in V spanned by x_i. Then there exists a constant C such that

(1) $F_2(\Delta)$ is contractible in C-thickning of γ_{x_i,x_j},

(2) $f(F_2(\Delta) \cap \Gamma)$ is contained in C neighbourhood of $f^{-1}(f(x_1))$,

Thus any simplex $x \in \Delta \in V$ is contractible in C thickning of $f^{-1}(f(x_1))$. With this property, we can make CBC as the case of 4.3. This completes the proof of 6.4.

Notice that we donot know whether $F_2|S$ is Lipschitz or not. However if we symmetrize for $h : V \mapsto \mathbb{R}^n$, the result $h'|S'$ becomes proper Lipschitz and h' is properly homotopic to the identity.

When we symmetrize using F_1 and F_2, we get a (Γ, S) APL $G : V \mapsto \mathbb{R}^n$, which satisfies:

$$(2)' \quad d(G(x), G(y)) \leq \exp(C|f(x)|_S),$$

$$(3)' \quad d(G(f(x)), G(f(y))) \leq C.$$

Moreover $G|B_\epsilon = exp$, $G(V \backslash B_\epsilon) \subset \mathbb{R}^n \backslash B_\epsilon$.

Thus combined with 1.6 and §5, we have completed to construct a family of (Γ, H) APL for fundamental groups of almst non positively curved manifolds. This is enough to conclude that any cohomology element of Γ can be represented by a Lipschitz class, using the results of the previous sections. This completes the proof of 6.2.

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Department of Mathematics
Faculty of Sciences
Kyoto University
Kyoto, 606-01, Japan

Proposition 6.1([FY]). *Let M be an almost non positively curved manifold, with $K \geq -1$. Then the following holds.*

(1) M is of $K(\Gamma, 1)$ type,

(2) there exists an orbifold fiber bundle $\mathbb{H}/\Lambda \to \tilde{M} \to X/H$ where X is a simply connected complete manifold of non positive curvature, and H acts on X proper discontinuously and the quotient is compact. i.e. X/H is an orbifold non positively curved manifold, $\mathbb{H}/\Lambda$ is a nil manifold. Moreover there is a short exact sequence $0 \to \Lambda \to \Gamma \to H \to 0$.

From the previous results, it follows that Γ has a rank n fiber structure with fiber almost nilpotent.

Corollary 6.2. *Γ satisfies the Novikov conjecture.*

Let Γ and S be discrete metric spaces, and consider a fiber structure $f : \Gamma \mapsto S$ as above.

Let us say that S has an *expanding section* into Γ if there exist a constant C and a section $s : S \mapsto \Gamma$ of f such that s^{-1} is proper Lipschitz. Moreover for any $x \in S$ and a fixed $x_0 \in S$,

$$C^{-1}d(x, x_0) \leq d(s(x), s(x_0)) \leq Cd(x, x_0).$$

Example 6.3. (1) Let us consider a surjective group homomorphism $f : \Gamma \mapsto H$ for finitely generated groups. In this case we always take identities as base points. By assigning generating sets, Γ and H become metric spaces.

Let $h_1, \ldots, h_n$ be a generating set of H and $g_1, \ldots, g_n$ be their lifts. By adding another $g_{n+1}, \ldots$, we get a generating set for Γ. For any $h \in H$, let us choose a geodesic γ_h from the base to h in the Cayley graph. Then for γ_h, a lift γ_g corresponds. Thus we get a section of f, $s : H \mapsto \Gamma$. We claim that s satisfies the above inequalities. Let C be a constant with $|f(g_i)| \leq C$ for any generators g_i. Then the first follows for this constant. A similar argument works for the second inequality.

(2) Let $0 \to H \to \Gamma \to \Lambda \to 0$ be a short exact sequence of finitely generated groups. As a generating set of Γ, we take ones of H and Λ respectively. Let $s : \Lambda \mapsto \Gamma$ be as above and $\Lambda' = s(\Lambda)$. There is a natural fiber structure $f' : \Gamma \mapsto \Lambda'$ by $f'(H\lambda') = \lambda'$ on which the quasi isometry $I_{\lambda,\lambda'}$ brings $h\lambda'$ to $h\lambda$ for $s, s' \in \Lambda'$.

Let $f : \Gamma \mapsto S$ be a fiber structure with an expanding section $s : S \mapsto \Gamma$.

Proposition 6.4. *(1) There exists $F_1 : V \mapsto V$ a (Γ, S) contraction.*

(2) There exists CBC $F_2 : V \mapsto V$ with respect to $(S', d_{S'})$.

Proof. (1) is the same as 3.2.

(2) Using convexity of S, there is λ contracting on the large scale $F_2' : S \mapsto S$. Let us put $S' = s(S)$ and $F_2 : S' \mapsto S'$ by $F_2 = s \circ F_2' \circ s^{-1}$. Let $F_2 : f^{-1}(1) \mapsto f^{-1}(1)$ be the contracting map on the large scale(see §3), and using quasi isometries, let us put $F_2|f^{-1}(s) = I_{F_2(s),1} \circ F_2 \circ I_{1,s}$. Thus we get $F_2 : \Gamma \mapsto \Gamma$.

If $d(x, y) \leq 1$, there exists a quasi path $\gamma_{x,y}$ in $f^{-1}(f(x))$ between $F_2(x)$ and $F_2(y)$. In V, we may regard $\gamma_{x,y}$ as a continuous path between $F_2(x)$ and $F_2(y)$.

for $k = 0, 1, 2$. As $k = 2$ is the trivial case, let us consider $k = 1$. Let us recall that for every $a_q^p, a_s^r \in S_m$, there exists $z_1, z_2 \in S_{m-1}$ such that

$$a_q^p a_s^r = z_1 a_s^r a_q^p = a_s^r a_q^p z_2.$$

Let M be the maximun of such $|z_1|_{S_{m-1}}, |z_2|_{S_{m-1}}, \dots$ for all m.

Let us take $a_m \in S_m \subset Z_i$ and $a_l \in S_{m-1}$, where a_m, a_l are elements of the generator. Let $a_m^n a_l = z_n a_l a_m^n$. We estimate $|z|_{S_{m-1}}$. Let $a_m a_l = z_0 a_l a_m$, $|z_0|_{S_{m-1}} \leq M$. Then

$$a_m a_m a_l = a_m z_0 a_l a_m = z_1 z_0 a_l a_m^2$$
$$|z_1 z_0|_{m-1} \leq M^2 + M \leq 2M^2.$$

By rewriting $z_1 z_0$ by z_1, we have $a_m^2 a_l = z_1 a_l a_m^2$, $|z_1|_{m-1} \leq 2M^2$. Similarly we have $(a_m)^n a_l = z a_l (a_m)^n = z_1^0 \dots z_{s_0}^0 a_l (a_m)^n$ where $z_i^0 \in S_{m-1}$, $s_0 \leq (2M)^n$.

This estimate implies that there exists a constant C such that $exp\{Cd^2\} \geq d_f$.

Moreover for $x, y \in \Gamma$, $d(x, y) \leq 1$, $d_S(f(x), f(y)) \leq exp\{C|x|_f\}$, $|y|_f \leq |x|_f + 1$. By the same step, we obtain a part of fiber structure,

$$S_0 \subset \dots S_{n_1} \subset S_{n_1+1} \subset \dots \subset S_{n_1+n_2-1} \quad S_{n_1+n_2-1} = Z_2.$$

Repeating this process, we conclude that Γ admits rank n hyperbolic fiber structure for some N. This completes the proof.

§6 The general case

6A. Non sectional base.

Let Γ and S be discrete metric spaces. Let us remove the assumption that S is a subset of Γ. Let $f : \Gamma \mapsto S$ be a fiber structure with base points $f(s_0) = x_0$. We will use the following modified formulation of the fiber structure; instead of (2) and (4) in 2.1, we have the following: There exists a constant C such that for any $sx \in f^{-1}(s)$, $sxy \in f^{-1}(s')$, $\quad |y| \leq 1$,

$$(2)' \quad d(I_{s',s}(sx), sxy) \leq \exp(C|f(x)|_S),$$

$$(4)' \quad d(s, s') \leq C.$$

The main application to introduce this is to treat an extension of groups. The above assumption is satisfied in this case. In particular we will show the following: Suppose Γ is a fundamental group of a compact aspherical manifold, and admits (rank n) H fiber structure in the generalized sense, which will be defined below. Then we show that any $\gamma \in H^*(\Gamma; \mathbb{R})$ is a proper Lipschitz class.

As an application, we consider fundamental groups of almost non positively curved manifolds. A compact Riemannian manifold is said to be *almost non positive* if its sectional curvature is less than ϵ where ϵ only depends on diameter of the manifold.

Let us recall fundamental results on such manifolds from [FY].

Proof. Let us fix a generator $\{s_1, \dots, s_m\}$ for Γ/T. Let $\{f_i : S_i \mapsto S_{i-1}\}_{i=0}^n$ be an rank n hyperbolic fiber structure for Γ/T. We choose representatives for each s_i in Γ and regard them as an elements of Γ. Let $\Gamma = \bar{\Gamma}t_1 \cup \dots \bar{\Gamma}t_l$ where $t_i \in T$ and $\bar{\Gamma}$ is the subgroup of Γ spanned by $\{s_1, \dots, s_m\}$. As a generator for Γ, we choose $\{t_1, \dots, t_l\} \cup \{s_1, \dots, s_m\}$.

We claim that $S_0 \subset \cdots \subset S_n \subset \Gamma$ forms $n+1$ fiber structure, where $f_{n+1}^{-1}(g) = \{gt_i\}_{i=0}^l$.

If $|\bar{x} - \bar{y}| \leq C$ on Γ/T, then $|\bar{x} - \bar{y}| \leq C + 1$ on Γ. Because $\pi : \Gamma \mapsto \Gamma/T$ is distance decreasing, if x and y are near in Γ, then $\pi(x)$ and $\pi(y)$ are near in Γ/T. From these remarks, it follows that the conditions of $n+1$ fiber structure are all satisfied. This completes the proof.

We need another lemmas. Let Γ be a finitely generated nilpotent group and $Z_{-1} = e \subset Z_0 \subset \dots Z_n = \Gamma$ be the upper central series of Γ. (i.e. Z_{i+1}/Z_i is the center of Γ/Z_i.)

Lemma 5.6. (1) Let $T \subset \Gamma$ be torsion subgroup of finitely generated nilpotent group Γ. Then T is normal and finite.

(2) If Γ is torsion free nilpotent, then Γ/Z_i is also torsion free.

Proof of 5.4. By 5.5 and 5.6, we may assume that Γ is torsion free.

Let us choose isomorphisms, $Z_i/Z_{i-1} = \oplus^{n_i}\mathbb{Z}$ and take $\{a_1^i, \dots, a_{n_i}^i; a_j^i \in \Gamma\}$ corresponding to the standard generator of $\oplus^{n_i}\mathbb{Z}$. As a generator of Γ, we choose $\{a_j^i; 0 \leq i \leq n, 1 \leq j \leq n_i\}$. Put $S_0 = \{(a_1^0)^{a_1} \dots (a_{n_0}^0)^{a_{n_0}}; a_{n_i} \in \Gamma\} = Z_0$,

$$S_1 = \{s_0(a_1^1)^i(a_2^1)^j; s_0 \in S_0, i, j \in \mathbb{Z}\}, \quad S_2 = \{s_1(a_3^1)^i; s_1 \in S_1, i \in \mathbb{Z}\}$$
$$\dots$$
$$S_i = \{s_{i-1}(a_{i+1}^1)^l; l \in \mathbb{Z}\} \quad 3 \leq i \leq n_1 - 1, \quad S_{n_1} = \{s_{n_1-1}(a_1^2)^l; l \in \mathbb{Z}\}$$
$$\dots$$
$$S_i = \{s_{i-1}(a_{i-n_1+1}^2)^l; l \in \mathbb{Z}\} \quad n_1 \leq i \leq n_1 + n_2 - 1 \quad \dots \quad S_N = \Gamma.$$

Let $f_i : S_i \mapsto S_{i-1} \quad N \geq 1$ be the canonical map. The definition (1) is automatically satisfied.

Notice that for every $a_l^i, a_k^i \in Z_i$, there exists $z_{l,k}^i \in Z_{i-1}$ such that $a_l^i a_k^i = z_{l,k}^i a_k^i a_l^i$. This case is essentially the same as the one of the Heisenberg group.

By the same step, we see

$$S_0 \subset S_1 \subset \cdots \subset S_{n_1-1} \quad S_i = \{s_{i-1}a_{i+1}^1; l \in \mathbb{Z}\}$$

admits a part of rank N hyperbolic fiber structure.($S_{n_1-1} = Z_1$)

So far, the situation is easy because Z_0 is the center of Γ. We want to apply the step above replacing $Z_0 \subset Z_1$ by $Z_1 \subset Z_2$. Because Z_1 is not abelian, in general, the direct application makes it difficult. In place we consider in Γ/Z_0. In this space, Z_1/Z_0 is the center and we can apply the same step as above. Notice that the quotient map $\pi : \Gamma \mapsto \Gamma/Z_0$ is distance decreasing.

Let us recall $S_{n_1} = \{s_{n_1-1}(a_1^2)^i; i \in \mathbb{Z}\}$. Let $I_{s,s'}(s'a_1^2)^i) = sa_1^2$. The definition (2) is clearly satisfied. Assume

$$d(s_{n_1-1}(a_1^2)^n), d(s_{n_1-1}(a_1^2)^n a_l^k) \leq 1$$

§5 Higher rank case

5A. Generalization.

We have an immediate generalization of the fiber structure to the higher rank case.

A sequence of subsets $S_0 \subset \cdots \subset S_N = \Gamma$ is *rank n fiber structure* if for each i, there is a fiber structure $f_i : S_{i+1} \mapsto S_i$. In §5 we always assume that $S_0 \subset \Gamma$ is a quasi isometric embedding.

If S_0 and each fiber of f_i are hyperbolic nets, then it is called rank n *H fiber structure*.

Let us denote:

$$N_\Gamma(r_i, \dots, r_{n-1}) = \{x \in \Gamma : |x|_{f_{n-1}} \le r_{n-1}, \dots, |f_{i+1} \circ \dots f_{n-1}(x)|_{f_i} \le r_i\}$$

and its C neighbourhood in V by $N(r_i, \dots, r_{n-1}, C)$.

Let us say that $F_i : V \mapsto V$ is $(S_i, \dots, S_n = \Gamma)$ *contraction* if

(1) $F_i|S_{i-1} =$ identity,

(2) $F_i|S_{i+1}$ is (S_{i+1}, S_i) contraction,

(3) for any $R \ge 0$, there exists $N = N_\Gamma(r_i, \dots, r_{n-1})$ containing $B_R(1) \subset \Gamma$ such that $F_i(N) \subset\subset N$,

(3) F_i is properly homotopic to the identity.

Proposition 5.1. *Let $(S_0, \dots, S_n = \Gamma)$ be rank n H-fiber structure. Then for each i, there exists $(S_i, \dots, S_n = \Gamma)$ contraction $F_i : V \mapsto V$.*

The construction is the combination of 3.2 and 4.3.

Let us denote $\alpha_j(x) = A|x|_{f_j} + B$. Using the above map, we get $(S_0, \dots, S_n)$ APL in the following sense.

Corollary 5.2. *There exists a proper degree 1 map $G : V \mapsto \mathbb{R}^n$, constants A, B, C, $\delta > 0$ and $\mu > 0$ such that*

(1) for any i, it is (S_{i+1}, S_i) APL,

(2) for $x, y \in S_{i+1}$, $d(x, y) \le 1$, $d(G(x) - G(f_i(x)), G(y) - G(f_i(y))) \le C$,

(3) $d(G(f_j \circ \dots f_i(x)), G(f_j \circ \dots f_i(y))) \le \alpha_j(f_{j-1} \circ \dots f_i(x)) \dots \alpha_i(x)$ and

(4) $|G(x) - G(f_i(x))| \ge C|x|^\delta_{f_i}$,

(5) the angle between $G(S_i)$ and $(G - G \circ f_{i+1})(S_{i+1})$ is at least μ,

(6) $G|B_\epsilon = exp$, $G(V \backslash B_\epsilon) \subset \mathbb{R}^n \backslash B_\epsilon$.

Remark 5.3. In fact we have a strict bound $|G(x)| \ge C\Sigma_{i=1}^l |f_i \circ \dots f_l(x)|^\delta_f$ for $x \in S_l$.

By iterating the deformation in 3.7, we can construct a proper homotopy between G and another proepr Lipschitz map. The process for families of maps, is the same as $4B$, and we omit the repetetion. This completes the proof of 4.1.

Example 5.4. Every nilpotent group admits rank n hyperbolic fiber structure for some n.

Sublemma 5.5. Let $T \subset \Gamma$ be a normal finite subgroup of any discrete group Γ. If Γ/T admits an rank n hyperbolic fiber structure, then Γ itself admits an rank $n+1$ hyperbolic fiber structure.

At this stage we have reached to the same situation as the case $S = \mathbb{Z}$. Let us take a proper degree 1 map $h : V \mapsto \mathbb{R}^n$, $N = N_\Gamma(r, R, C)$ such that

(1) $N_\Gamma(r, R, C)$ is a (Γ, S) net in $N(r, R, C)$, i.e. $I_{s,s'} \circ f^{-1}(s') \cap N_\Gamma(r, R, C) = f^{-1}(s) \cap N_\Gamma(r, R, C)$,

(2) $h|N_\Gamma(r, R, C)(x) = h(f(x)) + h(I_{1,f(x)}(x))$,

(3) $h(V \backslash B_\epsilon) \subset \mathbb{R}^n \backslash B_\epsilon$.

Proposition 4.6. *There exists a (Γ, S) APL $G : V \mapsto \mathbb{R}^n$ properly homotopic to h.*

Proof. Using 3.8, let us symmetrize $h|S$, by taking the base set $N \cap S$, and obtain $h'|S$ which is basewise proper Lipschitz. Notice that $h|S$ satisfies 3.6 (3). By the property (2) above, we can extend it as $h' : \Gamma \mapsto \mathbb{R}^n$, since any point $s \in S$ passes through another $s' \in N \cap S$ uniquely. Moreover using a (Γ, S) contraction F_1, we can symmetrize each fiber direction. Thus we get $h|\Gamma$, which is a restriction of the desired one. We may assume that $N_\Gamma(r, R, C)$ is a deformation retract of V. Thus at each step of deformations of the original h, we can extend to maps from V, since V is contractible. This completes the proof.

4B. Families.

We need to do the previous construction for families, namely a fiberwise proper Lipschitz map $V \times_\Gamma V \mapsto TM$. The method is parallel to the one in [CGM](see [K1]), so we only sketch the way to construct a *family* of (Γ, S) contraction $F_1 : V \times_\Gamma V \mapsto V \times_\Gamma V$ in the obvious sense.

Let us take a covering of M by sufficiently small open sets, and denote $\{Y_j\}$ for the lift of it to V. Thus we have a Γ covering of V.

Let Z be a large net in Γ and $F : Z \mapsto V$ be a fiberwise contracting from Z to V. Let us choose any Y_0 and put $F : Y_0 \times Z \mapsto Y_0 \times V$ by id $\times F$. Let us regard F as a *family of contraction.*

The essential step is to construct some metric d' on $Y_1 \times Z$ so that F itself is Lipschitz with respect to d'. More precisely one may take a smooth metric d' on $Y_1 \times V$ such that for any $y \in Y_1$, (1) $d'|y \times V = d_V$, (2) $F : (Y_0 \cap Y_1 \times Z, d') \mapsto (V, d_V)$ is Lipschitz.

From Lipschitz extension lemma 3.3, we get a fiberwise proper Lipschitz map $F : Y_1 \times V$.

Let us take Y_k such that we have already F on some subset $Y_k \backslash Y_k' \times V \subset Y_k \times V$, and try to extend it applying the previous argument. Notice that if $y \neq y' \in Y_k \backslash Y_k'$, then for $x \in \Gamma$, the distance between $F(y, x)$ and $F(y', x)$ may be far from each other. However there is a bound, $d(F(y, x), F(y', x)) \leq A|x|_f + B$ for some constants. This enables us to do the same process as the one variable case.

Using *families* of (Γ, S) contraction and CBC, we get a *family* of (Γ, S) APL $F : V \times_\Gamma V \mapsto TM$. Let us take a complementary bundle of TM, (i.e. $F \to M$, $TM \oplus F$ is trivial). In general the tangent space is non trivial, but by taking *suspension* with the universal covering space of F, one may reduce to the trivial case.

Thus combined with 3.7, we get 4.1 for H fiber structure.

Proposition 4.3. *Let $S \subset \Gamma$ admit H-fiber structure. Then there exists CBC $F_2 : V \mapsto V$.*

The construction of CBC $F_2 : V \mapsto V$ is as follows. First using hyperbolicity of S, we have a λ contraction on the large scale $F_2|S$ such that $F_2(S) \subset S$. For any $x \in \Gamma$, let us choose a geodesic in the fiber $\gamma_x \subset f^{-1}(f(x))$, and $p(x)$ be the map in 3.2, i.e. λ contracting point along γ_x. Then we define

$$F(x) = I_{F_2(f(x)),f(x)} \circ p(x).$$

By the definition of fiber structure, there exists a constant C, independent of λ such that for any $x, y \in \Gamma$, $d(x, y) = 1$, $F(x)$ and $F(y)$ is connected by a path $\gamma_{x,y}$ satisfying:

$$(1) \quad |\gamma_{x,y}| \leq C|x|_f, \quad (2) \quad |\gamma_{x,y}(t)|_f \geq C^{-1}|F(x)|_f - C.$$

Thus we have a map from 1 simplices of V to V which is concentraring. We want to extend this to higher dimensional simplices. First let us recall a notion, uniform contractibility.

A metric space is called *uniformly contractible* if R ball with any center is contractible in $s(R)$ ball with the same center, where $s : \mathbb{R}_+ \mapsto \mathbb{R}_+$.

Example 4.4. The universal covering space of a compact aspherical manifold is uniformly contractible. In fact there exists a constant C such that $s(r) = r + C$.

Proof of 4.4. Let us take a triangulation of M and $\{\sigma_l\}_l$ be its lift to V. Let $\cup_{1 \leq i \leq n} B_i$ be a partition of V into n subsets, where B_i is the subset of the simplices, each of which has distance more than Δ. $\cup_{1 \leq i \leq n} B_i$ is called Δ *cover*.

Let us take a subset $K \subset \{\sigma_l\}$ which contain $B(R)$. First we contract each simplex in $B_1 \cap K$ to a point in itself. When we try to contract $B_2 \cap K$ which extends the above, one can find a constant $C > 0$, only dependent on the triangulation, such that each simplex in $B_2 \cap K$ is contracted in C-thickning of $(B_1 \cup B_2) \cap K$. We proceed to construct inductively on $1 \leq i \leq n$, and finally we get a contraction of K in some thickning of itself, where the thickning constant is independent of K.

Proof of 4.3. The above shows that for any $x_1, x_2, x_3 \in \Gamma$, $d(x_i, x_j) = 1$, the triangle made by γ_{x_i,x_j} is contractible in the domain

$$\{x \in V : C^{-1}|x_1|_f^{\delta} - C \leq d_f(x, f(x)) \leq C\lambda|x_1|_f + C\}.$$

Using the above contraction, let us extend F_2 inductively on higher dimensional simplices from B_1 to B_n. The result $F_2 : V \mapsto V$ is CBC, properly homotopic to the identity. This completes the proof of 4.3.

Corollary 4.5. *Let X admit CBC. Then any proper map $h : X \mapsto \mathbb{R}^n$ is proper homotopic to another $h' : X \mapsto \mathbb{R}^n$ such that $h'|(S, d_S)$ is proper Lipschitz.*

Proof. Let $N = N(r, R, C) \in V$ be C neighbourhood of $N_\Gamma(r, R,) = \{x \in \Gamma : |f(x)|_S \leq r, |x|_f \leq R\}$. If we take these numbers sufficiently large, then it contains large ball, and $F_2(N) \subset\subset N$. Using N as the base set, we symmetrize h.

Lemma 3.8. *Let $h : V \mapsto \mathbb{R}^n$ has positive angle between $h(S)$ and $(h - h \circ f)(\Gamma)$. Suppose $h|\Gamma$ satisfies*

$$h(x) = h(I_{1,f(x)}(x)) + h(f(x)).$$

Then there exists (Γ, S) net Z such that h is properly homotopic to another, which is fiberwise proper Lipschitz on the large scale, and is so in the usual sense on Z, and satisfy (2), (4) and (5) in 3.6.

Proof. Let us choose a large neighbourhood L of $S \subset V$ such that

(1) $F(L)$ is strictly contained in L, and

(2) any $x \in V$ lies in L by finitely many times contraction by (Γ, S) contraction F.

We will say that L is the *base set* for the symmetrization.

Let $D_l = F^{-l}(L) \backslash F^{-(l-1)}(L)$. Then define G on D_l by

$$G(x) = \mu^{-l}(h(F^{l-1}(x)) - h(f(x)) + h(f(x)).$$

Then $G : V \mapsto \mathbb{R}^{n+1}$ has the desired properties.

Remark 3.9. There exists $C > 0$ such that $|G(x) - G(f(x))| \geq C|x|_f^\delta$, where $\delta = \mu^{-1}\lambda$.

Example 3.10. As the simplest case, we will treat $S = \mathbb{Z}$. Let $f : \Gamma \mapsto \mathbb{Z}$ be a fiber structure, where $\mathbb{Z}$ is the integer. As an example, we have a compact manifold M which admits a fibration $X \to M \to S^1$, where $K(X) \leq 0$.

Using contractibility of V, one can construct a proper degree 1 map $h : V \mapsto \mathbb{R}^{n+1}$ such that

(1) $h : \mathbb{Z} \mapsto 0 \times \mathbb{R} \subset \mathbb{R}^{n+1}$ is the standard embedding.

(2) $h : f^{-1}(1) \mapsto \mathbb{R}^n \times 0$.

(3) $h \circ I_{i+1,i}|(f^{-1}(a^i)) + (0, \ldots, 0, 1) = h|f^{-1}(a^{i+1})$.

In this case, we have $(\Gamma, \mathbb{Z})$ APL $G : V \mapsto \mathbb{R}^{n+1}$, properly homotopic to h with

(1) $G : \mathbb{Z} \mapsto \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R}$ is the standard embedding,

(2) $G|f^{-1}(a^i) : f^{-1}(a^i) \mapsto \mathbb{R}^n \times \{i\}$.

§4 H-fiber structure and the Novikov conjecture

As a generalization of the H fiber structure, we will formulate rank $n \geq 1$ fiber structure in §5. Let Γ be isomorphic to the fundamental group of compact aspherical manifold, and admits rank n H fiber structure. Our main aim in this and next sections is to produce the following theorem, using the preparation in §3:

Theorem 4.1. *Every $\gamma \in H^*(\Gamma; \mathbb{R})$ is a proper Lipschitz class.*

4A. Basewise contraction.

Definition 4.2. Let X be a geodesic space and $S \subset X$ be a subset with another metric d_S. $f : X \mapsto X$ is *concentrating and basewise contracting*(CBC) if

(1) $f|(S, d_S) : S \mapsto S$ is proper contracting on the large scale.

(2) For any $R \geq R_0$, there exists $N \subset X$ containing R ball $B_{x_0}(R)$ in X such that $f(N) \subset\subset N$.

(3) f is properly homotopic to the identity.

(4) there exists a net $Z_S \subset S$ such that $f|(Z_S, d_S)$ is proper contracting.

Definition 3.6. $G : V \mapsto \mathbb{R}^n$ proper degree 1 map is (Γ, S) *APL* if there exist constants A, B, C, $\delta > 0$, $\epsilon > 0$, and (Γ, S) net Z such that

(1) $G|Z$ is fiberwise and basewise proper C Lipschitz with respect to $|\quad|_{f(S)}$.
(2) For $x, y \in Z$, $d(x, y) \leq 1$, $d(G(x) - G(f(x)), G(y) - G(f(y))) \leq C$,
(3) and $d(G(f(x)), G(f(y))) \leq A|x|_f + B$.
(4) $|G(x) - G(f(x))| \geq C|x|_f^{\delta}$ for $x \in \Gamma$.
(5) The angle between $G(S)$ and $(G - G \circ f)(\Gamma)$ is at least ϵ.

Proposition 3.7 (see[CGM]3.C). *A (Γ, S) APL is properly homotopic to a proper Lipschitz map.*

Proof. Let us construct an almost proper Lipschitz map, using elementary operations on Euclidean space(take logarism or square root). Let us define $F : Z \mapsto \mathbb{R}^n$ by

$$F(x) = (\epsilon log C|G_i(x) - G_i(f(x))| + \epsilon(log|G_i(f(x))|)^{\frac{1}{2}})_{1 \leq i \leq n}$$

where ϵ are the signs of $G_i(x) - G_i(f(x))$ or $G_i(f(x))$. First let us estimate (local) Lipschitz constant on $f^{-1}(s)$. Let $\tilde{G}_i(x) = C(G_i(x) - G_i(f(x)))$ and take $y \in f^{-1}(s)$ with $1 \leq |\tilde{G}_i(x)| \leq |\tilde{G}_i(y)| \leq |\tilde{G}_i(x)|^2$. We may assume

$$sign\tilde{G}_i(x) = sign\tilde{G}_i(y), \quad |\tilde{G}_i(x)| \leq |\tilde{G}_i(y)| - |\tilde{G}_i(x)| \leq |\tilde{G}_i(y)|$$

for $1 \leq i \leq n$. If otherwise $1 \leq \frac{|\tilde{G}_i(y)|}{|\tilde{G}_i(x)|} \leq 2$ and $0 \leq log|\tilde{G}_i(y)| - log|\tilde{G}_i(x)| \leq log2$.

Now we have $|log|\tilde{G}_i(y)| - log|\tilde{G}_i(x)|| \leq log|\tilde{G}_i(y) - \tilde{G}_i(x)|$. Thus

$$\begin{aligned}\Sigma_{l=1}^n |log|\tilde{G}_i(x)| - log|\tilde{G}_i(y)|| &\leq \Sigma_{l=1}^n log|\tilde{G}_i(x) - \tilde{G}_i(y)| \\ &\leq C log d_f(x, y) \leq C d(x, y).\end{aligned}$$

Thus $F|f^{-1}(s)$ is fiberwise C Lipschitz with respect to the word metric. Similarly we can see Lipschitzness of $F|S$.

Now let $x, y \in \Gamma$, $d(x, y) \leq 1$. Then $d(\tilde{G}(x), \tilde{G}(y)) \leq C$, $d(G(f(x)), G(f(y))) \leq A|x|_f$. Thus F is $(logA|x|_f)^{\frac{1}{2}}$ Lipschitz around x. Notice that

$$\begin{aligned}C|F(x)| \geq C' log|G(x) - G(f(x))| \geq log|x|_f, \\ d(\epsilon(log|G_i(f(x))|)^{\frac{1}{2}}, \epsilon(log|G_i(f(y))|)^{\frac{1}{2}})) \leq C(log|x|_f)^{\frac{1}{2}}.\end{aligned}$$

Thus there exists C such that $F|F^{-1}(B(R))$ is $CR^{\frac{1}{2}}$ Lipschitz, namely F is almost proper Lipschitz.

The construction of F from $G|\Gamma$ is elementary, and clearly these are connected by a proper homotopy. From Corollary 3.4, we get $F : V \mapsto \mathbb{R}^n$ proper Lipschitz of degree 1, properly homotopic to G. This completes the proof of 3.7.

The following deformation of maps is called *symmetrization*.

The above forms a part of a family of contraction. We want to extend F to a map from V itself, where we regard Γ as a subset of V. [CGM] developed Lipschitz extension theory. They verified that if the target space satisfies uniform contractibility and uniformly boundedness, then any Lipschitz map can be extended, where they called these conditions as *uniformly Lipschitz contractible*(ULC) and *uniformly locally bounded*(ULB). A particular example is any universal covering spaces of compact aspherical manifolds, or any vector bundle over them.

Lemma 3.3([CGM]). *Let V be ULC and ULB. For any positive λ and δ, any metric subspace $Y_0 \subset Y$, there exists λ_0 such that any λ_0 Lipschitz map from Y_0 to V can be extended to λ Lipschitz map from Y, if Y_0 is $\epsilon \leq \lambda_0^{-1}\delta$ net.*

Corollary 3.4 of 3.3. *Let $F : V \mapsto \mathbb{R}^n$ be a proper map such that $F|\Gamma$ is almost proper Lispchitz. Then F is proper homotopic to a proper Lipschitz map.*

Proof of 3.2. Let us return to our λ contracting $F_1 : Z_1 \mapsto f^{-1}(1)$. Let us take a large constant C, and let $N(C,1)$ be C neighbourhood of $f^{-1}(1)$ in V. Let us choose a metric d_1 on $N(C,1)$ such that $d_1|f^{-1}(1)$ is quasi isometric to d_f. Then we have λ contracting map $F_1 : (Z_1, d_f) \mapsto (N(C,1), d_1)$. Using 3.3, we get an extension from $f^{-1}(1)$ to $N(C,1)$ which is $C'\lambda$ contracting. Let $F_1 : f^{-1}(1) \mapsto f^{-1}(1)$ be a modified map which assign $F_1(x)$ to the nearest point in $f^{-1}(1)$ for $x \in f^{-1}(1)$. Then it is λ Lipschitz on the large scale, namely there is a constant C such that $d_1(F_1(x), F_1(y)) \leq \lambda d_f(x,y) + C$. Here C is independent of choice of λ. Let $F : \Gamma = \cup_s f^{-1}(s) \mapsto V$ by $I_{i,1} \circ F_1 \circ I_{1,i}$ as before. Then it is fiberwise proper λ contracting map on the large scale, which is fiberwise properly homotopic to the identity. Notice that for $x, y \in \Gamma$, $d(x,y) \leq 1$, $d(F(x), F(y)) \leq A|x|_f + B$. Using contractibility of V, we have $F : V \mapsto V$, an extension of F, which is properly homotopic to F.

Notice that there is a constant C such that for any $x \in V$, there exists $s \in S$ such that $d_f(F^i(x), s) \leq \lambda^i d_f(x,s) + C$. Moreover for any $x, y \in f^{-1}(s)$, for $i \geq 0$, $d_f(F^i(x), F^i(y)) \leq \lambda^i d_f(x,y) + C$. This completes the proof of 3.2.

3B. Partial symmetrization.

To produce proper Lipschitz maps, we need growth conditions of maps in the fiber and base directions. In particular in order to get $|G(x)| \geq C|x|_f^\delta$ for $x \in \Gamma$ and some $1 > \delta > 0$, we introduce the following condition:

Let pr: $\mathbb{R}^n \backslash 0 \mapsto B = B_0(1)$ be the natural projection and take a proper map $h : \Gamma \mapsto \mathbb{R}^n \backslash 0$. We say that the *angle* between $h(S)$ and $h(S')$ is at least $\epsilon > 0$ if ϵ neighbourhood $N_S(\epsilon)$ of pr$\circ h(S) \subset B$ does not meet pr$\circ h(S')$.

Notice that for any $\epsilon < \delta$, there exists a constant $C = C(\delta)$ such that

$$C^{-1}|h(s)| \leq | < h(s), v > | \leq C|h(s)|$$

for any unit vector in $B \backslash N_S(\delta)$.

Remark 3.5. Let $h : [1, \infty) \mapsto [1, \infty)$ be a function whose graph Γ in $(x,y) = \mathbb{R}^2$ has a positive angle with x axis. Then $log\Gamma = (log(a), log(Ch(a))$, $C = \max\{\epsilon^{-1}, 1\}$, $\Gamma^\alpha = (a^\alpha, (h(a))^\alpha)$, $\alpha \in \mathbb{R}_+$ have also positive angles.

§3 Family of symmetrization

For the rest of this paper, we assume that Γ is isomorphic to a fundamental group of a compact aspherical manifold M, and denote its universal covering space $\tilde{M} = V$. We regard Γ as a subset of V. Notice that the induced metric on Γ and word metrics are quasi isometry.

Let $S \subset \Gamma$ admit H-fiber structure,

$$f : \Gamma \mapsto S$$

whose fiber and base are all δ hyperbolic net. In this section, we will prepare some basic method in order to construct a family of proper Lipschitz maps $f : V \times_\Gamma V \mapsto TV$.

We say that a map $f : S \mapsto S'$ is λ contracting(Lipschitz constant ≤ 1) *on the large scale* if there exists a conatsnt $C \geq 0$ such that for any $x, y \in S$, $d_{S'}(f(x), f(y)) \leq \lambda d_S(x, y) + C$.

3A. Family of contraction.

Definition 3.1. A (Γ, S) *contraction* is a map $F : \Gamma \mapsto \Gamma$ such that for some constants A, B and C,

(1) it is fiberwise proper contracting on the large scale, with respect to the fiber metric.

(2) For $x, y \in \Gamma$, $d(x, y) \leq 1$, $d(F(x), F(y)) \leq A|x|_f + B$,

(3) $I_{s,s'} \circ F \circ I_{s',s} | f^{-1}(s) = F | f^{-1}(s)$ for any $s, s' \in S$.

(4) For any $x \in \Gamma$, $d(F^i(x), S) \leq C$ for any $i \geq i(x)$.

(5) There exists a net Z in Γ such that $F|Z$ is fiberwise proper contracting and $I_{s,s'}(Z_{s'} = Z \cap f^{-1}(s')) = Z_s$.

$F : V \mapsto V$ is (Γ, S) contraction if $F|\Gamma : \Gamma \mapsto \Gamma$ is (Γ, S) contraction and F is properly homotopic to the identity.

We say $Z \subset \Gamma$ satisfying (5) is (Γ, S) *net*.

Proposition 3.2. *There exists* (Γ, S) *contraction* $F : V \mapsto V$,

Proof. For every $x \in f^{-1}(1)$, let us choose arbitrary geodesic (C quasi geodesic) in $f^{-1}(1)$ γ_x between 1 and x. Then for each $x \in \Gamma$, there exists $p(x) \in \text{image}\gamma_x$ such that $|d_f(1, p(x)) - \lambda d_f(1, x)| \leq C$ for some constant C, independent of λ and x. Let us denote this assignment by $F_\lambda : \Gamma \mapsto \Gamma$. From hyperbolicity, this map is $C\lambda$ contracting on the large scale.

Let us choose a large constant δ, which may depend on λ so that F_λ is 2λ Lipschitz on any maximal δ net $Z_1 \subset f^{-1}(1)$. Thus we have a small λ contracting map:

$$F_1 : Z_1 \mapsto f^{-1}(1) \subset V.$$

Define $F_s : Z_s = I_{s,1}(Z_1) \subset f^{-1}(s) \mapsto f^{-1}(s)$ by $F_s = I_{s,1} \circ F_1 \circ I_{1,s}$. Then by the assumption that $I_{s,s'}$ is C quasi isometry, F_s are $C\lambda$ contracting for some constant. Thus we have a family of λ contracting maps $F : Z = \cup_s Z_s \mapsto \Gamma$. Notice that F_1 and hence F is fiberwise properly homotopic to the identity.

We denote $|x|_f = d_f(x, f(x))$. If S is a subset of Γ, then we equip the base metric d_S similarly.

(2) For any $s, s' \in S$, there is a compatible C quasi isometry

$$I_{s,s'} : (f^{-1}(s'), d_f) \mapsto (f^{-1}(s), d_f)$$

such that $d(I_{s',s}(x), y) \leq C$ for any $x \in f^{-1}(s), y \in f^{-1}(s')$, $d(x, y) \leq 1$. Here compatibility implies $I_{s,s''} \circ I_{s'',s'} = I_{s,s'}$.

(3) $f^{-1}(1) \to \Gamma$ is a quasi isometric embedding.

(4) There are constants A, B such that for any $x, y \in \Gamma$, $d(x, y) \leq 1$,

$$d_S(f(x), f(y)), |y|_f \leq A|x|_f + B.$$

(5) For any $x, y \in f^{-1}(s)$,

$$exp\{Cd(x, y)\} \geq d_f(x, y).$$

We say a fiber structure admits *quasi isometric base* if

(6) S is a subset of Γ and $S \subset \Gamma$ is a quasi isometic embedding.

In §3 and §4, we always assume this property.

We say that $f : \Gamma \mapsto S$ is *H fiber structure* if for some $\delta > 0$, any fiber and the base are δ hyperbolic nets with respect to the fiber and base metric.

Remark 2.2. (a) From (5), we have $exp\{Cd(f(x), x)\} \geq |x|_f$. Thus if any $sx, sxa \in f^{-1}(s)$ implies $sa \in f^{-1}(s)$, then (5) follows from (4).

(b) In §5 and §6, we use a modified fiber structure. Instead of (5), we use

$$(5)' \quad \exp(Cd^2(x, y)) \geq d_f.$$

In the process of constructing Lipschitz maps (3.7), we will use an inequality $\log d_f \leq d$. In the above case, we only replace it by $(\log d_f)^{\frac{1}{2}} \leq d$.

(c) There is another way to regard H as a fibered space. Let $S = \{c^i | i \in \mathbb{Z}\}$ and $f : H \mapsto S$ by $f(a^i b^j c^k) = c^k$. $f^{-1}(c^k)$ is isomorphic to $\mathbb{Z}^2 \subset \mathbb{R}$ tree with respect to the natural metric on the fiber. Notice that $\{c^i | i \in \mathbb{Z}\} \subset \Gamma$ is not quasi isometric embedding, e.g. $a^i b^i a^{-i} b^{-i} = c^{i^2}$. Though $H \mapsto \{c^i | i \in \mathbb{Z}\}$ does not satisfy the assumption in (a), it still satisfies (5).

(d) Let us see $f : H \mapsto \{a^i | i \in \mathbb{Z}\}$ by $f(a^i b^j c^k) = a^i$ satisfies a fiber structure except uniformity of quasi isometry constant. The definition (1), (3), (4) are clear. For (5), notice that c is the only element which can be born or vanished. Thus it is obvious from the remark that $d^2 \geq Cd_f$. Let us define $I_{i+1,i}(a^i b^j c^k) = a^{i+1} b^j c^{j+k}$. Identifying $\mathbb{Z}^2 = \{b, c\}$, $I_{i+1,i}$ is expressed by a 2 by 2 matrix $\alpha_{i,j} = 0$ for $(i, j) = (1, 0)$ and 1 otherwise. It is in $SL_2(\mathbb{R})$ and clearly quasi isometry.

(e) Let $x \in f^{-1}(s)$, $y \in f^{-1}(s')$, $|x - y| \leq C$. Then from the definition (2), there exists a quasi path $\{x_i\}_{i=0}^m$, $|x_{i+1} - x_i| \leq C$, $x_i \in f^{-1}(s_i)$, $x_0 = x$ and $x_m = y$ such that $\{s_i\}$ is also a quasi path for S.

$f(a^i b^j c^k) = a^i$. Then its fiber $f^{-1}(a^i)$ is a group generated by b, c. Notice that for this structure, $d(a^i \omega a, a^{i+1}\omega) \leq |\omega|$.

H is realized as a fundamental group of a nil manifold $M = \mathbb{R}^3/H$. The actions of H on $\mathbb{R}^3$ is as follows. For $m = (x, y, z)$,

$$a(m) = (x, y+1, z) \quad b(m) = (x+1, y, z+y)$$
$$c(m) = (x, y, z+1).$$

Let us take 0 orbit of H in $\mathbb{R}^3$. Then the natural embedding has the following properties:

(a) $S \mapsto \mathbb{R} \subset \mathbb{R}^3$ is isometric,

(b) $f^{-1}(a^i) \mapsto \mathbb{R}^3$ is $i+1$ Lipschitz embedding.

Let us take any Riemannian metric on M and d_1 on $\mathbb{R}^3$ be its lift. Let d_2 be the standard metric on $\mathbb{R}^3$. Then the identity $(\mathbb{R}^3, d_1) \to (\mathbb{R}^3, d_2)$ is $C(i+1)$ Lipschiz on $\{(x, y, z) || y| \leq i\}$. By taking square root in x, z coordinates, namely $\delta(2) : (x, z) \mapsto (\epsilon(|x|)^{\frac{1}{2}}, (\epsilon(|z|)^{\frac{1}{2}})$, where ϵ is the sign, we have the following:

The Lipschitz constant of $\delta(2) : (\mathbb{R}^3, d_1) \to (\mathbb{R}^3, d_2)$ is bounded by $C(i^{\frac{1}{2}} + 1)$ on $\delta(2)^{-1}(B(i))$.

Thus from Proposition 1.3, we get a map $F : (\mathbb{R}^3, d_1) \to (\mathbb{R}^3, d_2)$ proper Lipschitz of degree 1, properly homotopic to the identity. By considering $family$, there exists $F : \mathbb{R}^3 \times_H \mathbb{R}^3 \mapsto T(\mathbb{R}^3/H)$ fiberwise proper Lipschitz of degree 1 such that

$$[F^*(\omega)] \cap : H_*(\mathbb{R}^3/H; \mathbb{R}) \mapsto H^*(\mathbb{R}^3/H; \mathbb{R})$$

induces the Poincaré duality isomorphism.

§2 Fiber structure on infinite groups

Let Γ be a finitely generated discrete group with a metric for a fixed generator. The metric is called word metric.

A metric space is called C *net* if for any two points x, y, there exists a finite set $\{x_i\}_{0 \leq i \leq m}$ such that $x_0 = x$, $x_m = y$, $d(x_i, x_{i+1}) \leq C$. Such a sequence of points is called C *quasi path*. $f : S \mapsto S'$ is C *quasi isometry* if for any $x, y \in S$,

$$C^{-1} d_S(x, y) \leq d_{S'}(f(x), f(y)) \leq C d_S(x, y)$$

Let Γ be a discrete group and S be a discrete metric space. We always have base points $1 \in \Gamma$ and $s_0 \in S$. When we consider maps between them, we assume it to be based, $f(1) = s_0$. If S is a subset of Γ, then we assume $1 \in S$, and $f(1) = 1$.

Definition 2.1. A *fiber structure* on Γ is a map $f : \Gamma \mapsto S$ which satisfies the following properties (1) to (5).

(1) there exists a constant $C \geq 1$ such that S and each fiber $f^{-1}(s)$ are C nets.

A *fiber metric* $d_{f^{-1}(s)}$ or simply d_f is defined as follows:

$$d(x, x')_{f^{-1}(s)} = \inf \{\Sigma_{i=1}^{m-1} |x_{i+1} - x_i|; \{x_i\}_{i=1}^m \text{ is } C \text{ quasi path}, x_1 = x, x_m = x'\}.$$

Example 1.2. (1) Let us equip two metrics on $\mathbb{Z}$ by $d = d_1$ and its square root, d_2. Then the identity $(\mathbb{Z}, d_2) \to (\mathbb{Z}, d_1)$ is almost proper Lipschitz, with $\delta = \frac{1}{2}$. If one takes log instead of square root, then it also APL, with δ arbitrary small number.

(2) If $F : X \mapsto \mathbb{R}_+$ is APL, then $(F(x))^\alpha$ is also APL for $0 \leq \alpha \leq 1$.

Proposition 1.3. *Let X be a quasi geodesic space. Let $f : X \mapsto \mathbb{R}^n$ be almost proper Lipschitz. Then f is properly homotopic to a proper Lipschitz map.*

Proof. The proof is a simple application of the method in [CGM]. Let $h : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a proper map such that for some $0 < \lambda < \mu < 1$ and $I_\lambda(x) = \lambda x$,

$$(1) \quad h|\mathbb{R}^n \backslash B(r_0) = id, \quad (2) \quad \mu^{-1} h \circ I_\lambda | \partial B(r_0) = h | \partial B(r_0).$$

Let $D_i = B(\lambda^{-i} r_0) \backslash B(\lambda^{-(i-1)} r_0)$ and define $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ by $F|D_i(x) = \mu^{-i} h(\lambda^i x)$. Then the local Lipschitz constant on D_i is bounded by $C(\mu^{-1}\lambda)^i = (\lambda^{\delta'})^i$. If we choose $1 > \delta' \geq \delta$, then

$$F \circ f : X \mapsto \mathbb{R}^n$$

is proper Lipschitz. It is clear that F is proper homotopic to the identity.

1B. Lipschitz cohomology.

Let us equip a word metric on Γ and take any proper Lipschitz map $f : \Gamma \mapsto \mathbb{R}^n$. Then by pulling back the Bott generator in the compactly supported cohomology of $\mathbb{R}^n$, one may find an element of group cohomology $f^*(\omega) \in H^n(\Gamma; \mathbb{R})$. It is called a proper Lipschitz cohomology class. To produce more elements of group cohomology, we consider a family of proper Lipschitz map. Let P be locally compact space (ex. simplicial complex) on which Γ acts proper discontinuously. Let $F : P \times E\Gamma \mapsto \mathbb{R}^n$ be a Γ invariant fiberwise proper map. Then by taking the slant product with $F^*(\omega)$, we have a duality map

$$[F^*(\omega)] \cap : H_*(P : \Gamma) \mapsto H^*(\Gamma; \mathbb{R}).$$

If F is a fiberwise proper Lipschitz map, then it is enough to have a map from $P \times \Gamma$. Let us recall the following notion ([CGM]):

Definition 1.4. Let $F : P \times \Gamma \mapsto \mathbb{R}^n$ be a fiberwise proper Lipschitz map. Then the image of $[F^*(\omega)] \cap$ in $H^*(\Gamma : \mathbb{R})$ is called *proper Lipschitz cohomology class.*

For many class of discrete groups, proper Lipschitz cohomology classes occupy all cohomology classes, including hyperbolic groups, $\pi_1(K \leq 0)$. For more large class of discrete groups, $H_L(\Gamma; \mathbb{R}) = H(\Gamma; \mathbb{R})$ asymptotically. This reflects the fact that we cannot expect to have finite dimensional $K(\Gamma, 1)$ spaces on this class([K1]).

Proposition 1.5([CGM]). *Proper Lipschitz cohomology classes satisfy Novikov conjecture.*

Example 1.6. Our first example is the Heisenberg group H, whose presentation has as follows:

$$H =< a, b, c | [a, c], [b, c], [a, b]c^{-1} > .$$

It has a subgroup, isomorphic to $\mathbb{Z}$, and thre is a natural maps from H to it which gives H as a structure of *family of* $\mathbb{Z}^2$. Let $S = \{a^i | i \in \mathbb{Z}\} \subset H$ and $f : H \mapsto S$,

paper we will use a more general version of this technique, in order to applicable to more abstruct discrete groups. Our main theorem is the following:

Theorem. *Let Γ be a fundamental group of almost non positively curved manifold. Then any $\gamma \in H^*(\Gamma;\mathbb{R})$ is a proper Lipschitz class. In particular Γ satisfies Novikov conjecture.*

Throughout this paper we assume the existence of closed $K(\Gamma,1)$ manifold. Though there are several approaches to remove this condition, we donot pursue these here. Our essential idea is to make construction of [CGM] for family of discrete metric spaces.

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§1 Lipschitz cohomology

1A. Almost Lipschitz maps.

Let X be a locally compact discrete metric space. For many X, it is a quasi geodesic space, i.e. there exists a constant C such that any $x, y \in X$ can be connected by a sequence $x_0 = x, \ldots, x_n = y$ with $d(x,y) = \Sigma d(x_i, x_{i+1})$, $d(x_i, x_{i+1}) \leq C$. Here we mean discrete by $d(x,y) \geq \delta > 0$ for any $x \neq y$.

Let us consider a proper map $f : X \mapsto \mathbb{R}^n$ and try to measure how far it is from *Lipschitz*. Let Map $(X : \mathbb{R})$ be the set of maps from X to $\mathbb{R}$ with $f(x_0) = 0$ for a fixed point x_0, and equip topology on it by convergence on any compact subsets. Let $\mathrm{Map}_L(X : \mathbb{R}) \subset \mathrm{Map}(X : \mathbb{R})$ be the subset consisted by Lipschitz maps. We want to measure its size. Since $\mathrm{Map}(X : \mathbb{R})$ is not compact, it may be natural to consider its subset by adding growth condition on maps, for example, by requiring $|f(a)| \leq \alpha(d(x_0, a))$ for some maps $\alpha : \mathbb{R}_+ \mapsto \mathbb{R}_+$. However one can find a trivial sequence by Lipschitz maps with the bound, such that they converge to f. Let us take any Lipschitz g with the bound, and define g_i to be also Lipschitz such that $g_i | B_{x_0}(i) = f$ and otherwise $g_i = g$, where $B_{x_0}(i)$ is i ball in X with the center x_0. Then clearly g_i converge to f. Thus $\mathrm{Map}_L(X;\mathbb{R})$ is dense in $\mathrm{Map}(X;\mathbb{R})$. In the case when $\alpha(a) = Ca^l$ for some $C \geq 0$ and $l \in \mathbb{R}_+$, let us note that f satisfies a growth condition of the Lipschitz constants on compact subsets, $\mathrm{Lip} f | B_{x_0}(i) \leq Ci^l$. Let $\mathrm{Map}^{l,C}(X : \mathbb{R})$ be the set of maps with the bound on the Lipschitz constants. Clearly it is compact. Moreover $\mathrm{Map}^{l,C}_L(X : \mathbb{R}) = \mathrm{Map}^{l,C}(X : \mathbb{R}) \cap \mathrm{Map}_L(X : \mathbb{R})$ is dense in $\mathrm{Map}^{l,C}(X : \mathbb{R})$. This follows from the fact that any Lipschitz map from a subset of X to $\mathbb{R}$ can be extended to the one from X with the same Lipschitz constant.

Definition 1.1. Let X be a metric space and $f : X \mapsto \mathbb{R}^n$ be a proper map. Then f is *almost proper Lipschitz* (APL) if the Lipschitz constant $\mathrm{Lip} f$ satisfies

$$\mathrm{Lip} f | f^{-1}(B_0(R)) \leq CR^{\delta}$$

for some $0 \leq \delta < 1$ and any $R \geq 1$.

FIBER STRUCTURE ON INFINITE GROUPS AND HIGHER SIGNATURES

T.Kato

Introduction

Let Γ and H be discrete metric spaces and consider a map between them. Using metric structure, one may add conditions on the maps, for example we can consider Lipschitz maps. If a map satisfies some metric conditions, then we say that it is *fiber structure*. There are many examples of discrete metric spaces which arise from geometry. One of the most important is a discrete group. There are canonical metrics on discrete groups unique up to quasi isometry, so-called word metric. Using this metric, there are many discrete groups which admit non trivial maps between them.

As a first example, we consider the Heisenberg group. It is known that the group does not satisfy convexity on the Cayley graph. However by taking a natural map, we can find a structure of fiberwise convexity. Heisenberg group is a model case for fiber structure.

We will extend this structure to higher rank case. After verifying that any Nilpotent group admits this, we will treat our main examples, fundamental groups of almost non positively curved manifolds.

Our main aim to introduce this is an application to the *Novikov conjecture*. Let Γ be a discrete group and consider a continuous map $f : M \mapsto K(\Gamma, 1)$ for any closed, oriented manifold. For any $x \in H^*(\Gamma; \mathbb{R})$, the associated characteristic number $< L(M)f^*(x), [M] >$ is called a higher signature. Novikov conjecture states homotopy invariance of higher signatures. At this time there are many partial solutions by posing various restrictions on fundamental groups.

In [CGM], A.Connes, M.Gromov and H.Moscovici introduced proper Lipschitz cohomology in group cohomology. In particular they verified that any proper Lipschitz element satisfies Novikov conjecture, i.e. homotopy invariance of the corresponding higher signature. It is known that for large class of discrete groups, proper Lipschitz cohomology classes occupy all the group cohomology in appropreate sense.

In this paper we study proper Lipschitz cohomology for fibered groups. In particular we will widely develop a technique introduced in [CGM] §3. Let G be any linear algebraic group over $\mathbb{R}$ and K its maximal compact subgroup. It is not immediate to have proper Lipschitz maps from G/K to $\mathbb{R}^n$, in the presence of Nilpotent subgroups of G. It turns out there that during trial to construct proper Lipschitz maps, one gets proper maps whose Lipschitz constants between any two points of bounded distances are controlled enough. Using some modification of the maps, they can be deformed to be proper Lipschitz. [CGM] used this technique on manifolds, such as linear algebraic groups, jet bundles of smooth manifolds, etc. In this

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