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because $c_1(\overline{H}_0)$ is positive on W, where $\mathcal{X}_w$ is the fiber of $\mathcal{X} \to B$ over w. This is a contradiction because the left hand side is positive, but the right hand side is not positive along the diagonal of $(\mathcal{X}_w)^m$.

Next let us consider a general case. Let $\mu : A_{\overline{K}} \to A' = A_{\overline{K}}/G(X)$ be the quotient of $A_{\overline{K}}$ by $G(X)$. We set $X' = \mu(X)$. Then, it is easy to see that $\mu^{-1}(X') = X$ and $G(X') = \{0\}$. Let L' be a symmetric and ample line bundle on A'. Let K' be a finite extension field of K such that μ, A', and L' are defined over K'. Let $\overline{B}'$ be the polarization of K' induced by $\overline{B}$. Since L is ample, by Proposition 3.3.1 and (2) of Proposition 3.4.2, there is a positive number a such that $\hat{h}^{\overline{B}'}_{\mu^*(L')} \leq a\hat{h}^{\overline{B}}_L$. Thus,

$$\begin{aligned}\{x \in X(\overline{K}) \mid \hat{h}^{\overline{B}}_L(x) \leq \epsilon/a\} &\subseteq \{x \in X(\overline{K}) \mid \hat{h}^{\overline{B}'}_{\mu^*(L')}(x) \leq \epsilon\} \\ &= \mu^{-1}\left(\{x' \in X'(\overline{K}) \mid \hat{h}^{\overline{B}'}_{L'}(x') \leq \epsilon\}\right).\end{aligned}$$

Therefore, the set $\{x' \in X'(\overline{K}) \mid \hat{h}^{\overline{B}'}_{L'}(x') \leq \epsilon\}$ is Zariski dense in X' for any $\epsilon > 0$. Thus, by the previous observation, $X' = \{x'\}$ for some torsion point x' of A'. Hence, X is a coset of $G(X)$ because $\mu^{-1}(X') = X$. In particular, $G(X)$ is an abelian subvariety. Thus, it is sufficient to show that there is a torsion point x of A with $\mu(x) = x'$. First, pick up x_1 of A with $\mu(x_1) = x'$. Since x' is a torsion point, there is a positive number n with $nx_1 \in G(X)$. Here $G(X)$ is a divisible group. Thus, we can find $x_2 \in G(X)$ with $nx_1 = nx_2$. Hence, if we set $x = x_1 - x_2$, then we have a desired torsion point. □

As corollary, we can recover the following Raynaud's result ([8] and [9]).

Corollary 8.2. *Let F be an algebraically closed field of characteristic zero. Let A be an abelian variety over F, and X a subvariety of A. If $X(F) \cap A(F)_{tor}$ is Zariski dense in $X(F)$, then X is a translation of an abelian subvariety of A by a torsion point.*

Proof. Let K be a subfield of F such that K is a finitely generated field over $\mathbb{Q}$, and that A and X are defined over K. Then, $A(F)_{tor} = A(\overline{K})_{tor}$. Thus, we can see that $X(\overline{K}) \cap A(\overline{K})_{tor}$ is Zariski dense in $X(\overline{K})$. Therefore, by Theorem 8.1, X is a translation of an abelian subvariety of A by a torsion point. □

Remark 8.3. We assume that $\mathrm{tr.deg}_{\mathbb{Q}}(K) = 1$. Then, as in the introduction, we have two types of heights $\hat{h}^{geom}_L$ and $\hat{h}^{arith}_L$. Then, by virtue of (3) of Proposition 3.4.2, there is a positive constant a with $\hat{h}^{geom}_L \leq a\hat{h}^{arith}_L$. This means that Bogomolov's conjecture for the geometric height $\hat{h}^{geom}_L$ is a subtle problem. Actually, if the trace of A is not zero, the conjecture does not holds in general. (For example, take X as a non-torsion point P with $\hat{h}^{geom}_L(P) = 0$.) However, we can expect the conjecture for $\hat{h}^{geom}_L$ if either the trace of A is zero, or X is a non-isotrivial curve and A is its jacobian. For example, in [7], the author gives an affirmative answer under the assumption of singular fibers on the stable model of X.

Let $\pi^m : \mathcal{A}^m \to B$ and $\pi^m_n : \mathcal{A}^m_n \to B$ be the canonical projections. Let $\mathcal{X}^m$ (resp. $\mathcal{X}^m_n$) be the closure of X^m in $\mathcal{A}^m$ (resp. $\mathcal{A}^m_n$). Further, let $\mathcal{Y}^m$ (resp. $\mathcal{Y}^m_n$) be the closure of $\phi_m(X^m)$ in $\mathcal{A}^{m-1}$ (resp $\mathcal{A}^{m-1}_n$). We set

$$\overline{\mathcal{P}}^m = p_1^*(\overline{\mathcal{L}}) \otimes \cdots \otimes p_m^*(\overline{\mathcal{L}}) \quad \text{and} \quad \overline{\mathcal{P}}^m_n = p_1^*(\overline{\mathcal{L}}_n) \otimes \cdots \otimes p_m^*(\overline{\mathcal{L}}_n),$$

where p_i is the projection to the i-th factor. Note that ϕ_m extends to $\mathcal{A}^m_U \to \mathcal{A}^{m-1}_U$. By abuse of notation, we denote this extension by ϕ_m. Then, ϕ_m induces a birational morphism $\mathcal{X}^m_U \to \mathcal{Y}^m_U$. Let V and V' be Zariski open sets of $\mathcal{X}^m_U$ and $\mathcal{Y}^m_U$ respectively such that ϕ_m gives rise to an isomorphism $V \xrightarrow{\sim} V'$.

Since X has only countably many subvarieties over $\overline{K}$, let $\{Y_t\}_{t=1}^{\infty}$ be the set of all proper subvarieties of X. By the hypothesis (c), we can find $x_t \in X(\overline{K})$ such that $x_t \notin \bigcup_{i=1}^t Y_i$ and $\hat{h}_L^{\overline{B}_0}(x_t) \leq 1/t$. Then, we have a generic sequence $\{x_t\}$ of $X(\overline{K})$ with $\lim_{t\to\infty} \hat{h}_L^{\overline{B}_0}(x_t) = 0$. Let us fix a bijection $\tau : \mathbb{N} \to \mathbb{N}^m$. We denote $(x_{\tau_1(t)}, \ldots, x_{\tau_m(t)}) \in X^m$ by $x_{\tau(t)}$, where $\tau(t) = (\tau_1(t), \ldots, \tau_m(t))$. Since $\{x_{\tau(t)}\}$ is Zariski dense in X^m, in the same way as before, we can find a generic subsequence of $\{x_{\tau(t)}\}$. Thus, we may assume that $\{x_{\tau(t)}\}$ is a generic sequence. Moreover, considering a subsequence of $\{x_t\}$, we may further assume that $x_{\tau(t)} \in V_K$. Further, we can see that $\lim_{t\to\infty} \hat{h}^{\overline{B}_0}_{P^m}(x_{\tau(t)}) = 0$ and $\hat{h}^{\overline{B}_0}_{P^{m-1}}(\phi_m(x_{\tau(t)})) = 0$, where $P^m = \mathcal{P}^m_K$ and $P^{m-1} = \mathcal{P}^{m-1}_K$. Thus, using the equidistribution theorem (cf. Theorem 6.1), over $(\pi^m)^{-1}(W) \cap \mathcal{X}^m$,

$$\lim_{t\to\infty} \frac{\delta_{\Delta_{x_{\tau(t)}}} \wedge (\pi^m)^*(c_1(\overline{H}_0)^d)}{\deg(\Delta_{x_{\tau(t)}} \to W)} = \left[\frac{(p_1^*(\omega) + \cdots + p_m^*(\omega))^{em} \wedge (\pi^m)^*(c_1(\overline{H}_0)^d)}{\deg\left(\left(P^m_K|_{X^m}\right)^{em}\right)} \right],$$

where $e = \dim X$. Moreover, if we denote α by the restriction of $p_1^*(\omega) + \cdots + p_{m-1}^*(\omega)$ to $(\pi_{m-1})^{-1}(W) \cap \mathcal{Y}^m$, then

$$\lim_{t\to\infty} \frac{\delta_{\Delta_{\phi_m(x_{\tau(t)})}} \wedge (\pi^{m-1})^*(c_1(\overline{H}_0))^d}{\deg(\Delta_{\phi_m(x_{\tau(t)})} \to W)} = \left[\frac{\alpha^{em} \wedge (\pi^{m-1})^*(c_1(\overline{H}_0)^d)}{\deg\left(\left(P^{m-1}_K|_{Y^m}\right)^{em}\right)} \right],$$

where $Y^m = \mathcal{Y}^m_K$. Note that on $V \cap (\pi^m)^{-1}(W)$,

$$\frac{\delta_{\Delta_{x_{\tau(t)}}} \wedge (\pi^m)^*(c_1(\overline{H}_0)^d)}{\deg(\Delta_{x_{\tau(t)}} \to W)} \quad \text{and} \quad \frac{\delta_{\Delta_{\phi_m(x_{\tau(t)})}} \wedge (\pi^{m-1})^*(c_1(\overline{H}_0))^d}{\deg(\Delta_{\phi_m(x_{\tau(t)})} \to W)}$$

give rise to the same current via the isomorphism $\phi_m : V \xrightarrow{\sim} V'$. Thus two limit C^∞-forms

$$\frac{(p_1^*(\omega) + \cdots + p_m^*(\omega))^{em} \wedge (\pi^m)^*(c_1(\overline{H}_0)^d)}{\deg\left(\left(P^m_K|_{X^m}\right)^{em}\right)} \quad \text{and} \quad \phi_m^* \left(\frac{\alpha^{em} \wedge (\pi^{m-1})^*(c_1(\overline{H}_0)^d)}{\deg\left(\left(P^{m-1}_K|_{Y^m}\right)^{em}\right)} \right)$$

are same forms on $V \cap (\pi^m)^{-1}(W)$. Hence we have

$$(p_1^*(\omega) + \cdots + p_m^*(\omega))^{em} \wedge (\pi^m)^*(c_1(\overline{H}_0)^d) = c(\phi_m^*(\alpha^{em})) \wedge (\pi^m)^*(c_1(\overline{H}_0)^d)$$

over $(\pi^m)^{-1}(W) \cap \mathcal{X}^m$ for some positive constant c. Let $\mathcal{X}$ be the closure of X in $\mathcal{A}$. We choose a general point w of W. Then, we can easily see that

$$(p_1^*(\omega) + \cdots + p_m^*(\omega))^{em}|_{(\mathcal{X}_w)^m} = c(\phi_m^*(\alpha^{em}))|_{(\mathcal{X}_w)^m}$$

Proof. First of all, note that in order to prove our theorem, we can replace the field K by a finite extension of K if it is necessary.

We set

$$G(X) = \{a \in A(\overline{K}) \mid a + X = X\}.$$

First, let us consider the case where $G(X)$ is trivial. In this case, we need to show that there is a torsion point x of A with $X = \{x\}$. For this purpose, it is sufficient to show that $\dim X = 0$. For, if we set $X = \{x\}$, then $\hat{h}_L^{\overline{B}}(x) = 0$. Thus, x is a torsion point by Proposition 3.4.1.

From now on, we assume that $\dim X \geq 1$. Changing K by a finite extension of K, if necessarily, by Proposition 3.1.1, we may assume that there is a polarization $\overline{B}_0 = (B, \overline{H}_0)$ of K such that $\widehat{\deg}(\widehat{c}_1(\overline{H}_0)^{d+1}) = 0$ and $\deg((H_0)_{\mathbb{Q}}^d) > 0$. Then, by virtue of (3) of Proposition 3.4.2, there is a positive constant a with $\hat{h}_L^{\overline{B}_0} \leq a\hat{h}_L^{\overline{B}}$. Thus, the set

$$\{x \in X(\overline{K}) \mid \hat{h}_L^{\overline{B}_0}(x) \leq \epsilon\}$$

is Zariski dense for any $\epsilon > 0$. Therefore, we will try to find a contradiction using hypotheses:

(a) $G(X) = \{0\}$.
(b) $\dim X \geq 1$.
(c) The set $\{x \in X(\overline{K}) \mid \hat{h}_L^{\overline{B}_0}(x) \leq \epsilon\}$ is Zariski dense for any $\epsilon > 0$.

Here we consider a morphism

$$\phi_m : A_{\overline{K}}^m \to A_{\overline{K}}^{m-1}$$

given by $\phi_m(x_1, \ldots, x_m) = (x_1 - x_2, x_2 - x_3, \ldots, x_{m-1} - x_m)$. Then, since $G(X) = \{0\}$, in the same way as in the proof of [15, Lemma 3.1], we can see that if m is sufficiently large, then ϕ_m induces a birational morphism $X^m \to \phi_m(X^m)$. Considering a finite extension of K, we may assume that X is defined over K, and that ϕ_m induces a birational morphism $X^m \to \phi_m(X^m)$ over K. Here, if it is necessary, we change $\overline{B}_0$ by the polarization induced by the extension of K accordingly.

We note that the above hypothesis (c) does not depend on the choice of the ample and symmetric line bundle L by virtue of (2) of Proposition 3.4.2. Hence, by Proposition 7.1, there is a sequence of C^∞-models $(\mathcal{A}_n, \overline{\mathcal{L}}_n)$ of (A, L) with the following properties.

(1) There is a Zariski open set U of B such that $(\mathcal{A}_n)_U = (\mathcal{A}_1)_U$ for all n, and that $(\mathcal{A}_1)_U \to U$ is an abelian scheme over U.
(2) If n is sufficiently large, then $\mathcal{L}_n$ is ample and $\overline{\mathcal{L}}_n$ is vertically nef.
(3) $\lim_{n\to\infty} \sup_{x \in A(\overline{K})} |\hat{h}_L^{\overline{B}_0}(x) - h^{\overline{B}_0}_{(\mathcal{A}_n, \overline{\mathcal{L}}_n)}(x)| = 0$.
(4) There are a connected open set W of $U(\mathbb{C})$, and a positive C^∞-form ω on $(\mathcal{A}_1)_W$ such that W is non-singular, $c_1(\overline{H}_0)$ is positive on W, and that $c_1(\overline{\mathcal{L}}_n) = \omega$ on $(\mathcal{A}_1)_W$ for all $n \gg 0$.

For simplicity, we denote $\mathcal{A}_1$ by $\mathcal{A}$. Let $\mathcal{A}^m$ (resp. $\mathcal{A}_n^m$) be the main component of

$$\underbrace{\mathcal{A} \times_B \cdots \times_B \mathcal{A}}_{m\text{-times}} \qquad \left(\text{resp.} \quad \underbrace{\mathcal{A}_n \times_B \cdots \times_B \mathcal{A}_n}_{m\text{-times}}\right).$$

To get an invariant metric $\|\cdot\|$ under F_∞, over $F_\infty(W_1)$, we replace $\|\cdot\|_0$ by $F_\infty^*(\|\cdot\|_1)$. In this way, we have a hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ with the following properties.

(a) $c_1(\overline{\mathcal{L}})$ is positive over $\mathcal{A}(\mathbb{C}) \setminus \pi^{-1}(W_2 \cup F_\infty(W_2))$.
(b) For any $\epsilon > 0$, there is a positive number n_0 such that, for all $n \geq n_0$,

$$2^{-2n}([2]^n)^*(c_1(\overline{\mathcal{L}})) + \epsilon\pi^* c_1(\overline{H})$$

is positive on $\pi^{-1}(W_2 \cup F_\infty(W_2))$.
(c) $2^{-2n}([2]^n)^* c_1(\overline{\mathcal{L}}) = c_1(\overline{\mathcal{L}})$ on W for all n, where $W = W_3$.

Let $f_n : \mathcal{A}_n \to \mathcal{A}$ be the normalization of

$$\mathcal{A}_U \xrightarrow{[2]^n} \mathcal{A}_U \hookrightarrow \mathcal{A}.$$

Then, by projection formula,

$$h^{\overline{B}}_{(\mathcal{A}_n, f_n^*(\overline{\mathcal{L}}))}(x) = h^{\overline{B}}_{(\mathcal{A}, \overline{\mathcal{L}})}(2^n x)$$

for all $x \in A(\overline{K})$. Thus, if we set $\overline{\mathcal{L}}'_n = 2^{-2n} f_n^*(\overline{\mathcal{L}})$, then,

$$h^{\overline{B}}_{(\mathcal{A}_n, \overline{\mathcal{L}}'_n)}(x) = 2^{-2n} h^{\overline{B}}_{(\mathcal{A}, \overline{\mathcal{L}})}(2^n x).$$

Therefore,

$$\lim_{n\to\infty} \sup_{x \in A(\overline{K})} |\hat{h}^{\overline{B}}_L(x) - h^{\overline{B}}_{(\mathcal{A}_n, \overline{\mathcal{L}}'_n)}(x)| = 0.$$

Moreover,

$$h^{\overline{B}}_{(\mathcal{A}_n, \overline{\mathcal{L}}'_n + \epsilon\pi_n^*(\overline{H}))} = h^{\overline{B}}_{(\mathcal{A}_n, \overline{\mathcal{L}}'_n)}$$

for any positive rational number ϵ, where $\pi_n : \mathcal{A}_n \to B$ is the canonical morphism. Thus, if we set $\overline{\mathcal{L}}_n = \overline{\mathcal{L}}'_n + \pi_n^*(\overline{H})$, then a sequence of models $(\mathcal{A}_n, \mathcal{L}_n)$ satisfies our desired properties. □

8. Bogomolov's conjecture over finitely generated fields

Let K be a finitely generated field over $\mathbb{Q}$ with $d = \mathrm{tr.deg}_{\mathbb{Q}}(K)$, and $\overline{B} = (B, \overline{H})$ a polarization of K. Let A be an abelian variety over K. In this section, we would like to prove the following theorem, which is a generalization of results due to Ullmo [13] and Zhang [15].

Theorem 8.1. *Let X be a subvariety of $A_{\overline{K}}$, and L a symmetric ample line bundle on A. We assume that $\overline{B}$ is big, i.e., $\overline{H}$ is nef and big. If the set*

$$\{x \in X(\overline{K}) \mid \hat{h}^{\overline{B}}_L(x) \leq \epsilon\}$$

is Zariski dense in X for any $\epsilon > 0$, then X is a translation of an abelian subvariety of $A_{\overline{K}}$ by a torsion point.

number λ such that $0 < \lambda < 1$ and $\omega' - \lambda\omega$ is positive on $T_{\mathcal{A}_{W_2}/W_2}$. To see our claim, clearly we may assume that

$$W_2 = \mathbb{D}^d = \{(t_1, \ldots, t_d) \in \mathbb{C}^d \mid |t_i| < 1\}.$$

Let $\mu : \mathbb{C}^g \times \mathbb{D}^d \to \mathcal{A}_{\mathbb{D}^d}$ be the universal covering of $\mathcal{A}_{\mathbb{D}^d}$ such that μ is a morphism over $\mathbb{D}^d$, and μ is a homomorphism on each fiber over $\mathbb{D}^d$. Let $(z_1, \cdots, z_g)$ be a coordinate of $\mathbb{C}^g$. Then, $\mu^*(\omega)$ and $\mu^*(\omega')$ can be written by the forms

$$\sum_{ij} b_{ij}(z,t) dz_i \wedge d\bar{z}_j,$$

where $b_{ij}(z,t)$'s are bounded C^∞-functions. Moreover, we set

$$\begin{cases} A_1 = \mu^*(c_1(\mathcal{L}, \|\cdot\|_{cub})) - \mu^*(\omega) \\ A_2 = \mu^*(c_1(\mathcal{L}, \|\cdot\|_0)) - \mu^*(\omega') \\ A_3 = \mu^*(c_1(\mathcal{L}, \|\cdot\|_1)) - (\pi \cdot \mu)^*(\phi)\mu^*(c_1(\mathcal{L}, \|\cdot\|_{cub})) - (1 - (\pi \cdot \mu)^*(\phi))\mu^*(c_1(\mathcal{L}, \|\cdot\|_0)). \end{cases}$$

Then, it is easy to see that each A_i's can written by the form

$$\sum_{ik} c_{ik}(z,t) dz_i \wedge d\bar{t}_k + \sum_{lj} c'_{lj}(z,t) dt_l \wedge d\bar{z}_j + \sum_{lk} c''_{lk}(z,t) dt_l \wedge d\bar{t}_k,$$

where c_{ik}'s, c'_{lj}'s and c''_{lk}'s are bounded C^∞-functions. Here let us see that $2^{-2n}[2^n]^*(A_i)$ $(i = 1, 2, 3)$ converges uniformly to 0. Indeed,

$$2^{-2n}[2^n]^* \left(\sum_{ik} c_{ik}(z,t) dz_i \wedge d\bar{t}_k + \sum_{lj} c'_{lj}(z,t) dt_l \wedge d\bar{z}_j + \sum_{lk} c''_{lk}(z,t) dt_l \wedge d\bar{t}_k \right) =$$
$$\sum_{ik} 2^{-n} c_{ik}(2^n z, t) dz_i \wedge d\bar{t}_k + \sum_{lj} 2^{-n} c'_{lj}(2^n z, t) dt_l \wedge d\bar{z}_j + \sum_{lk} 2^{-2n} c''_{lk}(2^n z, t) dt_l \wedge d\bar{t}_k$$

Thus, we have our assertion because c_{ik}'s, c'_{lj}'s and c''_{lk}'s are bounded.

Next, we try to see that $A_1 = 0$. For, $2^{-2n}[2^n]^*(\mu^*(c_1(\mathcal{L}, \|\cdot\|_{cub}))) = \mu^*(c_1(\mathcal{L}, \|\cdot\|_{cub}))$ for all n, which shows us that $2^{-2n}[2^n]^*(\omega) = \omega$ and $2^{-2n}[2^n]^*(A_1) = A_1$. Hence, A_1 must be zero.

Thus, if we set $A = (1 - (\pi \cdot \mu)^*(\phi))A_2 + A_3$ and

$$C = (1 - \lambda)(\pi \cdot \mu)^*(\phi)\mu^*(\omega) + (1 - (\pi \cdot \mu)^*(\phi))\mu^*(\omega' - \lambda\omega),$$

then

$$\mu^*(c_1(\mathcal{L}, \|\cdot\|_1)) = \lambda\mu^*(\omega) + A + C$$

and C is semipositive. Hence,

$$2^{-2n}[2^n]^*(\mu^*(c_1(\mathcal{L}, \|\cdot\|_1))) + \epsilon(\pi \cdot \mu)^*(c_1(\overline{H})) =$$
$$\lambda\mu^*(\omega) + \epsilon(\pi \cdot \mu)^*(c_1(\overline{H})) + 2^{-2n}[2^n]^*(A) + 2^{-2n}[2^n]^*(C).$$

On the other hand, $\lambda\mu^*(\omega) + \epsilon(\pi \cdot \mu)^*(c_1(\overline{H}))$ is positive and $2^{-2n}[2^n]^*(A)$ converges uniformly to 0. Thus,

$$\lambda\mu^*(\omega) + \epsilon(\pi \cdot \mu)^*(c_1(\overline{H})) + 2^{-2n}[2^n]^*(A)$$

is positive if n is sufficiently large. Hence we get our claim.

7. Construction of the canonical height in terms of Arakelov Geometry

Let K be a finitely generated field over $\mathbb{Q}$ with $\mathrm{tr.deg}_{\mathbb{Q}}(K) = d$. We assume that there is a polarization $\overline{B} = (B, \overline{H})$ of K such that $\widehat{\deg}(\widehat{c}_1(\overline{H})^{d+1}) = 0$ and $\deg(H_{\mathbb{Q}}^d) > 0$. Let A be an abelian variety over K of dimension g. Let us fix a projective embedding $\iota : A \hookrightarrow \mathbb{P}_K^N$, so that we have a new embedding $\iota' : A \to \mathbb{P}_K^N \times_K \mathbb{P}_K^N$ given by $\iota'(x) = (\iota(x), \iota([-1](x)))$. Then, $L = \iota'^*(p_1^*(\mathcal{O}(1)) \otimes p_2^*(\mathcal{O}(1)))$ is ample and symmetric, where p_i is the projection to the i-th factor. In this section, we would like to show the following proposition.

Proposition 7.1. *There is a sequence of C^∞-models $(\mathcal{A}_n, \overline{\mathcal{L}}_n)$ of (A, L) with the following properties.*

1. *There is a Zariski open set U of B such that $(\mathcal{A}_n)_U = (\mathcal{A}_1)_U$ for all n, and that $(\mathcal{A}_1)_U \to U$ is an abelian scheme over U.*
2. *If n is sufficiently large, then $\mathcal{L}_n$ is ample and $\overline{\mathcal{L}}_n$ is vertically nef.*
3. $\lim_{n\to\infty} \sup_{x \in A(\overline{K})} |\hat{h}_L^{\overline{B}}(x) - h^{\overline{B}}_{(\mathcal{A}_n, \overline{\mathcal{L}}_n)}(x)| = 0$.
4. *There are a connected open set W of $U(\mathbb{C})$, and a positive C^∞-form ω on $(\mathcal{A}_1)_W$ such that W is non-singular, $c_1(\overline{H})$ is positive on W, and that $c_1(\overline{\mathcal{L}}_n) = \omega$ on $(\mathcal{A}_1)_W$ for all $n \gg 0$.*

Proof. Let $\mathcal{A}$ be the closure of $\iota'(A)$ in $\mathbb{P}_B^N \times_B \mathbb{P}_B^N$, and $\mathcal{L} = p_1^*(\mathcal{O}(1)) \otimes p_2^*(\mathcal{O}(1))|_{\mathcal{A}}$. Then, $\mathcal{A}_K = A$, $L = \mathcal{L}_K$ is ample and symmetric, and $\mathcal{L}$ is π-ample, where $\pi : \mathcal{A} \to B$ is the canonical morphism. Replacing $\mathcal{L}$ by $\mathcal{L} \otimes \pi^*(Q)$ for some ample line bundle Q on B, we may assume that $\mathcal{L}$ is ample on $\mathcal{A}$. Let U be a Zariski open set of B such that $\mathcal{A}_U \to U$ is an abelian scheme and $[-1]^*(\mathcal{L}_U) = \mathcal{L}_U$ over $\mathcal{A}_U$. Shrinking U if necessarily, we may assume that $[2]^*(\mathcal{L}_U) = \mathcal{L}_U^{\otimes 4}$. Let $\|\cdot\|_0$ be a hermitian metric of $\mathcal{L}$ such that $c_1(\mathcal{L}, \|\cdot\|_0)$ is positive on $\mathcal{A}(\mathbb{C})$. We would like to slightly change $\|\cdot\|_0$ on $\mathcal{A}(\mathbb{C})$. Let $F_\infty : \mathcal{A}(\mathbb{C}) \to \mathcal{A}(\mathbb{C})$ be the Frobenius map given by the complex conjugation. Since $\deg(H_{\mathbb{Q}}^d) = \int_{B(\mathbb{C})} c_1(\overline{H})^d > 0$ and $c_1(\overline{H})$ is semipositive, we can find a small open set W_1 of $U(\mathbb{C})$ in the classical topology such that W_1 is non-singular, $c_1(\overline{H})$ is positive on W_1 and $W_1 \cap F_\infty(W_1) = \emptyset$. Here, we give a C^∞-family of cubic metrics $\|\cdot\|'_{cub}$ of $\mathcal{L}_{W_1}$ over W_1. Then, there is a positive C^∞-function ρ on W_1 with $[2]^*(\mathcal{L}_{W_1}, \|\cdot\|'_{cub}) = (\mathcal{L}_{W_1}^{\otimes 4}, \rho(\|\cdot\|'_{cub})^{\otimes 4})$. If we set $\|\cdot\|_{cub} = \rho^{1/3}\|\cdot\|'_{cub}$, then $[2]^*(\mathcal{L}_{W_1}, \|\cdot\|_{cub}) = (\mathcal{L}_{W_1}, \|\cdot\|_{cub})^{\otimes 4}$. Let us choose open sets W_3 and W_2 with $W_3 \Subset W_2 \Subset W_1$. We also choose a C^∞ function ϕ on $B(\mathbb{C})$ such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on W_3 and $\phi \equiv 0$ on $B(\mathbb{C}) \setminus W_2$. Let a be a positive C^∞-function on $\pi^{-1}(W_1)$ with $\|\cdot\|_{cub} = a\|\cdot\|_0$. Here we set $\|\cdot\|_1 = a^{\pi^*(\phi)}\|\cdot\|_0$. Then, $\|\cdot\|_1$ gives rise to a C^∞-metric of $\mathcal{L}$, which coincides with $\|\cdot\|_0$ on $\pi^{-1}(B(\mathbb{C}) \setminus W_2)$. Here we claim the following.

Claim 7.1.1. *For any $\epsilon > 0$, there is a positive integer n_0 such that, for all $n \geq n_0$,*

$$2^{-2n}([2]^n)^*(c_1(\mathcal{L}, \|\cdot\|_1)) + \epsilon\pi^*(c_1(\overline{H}))$$

is positive on $\mathcal{A}_{W_2}$.

Note that the relative tangent bundle $T_{\mathcal{A}_{W_2}/W_2}$ is a vector subbundle of the tangent bundle $T_{\mathcal{A}_{W_2}}$. Let ω and ω' be the restriction of $c_1(\mathcal{L}, \|\cdot\|_{cub})$ and $c_1(\mathcal{L}, \|\cdot\|_0)$ to $T_{\mathcal{A}_{W_2}/W_2}$. Then, ω and ω' are positive hermitian form on $T_{\mathcal{A}_{W_2}/W_2}$. Thus, since $\overline{W_2}$ is compact, there is a real

First of all, there is a positive number λ_0 such that for all λ with $|\lambda| \leq \lambda_0$, $\omega + \lambda dd^c(f)$ is positive on $\pi^{-1}(W)$. Let $\mathcal{O}_n(\lambda f)$ be the hermitian line bundle for $\mathcal{O}_{\mathcal{X}_n}$ such that the length of 1 at each point is given by $\exp(-\lambda f)$. Here, we set $\overline{\mathcal{L}}_n^{\lambda} = \overline{\mathcal{L}}_n \otimes \mathcal{O}_n(\lambda f)$. Then, by our construction, $\overline{\mathcal{L}}_n^{\lambda}$ is vertically nef and ample on $(\mathcal{X}_n)_{\mathbb{Q}}$ for $n \gg 0$. Moreover,

$$h^{\overline{B}}_{(\mathcal{X}_n, \overline{\mathcal{L}}_n^{\lambda})}(x) = h^{\overline{B}}_{(\mathcal{X}_n, \overline{\mathcal{L}}_n)}(x) + \frac{\lambda}{\deg(\Delta_x \to B)} \int_{\pi^{-1}(W)} f\delta_{\Delta_x} \wedge \pi^*(c_1(\overline{H}))^d$$

and

$$\frac{\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}_n^{\lambda})^{e+1} \cdot \widehat{c}_1(\pi^*(\overline{H}))^d)}{(e+1)\deg(L^e)} = \frac{\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}_n)^{e+1} \cdot \widehat{c}_1(\pi^*(\overline{H}))^d)}{(e+1)\deg(L^e)} + \frac{\lambda}{\deg(L^e)} \int_{\pi^{-1}(W)} f\omega^e \wedge \pi^*(c_1(\overline{H}))^d + O(\lambda^2),$$

where Δ_x is the closure of x in $\mathcal{X}_n$, and the term $O(\lambda^2)$ is independent on n. Let ϵ be an arbitrary positive number. Then, there is a positive integer n_1 such that, for all $n \geq n_1$,

$$h - \epsilon \leq h^{\overline{B}}_{(\mathcal{X}_n, \overline{\mathcal{L}}_n)} \leq h + \epsilon.$$

On the other hand, since $\{x_m\}$ is generic, if $n \gg 0$, by Corollary 5.2, we can see

$$\begin{aligned} \liminf_m h^{\overline{B}}_{(\mathcal{X}_n, \overline{\mathcal{L}}_n^{\lambda})}(x_m) &\geq \frac{\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}_n^{\lambda})^{e+1} \cdot \widehat{c}_1(\pi^*(\overline{H}))^d)}{(e+1)\deg(L^e)} \\ &\geq \inf_{x \in X(\overline{K})} h^{\overline{B}}_{(\mathcal{X}, \overline{\mathcal{L}}_n)}(x) + \frac{\lambda}{\deg(L^e)} \int_{\pi^{-1}(W)} f\omega^e \wedge \pi^*(c_1(\overline{H}))^d + O(\lambda^2), \end{aligned}$$

which implies

$$\epsilon + \lambda \liminf_m \frac{\int_{\pi^{-1}(W)} f\delta_{\Delta_{x_m}} \wedge f^*(c_1(\overline{H}))^d}{\deg(\Delta_{x_m} \to B)} \geq -\epsilon + \lambda \frac{\int_{\pi^{-1}(W)} f\omega^e \wedge \pi^*(c_1(\overline{H}))^d}{\deg(L^e)} + O(\lambda^2).$$

Thus,

$$\lambda \liminf_m \frac{\int_{\pi^{-1}(W)} f\delta_{\Delta_{x_m}} \wedge f^*(c_1(\overline{H}))^d}{\deg(\Delta_{x_m} \to B)} \geq \lambda \frac{\int_{\pi^{-1}(W)} f\omega^e \wedge \pi^*(c_1(\overline{H}))^d}{\deg(L^e)} + O(\lambda^2).$$

Therefore, taking $\lambda \to 0$, we obtain

$$\liminf_m \frac{\int_{\pi^{-1}(W)} f\delta_{\Delta_{x_m}} \wedge f^*(c_1(\overline{H}))^d}{\deg(\Delta_{x_m} \to B)} \geq \frac{\int_{\pi^{-1}(W)} f\omega^e \wedge \pi^*(c_1(\overline{H}))^d}{\deg(L^e)}.$$

The above inequality still holds even if we replace f by $-f$. Thus,

$$\limsup_m \frac{\int_{\pi^{-1}(W)} f\delta_{\Delta_{x_m}} \wedge f^*(c_1(\overline{H}))^d}{\deg(\Delta_{x_m} \to B)} \leq \frac{\int_{\pi^{-1}(W)} f\omega^e \wedge \pi^*(c_1(\overline{H}))^d}{\deg(L^e)}.$$

Therefore,

$$\lim_{m \to \infty} \frac{\int_{\pi^{-1}(W)} f\delta_{\Delta_{x_m}} \wedge f^*(c_1(\overline{H}))^d}{\deg(\Delta_{x_m} \to B)} = \frac{\int_{\pi^{-1}(W)} f\omega^e \wedge \pi^*(c_1(\overline{H}))^d}{\deg(L^e)}.$$

□

Next let μ be an arbitrary rational number with

$$\mu \leq \frac{\inf_{x \in X(\overline{K})} h^{\overline{B}}_{(\mathcal{X}, \overline{\mathcal{L}})}(x)}{\widehat{\deg}(\widehat{c}_1(\overline{A}) \cdot \widehat{c}_1(\overline{H})^d)}.$$

Then,

$$\inf_{x \in X(\overline{K})} h^{\overline{B}}_{(\mathcal{X}, \overline{\mathcal{L}} - \mu \pi^*(\overline{A}))}(x) \geq 0.$$

Thus, by (2) of Theorem 5.1,

$$\widehat{\deg}\left((\widehat{c}_1(\overline{\mathcal{L}}) - \mu \widehat{c}_1(\pi^*(\overline{A})))^{e+1} \cdot \widehat{c}_1(\pi^*(\overline{H}))^d\right) \geq 0,$$

which says us that

$$\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}})^{e+1} \cdot \widehat{c}_1(\pi^*(\overline{H}))^d\right) \geq \mu(e+1) \deg(\mathcal{L}_K^e) \widehat{\deg}(\widehat{c}_1(\overline{A}) \cdot \widehat{c}_1(\overline{H})^d).$$

Hence, we get the second inequality. □

6. Equidistribution theorem over finitely generated fields

Let K be a finitely generated field over $\mathbb{Q}$ with $d = \mathrm{tr.deg}_{\mathbb{Q}}(K)$, and $\overline{B} = (B, \overline{H})$ a polarization of K. Let X be an e-dimensional projective variety over K. Let $\{x_m\}_{m=1}^{\infty}$ be a sequence of elements of $X(\overline{K})$. We say $\{x_m\}$ is *generic* if any subsequences of $\{x_m\}$ are not contained in any proper closed subsets of $X(\overline{K})$.

Let L be a line bundle on X. Let $(\mathcal{X}, \overline{\mathcal{L}})$ be a C^∞-model of (X, L), and $\pi : \mathcal{X} \to B$ the canonical morphism. Then, we have the following equidistribution theorem, which is a generalization of Szpiro-Ullmo-Zhang's result (cf. [12], [13] and [15]).

Theorem 6.1. *We assume that* $\widehat{\deg}(\widehat{c}_1(\overline{H})^{d+1}) = 0$ *and* $\deg(H^d_{\mathbb{Q}}) > 0$. *Let* $h : X(\overline{K}) \to \mathbb{R}$ *be a height function associated with* L *and* $\overline{B}$ *such that* $h(x) \geq 0$ *for all* $x \in X(\overline{K})$. *We assume that there are a sequence of projective arithmetic varieties* $\mathcal{X}_n$ *over* B, *and a* C^∞*-hermitian* $\mathbb{Q}$*-line bundle* $\overline{\mathcal{L}}_n$ *on each* $\mathcal{X}_n$ *with the following properties.*

(1) *There is a Zariski open set* U *of* B *such that* $(\mathcal{X}_n)_U = \mathcal{X}_U$ *for all* n.
(2) $\sup_{x \in X(\overline{K})} |h(x) - h^{\overline{B}}_{(\mathcal{X}_n, \overline{\mathcal{L}}_n)}(x)|$ *converges to* 0 *as* n *tends to* ∞.
(3) *For* $n \gg 0$, $\overline{\mathcal{L}}_n$ *is vertically nef, and* $(\mathcal{L}_n)_{\mathbb{Q}}$ *is ample on* $(\mathcal{X}_n)_{\mathbb{Q}}$.
(4) *There are a connected open set* W *of* $U(\mathbb{C})$ *(in the topology as complex manifolds) and a positive* C^∞*-form* ω *on* $\pi^{-1}(W)$ *such that* $c_1(\overline{\mathcal{L}}_n) = \omega$ *on* $\pi^{-1}(W)$ *for* $n \gg 0$.

Let $\{x_m\}$ *be a generic sequence in* $X(\overline{K})$ *with* $\lim_{m \to \infty} h(x_m) = 0$. *Then, over* $\pi^{-1}(W)$, *we have the following weak convergence*

$$\lim_{m \to \infty} \frac{\delta_{\Delta_{x_m}} \wedge \pi^*(c_1(\overline{H}))^d}{\deg(\Delta_{x_m} \to B)} = \left[\frac{\omega^e \wedge \pi^*(c_1(\overline{H}))^d}{\deg(L^e)}\right]$$

as currents.

Proof. Let f be a C^∞-function on $\pi^{-1}(W)$ with compact support. We need to show that

$$\lim_{m \to \infty} \frac{\int_{\pi^{-1}(W)} f \delta_{\Delta_{x_m}} \wedge f^*(c_1(\overline{H}))^d}{\deg(\Delta_{x_m} \to B)} = \frac{\int_{\pi^{-1}(W)} f \omega^e \wedge \pi^*(c_1(\overline{H}))^d}{\deg(L^e)}.$$

Let $t_0 = \max\{t \in \mathbb{R} \mid P(t) = 0\}$. We assume $t_0 > 0$. Then, by the above claim, for any $t > t_0$,

$$P(t) \geq t^{e+1}\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{M}})^{e+1} \cdot \widehat{c}_1(\pi^*(\overline{H}))^d\right).$$

Thus, taking $t \to t_0$,

$$0 = P(t_0) \geq t_0^{e+1}\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{M}})^{e+1} \cdot \widehat{c}_1(\pi^*(\overline{H}))^d\right) > 0.$$

This is a contradiction. Therefore, $t_0 \leq 0$. In particular, $P(0) \geq 0$, which is nothing more than the assertion of (2). □

As corollary, we have the following generalization of [14, Theorem (5.2)].

Corollary 5.2. *We assume that* $\widehat{\deg}(\widehat{c}_1(\overline{H})^{d+1}) = 0$, $\deg(H_{\mathbb{Q}}^d) > 0$, *and that, for some rational number* a, $\overline{\mathcal{L}} + a\pi^*(\overline{H})$ *is vertically nef and* $(\mathcal{L} + a\pi^*(H))_{\mathbb{Q}}$ *is ample on* $\mathcal{X}_{\mathbb{Q}}$. *Then we have the following inequalities:*

$$\sup_{Y \subsetneq X}\left\{\inf_{x \in (X \setminus Y)(\overline{K})} h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(x)\right\} \geq \frac{\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{e+1} \cdot \widehat{c}_1(\pi^*(\overline{H}))^d)}{(e+1)\deg(\mathcal{L}_K^e)} \geq \inf_{x \in X(\overline{K})} h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(x).$$

Proof. Let c be a real number with $0 < c < 1$. We set $\overline{A} = (H, c\|\cdot\|)$, where $\|\cdot\|$ is the metric of $\overline{H}$. Then,

$$\widehat{\deg}(\widehat{c}_1(\overline{A}) \cdot \widehat{c}_1(\overline{H})^d) > 0$$

because $\deg(H_{\mathbb{Q}}^d) > 0$. Let λ be an arbitrary rational number with

$$\lambda < \frac{\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{e+1} \cdot \widehat{c}_1(\pi^*(\overline{H}))^d)}{(e+1)\deg(\mathcal{L}_K^e)\widehat{\deg}(\widehat{c}_1(\overline{A}) \cdot \widehat{c}_1(\overline{H})^d)}.$$

Then, it is easy to see that

$$\widehat{\deg}\left((\widehat{c}_1(\overline{\mathcal{L}}) - \lambda\widehat{c}_1(\pi^*(\overline{A})))^{e+1} \cdot \widehat{c}_1(\pi^*(\overline{H}))^d\right) > 0.$$

Here, note that

$$\overline{\mathcal{L}} - \lambda\pi^*(\overline{A}) + (a+\lambda)\pi^*(\overline{H}) = \overline{\mathcal{L}} + \lambda\pi^*(\overline{H} - \overline{A}) + a\pi^*(\overline{H})$$

is vertically nef and ample on $\mathcal{X}_{\mathbb{Q}}$ because $c_1(\overline{H}) = c_1(\overline{A})$. Thus, applying (1) of Theorem 5.1,

$$\sup_{Y \subsetneq X}\left\{\inf_{x \in (X \setminus Y)(\overline{K})} h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}} - \lambda\pi^*(\overline{A}))}(x)\right\} \geq 0,$$

which implies

$$\sup_{Y \subsetneq X}\left\{\inf_{x \in (X \setminus Y)(\overline{K})} h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(x)\right\} \geq \lambda\widehat{\deg}(\widehat{c}_1(\overline{A}) \cdot \widehat{c}_1(\overline{H})^d).$$

Thus, we get

$$\sup_{Y \subsetneq X}\left\{\inf_{x \in (X \setminus Y)(\overline{K})} h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(x)\right\} \geq \frac{\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{e+1} \cdot \widehat{c}_1(\pi^*(\overline{H}))^d)}{(e+1)\deg(\mathcal{L}_K^e)}.$$

for all $0 \leq i \leq \dim(\Gamma/B)+1$. Further,

$$\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{N}}\big|_{\Gamma})^{\dim(\Gamma/B)} \cdot \widehat{c}_1(\pi^*(\overline{A})\big|_{\Gamma}) \cdot \widehat{c}_1(\pi^*(\overline{H})\big|_{\Gamma})^d\right) \\ = \deg((\mathcal{N}_\eta)^{\dim(\Gamma/B)})\widehat{\deg}(\widehat{c}_1(\overline{A}) \cdot \widehat{c}_1(\overline{H})^d) > 0,$$

where η means the restriction to the generic fiber of $\Gamma \to B$. Thus, if we set $\overline{\mathcal{M}} = \overline{\mathcal{N}} + \pi^*(\overline{A})$, then we have the desired hermitian line bundle. □

Let us go back to the proof of (2). We prove (2) by induction on e. If $e = 0$, then the assertion is trivial. Thus, we assume $e > 0$.

In the same way as in the proof of (1), we may assume that $\overline{\mathcal{L}}$ is vertically nef and $\mathcal{L}_{\mathbb{Q}}$ is ample on $\mathcal{X}_{\mathbb{Q}}$. By Claim 5.1.3, there is an ample C^∞-hermitian line bundle $\overline{\mathcal{M}}$ on $\mathcal{X}$ such that

$$\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{M}}\big|_{\Gamma})^{\dim(\Gamma/B)+1} \cdot \widehat{c}_1(\pi^*(\overline{H})\big|_{\Gamma})^d\right) > 0$$

for any integral subschemes Γ on $\mathcal{X}$ with $\pi(\Gamma) = B$. Thus, by hypothesis of induction and Claim 5.1.1, for any integral subschemes Γ on X with $\pi(\Gamma) = B$ and $\dim(\Gamma/B) < e$,

$$\widehat{\deg}\left(\left(\widehat{c}_1(\overline{\mathcal{L}}\big|_{\Gamma}) + t\widehat{c}_1(\overline{\mathcal{M}}\big|_{\Gamma})\right)^{\dim(\Gamma/B)+1} \cdot \widehat{c}_1(\pi^*(\overline{H})\big|_{\Gamma})^d\right) > 0 \tag{5.1.4}$$

for all $t > 0$. Moreover, we can see

$$\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}})^{e+1-i} \cdot \widehat{c}_1(\overline{\mathcal{M}})^i \cdot \widehat{c}_1(\pi^*(\overline{H}))^d\right) \geq 0 \tag{5.1.5}$$

for all $1 \leq i \leq e+1$. We set

$$P(t) = \widehat{\deg}\left(\left(\widehat{c}_1(\overline{\mathcal{L}}) + t\widehat{c}_1(\overline{\mathcal{M}})\right)^{e+1} \cdot \widehat{c}_1(\pi^*(\overline{H}))^d\right)$$

Here we claim the following.

Claim 5.1.6. *If $t > 0$ and $P(t) > 0$, then $P(t) \geq t^{e+1}\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{M}})^{e+1} \cdot \widehat{c}_1(\pi^*(\overline{H}))^d\right)$.*

Clearly we may assume that t is a rational number. For simplicity, we set $\overline{\mathcal{N}} = \overline{\mathcal{L}} + t\overline{\mathcal{M}}$. Then, by (5.1.4) and $P(t) > 0$, we have

$$\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{N}}\big|_{\Gamma})^{\dim(\Gamma/B)+1} \cdot \widehat{c}_1(\pi^*(\overline{H})\big|_{\Gamma})^d\right) > 0$$

for any integral subschemes Γ on $\mathcal{X}$ with $\pi(\Gamma) = B$. Thus, by using Claim 5.1.1 and the assumption $\inf_{x \in X(\overline{K})} h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(x) \geq 0$, we obtain $\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}) \cdot \widehat{c}_1(\overline{\mathcal{N}})^e \cdot \widehat{c}_1(\pi^*(\overline{H}))\right) \geq 0$. Therefore, by using (5.1.5),

$$\begin{aligned} P(t) &= \widehat{\deg}\left(\left(\widehat{c}_1(\overline{\mathcal{L}}) + t\widehat{c}_1(\mathcal{M})\right) \cdot \widehat{c}_1(\overline{\mathcal{N}})^e \cdot \widehat{c}_1(\pi^*(\overline{H}))^d\right) \\ &\geq t\widehat{\deg}\left(\widehat{c}_1(\mathcal{M}) \cdot \widehat{c}_1(\overline{\mathcal{N}})^e \cdot \widehat{c}_1(\pi^*(\overline{H}))^d\right) \\ &\geq t^{e+1}\widehat{\deg}\left(\widehat{c}_1(\mathcal{M})^{e+1} \cdot \widehat{c}_1(\pi^*(\overline{H}))^d\right). \end{aligned}$$

Thus, we get the claim.

for a sufficiently large m. Thus, by Theorem 2.1, for a sufficiently large integer n, there is a non-zero section s of $H^0(\mathcal{X}, n(\mathcal{L}_2 + m\nu^*(H)))$ with $\|s\|_{\sup} < 1$. Let $\mathrm{div}(s) = \sum_i a_i\Gamma_i$ be the irreducible decomposition as cycles. Since $\widehat{\deg}\left(\widehat{c}_1(\overline{H})^{d+1}\right) = 0$, we have

$$\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}_1)^{s+1}\cdot\widehat{c}_1(\overline{\mathcal{L}}_2)^{\dim(\mathcal{Y}/B)-s}\cdot\widehat{c}_1(\nu^*(\overline{H}))^d\right) = \widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}_1)^{s+1}\cdot\widehat{c}_1(\overline{\mathcal{L}}_2)^{\dim(\mathcal{Y}/B)-s-1}\cdot\widehat{c}_1(\overline{\mathcal{L}}_2+m\nu^*(\overline{H}))\cdot\widehat{c}_1(\nu^*(\overline{H}))^d\right),$$

which implies that

$$\begin{aligned}n\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}_1)^{s+1}\cdot\widehat{c}_1(\overline{\mathcal{L}}_2)^{\dim(\mathcal{Y}/B)-s}\cdot\widehat{c}_1(\nu^*(\overline{H}))^d\right) = &\sum_i a_i\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}_1|_{\Gamma_i})^{s+1}\cdot\widehat{c}_1(\overline{\mathcal{L}}_2|_{\Gamma_i})^{\dim(\mathcal{Y}/B)-s-1}\cdot\widehat{c}_1(\nu^*(\overline{H})|_{\Gamma_i})^d\right)\\ &+\int_{\mathcal{X}(\mathbb{C})}(-\log\|s\|_{\sup})c_1(\overline{\mathcal{L}}_1)^{s+1}\cdot c_1(\overline{\mathcal{L}}_2)^{\dim(\mathcal{Y}/B)-s-1}\cdot c_1(\nu^*(\overline{H}))^d\end{aligned}$$

Thus, by Lemma 2.4, it is sufficient to show

$$\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}_1|_{\Gamma_i})^{s+1}\cdot\widehat{c}_1(\overline{\mathcal{L}}_2|_{\Gamma_i})^{\dim(\mathcal{Y}/B)-s-1}\cdot\widehat{c}_1(\nu^*(\overline{H})|_{\Gamma_i})^d\right)\geq 0 \tag{5.1.2}$$

for all i.

If Γ_i maps surjectively to B, then, by hypothesis of induction, we can see (5.1.2). Thus, we assume that Γ_i dose not map surjectively to B. Let T be the generic fiber of $\Gamma_i \to \nu(\Gamma_i)$. Since $\dim T \geq e$, we can see

$$\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}_1|_{\Gamma_i})^{s+1}\cdot\widehat{c}_1(\overline{\mathcal{L}}_2|_{\Gamma_i})^{\dim(\mathcal{Y}/B)-s-1}\cdot\widehat{c}_1(\nu^*(\overline{H})|_{\Gamma_i})^d\right) = \begin{cases}\deg(\mathcal{L}_1|_T^{s+1}\cdot\mathcal{L}_2|_T^{\dim(\mathcal{Y}/B)-s-1})\widehat{\deg}(\widehat{c}_1(\overline{H}|_{\nu(\Gamma_i)})^d) & \text{if } \dim T = \dim(\mathcal{Y}/B)\\ 0 & \text{if } \dim T > \dim(\mathcal{Y}/B).\end{cases}$$

Here $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$ are vertically nef and $\overline{H}$ is nef. Thus, by the above formula, we have (5.1.2) even if Γ_i dose not map surjectively to B. □

Claim 5.1.3. *If $\deg(H^d_{\mathbb{Q}}) > 0$, then there is an ample C^∞-hermitian line bundle $\overline{\mathcal{M}}$ on $\mathcal{X}$ such that*

$$\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{M}}|_\Gamma)^{\dim(\Gamma/B)+1}\cdot\widehat{c}_1(\pi^*(\overline{H})|_\Gamma)^d\right) > 0$$

for any integral subschemes Γ on $\mathcal{X}$ with $\pi(\Gamma) = B$.

Proof. Let $\overline{\mathcal{N}}$ be an ample C^∞-hermitian line bundle on $\mathcal{X}$, and let $\|\cdot\|$ be the metric of $\overline{H}$. For a positive number c with $0 < c < 1$, we set $\overline{A} = (H, c\|\cdot\|)$. Then, since $\deg(H^d_{\mathbb{Q}}) > 0$, we can see that $\widehat{\deg}(\widehat{c}_1(\overline{A})\cdot\widehat{c}_1(\overline{H})^d) > 0$. Let Γ be a subscheme of $\mathcal{X}$ with $\pi(\Gamma) = B$. Then, by (2) of Proposition 2.3,

$$\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{N}}|_\Gamma)^i\cdot\widehat{c}_1(\pi^*(\overline{A})|_\Gamma)^{\dim(\Gamma/B)+1-i}\cdot\widehat{c}_1(\pi^*(\overline{H})|_\Gamma)^d\right)\geq 0$$

Proof. (1) Since $\widehat{\deg}(\widehat{c}_1(\overline{H})^{d+1}) = 0$, by virtue of the projection formula (cf. (1) and (2) of Proposition 1.3), we can easily see that

$$h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})} = h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}}+m\pi^*(\overline{H}))}$$

and

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{e+1}\cdot\widehat{c}_1(\pi^*(\overline{H}))^d) = \widehat{\deg}\left((\widehat{c}_1(\overline{\mathcal{L}}+m\pi^*(\overline{H}))^{e+1}\cdot\widehat{c}_1(\pi^*(\overline{H}))^d\right).$$

Thus, we may assume that $\overline{\mathcal{L}}$ is vertically nef and $\mathcal{L}_{\mathbb{Q}}$ is ample because

$$\overline{\mathcal{L}}+m\pi^*(\overline{H}) = \overline{\mathcal{L}}+a\pi^*(\overline{H})+(m-a)\pi^*(\overline{H}).$$

Moreover, since

$$\widehat{\deg}\left(\left(\widehat{c}_1(\overline{\mathcal{L}})+m\widehat{c}_1(\pi^*(\overline{H}))\right)^{e+d+1}\right) = \binom{e+d+1}{d}\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}})^{e+1}\cdot\widehat{c}_1(\pi^*(\overline{H}))^d\right)m^d + O(m^{d-1}),$$

we may further assume that $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{e+d+1}) > 0$. Thus, by Theorem 2.1, for a sufficiently large integer n, there is a non-zero section s of $H^0(\mathcal{X},\mathcal{L}^{\otimes n})$ with $\|s\|_{\sup} < 1$. We set $Y = (\mathrm{div}(s))_K$. Then, for any $x \in (X\setminus Y)(\overline{K})$,

$$h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(x) = \frac{1}{n\deg(\Delta_x\to B)}\widehat{\deg}\left(\widehat{\mathrm{div}(s)}\Big|_{\Delta_x}\cdot\widehat{c}_1(\,\pi^*(\overline{H})\big|_{\Delta_x})^d\right).$$

Here, $\widehat{\mathrm{div}(s)}\Big|_{\Delta_x}$ is effective, and $\pi^*(\overline{H})\big|_{\Delta_x}$ is nef. Thus, by (3) of Proposition 2.3, we have $h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(x) \geq 0$ for all $x \in (X\setminus Y)(\overline{K})$.

(2) The proof is very similar to [14, Lemma 5.4]. First of all, we need the following two claims.

Claim 5.1.1. *Let $\nu : \mathcal{Y}\to B$ be a surjective morphism of projective arithmetic varieties, and let $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$ be C^∞-hermitian $\mathbb{Q}$-line bundles on $\mathcal{Y}$ such that $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$ are vertically nef, and that $(\mathcal{L}_1)_{\mathbb{Q}}$ and $(\mathcal{L}_2)_{\mathbb{Q}}$ are ample on $\mathcal{Y}_{\mathbb{Q}}$. Let us fix an integer s with $0\leq s\leq \dim(\mathcal{Y}/B)$, where $\dim(\mathcal{Y}/B)$ is the dimension of the generic fiber of $\mathcal{Y}\to B$. We assume the following.*

(a) $\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}_1\big|_\Gamma)^{s+1}\cdot\widehat{c}_1(\nu^*(\overline{H})\big|_\Gamma)^d\right)\geq 0$ *for any integral subschemes Γ on $\mathcal{Y}$ with $\nu(\Gamma)=B$ and $\dim(\Gamma/B)=s$.*

(b) $\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}_2\big|_\Gamma)^{\dim(\Gamma/B)+1}\cdot\widehat{c}_1(\nu^*(\overline{H})\big|_\Gamma)^d\right) > 0$ *for any integral subschemes Γ on $\mathcal{Y}$ with $\nu(\Gamma)=B$.*

Then, $\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}_1)^{s+1}\cdot\widehat{c}_1(\overline{\mathcal{L}}_2)^{\dim(\mathcal{Y}/B)-s}\cdot\widehat{c}_1(\nu^*(\overline{H}))^d\right)\geq 0$.

Proof. We prove this claim by induction on $\dim(\mathcal{Y}/B)$. If $s=\dim(\mathcal{Y}/B)$, then our assertion is trivial. Hence we may assume that $\dim(\mathcal{Y}/B) > s$. Since

$$\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}_2)^{\dim(\mathcal{Y}/B)+1}\cdot\widehat{c}_1(\nu^*(\overline{H}))^d\right) > 0,$$

in the same way as in the proof of (1),

$$\widehat{\deg}\left(\left(\widehat{c}_1(\overline{\mathcal{L}}_2)+m\widehat{c}_1(\nu^*(\overline{H}))\right)^{\dim(\mathcal{Y}/B)+d+1}\right) > 0$$

i. Let $\mathcal{Y}$ be the main component of $\underbrace{\mathcal{X} \times_B \cdots \times_B \mathcal{X}}_{e\text{-times}}$, and $\overline{\mathcal{M}} = p_1^*(\overline{\mathcal{L}}) \otimes \cdots \otimes p_e^*(\overline{\mathcal{L}})$, where p_i is the projection to the i-th factor. Moreover, let $f : \mathcal{X} \to B$ and $f' : \mathcal{Y} \to B$ be the canonical morphism. Then, using the projection formula,

$$\begin{aligned} h^{\overline{B}}_{(\mathcal{Y},\overline{\mathcal{M}})}(P_1, \ldots, P_e) &= \frac{\widehat{\deg}\left((f')^*(\widehat{c}_1(\overline{H})^d) \cdot \widehat{c}_1(\overline{\mathcal{M}}) \cdot (\Delta_{(P_1,\ldots,P_e)}, 0)\right)}{\deg(\Delta_{(P_1,\ldots,P_e)} \to B)} \\ &= \frac{\sum_{i=1}^e \widehat{\deg}\left(p_i^* f^*(\widehat{c}_1(\overline{H})^d) \cdot p_i^* \widehat{c}_1(\overline{\mathcal{L}}) \cdot (\Delta_{(P_1,\ldots,P_e)}, 0)\right)}{\deg(\Delta_{(P_1,\ldots,P_e)} \to B)} \\ &= \frac{\sum_{i=1}^e \deg(\Delta_{(P_1,\ldots,P_e)} \to \Delta_{P_i}) \widehat{\deg}\left(f^*(\widehat{c}_1(\overline{H})^d) \cdot \widehat{c}_1(\overline{\mathcal{L}}) \cdot (\Delta_{P_i}, 0)\right)}{\deg(\Delta_{(P_1,\ldots,P_e)} \to B)} \\ &= e h^{\overline{B}}_{(\mathcal{X},\mathcal{L})}(P). \end{aligned}$$

Let $\rho : X^e \to Z = X^e/\mathfrak{S}_e$ be the quotient of X^e by the symmetric group $\mathfrak{S}_e$. Since $M = \mathcal{M}_K$ is invariant under the action of $\mathfrak{S}_e$, there is a line bundle N on Z with $\rho^*(N) = M$. Thus, $\rho^*(h^{\overline{B}}_N) = h^{\overline{B}}_{(\mathcal{Y},\overline{\mathcal{M}})} + O(1)$. Hence,

$$h^{\overline{B}}_N(\rho(P_1, \ldots, P_e)) \leq e h^{\overline{B}}_L(P) + C \leq eM + C$$

for some constant C independent on P. Moreover, $\rho(P_1, \ldots, P_e)$ is defined over K. Thus, if our theorem holds for the case $e = 1$, there are finitely many $\rho(P_1, \ldots, P_e)$ with

$$h^{\overline{B}}_N(\rho(P_1, \ldots, P_e)) \leq eM + C.$$

Here the number of the fiber of ρ is $e!$ at most. Hence, we have our claim.

Let us start the proof of the theorem. First, by Claim 4.3.2, we may assume $(B, \overline{H}) = (B_0, \overline{H}_0)$. Thus, Claim 4.3.1 and Claim 4.3.3 implies our theorem. □

5. Estimate of height functions in terms of intersection numbers

Let K be a finitely generated field over $\mathbb{Q}$ with $d = \mathrm{tr.deg}_{\mathbb{Q}}(K)$, and let $\overline{B} = (B, \overline{H})$ be a polarization of K. Let X be an e-dimensional projective variety over K, and L a line bundle on X. Let $(\mathcal{X}, \overline{\mathcal{L}})$ be a C^∞-model of (X, L), and $\pi : \mathcal{X} \to B$ the canonical morphism. Then, we have the following.

Theorem 5.1. *We assume that $\widehat{\deg}(\widehat{c}_1(\overline{H})^{d+1}) = 0$, and that, for some rational number a, $\overline{\mathcal{L}} + a\pi^*(\overline{H})$ is vertically nef and $(\mathcal{L} + a\pi^*(H))_{\mathbb{Q}}$ is ample on $\mathcal{X}_{\mathbb{Q}}$. Then, we have the following.*

(1) *If $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{e+1} \cdot \widehat{c}_1(\pi^*(\overline{H}))^d) > 0$, then*

$$\sup_{Y \subsetneq X} \left\{ \inf_{x \in (X \setminus Y)(\overline{K})} h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(P) \right\} \geq 0.$$

(2) *If $\deg(H^d_{\mathbb{Q}}) > 0$ and $\inf_{x \in X(\overline{K})} h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(x) \geq 0$, then*

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{e+1} \cdot \widehat{c}_1(\pi^*(\overline{H}))^d) \geq 0.$$

because $c_1(\overline{A}_i)^d = 0$ and $f_1, \cdots, f_n$ are relatively prime. Thus, there is a constant M_1 independent on P such that $\deg_i(f_j) \leq M_1$ for all i, j. On the other hand,

$$h_{nv}^{(B,\overline{H})}(P) = \sum_i \max\{\deg_i(f_1), \ldots, \deg_i(f_n)\}\widehat{\deg}\left(\widehat{c}_1(\overline{H}|_{\Delta_\infty^{(i)}})^d\right) + \int_{(\mathbb{P}^1)^d} \log\left(\max_i\{|f_i|\}\right) c_1(\overline{H})^d.$$

Hence, there is a constant M_2 independent on P such that

$$\int_{(\mathbb{P}^1)^d} \log(|f_i|) c_1(\overline{H})^d \leq M_2$$

for all i. Thus, by Lemma 4.1, we have only finitely many f_i's as above. □

Theorem 4.3. *Let K be a finitely generated field over $\mathbb{Q}$ with $\mathrm{tr.deg}_{\mathbb{Q}}(K) = d$, and let $\overline{B} = (B, \overline{H})$ be a big polarization of K, i.e., $\overline{H}$ is nef and big. Let X be a projective variety over K, and L an ample line bundle on X. Then, for any numbers M and any positive integers e, the set*

$$\{P \in X(\overline{K}) \mid h_L^{\overline{B}}(P) \leq M, \quad [K(P) : K] \leq e\}$$

is finite.

Proof. We set $B_0 = (\mathbb{P}^1_{\mathbb{Z}})^d$ and $\overline{H}_0 = p_1^*((\mathcal{O}(1), \|\cdot\|_{FS})) \otimes \cdots \otimes p_d^*((\mathcal{O}(1), \|\cdot\|_{FS}))$.

Claim 4.3.1. *If $(B, \overline{H}) = (B_0, \overline{H}_0)$ and $e = 1$, then the theorem holds.*

If m is sufficiently large, then we have an embedding $X \hookrightarrow \mathbb{P}^n$ by $L^{\otimes m}$. Thus, we may assume $(X, L) = (\mathbb{P}^n_{\mathbb{Q}(z_1,\ldots,z_d)}, \mathcal{O}(1))$. Hence, this claim follows from Lemma 4.2.

Claim 4.3.2. *To prove the theorem, we may assume that $(B, \overline{H}) = (B_0, \overline{H}_0)$.*

As in the previous claim, we may assume $(X, L) = (\mathbb{P}^n_K, \mathcal{O}(1))$. Since $\mathrm{tr.deg}_{\mathbb{Q}}(K) = d$, K contains $\mathbb{Q}(z_1, \ldots, z_d)$, which means that there is a rational map $B \dashrightarrow B_0$. Thus, replacing B by the graph of $B \dashrightarrow B_0$, we may assume that there is a generically finite morphism $\mu : B \to B_0$. Then, since $\overline{H}$ is big, there is a positive integer n_0 such that $\overline{H}^{\otimes n_0} \otimes \mu^*(\overline{H}_0)^{\otimes -1}$ is effective by Proposition 2.2. Thus, using Proposition 3.3.8, $h_L^{(B,\mu^*(\overline{H}_0))} \leq a h_L^{(B,\overline{H})} + b$ for some positive constants a and b. Hence, we may assume that $\overline{H} = \mu^*(\overline{H}_0)$. Therefore, our claim follows from Proposition 3.3.1.

Claim 4.3.3. *In order to prove our theorem, we may assume that $e = 1$.*

It is sufficient to show that the set

$$\{P \in X(\overline{K}) \mid h_L^{\overline{B}}(P) \leq M, \quad [K(P) : K] = e\}$$

is finite for any numbers M and any integers $e \geq 1$. Let $(\mathcal{X}, \mathcal{L})$ be a C^∞-model of (X, L). Let P be a point of $X(\overline{K})$ with $h_L^{\overline{B}}(P) \leq M$ and $[K(P) : K] = e$. Let $\{\sigma_1, \ldots, \sigma_e\}$ be the set of all embeddings of $K(P)$ into $\overline{K}$. Let $P_i \in X(\overline{K})$ be a point given by $\mathrm{Spec}(\overline{K}) \xrightarrow{\sigma_i^*} \mathrm{Spec}(K(P)) \xrightarrow{P} X$, and let Δ_{P_i} be the closure of P_i in $\mathcal{X}$. Then, we have $\Delta_P = \Delta_{P_i}$ for all

Let $q_i : (\mathbb{P}^1_{\mathbb{Z}})^{d-1} \to \mathbb{P}^1_{\mathbb{Z}}$ be the projection to the i-th factor. Then, $\phi^*(p_i^*(\overline{L}_i)) = q_i^*(\overline{L}_i)$ for $i = 1, \ldots, d-1$. Moreover, if we set $c = \exp(-\widehat{\deg}(\overline{L}_d|_{\Delta_\infty}))$, then $\phi^*(p_d^*(\overline{L}_d)) = (\mathcal{O}, c\|\cdot\|_{can})$. We need to calculate $\widehat{\deg}(\widehat{c}_1(\phi^*(\overline{L}|_{\Delta_\infty^{(d)}}))^d)$. Since

$$\widehat{c}_1(\phi^*(\overline{L}|_{\Delta_\infty^{(d)}})) = q_1^*(\widehat{c}_1(\overline{L}_1)) + \cdots + q_{d-1}^*(\widehat{c}_1(\overline{L}_{d-1})) + (0, -2\log(c)),$$

it is equal to

$$\sum_{a_1+\cdots+a_d=d} \frac{d!}{a_1!\cdots a_d!} \widehat{\deg}\left(q_1^*(\widehat{c}_1(\overline{L}_1))^{a_1} \cdots q_{d-1}^*(\widehat{c}_1(\overline{L}_{d-1}))^{a_{d-1}} \cdot (0, -2\log(c))^{a_d}\right).$$

Let us consider non-zero terms in the above equation. Clearly, a_d must be 0 or 1. If $a_d = 0$, then one of $a_1, \ldots, a_{d-1}$ is 2, and others are 1. If $a_d = 1$, then $a_1 = \cdots = a_{d-1} = 1$. Thus, $\widehat{\deg}(\widehat{c}_1(\phi^*(\overline{L}|_{\Delta_\infty^{(d)}}))^d)$ is equal to

$$\frac{d!}{2} \sum_{j=1}^{d-1} \left(\prod_{\substack{1 \leq k \leq d-1 \\ k \neq j}} \deg(L_k) \right) \widehat{\deg}(\widehat{c}_1(\overline{L}_j)^2) - d! \log(c) \deg(L_1) \cdots \deg(L_{d-1}).$$

Therefore, we get our claim.

Let us go back to the proof of our proposition. We fix a number c with $0 < c < 1$. We set

$$\overline{A}_i = p_1^*((\mathcal{O}_{\mathbb{P}}(1), \|\cdot\|_{FS})) \otimes \cdots \otimes p_i^*(\mathcal{O}_{\mathbb{P}}, c\|\cdot\|_{can}) \otimes \cdots \otimes p_d^*((\mathcal{O}_{\mathbb{P}}(1), \|\cdot\|_{FS}))$$

Then, by the above claim, if we set

$$e = d! \left(-\log(c) + \frac{d-1}{2} \widehat{\deg}(\widehat{c}_1(\mathcal{O}(1), \|\cdot\|_{FS})^2) \right),$$

then

$$\widehat{\deg}\left(\widehat{c}_1(\overline{A}_i|_{\Delta_\infty^{(j)}})\right) = \begin{cases} e & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Moreover, since $\overline{H}$ is ample, there is a positive integer n_0 such that $\overline{H}^{\otimes n_0} \otimes \overline{A}_i^{\otimes -1}$ is effective for every i. Thus, by Proposition 3.3.8, there are positive constants a and b such that $h_{nv}^{(B,\overline{A}_i)} \leq a h_{nv}^{(B,\overline{H})} + b$ for all i. We set

$$S = \{P \in \mathbb{P}^n(\mathbb{Q}(z_1, \cdots, z_d)) \mid h_{nv}^{\overline{B}}(P) \leq M\}$$

Then, for any $P \in S$, $h_{nv}^{(B,\overline{A}_i)}(P) \leq aM + b$. Pick up an arbitrary $P \in S$. Then there are $f_1, \cdots, f_n \in \mathbb{Z}[z_1, \cdots, z_d]$ such that $f_1, \cdots, f_n$ are relatively prime and $P = (f_1 : \cdots : f_n)$. Then,

$$h_{nv}^{(B,\overline{A}_i)}(P) = \max\{\deg_i(f_1), \ldots, \deg_i(f_n)\}e,$$

Proof. First, let us consider the case $n = 1$. By straightforward calculations together with Jensen's formula, we can see that $v(z - \alpha) = \sqrt{1 + |\alpha|^2}$ for all $\alpha \in \mathbb{C}$. Thus, if we set $f(z) = c(z - \alpha_1) \cdots (z - \alpha_d)$, then $v(f) = c\sqrt{1 + |\alpha_1|^2} \cdots \sqrt{1 + |\alpha_d|^2}$. Therefore, we can easily see that $|f| \leq 2^d v(f)$.

In general, we will prove this lemma by induction on n. We set

$$f = a_0(z_2, \ldots, z_n) z_1^{d_1} + a_1(z_2, \ldots, z_n) z_1^{d_1 - 1} + \cdots + a_{d_1}(z_2, \ldots, z_n),$$

where $d_1 = \deg_1(f)$. Then, by the case $n = 1$,

$$\max_i \{|a_i(c_2, \ldots, c_n)|\} \leq 2^{\deg_1(f)} \exp\left(\int_{\mathbb{C}} \log(|f(z_1, c_2, \cdots, c_n)|) \omega_1 \right).$$

for all $(c_2, \ldots, c_n) \in \mathbb{C}^{n-1}$. Thus, by hypothesis of induction,

$$\begin{aligned}
\log(v(f)) &= \int_{\mathbb{C}^{n-1}} \left(\int_{\mathbb{C}} \log(|f(z_1, \ldots, z_n)|) \omega_1 \right) \omega_2 \wedge \cdots \wedge \omega_n \\
&\geq -\log(2) \deg_1(f) + \max_{a_i \neq 0} \int_{\mathbb{C}^{n-1}} \log(|a_i(z_2, \ldots, z_n)|) \omega_2 \wedge \cdots \wedge \omega_n \\
&\geq -\log(2) \deg_1(f) - \log(2)(\deg_2(f) + \cdots + \deg_n(f)) + \max_{a_i \neq 0} \log |a_i| \\
&= -\log(2)(\deg_1(f) + \cdots + \deg_n(f)) + \log(|f|).
\end{aligned}$$

Therefore, we get our lemma. □

Next let us consider the following lemma.

Lemma 4.2. *Let $B = (\mathbb{P}^1_{\mathbb{Z}})^d$ and $\overline{H} = p_1^*((\mathcal{O}_{\mathbb{P}}(1), \|\cdot\|_{FS})) \otimes \cdots \otimes p_d^*((\mathcal{O}_{\mathbb{P}}(1), \|\cdot\|_{FS}))$, where p_i is the projection to the i-th factor. Then, for any numbers M, the set*

$$\{P \in \mathbb{P}^n(\mathbb{Q}(z_1, \cdots, z_d)) \mid h^{\overline{B}}_{nv}(P) \leq M\}$$

is finite, where $\overline{B} = (B, \overline{H})$.

Proof. Let $\overline{L}_1, \ldots, \overline{L}_d$ be C^∞-hermitian line bundles on $\mathbb{P}^1_{\mathbb{Z}}$. We set $\overline{L} = p_1^*(\overline{L}_1) \otimes \cdots \otimes p_d^*(\overline{L}_d)$. Let Δ_∞ be the closure of $\infty \in \mathbb{P}^1_{\mathbb{Q}}$ in $\mathbb{P}^1_{\mathbb{Z}}$. We set $\Delta^{(i)}_\infty = p_i^*(\Delta_\infty)$. First of all, we claim the following.

Claim 4.2.1.

$$\widehat{\deg}(\widehat{c}_1(\overline{L}|_{\Delta^{(i)}_\infty})^d) =$$

$$d! \left(\prod_{\substack{1 \leq k \leq d \\ k \neq i}} \deg(L_k) \right) \widehat{\deg}(\overline{L}_i|_{\Delta_\infty}) + \frac{d!}{2} \sum_{\substack{1 \leq j \leq d \\ j \neq i}} \left(\prod_{\substack{1 \leq k \leq d \\ k \neq i, j}} \deg(L_k) \right) \widehat{\deg}(\widehat{c}_1(\overline{L}_j)^2).$$

For simplicity, we assume $i = d$. Since $\Delta_\infty \simeq \mathrm{Spec}(\mathbb{Z})$, there is an isomorphism

$$\phi : \underbrace{\mathbb{P}^1_{\mathbb{Z}} \times_{\mathbb{Z}} \cdots \times_{\mathbb{Z}} \mathbb{P}^1_{\mathbb{Z}}}_{(d-1)\text{-times}} \xrightarrow{\sim} \Delta^{(d)}_\infty.$$

3.5. **Remarks.**

Remark 3.5.3. After writing the first draft of this paper, Prof. Silverman kindly informed me the work of Altman. In [1], he gave the size function similar to our height functions. On an abelian variety A over a field K, he proved that there is a quadratic function $A(K) \to \mathbb{R}$ with $\mathrm{size}(x) \leq Q(x)$ for all $x \in A(K)$. Compared with his method, our way gives rise to the point of view of geometry, so that it is easy to handle it in the functorial framework.

Remark 3.5.4. Here, we would like to point out a similarity between our height functions and the characteristic function in Nevanlinna theory. Let us choose $f \in \mathbb{Q}(z) \setminus \{0\}$. If f has no pole at 0, then the characteristic function T_f is given by

$$T_f(r) = \sum_{|x|<r} \max\{-\operatorname{ord}_x(f), 0\} \log(r/|x|) + \int_0^{2\pi} \log^+ \left(|f(re^{\sqrt{-1}\theta})| \right) \frac{d\theta}{2\pi},$$

where $\log^+(x) = \log(\max\{x, 1\})$.

On the other hand, if we fix a polarization $(\mathbb{P}^1_{\mathbb{Z}}, (\mathcal{O}(1), \|\cdot\|_{FS}))$ of $\mathbb{Q}(z)$, then the naive height of $(f : 1) \in \mathbb{P}^1(\mathbb{Q}(z))$ is given by

$$\begin{aligned} h_{nv}(f : 1) &= \sum_{\Gamma} \max\{-\operatorname{ord}_\Gamma(f), 0\} \widehat{\deg} \left((\mathcal{O}(1), \|\cdot\|_{FS})|_\Gamma \right) + \int_{\mathbb{C}} \log^+(|f|) \frac{\sqrt{-1} dz \wedge d\bar{z}}{2\pi(1+|z|^2)^2} \\ &= \sum_{\Gamma} \max\{-\operatorname{ord}_\Gamma(f), 0\} \widehat{\deg} \left((\mathcal{O}(1), \|\cdot\|_{FS})|_\Gamma \right) \\ &\qquad + \int_0^\infty \left[\frac{2r}{(1+|r|^2)^2} \int_0^{2\pi} \log^+ \left(|f(re^{\sqrt{-1}\theta})| \right) \frac{d\theta}{2\pi} \right] dr. \end{aligned}$$

4. Northcott's theorem over finitely generated fields

Let $\mathbb{C}[z_1, \ldots, z_n]$ be the ring of n-variables polynomials over $\mathbb{C}$. For $f \in \mathbb{C}[z_1, \ldots, z_n]$, we denote by $|f|$ the maximal of the absolute values of coefficients of f. Moreover, we denote by $\deg_i(f)$ the degree of f with respect to z_i.

Let us consider the following $(1,1)$-form ω_i on $\mathbb{C}^n$ for each i;

$$\omega_i = \frac{\sqrt{-1} dz_i \wedge d\bar{z}_i}{2\pi(1+|z_i|^2)^2}.$$

For $f \in \mathbb{C}[z_1, \ldots, z_n]$, we define $v(f)$ to be

$$v(f) = \exp\left(\int_{\mathbb{C}^n} \log(|f(z_1, \ldots, z_n)|) \omega_1 \wedge \cdots \wedge \omega_n \right).$$

Then we have the following lemma.

Lemma 4.1. *For any $f \in \mathbb{C}[z_1, \ldots, z_n]$, $|f| \leq 2^{\deg_1(f) + \cdots + \deg_n(f)} v(f)$. In particular, for any numbers M and any non-negative integers $d_1, \ldots, d_n$, the set*

$$\{ f \in \mathbb{Z}[z_1, \ldots, z_n] \mid v(f) \leq M, \quad \deg_i(f) \leq d_i \ (i = 1, \ldots, n) \}$$

is finite.

is a bounded function on $A(\overline{K}) \times A(\overline{K}) \times A(\overline{K})$. Thus, there are a unique bilinear form $q_L^{\overline{B}} : A(\overline{K}) \times A(\overline{K}) \to \mathbb{R}$ and a unique linear function $l_L^{\overline{B}} : A(\overline{K}) \to \mathbb{R}$ such that

$$h_L^{\overline{B}}(x) = q_L^{\overline{B}}(x, x) + l_L^{\overline{B}}(x) + O(1)$$

(cf. [5, Chapter 5, §1]). Actually, $q_L^{\overline{B}}(x, x)$ and $l_L^{\overline{B}}(x)$ are given by the following formula:

$$q_L^{\overline{B}}(x, x) = \lim_{n\to\infty} 2^{-2n} h_L^{\overline{B}}(2^n x) \quad \text{and} \quad l_L^{\overline{B}}(x) = \lim_{n\to\infty} 2^{-n} \left(2^{-2n} h_L^{\overline{B}}(2^n x) - q_L^{\overline{B}}(x, x)\right).$$

$q_L^{\overline{B}} + l_L^{\overline{B}}$ is denoted by $\hat{h}_L^{\overline{B}}$, and is called the *canonical height function of* L with respect to the polarization $\overline{B}$. Moreover, it is easy to see that $q = 0$ if $[-1]^*(L) = L^{\otimes -1}$, and $l = 0$ if $[-1]^*(L) = L$. Thus, if L is symmetric, then $\hat{h}_L^{\overline{B}}(x) = \lim_{n\to\infty} 2^{-2n} h_L^{\overline{B}}(2^n x)$. Here let us consider the following two propositions.

Proposition 3.4.1. *Let L be a symmetric ample line bundle on A. Then, we have the following.*

(1) $\hat{h}_L^{\overline{B}}(x) \geq 0$ *for all* $x \in A(\overline{K})$.
(2) *If x is a torsion point, then* $\hat{h}_L^{\overline{B}}(x) = 0$.
(3) *We assume that $\overline{B}$ is big, i.e., $\overline{H}$ is nef and big. Then, $\hat{h}_L^{\overline{B}}(x) = 0$ if and only if x is a torsion point.*

Proof. (1) This is a consequence of Corollary 3.3.7.

(2) We assume that x is a torsion point. Then, there is a positive number n with $nx = 0$. Thus, $0 = \hat{h}_L^{\overline{B}}(nx) = n^2 \hat{h}_L^{\overline{B}}(x)$. Hence $\hat{h}_L^{\overline{B}}(x) = 0$.

(3) We assume that $\hat{h}_L^{\overline{B}}(x) = 0$. Let us consider the subgroup G generated by x. If x is defined over a finite extension field K', then every element of G is defined over K'. Moreover, the height of every element of G is zero. Thus, by Theorem 4.3 in the next section, G is a finite group, namely, x is a torsion point. □

Proposition 3.4.2. *Let L and M be symmetric line bundles on A. Then we have the following.*

(1) $\hat{h}_{L\otimes M}^{\overline{B}} = \hat{h}_L^{\overline{B}} + \hat{h}_M^{\overline{B}}$.
(2) *If L is ample, then there is a positive number a with* $\hat{h}_M^{\overline{B}} \leq a\hat{h}_L^{\overline{B}}$.
(3) *Let $\overline{B}' = (B, \overline{H}')$ be another polarization of K. If L is ample and $\overline{B}'$ is big, then there is a positive number b with* $\hat{h}_L^{\overline{B}} \leq b\hat{h}_L^{\overline{B}'}$.

Proof. (1) This is obvious.

(2) There is a positive integer a such that $L^{\otimes a} \otimes M^{\otimes -1}$ is ample. Thus, by (1) of Proposition 3.4.1, $\hat{h}_{L^{\otimes a}\otimes M^{\otimes -1}}^{\overline{B}} \geq 0$. Hence, our assertion follows from (1).

(3) By Proposition 2.2, there is a positive integer n such that $\overline{H}'^{\otimes n} \otimes \overline{H}^{\otimes -1}$ is effective. Thus it follows from Proposition 3.3.8. □

On the other hand, since $\mu^*(\overline{\mathcal{L}})$ coincides with $(\mu')^*(\overline{\mathcal{L}}')$ on the generic fiber of $\mathcal{X}'' \to B$. Thus, by Corollary 3.3.4, there is a constant C with

$$|h^{\overline{B}}_{(\mathcal{X}'',\mu^*(\overline{\mathcal{L}}))} - h^{\overline{B}}_{(\mathcal{X}'',(\mu')^*(\overline{\mathcal{L}}'))}| \leq C.$$

Therefore, we get our corollary. □

By the above corollary, the height associated with L and $\overline{B}$ is well defined up to bounded functions, so that $h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}$ modulo the set of all bounded functions on $X(\overline{K})$ is denoted by $h^{\overline{B}}_L$. Using this notation $h^{\overline{B}}_L$, we can rephrase Proposition 3.3.3 as follows, and we also have its corollary.

Proposition 3.3.6. *A height function $h^{\overline{B}}_L$ is bounded below on $(X \setminus \mathrm{Bs}(L))(\overline{K})$.*

Corollary 3.3.7. *If L is ample, then $h^{\overline{B}}_L$ is bounded below.*

Finally let us consider the following proposition.

Proposition 3.3.8. *Let $\overline{H}'$ be a nef C^∞-hermitian line bundle on B such that $\overline{H}' \otimes \overline{H}^{\otimes -1}$ is $\mathbb{Q}$-effective. If L is ample, then $h^{(B,\overline{H})}_L \leq h^{(B,\overline{H}')}_L + O(1)$.*

Proof. Clearly, we may assume that L is very ample. Let $\phi_L : X \to \mathbb{P}^N_K$ be the embedding by L. Let $\mathcal{X}$ be the closure of X in $\mathbb{P}^N_B$, and $\mathcal{O}_{\mathcal{X}}(1)$ the restriction of $\mathcal{O}(1)$ on $\mathbb{P}^N_B$. Then, $\mathcal{O}_{\mathcal{X}}(1)$ is f-ample, where f is the canonical morphism $\mathcal{X} \to B$. Thus, there is an ample line bundle Q on B such that $\mathcal{O}_{\mathcal{X}}(1) \otimes f^*(Q)$ is ample. We set $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(1) \otimes f^*(Q)$. Then, $\mathcal{L}_K = L$. Moreover, we give a C^∞-hermitian metric to $\mathcal{L}$ such that $\overline{\mathcal{L}}$ is ample.

Let us pick up $P \in X(\overline{K})$ and let Δ_P be the closure in $\mathcal{X}$. For simplicity, we set $A = \widehat{c}_1(f^*\overline{H}')\big|_{\Delta_P}$, $B = \widehat{c}_1(f^*\overline{H})\big|_{\Delta_P}$, and $C = \widehat{c}_1(\overline{\mathcal{L}})\big|_{\Delta_P}$, Then,

$$A^d \cdot C - B^d \cdot C = \sum_{i=0}^{d-1} A^i \cdot (A - B) \cdot B^{d-i-1} \cdot C.$$

On the other hand, since A, B, and C are nef, and $A - B$ is $\mathbb{Q}$-effective, by (3) of Proposition 2.3, we have

$$A^i \cdot (A - B) \cdot B^{d-i-1} \cdot C \geq 0.$$

Thus, we get $A^d \cdot C \geq B^d \cdot C$, which says us that

$$h^{(B,\overline{H})}_{(\mathcal{X},\overline{\mathcal{L}})}(P) \leq h^{(B,\overline{H}')}_{(\mathcal{X},\overline{\mathcal{L}})}(P).$$

Hence, we obtain our proposition by Corollary 3.3.5. □

3.4. **Canonical height functions on abelian varieties.** Let K be a finitely generated field over $\mathbb{Q}$, and $\overline{B} = (B, \overline{H})$ a polarization of K. Let A be an abelian variety over K, and L a line bundle on A. Then, by virtue of the cubic theorem and Corollary 3.3.4,

$$h^{\overline{B}}_L(x+y+z) - h^{\overline{B}}_L(x+y) - h^{\overline{B}}_L(y+z) - h^{\overline{B}}_L(z+x) + h^{\overline{B}}(x) + h^{\overline{B}}_L(y) + h^{\overline{B}}_L(z)$$

Note that combining the above claim with Proposition 3.3.1, we can see that $h^{\overline{B}}_{nv,K} = [K' : K]h^{\overline{B}}_{nv,K'}$ as we remarked in the previous subsection 3.2. □

Next let us consider the following proposition.

Proposition 3.3.3. *If we denote* $\mathrm{Supp}\,(\mathrm{Coker}(H^0(X,L)\otimes\mathcal{O}_X \to L))$ *by* $\mathrm{Bs}(L)$, *then there is a constant* C *with* $h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(P) \geq C$ *for all* $P \in (X \setminus \mathrm{Bs}(L))(\overline{K})$.

Proof. Let A be an ample line bundle on B such that $f_*(\mathcal{L})\otimes A$ is generated by global sections, i.e.,

$$H^0(B, f_*(\mathcal{L})\otimes A)\otimes\mathcal{O}_B \to f_*(\mathcal{L})\otimes A$$

is surjective. Hence, $H^0(\mathcal{X}, \mathcal{L}\otimes f^*(A))\otimes\mathcal{O}_{\mathcal{X}} \to f^*(f_*(\mathcal{L}))\otimes f^*(A)$ is surjective. Therefore, if we set

$$\mathcal{Z} = \mathrm{Supp}\left(\mathrm{Coker}(H^0(\mathcal{X}, \mathcal{L}\otimes f^*(A))\otimes\mathcal{O}_{\mathcal{X}} \to \mathcal{L}\otimes f^*(A))\right),$$

then $\mathcal{Z}_K = \mathrm{Bs}(L)$.

Let $\{s_1, \ldots, s_l\}$ be a free basis of $H^0(\mathcal{X}, \mathcal{L}\otimes f^*(A))$ as $\mathbb{Z}$-modules. Let us choose a metric of A such that $\|s_i\|_{\sup} < 1$ for all i. Pick up an arbitrary $P \in (X \setminus \mathrm{Bs}(L))(\overline{K})$. Then, there is a s_i with $s_i(P) \neq 0$, which shows us that $\overline{\mathcal{L}}\otimes f^*\overline{A}|_{\Delta_P}$ is effective. Thus, by (3) of Proposition 2.3,

$$\widehat{\deg}\left(\widehat{c}_1(f^*\overline{H}|_{\Delta_P})^d\cdot\widehat{c}_1(\overline{\mathcal{L}}\otimes f^*\overline{A}|_{\Delta_P})\right) \geq 0.$$

Hence, by virtue of the projection formula ((2) of Proposition 1.3), we have

$$\widehat{\deg}\left(\widehat{c}_1(f^*\overline{H}|_{\Delta_P})^d\cdot\widehat{c}_1(\overline{\mathcal{L}}|_{\Delta_P})\right) + \deg(\Delta_P \to B)\widehat{\deg}(\widehat{c}_1(\overline{H})^d\cdot\widehat{c}_1(\overline{A})) \geq 0.$$

Therefore,

$$h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(P) \geq -\widehat{\deg}(\widehat{c}_1(\overline{H})^d\cdot\widehat{c}_1(\overline{A})).$$

□

Corollary 3.3.4. *If* $\mathcal{L}_K = \mathcal{O}_X$, *then there is a constant* C *with* $|h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(P)| \leq C$ *for all* $P \in X(\overline{K})$.

Proof. Apply Proposition 3.3.3 to $\overline{\mathcal{L}}$ and $\overline{\mathcal{L}}^{\otimes -1}$. □

Corollary 3.3.5. *Let* $(\mathcal{X}, \overline{\mathcal{L}})$ *and* $(\mathcal{X}', \overline{\mathcal{L}}')$ *be two models of* (X, L). *Then, there is a constant* $C > 0$ *with*

$$|h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(P) - h^{\overline{B}}_{(\mathcal{X}',\overline{\mathcal{L}}')}(P)| \leq C$$

for all $P \in X(\overline{K})$.

Proof. Let us consider the graph $\mathcal{X}''$ of the birational map $\mathcal{X} \dashrightarrow \mathcal{X}'$. Let $\mu : \mathcal{X}'' \to \mathcal{X}$ and $\mu' : \mathcal{X}'' \to \mathcal{X}'$ be the canonical morphisms. Then, by the projection formula, we can see that

$$h^{\overline{B}}_{(\mathcal{X}'',\mu^*(\overline{\mathcal{L}}))} = h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})} \quad\text{and}\quad h^{\overline{B}}_{(\mathcal{X}'',(\mu')^*(\overline{\mathcal{L}}'))} = h^{\overline{B}}_{(\mathcal{X}',\overline{\mathcal{L}}')}.$$

Let $\mathbb{P}^n_{\mathbb{C}}$ be the n-dimensional projective space over $\mathbb{C}$, and $\mathcal{O}(1)$ the tautological line bundle on $\mathbb{P}^n_{\mathbb{C}}$. We fix a homogeneous coordinate $(X_0, \ldots, X_n)$ of $\mathbb{P}^n_{\mathbb{C}}$, i.e., a basis of $H^0(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(1))$. For a real number $l \geq 1$, we define the metric $\|\cdot\|_l$ of $\mathcal{O}(1)$ to be

$$\|X_i\|_l = \frac{|X_i|}{(|X_0|^l + \cdots + |X_n|^l)^{1/l}}.$$

$\|\cdot\|_2$ is called the *Fubini-Study metric* and is denoted by $\|\cdot\|_{FS}$. Moreover, the metric $\|\cdot\|_\infty$ is defined by

$$\|X_i\|_\infty = \frac{|X_i|}{\max\{|X_0|, \ldots, |X_n|\}}.$$

Note that $\lim_{l\to\infty} \|\cdot\|_l = \|\cdot\|_\infty$.

Let us consider $\mathbb{P}^n_B$ and the natural projection $p : \mathbb{P}^n_B \to \mathbb{P}^n_{\mathbb{Z}}$. By abuse of notation, $p^*(\mathcal{O}(1), \|\cdot\|_l)$ is denoted by $(\mathcal{O}(1), \|\cdot\|_l)$ for $1 \leq l \leq \infty$. Then, we have the following proposition.

Proposition 3.3.2. *Let $h^{\overline{B}}_{nv}$ be the naive height on $\mathbb{P}^n(\overline{K})$ defined in the previous subsection 3.2. Then, $h^{\overline{B}}_{nv}$ coincides with $h^{\overline{B}}_{(\mathbb{P}^n_B, (\mathcal{O}(1), \|\cdot\|_\infty))}$.*

Proof. By virtue of Proposition 3.3.1 (actually in the same way as the proof of it), it is sufficient to show the following claim.

Claim 3.3.2.1. $h^{\overline{B}}_{(\mathbb{P}^n_B, (\mathcal{O}(1), \|\cdot\|_\infty))}(P) = h^{\overline{B}}_{nv}(P)$ *for all* $P \in X(K)$.

First, we fix a basis $\{X_0, \ldots, X_n\}$ of $H^0(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{O}(1))$. Let Δ_P be the section corresponding with P, and $s_P : B \to \Delta_P \hookrightarrow \mathbb{P}^n_B$ the canonical morphism. For simplicity, we assume that $s_P^*(X_0) \neq 0$. If we set $a_i = s_P^*(X_i)/s_P^*(X_0)$ for $i = 0, \ldots, n$, then $a_i \in K$ and $h^{\overline{B}}_{nv}(P) = h^{\overline{B}}_{nv}(a_0, a_1, \ldots, a_n)$. Since $\mathrm{div}(s_P^*(X_0)) = \sum_\Gamma \mathrm{ord}_\Gamma(s_P^*(X_0))\Gamma$,

$$\widehat{\deg}\left(\widehat{c}_1(\overline{H})^d \cdot \widehat{c}_1(s_P^*(\mathcal{O}(1), \|\cdot\|_\infty))\right) = \sum_\Gamma \mathrm{ord}_\Gamma(s_P^*(X_0))\widehat{\deg}\left(\widehat{c}_1(\overline{H}|_\Gamma)^d\right) + \int_{B(\mathbb{C})} (-\log s_P^*(\|X_0\|_\infty))\, c_1(\overline{H})^d$$

By the definition of $\|\cdot\|_\infty$, we can see that $-\log s_P^*(\|X_0\|_\infty) = \log(\max_i\{|a_i|\})$. On the other hand, since $s_P^*(X_i)$'s generate $s_P^*(\mathcal{O}(1))$, we have

$$\begin{aligned}
\mathrm{ord}_\Gamma(s_P^*(X_0)) &= \mathrm{length}_\Gamma\left(\frac{s_P^*(\mathcal{O}(1))}{s_P^*(X_0)}\right) \\
&= \mathrm{length}_\Gamma\left(\frac{\mathcal{O}_B s_P^*(X_0) + \cdots + \mathcal{O}_B s_P^*(X_n)}{s_P^*(X_0)}\right) \\
&= \mathrm{length}_\Gamma\left(\frac{\mathcal{O}_B a_0 + \cdots + \mathcal{O}_B a_n}{\mathcal{O}_B}\right) \\
&= \max_i\{-\mathrm{ord}_\Gamma(a_i)\}.
\end{aligned}$$

Thus, we get our claim.

Let K' be a finite extension field of K, and $\overline{B}'$ the polarization of K' induced by $\overline{B}$. Then, for $(x_0, \ldots, x_n) \in K^{n+1} \setminus \{0\}$, it is easy to see that

$$h^{\overline{B}'}_{nv,K'}(x_0, \ldots, x_n) = [K' : K] h^{\overline{B}}_{nv,K}(x_0, \ldots, x_n).$$

(Of course, this can be checked directly. In the next subsection, we give an alternative definition of $h^{\overline{B}}_{nv,K}$ in terms of Arakelov intersection theory, which also shows the above formula by virtue of the projection formula.) Thus, a family $\left\{ \dfrac{1}{[K' : K]} h^{\overline{B}'}_{nv,K'} \right\}_{K'}$ of functions gives rise to a naive height function

$$h^{\overline{B}}_{nv} : \mathbb{P}^n(\overline{K}) \to \mathbb{R}$$

associated with $\overline{B}$.

3.3. **Height functions in terms of Arakelov intersection theory.** Let K be a finitely generated field over $\mathbb{Q}$ with $\mathrm{tr.deg}_{\mathbb{Q}}(K) = d$, and $\overline{B} = (B, \overline{H})$ a polarization of K. Let X be a projective variety over K, and L a line bundle on X. Let $f : \mathcal{X} \to B$ be a projective morphism of arithmetic varieties, and $\overline{\mathcal{L}}$ a continuous hermitian $\mathbb{Q}$-line bundle on $\mathcal{X}$ such that $\mathcal{X}_K = X$ and $\mathcal{L}_K = L$. We say a pair $(\mathcal{X}, \overline{\mathcal{L}})$ is called a *model of* (X, L) over B. Moreover, if $\overline{\mathcal{L}}$ is a C^∞-hermitian $\mathbb{Q}$-line bundle, then the pair $(\mathcal{X}, \overline{\mathcal{L}})$ is called a *C^∞-model* of (X, L). For $P \in X(\overline{K})$, we denote by Δ_P the Zariski closure of $\mathrm{Image}(\mathrm{Spec}(\overline{K}) \to \mathcal{X})$ in $\mathcal{X}$. Then, we define the height of P with respect to $(\mathcal{X}, \overline{\mathcal{L}})$ and $\overline{B}$ to be

$$h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(P) = \frac{1}{[K(P) : K]} \widehat{\deg} \left(\widehat{c}_1(f^* \overline{H}|_{\Delta_P})^d \cdot \widehat{c}_1(\overline{\mathcal{L}}|_{\Delta_P}) \right).$$

Let us begin with the following proposition.

Proposition 3.3.1. *Let K' be a finite extension field of K, and let $\pi : B' \to B$ be a morphism of projective normal arithmetic varieties such that the function field of B' is K'. Let $\mathcal{X}'$ be the main component of $\mathcal{X} \times_B B'$. We set the induced morphism as follows.*

$$\begin{array}{ccc} \mathcal{X} & \xleftarrow{\pi'} & \mathcal{X}' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \xleftarrow{\pi} & B'. \end{array}$$

Then, $h^{(B', \pi^*(\overline{H}))}_{(\mathcal{X}', \pi'^*(\overline{\mathcal{L}}))} = [K' : K] h^{(B,\overline{H})}_{(\mathcal{X},\overline{\mathcal{L}})}$.

Proof. Pick up $P \in X(\overline{K})$. Let Δ_P (resp. Δ'_P) be the closure in $\mathcal{X}$ (resp. $\mathcal{X}'$). Then, by the projection formula (cf. (2) of Proposition 1.3),

$$\begin{aligned} h^{(B', \pi^*(\overline{H}))}_{(\mathcal{X}', \pi'^*(\overline{\mathcal{L}}))}(P) &= \frac{1}{\deg(\Delta'_P \to B')} \widehat{\deg} \left(\widehat{c}_1(\pi'^* f^* \overline{H}|_{\Delta'_P})^d \cdot \widehat{c}_1(\pi'^* \overline{\mathcal{L}}|_{\Delta'_P}) \right) \\ &= \frac{\deg(\Delta'_P \to \Delta_P)}{\deg(\Delta'_P \to B')} \widehat{\deg} \left(\widehat{c}_1(f^* \overline{H}|_{\Delta_P})^d \cdot \widehat{c}_1(\overline{\mathcal{L}}|_{\Delta_P}) \right) \\ &= [K' : K] h^{(B,\overline{H})}_{(\mathcal{X},\overline{\mathcal{L}})}(P). \end{aligned}$$

□

Proposition 3.1.1. *Let K be a finitely generated field over $\mathbb{Q}$ with* $\mathrm{tr.deg}_{\mathbb{Q}} K = d$. *Then, there are a finite extension field K' of K, and a polarization $(B, \overline{H})$ of K' such that H is ample and* $\widehat{\deg}(\widehat{c}_1(\overline{H})^{d+1}) = 0$.

Proof. First, we need a special arithmetic surface. Let us consider the following elliptic curve due to J. Tate (cf. [10, 5.10]);

$$y^2 + xy + \epsilon^2 y = x^3,$$

where $\epsilon = (5 + \sqrt{29})/2$ is the fundamental unit of $\mathbb{Q}(\sqrt{29})$. Then, the discriminant of this curve is $-\epsilon^{10}$. We denote $\mathbb{Q}(\sqrt{29})$ by k, and the ring of integers by O_k, i.e., $O_k = \mathbb{Z}[\epsilon]$. Here we set

$$E = \mathrm{Proj}\left(O_k[X, Y, Z]/(Y^2Z + XYZ + \epsilon^2 YZ^2 - X^3)\right)$$

and $E^d = E \times_{O_k} \cdots \times_{O_k} E$. Then, since E is smooth over O_k, E^d is an abelian scheme over O_k. For an ample line bundle L on E^d, we set $H_0 = [-1]^*(L) \otimes L$. Then, H_0 is symmetric on each fiber of $E^d \to \mathrm{Spec}(O_k)$. Thus, $[2]^*(H_0) = H_0^{\otimes 4}$ because the class group of k is trivial. Moreover, on each infinite fiber, we give the cubic metric of H_0 with $[2]^*(\overline{H}_0) = \overline{H}_0^{\otimes 4}$ (cf. [6]), so that $c_1(\overline{H}_0)$ is positive on each infinite fiber. Thus, the height function $h_{\overline{H}_0}$ given by $\overline{H}_0$ is nothing more than the Néron-Tate height associated with $(H_0)_k$. Hence, we can see that $\overline{H}_0$ is nef and $\widehat{\deg}(\widehat{c}_1(\overline{H}_0)^{d+1}) = 0$ by virtue of [14, Theorem (5.2)].

Let K_0 be the function field of E^d. Then, $(E^d, \overline{H}_0)$ is a polarization of K_0. Here we take a finite extension K' of K with $K_0 \subseteq K'$. Then, the polarization of K' induced by $(E^d, \overline{H}_0)$ is our desired polarization. □

3.2. **Naive height functions.** Let K be a finitely generated field over $\mathbb{Q}$ with $\mathrm{tr.deg}_{\mathbb{Q}}(K) = d$, and $\overline{B} = (B, \overline{H})$ a polarization of K. For $(x_0, \ldots, x_n) \in K^{n+1} \setminus \{0\}$, we set

$$h^{\overline{B}}_{nv,K}(x_0, \ldots, x_n) = \sum_{\Gamma} \max_i\{-\mathrm{ord}_\Gamma(x_i)\}\widehat{\deg}\left(\widehat{c}_1(\overline{H}|_\Gamma)^d\right) + \int_{B(\mathbb{C})} \log\left(\max_i\{|x_i|\}\right) c_1(\overline{H})^d,$$

where Γ runs over all prime divisors on B. Note that if $d = 0$, then

$$\widehat{\deg}\left(\widehat{c}_1(\overline{H}|_\Gamma)^d\right) = \log \#\left(\mathcal{O}_{B,\Gamma}/\mathfrak{m}_\Gamma\right),$$

where $\mathfrak{m}_\Gamma$ is the maximal ideal at Γ. For $x \in K \setminus \{0\}$,

$$0 = \widehat{\deg}\left(\widehat{c}_1(\overline{H})^d \cdot \widehat{(x^{-1})}\right) = \sum_{\Gamma}(-\mathrm{ord}_\Gamma(x))\widehat{\deg}\left(\widehat{c}_1(\overline{H}|_\Gamma)^d\right) + \int_{B(\mathbb{C})} \log(|x|)c_1(\overline{H})^d.$$

Thus, we can see that

$$h^{\overline{B}}_{nv,K}(ax_0, \ldots, ax_n) = h^{\overline{B}}_{nv,K}(x_0, \ldots, x_n)$$

for all $(x_0, \ldots, x_n) \in K^{n+1} \setminus \{0\}$ and all $a \in K \setminus \{0\}$. Hence, we have a function

$$h^{\overline{B}}_{nv,K} : \mathbb{P}^n(K) \to \mathbb{R}.$$

Proof. First we claim the following.

Claim 2.4.1. *Let $x_1, \ldots, x_n$ be elements of T. If we set $x_i = \sum_j a_{ij} e_j$ and $A = (a_{ij})$, then*

$$(x_1 \wedge \bar{x}_1) \wedge \cdots \wedge (x_n \wedge \bar{x}_n) = |\det(A)|^2 (e_1 \wedge \bar{e}_1) \wedge \cdots \wedge (e_n \wedge \bar{e}_n).$$

Since

$$x_1 \wedge \cdots \wedge x_n = \det(A)(e_1 \wedge \cdots \wedge e_n) \quad \text{and} \quad \bar{x}_1 \wedge \cdots \wedge \bar{x}_n = \overline{\det(A)}(\bar{e}_1 \wedge \cdots \wedge \bar{e}_n),$$

we have

$$(x_1 \wedge \cdots \wedge x_n) \wedge (\bar{x}_1 \wedge \cdots \wedge \bar{x}_n) = |\det(A)|^2 (e_1 \wedge \cdots \wedge e_n) \wedge (\bar{e}_1 \wedge \cdots \wedge \bar{e}_n),$$

which shows us the claim.

By our assumption, there are a unitary matrix U_i and non-negative real numbers $\lambda_1^i, \ldots, \lambda_n^i$ such that $U_i^* H_i U_i = \mathrm{diag}(\lambda_1^i, \ldots, \lambda_n^i)$, where $A^* = \bar{A}^t$. Thus, if we take a new basis $e_1^i, \ldots, e_n^i$ according to U_i, then

$$\omega(H_i) = \sqrt{-1} \sum_{a=1}^{n} \lambda_a^i (e_a^i \wedge \bar{e}_a^i).$$

Thus, we obtain

$$\omega(H_1) \wedge \cdots \wedge \omega(H_n) = (\sqrt{-1})^n \sum_{1 \leq a_1, \ldots, a_n \leq n} \lambda_{a_1}^1 \cdots \lambda_{a_n}^n (e_{a_1}^1 \wedge \bar{e}_{a_1}^1) \wedge \cdots \wedge (e_{a_n}^n \wedge \bar{e}_{a_n}^n).$$

On the other hand, by the above claim, there is a non-negative real number $\tau_{a_1, \ldots, a_n}$ with

$$(e_{a_1}^1 \wedge \bar{e}_{a_1}^1) \wedge \cdots \wedge (e_{a_n}^n \wedge \bar{e}_{a_n}^n) = \tau_{a_1, \ldots, a_n} (e_1 \wedge \bar{e}_1) \wedge \cdots \wedge (e_n \wedge \bar{e}_n).$$

Therefore,

$$\omega(H_1) \wedge \cdots \wedge \omega(H_n) = \left(\sum_{1 \leq a_1, \ldots, a_n \leq n} \tau_{a_1, \ldots, a_n} \lambda_{a_1}^1 \cdots \lambda_{a_n}^n \right) (\sqrt{-1})^n (e_1 \wedge \bar{e}_1) \wedge \cdots \wedge (e_n \wedge \bar{e}_n).$$

Hence, we get our lemma. □

3. Arithmetic height functions over finitely generated fields

3.1. **Polarization of finitely generated fields over $\mathbb{Q}$.** Let K be a finitely generated field over $\mathbb{Q}$ with $\mathrm{tr.deg}_{\mathbb{Q}}(K) = d$. A pair $(B, \overline{H})$ is called a *polarization* of K if B is a projective normal arithmetic variety whose function field is K, and $\overline{H}$ is a nef C^∞-hermitian $\mathbb{Q}$-line bundle on B. For simplicity, the pair $(B, \overline{H})$ is sometimes denoted by $\overline{B}$. Moreover, the polarization $(B, \overline{H})$ is said to be *big* (resp. *of pencil type*) if $\overline{H}$ is big (resp. $\widehat{\deg}(\widehat{c}_1(\overline{H})^{d+1}) = 0$).

Let K' be a finite extension field of K, and $\mu : B' \to B$ the normalization of B in K'. Then, we have a polarization $(B', \mu^*(\overline{H}))$ of K'. This polarization is denoted by $\overline{B}_{K'}$, and is called the *polarization of K' induced by $\overline{B}$*. Clearly, if $\overline{B}$ is big (resp. of pencil type), then so is $\overline{B}_{K'}$.

Here let us consider the existence of a non-trivial polarization of pencil type.

(2) *If* $\overline{L}_1, \ldots, \overline{L}_{d+1}$ *are nef* C^∞*-hermitian* $\mathbb{Q}$*-line bundles, then*

$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_{d+1})\right) \geq 0.$$

(3) *If* $\overline{L}_1, \ldots, \overline{L}_d$ *are nef* C^∞*-hermitian* $\mathbb{Q}$*-line bundles and* $\overline{M}$ *is a* $\mathbb{Q}$*-effective continuous hermitian* $\mathbb{Q}$*-line bundle, then*

$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_d)\cdot\widehat{c}_1(\overline{M})\right) \geq 0.$$

Proof. (1) First of all, let us see $\widehat{\deg}\left((\widehat{c}_1(\overline{L})+\epsilon\widehat{c}_1(\overline{A}))^{d+1}\right) > 0$. For this purpose, it is sufficient to see that $\widehat{\deg}\left(\widehat{c}_1(\overline{L})^i\cdot\widehat{c}_1(\overline{A})^{d+1-i}\right) \geq 0$ for $0 < i < d+1$ because $\widehat{\deg}\left(\widehat{c}_1(\overline{A})^{d+1}\right) > 0$ and $\widehat{\deg}\left(\widehat{c}_1(\overline{L})^{d+1}\right) \geq 0$ by [14, Lemma 5.4]. This can be proved by induction on d. Indeed, we take a non-zero section $s \in H^0(B, A^{\otimes n})$ with $\|s\|_{\sup} < 1$. Let $\mathrm{div}(s) = a_1\Gamma_1 + \cdot + a_r\Gamma_r$ be the irreducible decomposition as cycles. Then,

$$n\widehat{\deg}\left(\widehat{c}_1(\overline{L})^i\cdot\widehat{c}_1(\overline{A})^{d+1-i}\right) = \sum_i a_i\widehat{\deg}\left(\widehat{c}_1(\overline{L}|_{\Gamma_i})^i\cdot\widehat{c}_1(\overline{A}|_{\Gamma_i})^{d-i}\right) + \int_B(-\log\|s\|)\,c_1(\overline{L})^i c_1(\overline{A})^{d-i}.$$

Thus, by hypothesis of induction and Lemma 2.4, we can see $\widehat{\deg}\left(\widehat{c}_1(\overline{L})^i\cdot\widehat{c}_1(\overline{A})^{d+1-i}\right) \geq 0$.

Let us go back to the proof of (1). We prove (1) by induction on d. Thus, it follows from Nakai-Moishezon's criterion on arithmetic varieties (cf. [14, Corollary 4.8]) and the previous observation.

(2) We prove this by induction on d. Let $\overline{A}$ be an ample line bundle on B. Let ϵ is a positive rational number. Then, $\overline{L}_{d+1}+\epsilon\overline{A}$ is ample by (1). Thus, using small section of $\overline{L}_{d+1}+\epsilon\overline{A}$, hypothesis of induction and Lemma 2.4, we can see

$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_d)\cdot(\overline{L}_{d+1}+\epsilon\overline{A})\right) \geq 0.$$

Thus, we have our assertion taking $\epsilon \to 0$.

(3) This is a consequence of (2). □

Here let us consider the following lemma, which was used in the proof of the previous proposition.

Lemma 2.4. *Let* V *be a vector space over* $\mathbb{R}$ *with a complex structure* J*, i.e., an endomorphism* $J : V \to V$ *with* $J^2 = -\mathrm{id}$*. Let* T *(resp.* T'*) be the eigenspace of* J *with respect to* $\sqrt{-1}$ *(resp.* $-\sqrt{-1}$*) in* $V\otimes_{\mathbb{R}}\mathbb{C}$*. Note that the complex conjugation in* $V\otimes_{\mathbb{R}}\mathbb{C}$ *gives rise to the anti-isomorphism of* T *and* T' *over* $\mathbb{C}$*. Let us fix a basis* $\{e_1, \ldots, e_n\}$ *of* T *over* $\mathbb{C}$*. For a hermitian* $n\times n$*-matrix* $H = (h_{ij})$*, we set*

$$\omega(H) = \sqrt{-1}\sum_{i,j} h_{ij}(e_i\wedge\bar{e}_j).$$

If $H_1, \ldots, H_n$ *are semipositive hermitian* $n\times n$*-matrices, then there is a non-negative real number* λ *with*

$$\omega(H_1)\wedge\cdots\wedge\omega(H_n) = \lambda(\sqrt{-1})^n(e_1\wedge\bar{e}_1)\wedge\cdots\wedge(e_n\wedge\bar{e}_n).$$

$\widehat{\deg}\left(\overline{H}\big|_{\Gamma}\right) \geq 0$. Moreover, if $\overline{H}$ is vertically nef and horizontally nef, then we say $\overline{H}$ is *nef*. Further, $\overline{H}$ is said to be *big* if $\operatorname{rk} H^0(B, H^{\otimes m}) = O(m^d)$, and there is a non-zero section s of $H^0(B, H^{\otimes n})$ with $\|s\|_{\sup} < 1$ for some positive integer n. It is easy to see that if $\overline{H}$ is ample, then $\overline{H}$ is nef and big. The following theorem due to Faltings, Gillet-Soulé and Zhang is a very useful criterion for the bigness of C^∞-hermitian line bundles (cf. [14, Theorem 1.4]).

Theorem 2.1. *Let B be a projective arithmetic variety with $d = \dim B_{\mathbb{Q}}$, and $\overline{L}$ a C^∞-hermitian line bundle on B. If $\overline{L}$ is vertically nef, $L_{\mathbb{Q}}$ is ample on $B_{\mathbb{Q}}$, and $\widehat{\deg}\left(\widehat{c}_1(\overline{L})^{d+1}\right) > 0$, then $\overline{L}$ is big.*

Moreover, we have the following.

Proposition 2.2. *Let B be a projective arithmetic variety with $d = \dim B_{\mathbb{Q}}$, and $\overline{L}$ a C^∞-hermitian line bundle on B. Then, the following are equivalent.*

(1) *$\overline{L}$ is big.*
(2) *For any C^∞-hermitian line bundles $\overline{M}$ on B, there are a positive integer n and a non-zero section s of $H^0(B, L^{\otimes n} \otimes M)$ with $\|s\|_{\sup} < 1$.*

Proof. First, we assume (1). Then, there is a non-zero section s_1 of $H^0(B, L^{\otimes n_1})$ with $\|s_1\|_{\sup} < 1$ for some n_1. Moreover, since $\operatorname{rk} H^0(B, L^{\otimes m}) = O(m^d)$, we can find a non-zero section s_2 of $H^0(B, L^{\otimes n_2} \otimes M)$. Let n_3 be a sufficiently large integer with

$$(\|s_1\|_{\sup})^{n_3} \|s_2\|_{\sup} < 1.$$

Then, $s_1^{\otimes n_3} \otimes s_2 \in H^0(B, L^{\otimes n_3 n_1 + n_2} \otimes M)$, and

$$\|s_1^{\otimes n_3} \otimes s_2\|_{\sup} \leq (\|s_1\|_{\sup})^{n_3} \|s_2\|_{\sup} < 1.$$

Thus, we get (2).

Next, we assume (2). It is sufficient to show that $\operatorname{rk} H^0(B, L^{\otimes m}) = O(m^d)$. Let $\overline{A}$ be an ample C^∞-hermitian line bundle on B. Then, there are a positive integer n_1 and a non-zero section s of $H^0(B, L^{\otimes n_1} \otimes A^{\otimes -1})$. Thus, we have an injection $A \to L^{\otimes n_1}$. Therefore, $\operatorname{rk} H^0(B, L^{\otimes m}) = O(m^d)$. □

Let $\widehat{\operatorname{Pic}}_{C^\infty}(B)$ (resp. $\widehat{\operatorname{Pic}}_{C^0}(B)$)be the isomorphic classes of C^∞-hermitian (resp. continuous hermitian) line bundles on B. An element of $\widehat{\operatorname{Pic}}_{C^\infty}(B) \otimes \mathbb{Q}$ (resp. $\widehat{\operatorname{Pic}}_{C^0}(B) \otimes \mathbb{Q}$) is called a C^∞-hermitian (resp. continuous hermitian) $\mathbb{Q}$-line bundle on B. For simplicity, the group structure of $\widehat{\operatorname{Pic}}_{C^\infty}(B) \otimes \mathbb{Q}$ (or $\widehat{\operatorname{Pic}}_{C^0}(B) \otimes \mathbb{Q}$) is often written additively. Note that the previous definitions of 'ample', 'vertically nef', 'horizontally nef', 'nef' and 'big' work for C^∞-hermitian $\mathbb{Q}$-line bundles on B. Let $\overline{L}$ be a continuous hermitian $\mathbb{Q}$-line bundle. We say $\overline{L}$ is *effective* if $\overline{L} \in \operatorname{Pic}_{C^0}(B)$ and $H^0(B, L)$ contains a non-zero section s with $\|s\|_{\sup} \leq 1$. Moreover, $\overline{L}$ is said to be $\mathbb{Q}$-*effective* if $n\overline{L}$ is effective for some positive integer n.

Proposition 2.3. *Let B be a projective arithmetic variety with $d = \dim B_{\mathbb{Q}}$. Then, we have the following.*

(1) *If $\overline{L}$ is a nef C^∞-hermitian $\mathbb{Q}$-line bundle and $\overline{A}$ is an ample C^∞-hermitian $\mathbb{Q}$-line bundle, then $\overline{L} + \epsilon\overline{A}$ is ample for all positive rational numbers ϵ.*

T. We say $c_1(\overline{L})$ is *semipositive* if, for any analytic morphisms $h : M \to T$ from any complex manifolds M to T, $c_1(h^*(\overline{L}))$ is a semipositive form on M.

Let X be a projective arithmetic variety of $d = \dim X_{\mathbb{Q}}$. Let $\mu : X' \to X$ be a generic resolution of singularities of X, namely, $\mu : X' \to X$ is a birational morphism of projective arithmetic varieties such that X' is generically smooth. Note that a generic resolution of singularities exists for any arithmetic varieties by using Hironaka's resolution of singularities [3]. Let $\overline{L}_1, \cdots, \overline{L}_d$ be C^∞-hermitian line bundles on X, and $\overline{M}$ a continuous hermitian line bundle on X. Then, the intersection number $\widehat{\deg}\left(\widehat{c}_1(\mu^*\overline{L}_1)\cdots\widehat{c}_1(\mu^*\overline{L}_d)\cdot\widehat{c}_1(\mu^*\overline{M})\right)$ does not depend on the choice of the generic resolution of singularities $\mu : X' \to X$ by virtue of Proposition 1.2, so that we define

$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_d)\cdot\widehat{c}_1(\overline{M})\right)$$

by $\widehat{\deg}\left(\widehat{c}_1(\mu^*\overline{L}_1)\cdots\widehat{c}_1(\mu^*\overline{L}_d)\cdot\widehat{c}_1(\mu^*\overline{M})\right)$. Then, we have the following proposition.

Proposition 1.3. *Let $f : X \to Y$ be a morphism of projective arithmetic varieties with $d = \dim X_{\mathbb{Q}}$ and $n = \dim Y_{\mathbb{Q}}$*

(1) *Let $\overline{L}_1, \ldots, \overline{L}_r$ be C^∞-hermitian line bundles on X, and $\overline{M}_1, \ldots, \overline{M}_s$ C^∞-hermitian line bundles on Y. We assume that $r + s = d + 1$. Then,*

$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_r)\cdot\widehat{c}_1(f^*\overline{M}_1)\cdots\widehat{c}_1(f^*\overline{M}_s)\right) = \begin{cases} 0 & \text{if } s > n+1 \\ \deg((L_1)_\eta\cdots(L_r)_\eta)\widehat{\deg}\left(\widehat{c}_1(\overline{M}_1)\cdots\widehat{c}_1(\overline{M}_s)\right) & \text{if } s = n+1, \end{cases}$$

where the η means the restriction to the generic fiber of f.

(2) *We assume that f is generically finite. Let $\overline{L}_1, \ldots, \overline{L}_n$ be C^∞-hermitian line bundles on Y, and $\overline{M}$ a continuous hermitian line bundle on Y. Then,*

$$\widehat{\deg}\left(\widehat{c}_1(f^*\overline{L}_1)\cdots\widehat{c}_1(f^*\overline{L}_n)\cdot\widehat{c}_1(f^*\overline{M})\right) = \deg(f)\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_n)\cdot\widehat{c}_1(\overline{M})\right).$$

Proof. Let us consider the following commutative diagram:

$$\begin{array}{ccc} X & \xleftarrow{\mu} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow{\nu} & Y' \end{array}$$

where $\mu : X' \to X$ and $\nu : Y' \to Y$ are generic resolutions of singularities of X and Y respectively. Then, our assertions are consequences of Proposition 1.2. □

2. Arithmetically positive hermitian line bundles

Let B be a projective arithmetic variety with $d = \dim B_{\mathbb{Q}}$, and $\overline{H}$ a C^∞-hermitian line bundle on B. From now on, we introduce several kinds of positivity of $\overline{H}$. First of all, we say $\overline{H}$ is *ample* if H is ample, $c_1(\overline{H})$ is semipositive on $B(\mathbb{C})$, and, for a sufficiently large n, $H^0(B, H^{\otimes n})$ is generated by $\{s \in H^0(B, H^{\otimes n}) \mid \|s\|_{\sup} < 1\}$. $\overline{H}$ is said to be *vertically nef* if H is relatively nef with respect to $B \to \mathrm{Spec}(\mathbb{Z})$, and $c_1(\overline{H})$ is semipositive on $B(\mathbb{C})$. We say $\overline{H}$ is *horizontally nef* if, for all one-dimensional integral closed subschemes Γ flat over $\mathbb{Z}$,

Here we define

$$\widehat{\mathrm{CH}}^p(X) = \widehat{Z}^p(X)/\widehat{R}^p(X), \quad \text{and} \quad \widehat{\mathrm{CH}}^p_D(X) = \widehat{Z}^p_D(X)/\widehat{R}^p(X).$$

Let $\overline{L} = (L, \|\cdot\|)$ be a C^∞-hermitian line bundle on X. We would like to define a homomorphism

$$\widehat{c}_1(\overline{L})\cdot \ : \widehat{\mathrm{CH}}^p_D(X) \to \widehat{\mathrm{CH}}^{p+1}_D(X). \tag{1.1}$$

This can be defined as follows. Let (Z, g) be an element of $\widehat{Z}^p_D(X)$. We assume that Z is integral. Then, taking a rational section s of $L|_Z$, we consider an arithmetic D-cycle

$$(\mathrm{div}(s), -\log\left(\|s\|^2_Z\right) + c_1(\overline{L}) \wedge g),$$

where $\|s\|_Z$ is a function on $Z(\mathbb{C})$. The class of the above cycle in $\widehat{\mathrm{CH}}^{p+1}_D(X)$ does not depend on the choice of the rational section s. Thus, by linearity, we have a homomorphism

$$\widehat{c}_1(\overline{L})\cdot \ : \widehat{Z}^p_D(X) \to \widehat{\mathrm{CH}}^{p+1}_D(X).$$

On the other hand, it is well known that $\widehat{c}_1(\overline{L}) \cdot \widehat{R}^p(X) \subset \widehat{R}^{p+1}(X)$. Thus, we obtain our desired homomorphism (1.1).

Now let $\overline{M} = (M, \|\cdot\|)$ be a continuous hermitian line bundle on X, namely, $\|\cdot\|$ is a continuous metric. Then, $\widehat{c}_1(\overline{M})$ is defined by the class of $(\mathrm{div}(s), -\log\|s\|^2)$ in $\widehat{\mathrm{CH}}^1_D(X)$, where s is a rational section of M. This is actually well defined because the class does not depend on the choice of the rational section s. Then, using the scalar product (1.1), for C^∞-hermitian line bundles $\overline{L}_1, \ldots, \overline{L}_d$ on X, we can define

$$\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_d) \cdot \widehat{c}_1(\overline{M}) \in \widehat{\mathrm{CH}}^{d+1}_D(X),$$

where $d = \dim X_{\mathbb{Q}}$. In particular, if X is projective, then we get the intersection number

$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_d) \cdot \widehat{c}_1(\overline{M})\right).$$

Next, let us consider the push-forward of cycles. Let $f : X \to Y$ be a projective morphism of generically smooth arithmetic varieties. Then, $f_* : \widehat{Z}^p_D(X) \to \widehat{Z}^{p+\dim Y - \dim X}_D(Y)$ is defined by $f_*(Z, g) = (f_*(Z), f_*(g))$. This induces

$$f_* : \widehat{\mathrm{CH}}^p_D(X) \to \widehat{\mathrm{CH}}^{p+\dim Y - \dim X}_D(Y).$$

Then, we have the following projection formula (cf. [4, Proposition 2.4.1]).

Proposition 1.2. *Let $\overline{L}$ be a C^∞-hermitian line bundle on Y, and z an element of $\widehat{\mathrm{CH}}^p_D(X)$. Then,*

$$f_*(\widehat{c}_1(f^*(\overline{L})) \cdot z) = \widehat{c}_1(\overline{L}) \cdot f_*(z).$$

Finally, let us consider intersections on a general projective arithmetic variety. Let T be a reduced complex space, and $\overline{L} = (L, \|\cdot\|)$ a continuous hermitian line bundle on T. According to [14], we say $\overline{L}$ is C^∞ if, for any analytic morphisms $h : M \to T$ from any complex manifolds M to T, $h^*(\overline{L})$ is a C^∞-hermitian line bundle on M. From now on, we assume that $\overline{L}$ is C^∞. Then, $c_1(\overline{L})$ is a C^∞-form on T_{reg}, where T_{reg} is the smooth locus of

Theorem A (cf. Theorem 4.3). *If L is ample, then, for any numbers M and any positive integers e, the set*

$$\{P \in X(\overline{K}) \mid h_L^{arith}(P) \leq M, \quad [K(P) : K] \leq e\}$$

is finite.

Now let A be an abelian variety over K, and L a symmetric ample line bundle on A. Then, in the same way as the usual height theory, we can assign the canonical height (Néron-Tate height) $\hat{h}_L^{arith}$ to (A, L). Then, $\hat{h}_L^{arith}(x) \geq 0$ for all $x \in A(\overline{K})$, and $\hat{h}_L^{arith}(x) = 0$ if and only if x is a torsion point as a corollary of Theorem A (cf. Proposition 3.4.1). Moreover, in terms of $\hat{h}_L^{arith}$, we have the following solution of Bogomolov's conjecture over K, which is a generalization of results due to Ullmo [13] and Zhang [15].

Theorem B (cf. Theorem 8.1). *Let X be a subvariety of $A_{\overline{K}}$. If the set*

$$\{P \in X(\overline{K}) \mid \hat{h}_L^{arith}(P) \leq \epsilon\}$$

is Zariski dense in X for any positive numbers ϵ, then X is a translation of an abelian subvariety of $A_{\overline{K}}$ by a torsion point.

As corollary, we can recover the original Raynaud's theorem ([8] and [9]) conjectured by Manin and Mumford.

Corollary C (cf. Corollary 8.2). *Let A be an abelian variety over the complex number field $\mathbb{C}$, and X a subvariety of A. If $X(\mathbb{C}) \cap A(\mathbb{C})_{tor}$ is Zariski dense in X, then X is a translation of an abelian subvariety of A by a torsion point.*

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1. Arakelov intersection theory

In this paper, an arithmetic variety means a flat and quasi-projective integral scheme over $\mathbb{Z}$. Moreover, we say an arithmetic variety is *generically smooth* if it is smooth over $\mathbb{Q}$. For basic materials of Arakelov intersection theory, we refer to [2] and [11].

Let X be a generically smooth arithmetic variety. According to [4], a pair (Z, g) is called an *arithmetic cycle of codimension p* (resp. *arithmetic D-cycle of codimension p*) if Z is a cycle of codimension p on X, and g is a Green current for $Z(\mathbb{C})$ (resp. g is a current of type $(p-1, p-1)$ on $X(\mathbb{C})$). The set of all arithmetic cycles (resp. D-cycles) of codimension p is denoted by $\widehat{Z}^p(X)$ (resp. $\widehat{Z}^p_D(X)$). Let $\widehat{R}^p(X)$ be the subgroup of $\widehat{Z}^p(X)$ generated by the following elements:

(i) $((f), -[\log|f|^2])$, where f is a rational function on some subvariety Y of codimension $p-1$ and $[\log|f|^2]$ is the current defined by

$$[\log|f|^2](\gamma) = \int_{Y(\mathbb{C})} (\log|f|^2)\gamma.$$

(ii) $(0, \partial(\alpha) + \bar{\partial}(\beta))$, where α and β are currents of type $(p-2, p-1)$ and $(p-1, p-2)$ respectively.

It is well know that this height function can be given in terms of the usual intersection theory, so that it is rather easy to handle it. However, compared with height functions over number fields, it does not reflect the exact state of points on $\mathbb{P}^n(\overline{K})$. For example, Northcott's theorem does not hold for the geometric height function. A reason for this, we can consider, is that h^{geom} does not take care of data coming from the constant field. In this paper, we would like to propose a new kind of height functions for finitely generated fields over $\mathbb{Q}$, and unify them with the geometric height functions.

A key idea to get a new height function is to fix a polarization $\overline{B} = (B, \overline{H})$ of K, where B is a projective normal arithmetic variety whose function field is K, and $\overline{H}$ is a nef C^∞-hermitian line bundle on B, namely, $c_1(\overline{H})$ is semipositive, and, for any one-dimensional integral subschemes Γ of B, $\widehat{\deg}\left(\overline{H}\big|_\Gamma\right) \geq 0$. Once we fix the polarization $\overline{B}$ of K, then we can define a height function

$$h_K^{\overline{B}} : \mathbb{P}^n(K) \to \mathbb{R}$$

associated with $\overline{B}$ to be

$$h_K^{\overline{B}}(\phi_0, \ldots, \phi_n) = \sum_\Gamma \max_i\{-\operatorname{ord}_\Gamma(\phi_i)\}\widehat{\deg}\left(\widehat{c}_1\left(\overline{H}\big|_\Gamma\right)^d\right) + \int_{B(\mathbb{C})} \log\left(\max_i\{|\phi_i|\}\right) c_1(\overline{H})^d$$

where Γ runs over all prime divisors on B. Moreover, we can easily see that $h_K^{\overline{B}}$ extends to

$$h^{\overline{B}} : \mathbb{P}^n(\overline{K}) \to \mathbb{R}.$$

For example, if $d = 1$ and $\overline{H}$ is given by the infinite fibers of B, then $h^{\overline{B}}$ is nothing more than h^{geom} up to the multiplication of a positive constant. Moreover, note that if $d = 0$, then $h^{\overline{B}}$ is the usual height function over a number field.

Further, we can give these height functions in terms of Arakelov intersection theory. Let X be a projective variety over K, and L a line bundle on X. Let us take a model $(\mathcal{X}, \overline{\mathcal{L}})$ of (X, L), namely, $\mathcal{X}$ is a projective arithmetic variety over B and $\overline{\mathcal{L}}$ is a hermitian line bundle on $\mathcal{X}$ with $\mathcal{X}_K = X$ and $\mathcal{L}_K = L$. For a point $P \in X(\overline{K})$, we denote by Δ_P the closure of the image of $\operatorname{Spec}(\overline{K}) \to X \hookrightarrow \mathcal{X}$. Then, we define

$$h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})} : X(\overline{K}) \to \mathbb{R}$$

to be

$$h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(P) = \frac{1}{[K(P) : K]}\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}\big|_{\Delta_P}) \cdot \widehat{c}_1(f^*\overline{H}\big|_{\Delta_P})^d\right),$$

where $f : \mathcal{X} \to B$ is the canonical morphism. We can see that $h^{\overline{B}}_{(\mathcal{X},\mathcal{L})}$ modulo the set of bounded functions on $X(\overline{K})$ does not depend on the choice of the model $(\mathcal{X}, \overline{\mathcal{L}})$ of (X, L) (cf. Corollary 3.3.5), so that we may denote $h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}$ by $h^{\overline{B}}_L$.

Since our height functions include the geometric height functions, Northcott's theorem does not hold in general. However, if the polarization $\overline{B}$ is big, we can expect a certain kind of affirmative answers, so that we introduce the following notation. If $H_\mathbb{Q}$ is big on $B_\mathbb{Q}$ and $H^0(B, H^{\otimes n})$ has a strictly small section for some $n > 0$, then $h^{\overline{B}}_L$ is called an *arithmetic height function* and is denoted by h^{arith}_L for simplicity. Then, we have the following Northcott's theorem for the arithmetic height function.

ARITHMETIC HEIGHT FUNCTIONS OVER FINITELY GENERATED FIELDS

ATSUSHI MORIWAKI

Abstract. In this paper, we propose a new height function for a variety defined over a finitely generated field over $\mathbb{Q}$. For this height function, we will prove Northcott's theorem and Bogomolov's conjecture, so that we can recover the original Raynaud's theorem (Manin-Mumford's conjecture).

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Introduction

Let K be a finitely generated field over $\mathbb{Q}$ with $\mathrm{tr.deg}_{\mathbb{Q}}(K) = d$. If we assume $d = 1$, then there is a smooth projective curve C over a number field such that the function field of C is K. Using non-archimedean valuations arising from points of C, we can define a geometric height function

$$h^{geom} : \mathbb{P}^n(\overline{K}) \to \mathbb{R}.$$

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