$D^0 = D \cap Y^0$, $Y_o^0 = Y_o \cap Y^0$, $S^0 = S \cap Y^0$, and $\Gamma_i^0 = \Gamma_i \cap Y^0$ for each i. Then,

$$D^0 \sqcap S^0 = \sum_i \alpha_i(\Gamma_i^0 \sqcap S^0) \quad \text{and} \quad Y_o^0 \sqcap S^0 = \sum_i \beta_i(\Gamma_i^0 \sqcap S^0)$$

in $Z^2(Y^0)$. Thus,

$$\sum_i \alpha_i(\Gamma_i^0 \sqcap S^0) = \sum_i \alpha\beta_i(\Gamma_i^0 \sqcap S^0)$$

in $Z^2(Y^0)_{\mathbb{Q}}$. Here $H^{\otimes m}$ is very ample and $\mathrm{div}(s_0|_{(Y_o)_{red}})$ intersects with $(\Gamma_i \cap \mathrm{Sing}(Y))_{red}$ and $(\Gamma_i \cap \Gamma_j)_{red}$ properly for all $i \neq j$. Therefore, $\Gamma_i^0 \sqcap S^0 \neq 0$ for all i, and $\Gamma_i^0 \sqcap S^0$ and $\Gamma_j^0 \sqcap S^0$ have no common component for all $i \neq j$. Thus, we have $\alpha_i = \alpha\beta_i$ for all i. Hence $[D] = \alpha[Y_o]$ in $Z^1(Y)_{\mathbb{Q}}$. □

References

[1] J.-B. Bost, Potential Theory and Lefschetz Theorems for Arithmetic Surfaces, preprint.
[2] G. Faltings, Calculus on arithmetic surfaces, Ann. of Math., 119 (1984), 387–424.
[3] W. Fulton, Intersection Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 2, (1984), Springer-Verlag.
[4] H. Gillet and C. Soulé, Arithmetic Intersection Theory, Publ. Math. (IHES), 72 (1990), 93–174.
[5] H. Gillet and C. Soulé, An arithmetic Riemann-Roch theorem, Invent. Math., 110 (1992), 473–543.
[6] H. Gillet and C. Soulé, Arithmetic analogs of the standard conjectures, Proceedings of Symposia in Pure Mathematics, 55-I (1994), 129–140.
[7] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, Ann. of Math. 79 (1964), 109–326.
[8] P. Hriljac, Heights and Arakelov's intersection theory, Amer. J. Math., 107 (1985), 23–38.
[9] J.-P. Jouanolou, Théorèmes de Bertini et Applications, Birkhäuser (1983).
[10] S. Kawaguchi and A. Moriwaki, Inequalities for semistable families of arithmetic varieties, preprint (alg-geom/9710007).
[11] J. Lipman, Desingularization of two dimensional schemes, Ann. Math., 107 (1978), 151–207.
[12] A. Moriwaki, Inequality of Bogomolov-Gieseker type on arithmetic surfaces, Duke Math. J., 74 (1994), 713–761.
[13] A. Moriwaki, Bogomolov unstability on arithmetic surfaces, Math. Research Letters, 1 (1994), 601–611.
[14] A. Moriwaki, Arithmetic Bogomolov-Gieseker's inequality, Amer. J. of Math., 117 (1995), 1325–1347.
[15] A. Moriwaki, Hodge index theorem for arithmetic cycles of codimension one, Mathematical Research Letter, 3 (1996), 173–183.

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claim that if (a) and (b) hold for Y', then they hold for Y. Note that if $d = 1$, then the lemma does not involve H. By virtue of projection formula (cf. [3, (c) of Proposition 2.4]),

$$\deg(\mu^*(H)^{d-1} \cdot \mathcal{O}_{Y'}(\mu^*(D)) \cdot [\mu^*(D)]) = \deg(H^{d-1} \cdot \mathcal{O}_Y(D) \cdot [D]).$$

Thus, if $Q_H([\mu^*(D)], [\mu^*(H)]) \leq 0$, then $Q_H([D], [D]) \leq 0$. Moreover, if there is a rational number α such that $[\mu^*(D)] = \alpha[Y'_o]$, then $[\mu^*(D)] = \alpha[\mu^*(Y_o)]$. Thus, taking the push-forward μ_*, we can see that $[D] = \alpha[Y_o]$ in $Z^1(Y)_{\mathbb{Q}}$. Hence, we get our claim.

By the above claim, considering the normalization of Y, we may assume that Y is normal. Moreover, if $d = 1$, there is a resolution of singularities $\mu : Y' \to Y$ of Y (cf. [11]). Thus it holds for $d = 1$. Hence we may assume $d \geq 2$.

Let $(Y_o)_{red} = \Gamma_1 \cup \cdots \cup \Gamma_l$ be the irreducible decomposition of $(Y_o)_{red}$ and I the defining ideal of $(Y_o)_{red}$. Since H is ample, there is a positive integer m such that $H^{\otimes m}$ is very ample and $H^1(Y, H^{\otimes m} \otimes I) = 0$. Thus, $H^0(Y, H^{\otimes m}) \to H^0((Y_o)_{red}, H^{\otimes m}|_{(Y_o)_{red}})$ is surjective. Here note that $\mathrm{codim}(Y \setminus \mathrm{Sing}(Y)) \geq 2$ because Y is normal. Hence, there is a section s_0 of $H^0(X, H^{\otimes m})$ such that $s_0|_{\Gamma_i} \neq 0$ for every i, and that $\mathrm{div}(s_0|_{(Y_o)_{red}})$ intersects with $(\Gamma_i \cap \mathrm{Sing}(Y))_{red}$ and $(\Gamma_i \cap \Gamma_j)_{red}$ properly for all $i \neq j$. Let t be a element of R such that t is a generator of the maximal ideal of R. Since $H^0(Y, H^{\otimes m})$ is a free R-module, there is a basis $\{e_1, \ldots, e_n\}$ of $H^0(Y, H^{\otimes m})$ as R-module. Then, there are $c_1, \ldots, c_n \in R$ with $s_0 = c_1e_1 + \cdots + c_ne_n$. For each $a_1, \ldots, a_r$ of R, let us consider the following element s of $H^0(Y, H^{\otimes m})$;

$$s = s_0 + t(a_1e_1 + \cdots + a_ne_n) = (c_1 + ta_1)e_1 + \cdots + (c_n + ta_n)e_n.$$

Since $\#(R) = \infty$, it is easy to see that the set

$$\{(c_1 + ta_1, \ldots, c_n + ta_n) \mid a_1, \ldots, a_n \in R\}$$

is Zariski dense in $\mathbb{A}^n(K)$, where K is the quotient field of R. Thus, we can find $a_1, \ldots, a_n \in R$ such that $\mathrm{div}(s_\eta)$ is geometrically reduced and irreducible divisor on X_η (cf. [9, Theorem 6.10]). Let $[\mathrm{div}(s)] = S + T$ be the decomposition as cycles such that S is horizontal and T is vertical with respect to f. Then, by our choice of s, $s|_{\Gamma_i} = s_0|_{\Gamma_i}$ for all i. Thus, $T = 0$ and S is integral. Hence, by the proof of [3, Theorem 2.4], we can see that $\mathrm{div}(s) \sqcap [D] = D \sqcap S$ in $Z^2(Y)$. Therefore, if we set $H_S = H|_S$ and $D_S = D|_S$, then

$$\begin{aligned}
\deg(H_S^{d-2} \cdot \mathcal{O}_S(D_S) \cdot [D_S]) &= \deg(H^{d-2} \cdot \mathcal{O}_Y(D) \cdot (D \sqcap S)) \\
&= \deg(H^{d-2} \cdot \mathcal{O}_Y(D) \cdot (\mathrm{div}(s) \sqcap [D])) \\
&= \deg(H^{d-2} \cdot \mathcal{O}_Y(D) \cdot H^{\otimes m} \cdot [D])) \\
&= m \deg(H^{d-1} \cdot \mathcal{O}_Y(D) \cdot [D])).
\end{aligned}$$

Thus, by hypothesis of induction, $Q_H([D], [D]) \leq 0$.

Further, we assume that $Q_H([D], [D]) = 0$. Then, $\deg(H_S^{d-2} \cdot \mathcal{O}_S(D_S) \cdot [D_S]) = 0$. Thus, by hypothesis of induction, there is a rational number α with $D \sqcap S = \alpha(Y_o \sqcap S)$ in $Z^2(Y)_{\mathbb{Q}}$. We set $[D] = \sum_i \alpha_i \Gamma_i$ and $[Y_o] = \sum_i \beta_i \Gamma_i$ as cycles. Moreover, we set

$$Y^0 = Y \setminus \left(\mathrm{Sing}(Y) \cup \bigcup_{i \neq j} (\Gamma_i \cap \Gamma_j) \right),$$

For a Cartier divisor D on Y with $[D] \in F_c(Y)$, and $y \in F_c(Y)$, $D \cdot y$ depend only on $[D]$. For, if D' is a Cartier divisor on Y with $[D'] = [D]$, and E is a Cartier divisor on Y with $y = [E]$, then, by [3, Theorem 2.4],

$$D \cdot y = E \cdot [D] = E \cdot [D'] = D' \cdot y.$$

Thus, we can define a bi-linear map

$$q : F_c(Y) \times F_c(Y) \to \mathrm{CH}^1(Y_o)$$

by $q([D], y) = D \cdot y$. Moreover, [3, Theorem 2.4] says us that q is symmetric, i.e., $q(x, y) = q(y, x)$ for all $x, y \in F_c(Y)$. Let H be an ample line bundle on Y. Using q and H, we have a quadratic form Q_H on $F_c(Y)$ given by

$$Q_H(x, y) = \deg(H^{d-1} \cdot q(x, y)),$$

where $d = \dim Y_\eta$. Then, we have the following Zariski's lemma on integral schemes, which is well known if X is regular.

Lemma B.1 (Zariski's lemma for integral scheme)**.** (1) $Q_H([Y_o], x) = 0$ *for all* $x \in F_c(Y)_{\mathbb{Q}}$.
(2) $Q_H(x, x) \leq 0$ *for any* $x \in F_c(Y)_{\mathbb{Q}}$.
(3) $Q_H(x, x) = 0$ *if and only if* $x \in \mathbb{Q} \cdot [Y_o]$.

Proof. (1): This is obvious because $\mathcal{O}_Y(Y_o) \simeq \mathcal{O}_Y$.

(2) and (3): If $x \in \mathbb{Q} \cdot [Y_o]$, then by (1), $Q_H(x, x) = 0$. Thus, it is sufficient to prove that (a) $Q_H(x, x) \leq 0$ for any $x \in F_c(Y)_{\mathbb{Q}}$, and that (b) if $Q_H(x, x) = 0$, then $x \in \mathbb{Q} \cdot [Y_o]$. For this purpose, we may assume that $x \in F_c(Y)$, i.e., $x = [D]$ for some Cartier divisor D on Y. We prove (a) and (b) by induction on d. If $d = 1$ and Y is regular, the lemma follows from the following sublemma.

Sublemma B.1.1. *Let V be a finite dimensional vector space over $\mathbb{R}$, and Q a quadratic form on V. We assume that there are $e \in V$ and a basis $\{e_1, \ldots, e_n\}$ with the following properties:*

(i) *If we set $e = a_1 e_1 + \cdots + a_n e_n$, then $a_i > 0$ for all i.*
(ii) *$Q(x, e) \leq 0$ for all $x \in V$.*
(iii) *$Q(e_i, e_j) \geq 0$ for all $i \neq j$.*
(iv) *If we set $S = \{(i, j) \mid i \neq j$ and $Q(e_i, e_j) > 0\}$, then, for any $i \neq j$, there is a sequence $i_1, \ldots, i_l$ such that $i_1 = i$, $i_l = j$, and $(i_t, i_{t+1}) \in S$ for all $1 \leq t < l$.*

Then, $Q(x, x) \leq 0$ for all $x \in V$. Moreover, if $Q(x, x) = 0$ for some $x \neq 0$, then $x \in \mathbb{R}e$ and $Q(y, e) = 0$ for all $y \in V$.

Proof. Replacing e_i by $a_i e_i$, we may assume that $a_1 = \cdots = a_n = 1$. If we set $x = x_1 e_1 + \cdots + x_n e_n$, then, by an easy calculation, we can show

$$Q(x, x) = \sum_i x_i^2 Q(e_i, e) - \sum_{i<j} (x_i - x_j)^2 Q(e_i, e_j).$$

Thus, we can easily see our assertions. □

Let us go back to the proof of Lemma B.1. Let $\mu : Y' \to Y$ be a proper birational morphism of projective integral schemes over R. We assume that if $d \geq 2$, then μ is finite. Here we

On the other hand, let F_P be the kernel of the natural homomorphism $\pi^*(E) \to \mathcal{O}_P(1)$. We give F_P and $\mathcal{O}_P(1)$ the metrics induced from the metric of $\pi^*\overline{E}$, so that we get an exact sequence of Hermitian vector bundles:

$$\mathcal{E}_Q : 0 \to \overline{F_P} \to \pi^*\overline{E} \to \overline{\mathcal{O}_P(1)} \to 0.$$

Then,

$$\widehat{\mathrm{ch}}(\pi^*\overline{E}) = \widehat{\mathrm{ch}}(\overline{F_P}) + \widehat{\mathrm{ch}}(\overline{\mathcal{O}_P(1)}) - a(\widetilde{\mathrm{ch}}(\mathcal{E}_P)),$$

which implies

$$\widehat{\mathrm{ch}}(\nu^* f^* \overline{E}) = \widehat{\mathrm{ch}}(g^*\pi^*\overline{E}) = \widehat{\mathrm{ch}}(g^*\overline{F_P}) + \widehat{\mathrm{ch}}(g^*\overline{\mathcal{O}_P(1)}) - a(g^*\widetilde{\mathrm{ch}}(\mathcal{E}_P)).$$

Thus, since the rank of F_P is less than r and π is smooth, using hypothesis of induction and [5, 6 of Theorem 4], we have

$$\begin{aligned} f_*\nu_*(\widehat{\mathrm{ch}}(\nu^* f^* \overline{E}) \cdot \beta) &= \pi_* g_* \left((\widehat{\mathrm{ch}}(g^*\overline{F_P}) + \widehat{\mathrm{ch}}(g^*\overline{\mathcal{O}_P(1)}) - a(g^*\widetilde{\mathrm{ch}}(\mathcal{E}_P))) \cdot \beta \right) \\ &= \pi_* \left((\widehat{\mathrm{ch}}(\overline{F_P}) + \widehat{\mathrm{ch}}(\overline{\mathcal{O}_P(1)}) - a(\widetilde{\mathrm{ch}}(\mathcal{E}_P))) \cdot g_*\beta \right) \\ &= \pi_*(\widehat{\mathrm{ch}}(\pi^*\overline{E}) \cdot g_*\beta) = \widehat{\mathrm{ch}}(\overline{E}) \cdot \pi_* g_* \beta. \end{aligned}$$

Therefore,

$$f_*(\widehat{\mathrm{ch}}(f^*\overline{E}) \cdot z) = f_*\nu_*(\widehat{\mathrm{ch}}(\nu^* f^* \overline{E}) \cdot \beta) = \widehat{\mathrm{ch}}(\overline{E}) \cdot f_* z$$

because $\pi_* g_* \beta = f_* \nu_* \beta = f_* z$. □

Appendix B. Zariski's lemma for integral scheme

Let R be a discrete valuation ring, and $f : Y \to \mathrm{Spec}(R)$ a flat and projective integral scheme over R. Let η be the generic point of $\mathrm{Spec}(R)$ and o the special point of $\mathrm{Spec}(R)$. We assume that the genetic fiber Y_η of f is geometrically reduced and irreducible. Let Y_o be the special fiber of f, i.e., $Y_o = f^*(o)$. Let us consider a paring

$$\mathrm{Pic}(Y) \otimes \mathrm{CH}^p(Y_o) \to \mathrm{CH}^{p+1}(Y_o)$$

given by the composition of homomorphisms

$$\mathrm{Pic}(Y) \otimes \mathrm{CH}^p(Y_o) \to \mathrm{Pic}(Y_o) \otimes \mathrm{CH}^p(Y_o) \to \mathrm{CH}^{p+1}(Y_o).$$

We denote by $x \cdot z$ the image of $x \otimes z$ by the above homomorphism. Let D be a Cartier divisor on Y, and Z a cycle of codimension p on Y_o, i.e., $Z \in Z^p(Y_o)$. We assume that D and Z intersect properly. Let s be a rational section of $\mathcal{O}_Y(D)$ with $\mathrm{div}(s) = D$, and let $Z = a_1 Z_1 + \cdots + a_r Z_r$ be the irreducible decomposition as cycles. Then, since $s|_{Z_i}$ is a rational section of $\mathcal{O}_{Z_i}(D)$, we define $D \sqcap Z \in Z^{p+1}(Y_o)$ to be

$$D \sqcap Z = a_1 \,\mathrm{div}(s|_{Z_i}) + \cdots + a_r \,\mathrm{div}(s|_{Z_r}).$$

Then, the class of $D \sqcap Z$ is equal to $\mathcal{O}_Y(D) \cdot$ (the class of Z). Moreover, for a Cartier divisor D on Y, the associated cycle of D is denoted by $[D]$, which is an element of $Z^1(Y)$. Let us consider the following subgroup $F_c(Y)$ of $Z^0(Y_o)$:

$$F_c(Y) = \{x \in Z^0(Y_o) \mid x = [D] \text{ for some Cartier divisor } D \text{ on } Y\}.$$

Thus, our proposition is a consequence of Lemma A.2. Thus, we may assume that $r > 1$. Moreover, if $z = (0, T)$, then

$$\begin{aligned} f_*(\widehat{\mathrm{ch}}(f^*\overline{E}) \cdot z) &= f_*(0, \mathrm{ch}(f^*\overline{E}) \wedge T) = (0, f_*(f^* \mathrm{ch}(\overline{E}) \wedge T)) \\ &= (0, \mathrm{ch}(\overline{E}) \wedge f_*T) = \widehat{\mathrm{ch}}(\overline{E}) \cdot f_*z. \end{aligned}$$

Hence, we may further assume that z is an usual arithmetic cycle, i.e., $z \in \widehat{\mathrm{CH}}^p(X)$ for some p.

Let $\pi : P = \mathrm{Proj}(\oplus_{n \geq 0} \mathrm{Sym}^n(E)) \to Y$ and $\nu : Q = \mathrm{Proj}(\oplus_{n \geq 0} \mathrm{Sym}^n(f^*E)) \to X$ be the projective bundles of E and f^*E, and let $\mathcal{O}_P(1)$ and $\mathcal{O}_Q(1)$ be the tautological line bundles of P and Q respectively. We set the induced morphisms as following diagram.

$$\begin{array}{ccc} X & \xleftarrow{\nu} & Q \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xleftarrow[\pi]{} & P \end{array}$$

We give $\mathcal{O}_Q(1)$ the Hermitian metric induced from $\nu^* f^* \overline{E}$. Since $\nu : Q \to X$ is smooth, we can consider the pull-back $\nu^*(z)$ of z. Here we claim the following.

Claim A.3.2. $\nu_*(\widehat{c}_1(\overline{\mathcal{O}_Q(1)})^{r-1} \cdot \nu^*(z)) = z$.

Proof. Let (Z, g) be a representative of z. Clearly, we may assume that Z is integral. If Z is vertical, then our assertion is trivial. So we may assume that Z is horizontal. Let $\mu : Z' \to Z$ be a generic resolution of singularities of Z. Let $\nu' : T = \mathrm{Proj}(\oplus_{n \geq 0} \mathrm{Sym}^n(\mu^* f^*(E))) \to Z'$ be the projective bundle of $\mu^* f^*(E)$, and $\mathcal{O}_T(1)$ the tautological line bundle on T. We set the induced morphism as follows.

$$\begin{array}{ccc} T & \xrightarrow{\mu'} & Q \\ {\scriptstyle \nu'}\downarrow & & \downarrow{\scriptstyle \nu} \\ Z' & \xrightarrow[\mu]{} & X \end{array}$$

Then, μ' gives rise to a generic resolution of singularities of $\nu^*(Z)$. Thus, by virtue of Lemma A.3,

$$\widehat{c}_1(\overline{\mathcal{O}_Q(1)})^{r-1} \cdot \nu^*(z) = \mu'_*(\widehat{c}_1(\overline{\mathcal{O}_T(1)})^{r-1}) + a(c_1(\overline{\mathcal{O}_Q(1)})^{r-1} \wedge \nu^*(g)).$$

Here since

$$\nu_* \mu'_*(\widehat{c}_1(\overline{\mathcal{O}_T(1)})^{r-1}) = \mu_* \nu'_*(\widehat{c}_1(\overline{\mathcal{O}_T(1)})^{r-1}) = \mu_*([Z']) = (Z, 0)$$

and $\nu_*(c_1(\overline{\mathcal{O}_Q(1)})^{r-1} \wedge \nu^*(g)) = g$, we can see that

$$\nu_*(\widehat{c}_1(\overline{\mathcal{O}_Q(1)})^{r-1} \cdot \nu^*(z)) = (Z, 0) + a(g) = (Z, g).$$

Hence, we get our claim. □

Let us go back to the proof of Proposition A.1. We set $\beta = \widehat{c}_1(\overline{\mathcal{O}_Q(1)})^{r-1} \cdot \nu^*(z)$. Then, by the above claim, $\nu_*(\beta) = z$. Thus, since ν is smooth, using [5, 6 of Theorem 4],

$$f_* \nu_*(\widehat{\mathrm{ch}}(\nu^* f^* \overline{E}) \cdot \beta) = f_*(\widehat{\mathrm{ch}}(f^* \overline{E}) \cdot z).$$

Proof. If $\widehat{\deg}\left(\widehat{c}_1(\overline{H})^{d-1}\cdot\left(4\widehat{c}_2(\overline{E})-\widehat{c}_1(\overline{E})^2\right)\right)<0$, then, by Theorem 6.2, there is a saturated subsheaf L of E with

$$2\widehat{c}_1(L,h_L)-\widehat{c}_1(E,h)\in\widehat{C}_{++}(X;\overline{H}).$$

Thus,

$$(2\widehat{c}_1(L,h_L)-\widehat{c}_1(E,h)\cdot\lambda)_{\overline{H}}>0.$$

which contradicts to the semistability of $\overline{E}$. □

Appendix A. Projection formula for the arithmetic Chern Character

In this section, we will show the following projection formula in a general context.

Proposition A.1. *Let $X\to Y$ be a proper morphism of arithmetic varieties, $\overline{E}=(E,h)$ a Hermitian vector bundle on Y, and z an arithmetic D-cycle on X. Then,*

$$f_*(\widehat{\mathrm{ch}}(f^*\overline{E})\cdot z)=\widehat{\mathrm{ch}}(\overline{E})\cdot f_*z.$$

Proof. For the proof of the projection formula above, we need the following two lemmas. The proof of these lemmas can be found in [10, Proposition 2.4.1 and Proposition 2.4.2]. Here we fix notation. Let Z be a quasi-projective integral scheme over $\mathbb{Z}$. Then, by virtue of Hironaka's resolution of singularities [7], there is a proper birational morphism $\mu:Z'\to Z$ of quasi-projective integral schemes over $\mathbb{Z}$ such that $Z'_{\mathbb{Q}}$ is non-singular. The above $\mu:Z'\to Z$ is called a *generic resolution of singularities of Z.*

Lemma A.2. *Let $f:X\to Y$ be a proper morphism of arithmetic varieties. Let (L,h) be a Hermitian line bundle on Y, and $z\in\widehat{\mathrm{CH}}^p_D(X)$. Then*

$$f_*(\widehat{c}_1(f^*L,f^*h)\cdot z)=\widehat{c}_1(L,h)\cdot f_*(z).$$

Lemma A.3. *Let X be a arithmetic variety, and $\overline{L}_1=(L_1,h_1),\dots,\overline{L}_n=(L_n,h_n)$ be Hermitian line bundles on X. Let (Z,g) be an arithmetic D-cycle on X, and $Z=a_1Z_1+\cdots+a_rZ_r$ the irreducible decomposition as cycles. For each i, let $\tau_i:Z'_i\to Z_i$ be a proper birational morphism of quasi-projective integral schemes. We assume that if Z_i is horizontal with respect to $X\to\mathrm{Spec}(\mathbb{Z})$, then τ_i is a generic resolution of singularities of Z_i. Then, we have*

$$\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_n)\cdot(Z,g)=\sum_{i=1}^r a_i\mu_{i*}\left(\widehat{c}_1(\mu_i^*\overline{L}_1)\cdots\widehat{c}_1(\mu_i^*\overline{L}_n)\right)+a(c_1(\overline{L}_1)\wedge\cdots\wedge c_1(\overline{L}_n)\wedge g)$$

in $\widehat{\mathrm{CH}}^*_D(X)_{\mathbb{Q}}$, where μ_i is the composition of $Z'_i\xrightarrow{\tau_i}Z_i\hookrightarrow X$ for each i.

Let us start the proof of Proposition A.1. We will prove this proposition by induction on $r=\mathrm{rk}\,E$. If $r=1$, then

$$\widehat{\mathrm{ch}}(f^*\overline{E})=\sum_{n\geq0}\frac{1}{n!}\widehat{c}_1(f^*\overline{E})^n\quad\text{and}\quad\widehat{\mathrm{ch}}(\overline{E})=\sum_{n\geq0}\frac{1}{n!}\widehat{c}_1(\overline{E})^n.$$

be homomorphisms given in §4. Here, we set

$$\widehat{C}_{++}(X;\overline{H}) = \{x \in \widehat{\mathrm{CH}}^1_{\mathbf{Br}}(X)_{\mathbb{Q}} \mid (x\cdot x)_{\overline{H}} > 0 \text{ and } \deg_K(x) > 0\}$$

and

$$\widehat{C}_{+}(X;\overline{H}) = \{x \in \widehat{\mathrm{CH}}^1_{\mathbf{Br}}(X)_{\mathbb{Q}} \mid (x\cdot y)_{\overline{H}} > 0 \text{ for all } y \in \widehat{C}_{++}(X;\overline{H})\}.$$

By virtue of the Hodge index theorem (cf. Theorem 4.1), we have the following in the same way as [13, §1].

Proposition 6.1. $\widehat{C}_{++}(X;\overline{H}) \subset \widehat{C}_{+}(X;\overline{H})$.

Let $\overline{E} = (E,h)$ be a Hermitian vector bundle of rank 2 on X. Let L be a saturated subsheaf of E with $\operatorname{rk} L = 1$. Since X is regular and L is reflexive, L is an invertible sheaf. Let h_L be the metric induced by h of E. Then, h_L is a birationally Hermitian metric. In this notation, we have the following theorem.

Theorem 6.2. *If* $\widehat{\deg}\left(\widehat{c}_1(\overline{H})^{d-1}\cdot\left(4\widehat{c}_2(\overline{E}) - \widehat{c}_1(\overline{E})^2\right)\right) < 0$, *there is a saturated rank* 1 *subsheaf* L *of* E *with*

$$2\widehat{c}_1(L,h_L) - \widehat{c}_1(E,h) \in \widehat{C}_{++}(X;\overline{H}).$$

Proof. First of all, by virtue of [14], $E_{\overline{K}}$ is not μ-semistable with respect to $H_{\overline{K}}$. Thus, there is the destabilizing subsheaf L' of $E_{\overline{K}}$. Using the uniqueness of L', in the same way as [13, Claim 3.2], we can see that L' is defined over K. Hence, we can find a saturated subsheaf L of E with $L_K = L'$. Let $Q = (E/L)^{**}$, and let h_L and h_Q be birationally Hermitian metrics of L and Q induced by h. Then, by Corollary 5.5,

$$\widehat{\deg}\left(\widehat{c}_1(\overline{H})^{d-1}\cdot\widehat{c}_2(E,h)\right) \geq \widehat{\deg}\left(\widehat{c}_1(\overline{H})^{d-1}\cdot\widehat{c}_1(L,h_L)\cdot\widehat{c}_1(Q,h_Q)\right).$$

Therefore, since $\widehat{c}_1(E,h) = \widehat{c}_1(L,h_L) + \widehat{c}_1(Q,h_Q)$,

$$\begin{aligned}\left((2\widehat{c}_1(L,h_L) - \widehat{c}_1(E,h))^2\right)_{\overline{H}} &= (\widehat{c}_1(E,h)^2)_{\overline{H}} - 4(\widehat{c}_1(L,h_L)\cdot\widehat{c}_1(Q,h_Q))_{\overline{H}} \\ &\geq \widehat{\deg}\left(\widehat{c}_1(\overline{H})^{d-1}\cdot\left(\widehat{c}_1(\overline{E})^2 - 4\widehat{c}_2(\overline{E})\right)\right) > 0.\end{aligned}$$

Thus, we get our theorem. □

Let us fix $\lambda \in \widehat{C}_{+}(X;\overline{H})$. Let $\overline{E} = (E,h)$ be a Hermitian vector bundle on X of rank 2. $\overline{E}$ is said to be *arithmetically semistable with respect to* λ if, for any saturated rank 1 subsheaves L of E,

$$(\widehat{c}_1(L,h_L)\cdot\lambda)_{\overline{H}} \leq \frac{(\widehat{c}_1(E,h)\cdot\lambda)_{\overline{H}}}{2}.$$

With notation as above, we have the following corollary.

Corollary 6.3. *If* $\overline{E}$ *is arithmetically semistable with respect to* λ, *then,*

$$\widehat{\deg}\left(\widehat{c}_1(\overline{H})^{d-1}\cdot\left(4\widehat{c}_2(\overline{E}) - \widehat{c}_1(\overline{E})^2\right)\right) \geq 0.$$

Proof. First of all, there is an ideal sheaf I on X such that the image of $E \to Q$ is $Q \otimes I$. Then, $\operatorname{codim} \operatorname{Spec}(\mathcal{O}_X/I) \geq 2$. Let $p_I : X_I \to X$ be the blowing-up by the ideal sheaf I, and $\nu : Y \to X_I$ a generic resolution of singularities of X_I. Then, by our construction, there is an effective Cartier divisor Σ on Y with $I\mathcal{O}_Y = \mathcal{O}_Y(-\Sigma)$. Then, $\mu^*(E) \to \mu^*(Q) \otimes \mathcal{O}_Y(-\Sigma)$ is surjective and its kernel is $\mu^*(L) \otimes \mathcal{O}_Y(\Sigma)$, where $\mu = p_I \cdot \nu : Y \to X$. Thus, we get an exact sequence

$$0 \to \mu^*(L) \otimes \mathcal{O}_Y(\Sigma) \to \mu^*(E) \to \mu^*(Q) \otimes \mathcal{O}_Y(-\Sigma) \to 0.$$

Let h'_L and h'_Q be Hermitian metrics of $\mu^*(L) \otimes \mathcal{O}_Y(\Sigma)$ and $\mu^*(Q) \otimes \mathcal{O}_Y(-\Sigma)$ induced by $\mu^*(h)$ of $\mu^*(E)$. Let $\overline{A} = (A, e)$ be an arithmetically ample Hermitian line bundle on Y. Then, by [12, Proposition 7.3], for all $n > 0$,

$$\widehat{\deg}\left(\widehat{c}_1(\mu^*(\overline{H}^{\otimes n}) \otimes \overline{A})^{d-1} \cdot \widehat{c}_2(\mu^*(E), \mu^*(h))\right) \geq \widehat{\deg}\left(\widehat{c}_1(\mu^*(\overline{H}^{\otimes n}) \otimes \overline{A})^{d-1} \cdot \widehat{c}_1(\mu^*(L) \otimes \mathcal{O}_Y(\Sigma),\ h'_L) \cdot \widehat{c}_1(\mu^*(Q) \otimes \mathcal{O}_Y(-\Sigma),\ h'_Q)\right).$$

Taking $n \to \infty$ of the above inequality, we have

$$\widehat{\deg}\left(\widehat{c}_1(\mu^*(\overline{H}))^{d-1} \cdot \widehat{c}_2(\mu^*(E), \mu^*(h))\right) \geq \widehat{\deg}\left(\widehat{c}_1(\mu^*(\overline{H}))^{d-1} \cdot \widehat{c}_1(\mu^*(L) \otimes \mathcal{O}_Y(\Sigma),\ h'_L) \cdot \widehat{c}_1(\mu^*(Q) \otimes \mathcal{O}_Y(-\Sigma),\ h'_Q)\right).$$

Here, by the projection formula (cf. Proposition A.1),

$$\widehat{\deg}\left(\widehat{c}_1(\mu^*(\overline{H}))^{d-1} \cdot \widehat{c}_2(\mu^*(E), \mu^*(h))\right) = \widehat{\deg}\left(\widehat{c}_1(\overline{H})^{d-1} \cdot \widehat{c}_2(E, h)\right).$$

Thus, it is sufficient to show that

$$\widehat{\deg}\left(\widehat{c}_1(\mu^*(\overline{H}))^{d-1} \cdot \widehat{c}_1(\mu^*(L) \otimes \mathcal{O}_Y(\Sigma),\ h'_L) \cdot \widehat{c}_1(\mu^*(Q) \otimes \mathcal{O}_Y(-\Sigma),\ h'_Q)\right) \geq \widehat{\deg}\left(\widehat{c}_1(\overline{H})^{d-1} \cdot \widehat{c}_1(L, h_L) \cdot \widehat{c}_1(Q, h_Q)\right).$$

Namely,

$$\Delta_\mu(\widehat{c}_1(\mu^*(L) \otimes \mathcal{O}_Y(\Sigma),\ h'_L),\ \widehat{c}_1(\mu^*(Q) \otimes \mathcal{O}_Y(-\Sigma),\ h'_Q)) \geq 0.$$

Let e be a Hermitian metric of $\mathcal{O}_Y(\Sigma)$. Then, by Proposition 5.1,

$$\Delta_\mu(\widehat{c}_1(\mu^*(L) \otimes \mathcal{O}_Y(\Sigma),\ h'_L),\ \widehat{c}_1(\mu^*(Q) \otimes \mathcal{O}_Y(-\Sigma),\ h'_Q)) = -\Delta_\mu(\widehat{c}_1(\mathcal{O}_Y(\Sigma), e), \widehat{c}_1(\mathcal{O}_Y(\Sigma), e)).$$

Therefore, by Proposition 5.2, it suffices to show that $-\Sigma$ is μ-nef. This is obvious because $I\mathcal{O}_{X_I}$ is μ_I-ample and $\mathcal{O}_Y(-\Sigma) = \nu^*(I\mathcal{O}_{X_I})$. □

6. Bogomolov's instability for rank 2 bundles

Let K be a number field, and O_K the ring of integers of K. Let $f : X \to \operatorname{Spec}(O_K)$ be a regular projective arithmetic variety with the geometrically irreducible generic fiber and $d = \dim X_K$. Let $\overline{H} = (H, k)$ be an arithmetically ample Hermitian line bundle on X. Let

$$(\ \cdot\ )_{\overline{H}} : \widehat{\mathrm{CH}}^1_{\mathbf{Br}}(X)_{\mathbb{Q}} \times \widehat{\mathrm{CH}}^1_{\mathbf{Br}}(X)_{\mathbb{Q}} \to \mathbb{R} \quad \text{and} \quad \deg_K : \widehat{\mathrm{CH}}^1_{\mathbf{Br}}(X)_{\mathbb{Q}} \to \mathbb{Q}$$

Here, since Γ_k is not a component of D_i, $\deg_\mu(\mathcal{O}_Y(D_i)|_{\Gamma_k}) \geq 0$ for every k. Therefore, by Lemma 5.3, we get $\Delta_\mu(e_i, e_j) \geq 0$.

Let h be a Hermitian metric of $\mathcal{O}_Y(D)$ with $e = \widehat{c}_1(\mathcal{O}_Y(D), h)$. In the same way as above, by using Lemma A.3, we can see that

$$\Delta_\mu(e, e_j) = \sum_{k=1}^{s} b_k \widehat{\deg}\left(\widehat{c}_1(\nu_k^* \mu^* \overline{H})^{d-1} \cdot \widehat{c}_1(\nu_k^*(\mathcal{O}_Y(D), h))\right).$$

Here, by our assumption, $\deg_\mu(\mathcal{O}_Y(-D)|_{\Gamma_k}) \geq 0$ for all k. Thus, by Lemma 5.3, $\Delta_\mu(e, e_j) \leq 0$.

Finally, let us see that $\Delta_\mu(x, x) \leq 0$ for all $x \in V'$. We set $x = x_1 e_1 + \cdots + x_n e_n$. It is easy to see that

$$\Delta_\mu(x, x) = \sum_i x_i^2 \Delta_\mu(e_i, e) - \sum_{i<j} (x_i - x_j)^2 \Delta_\mu(e_i, e_j).$$

Thus, Claim 5.2.2 implies $\Delta_\mu(x, x) \leq 0$. □

Corollary 5.4. *We assume that X and Y are $\mathbb{Q}$-factorial, i.e., the natural homomorphisms $\mathrm{Div}(X)_\mathbb{Q} \to Z^1(X)_\mathbb{Q}$ and $\mathrm{Div}(Y)_\mathbb{Q} \to Z^1(Y)_\mathbb{Q}$ are isomorphisms. Then, $\Delta_\mu(x, y) \leq 0$ for any $x, y \in \widehat{\mathrm{CH}}^1(Y)_\mathbb{Q}$ with $\delta_\mu(x) = \delta_\mu(y)$. In particular, Δ_μ is negative semi-definite on $\widehat{\mathrm{CH}}^1(Y)_\mathbb{Q}$.*

Proof. Let $\Gamma_1, \ldots, \Gamma_n$ be all prime divisors on Y contracted by μ. By virtue of Proposition 5.1 and 5.2, it is sufficient to show that there are positive rational numbers $a_1, \ldots, a_n$ such that $-\sum_{i=1}^n a_i \Gamma_i$ is μ-ample.

Let A be an ample line bundle on Y. Then, there are a positive integer m and a section s of $H^0(Y, A^{\otimes m})$ such that Γ_i is not a component of $\mathrm{div}(s)$ for every i. We set $D = \mu^*(\mu_*(\mathrm{div}(s))) - \mathrm{div}(s)$. (Note that $\mu_*(\mathrm{div}(s)) \in \mathrm{Div}(X)_\mathbb{Q}$.) Then, by our choice of s, D is effective, $-D$ is μ-ample, and D is contracted by μ. Thus, there are non-negative rational numbers $a_1, \ldots, a_n$ with $D = \sum_{i=1}^n a_i \Gamma_i$. Here we suppose $a_i = 0$ for some i. Let F be the generic fiber of $\Gamma_i \to \mu(\Gamma_i)$. Then, $\dim F \geq 1$, $D|_F$ is effective, and $-D|_F$ is ample. This is a contradiction. Thus, $a_i > 0$ for all i. □

Let M be a d-dimensional smooth projective variety over an algebraically closed field, E a vector bundle bundle of rank 2 on M, and H an ample line bundle on M. It is well known that if L is a rank 1 saturated subsheaf of E, then

$$\deg(c_2(E) \cdot c_1(H)^{d-2}) \geq \deg(c_1(L) \cdot c_1(E/L) \cdot c_1(H)^{d-2}).$$

This is very useful formula to estimate the degree of $c_2(E)$. The following is an arithmetic analogue of the above formula.

Corollary 5.5. *Let X be a regular projective arithmetic variety with $d = \dim X_\mathbb{Q}$, (E, h) a Hermitian vector bundle of rank 2 on X, and $\overline{H} = (H, k)$ an arithmetically ample Hermitian line bundle on X. Let L be a saturated subsheaf of E with $\mathrm{rk}\, L = 1$, and let $Q = (E/L)^{**}$. Let h_L and h_Q be metrics of L and Q induced by h. Then, h_L and h_Q are birationally Hermitian metrics, and*

$$\widehat{\deg}\left(\widehat{c}_1(\overline{H})^{d-1} \cdot \widehat{c}_2(E, h)\right) \geq \widehat{\deg}\left(\widehat{c}_1(\overline{H})^{d-1} \cdot \widehat{c}_1(L, h_L) \cdot \widehat{c}_1(Q, h_Q)\right).$$

Let us choose a Green function g_i for each D_i. We set $e_i = (D_i, g_i)$ for $i = 1, \ldots, n$, and $V' = \mathbb{Q}e_1 + \cdots + \mathbb{Q}e_n$. By virtue of Proposition 5.1, if $\Delta_\mu(x, x) \leq 0$ for all $x \in V'$, then the assertion of our proposition holds. Here let us consider the following claim, which is the crucial part of the proof of our proposition.

Claim 5.2.2. (i) $\Delta_\mu(e_i, e_j) \geq 0$ *for all* $i \neq j$.
(ii) $\Delta_\mu(e, e_j) \leq 0$ *for all* j, *where* $e = e_1 + \cdots + e_n$.

To prove the above claim, we need the following lemma.

Lemma 5.3. *Let* $\overline{L}$ *be a Hermitian line bundle on* Y, *and* Γ *a prime divisor on* Y *with* $\mu_*(\Gamma) = 0$. *Let* $\nu : \Gamma' \to \Gamma$ *be a birational morphism of projective integral schemes. We assume that if* Γ *is horizontal with respect to* $Y \to \mathrm{Spec}(\mathbb{Z})$, *then* ν *is a generic resolution of singularities of* Γ; *otherwise,* $\nu = \mathrm{id}_\Gamma$. *If* $\deg_\mu(L|_\Gamma) \geq 0$, *then*

$$\widehat{\deg}\left(\widehat{c}_1(\nu^*(\overline{L})) \cdot \widehat{c}_1(\nu^*\mu^*(\overline{H}))^{d-1}\right) \geq 0.$$

(*For the definition of generic resolution of singularities, see Appendix A.*)

Proof. We set $\Sigma = f(\Gamma)$. Let $\pi : \Sigma' \to \Sigma$ be a proper birational morphism of projective integral schemes. We assume that if Σ is horizontal, then π is a generic resolution of singularities of Σ; otherwise, $\pi = \mathrm{id}_\Sigma$. Changing a model of Γ', if necessarily, we may assume that there is a morphism $f : \Gamma' \to \Sigma'$ with $\pi \cdot f = \mu \cdot \nu$.

$$\begin{array}{ccc} \Gamma' & \xrightarrow{\nu} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mu} \\ \Sigma' & \xrightarrow{\pi} & X \end{array}$$

Thus, using projection formula,

$$\begin{aligned} \widehat{\deg}\left(\widehat{c}_1(\nu^*(\overline{L})) \cdot \widehat{c}_1(\nu^*\mu^*(\overline{H}))^{d-1}\right) &= \widehat{\deg}\left(\widehat{c}_1(\nu^*(\overline{L})) \cdot \widehat{c}_1(f^*\pi^*(\overline{H}))^{d-1}\right) \\ &= \widehat{\deg}\left(f_*(\widehat{c}_1(\nu^*(\overline{L}))) \cdot \widehat{c}_1(\pi^*(\overline{H}))^{d-1}\right) \\ &= \deg_\mu(L|_\Gamma)\widehat{\deg}\left(\widehat{c}_1(\pi^*(\overline{H}))^{d-1}\right) \geq 0. \end{aligned}$$

□

Let us go back to the proof of Claim 5.2.2. Let $D_j = b_1\Gamma_1 + \cdots + b_s\Gamma_s$ be the irreducible decomposition as cycles. For each k, let $\nu_k : \Gamma'_k \to \Gamma_k$ be a proper birational morphism of projective integral schemes. We assume that if Γ_k is horizontal with respect to $Y \to \mathrm{Spec}(\mathbb{Z})$, then ν_k is a generic resolution of singularities of Γ_k. Then, by Lemma A.3,

$$\begin{aligned} &\widehat{\deg}\left(\widehat{c}_1(\mu^*\overline{H})^{d-1} \cdot (D_i, g_i) \cdot (D_j, g_j)\right) = \\ &\qquad \sum_{k=1}^{s} b_k \widehat{\deg}\left(\widehat{c}_1(\nu_k^*\mu^*\overline{H})^{d-1} \cdot \nu_k^*(D_i, g_i)\right) + \frac{1}{2}\int_{Y(\mathbb{C})} g_j\omega(g_i)\mu^*(\Omega^{d-1}), \end{aligned}$$

where $\Omega = c_1(H, k)$. Thus, combining the above with (5.2.1), we can see that

$$\Delta_\mu(e_i, e_j) = \sum_{k=1}^{s} b_k \widehat{\deg}\left(\widehat{c}_1(\nu_k^*\mu^*\overline{H})^{d-1} \cdot \nu_k^*(D_i, g_i)\right).$$

Hence, $\Delta_\mu(x, a(\phi)) = 0$.

Let us pick up $x, y \in \widehat{\mathrm{Pic}}_\mu(Y)_{\mathbb{Q}}$. In order to see that $\Delta_\mu(x, y)$ depends only on $\delta_\mu(x)$ and $\delta_\mu(y)$, by replacing x and y by nx and ny for some positive integer n, we may assume that $x, y \in \widehat{\mathrm{Pic}}_\mu(Y)$ and $\mu_*(x), \mu_*(y) \in \widehat{\mathrm{Pic}}_{\mathbf{Br}}(X)$. Let (L, h_L) and (Q, h_Q) be Hermitian line bundles on Y with $\widehat{c}_1(L, h_L) = x$ and $\widehat{c}_1(Q, h_Q) = y$, and let L' and Q' be line bundles on X with $c_1(L') = \mu_*(z(x))$ and $c_1(Q') = \mu_*(z(y))$. Here we can find Cartier divisors Σ_1 and Σ_2 on Y such that $\mu^*(L') \otimes \mathcal{O}_Y(\Sigma_1) = L$, $\mu^*(Q') \otimes \mathcal{O}_Y(\Sigma_2) = Q$, and $\mu_*(\Sigma_1) = \mu_*(\Sigma_2) = 0$ in $\mathrm{Div}(X)$. Then, $\delta_\mu(x) = \Sigma_1$ and $\delta_\mu(y) = \Sigma_2$. Let $h_{L'}$ and $h_{Q'}$ be C^∞-Hermitian metrics of L' and Q' respectively. Then, there are C^∞-Hermitian metrics e_1 and e_2 of $\mathcal{O}_Y(\Sigma_1)$ and $\mathcal{O}_Y(\Sigma_2)$ respectively such that $\mu^*(L', h_{L'}) \otimes (\mathcal{O}_Y(\Sigma_1), e_1) = (L, h_L)$ and $\mu^*(Q', h_{Q'}) \otimes (\mathcal{O}_Y(\Sigma_2), e_2) = (Q, h_Q)$. Therefore, since

$$\begin{cases} \mu_*(x) = \widehat{c}_1(L', h_{L'}) + \mu_*(\widehat{c}_1(\mathcal{O}_Y(\Sigma_1), e_1)) \\ \mu_*(y) = \widehat{c}_1(Q', h_{Q'}) + \mu_*(\widehat{c}_1(\mathcal{O}_Y(\Sigma_2), e_2)), \end{cases}$$

using projection formula (cf. Proposition A.1), we can easily see that

$$\Delta_\mu(x, y) = \Delta_\mu(\widehat{c}_1(\mathcal{O}_Y(\Sigma_1), e_1), \widehat{c}_1(\mathcal{O}_Y(\Sigma_2), e_2)).$$

Thus, combining the first assertion, we have our proposition. □

Before starting the next property, we would like to fix a terminology. Let L be a line bundle on Y, and let Γ be a prime divisor on Y with $\mu_*(\Gamma) = 0$. We define $\deg_\mu(L|_\Gamma)$ to be

$$\deg_\mu(L|_\Gamma) = \begin{cases} \text{the degree of } L \text{ on the generic fiber of } \Gamma \to \mu(\Gamma) & \text{if } \dim \Gamma - \dim \mu(\Gamma) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $D_1, \ldots, D_n$ be effective Cartier divisors on Y with the following properties.

(1) $\mu_*(D_i) = 0$ for all i.
(2) D_i and D_j have no common component for all $i \neq j$.
(3) There are positive integers $a_1, \ldots, a_n$ such that if we set $D = -\sum_{i=1}^n a_i D_i$, then $\deg_\mu(\mathcal{O}_Y(-D)|_\Gamma) \geq 0$ for all prime divisors Γ in $\mathrm{Supp}(D)$.

Note that if $-D$ is μ-nef, then (3) is satisfied.

Here we define the subspace V of $\widehat{\mathrm{Div}}(X)_{\mathbb{Q}}$ to be

$$V = \left\{ (D, g) \in \widehat{\mathrm{Div}}(X)_{\mathbb{Q}} \mid D = \sum_{i=1}^n x_i D_i \text{ for some } x_1, \ldots, x_n \in \mathbb{Q} \right\}.$$

Then, we have the following proposition.

Proposition 5.2. *$\Delta_\mu(x, x') \leq 0$ for any $x, x' \in V$ with $z(x) = z(x')$. In particular, Δ_μ is negative semi-definite on V.*

Proof. First of all, note that

$$\widehat{\deg}\left(\widehat{c}_1(\overline{H})^{d-1} \cdot \mu_*(D, g) \cdot \mu_*(D', g')\right) = \frac{1}{2} \int_Y g' \omega(D, g) \mu^*(\Omega^{d-1}) \tag{5.2.1}$$

because $\mu_*(D, g) = (0, \mu_*(g))$, $\mu_*(D', g') = (0, \mu_*(g'))$ and Remark 3.3.4. Further, in order to prove our proposition, we may assume $a_1 = \cdots = a_n = 1$ by replacing D_i by $a_i D_i$.

5. Comparison of intersection numbers via birational morphism

Let $\mu : Y \to X$ be a birational morphism of projective arithmetic varieties with $d = \dim X_{\mathbb{Q}} = \dim Y_{\mathbb{Q}}$. We assume that X is normal. We set

$$\mathrm{Pic}_{\mu}(Y)_{\mathbb{Q}} = \{x \in \mathrm{Pic}(Y)_{\mathbb{Q}} \mid \mu_*(x) \in \mathrm{Pic}(X)_{\mathbb{Q}}\}$$

and

$$\widehat{\mathrm{Pic}}_{\mu}(Y)_{\mathbb{Q}} = \{x \in \widehat{\mathrm{Pic}}(Y)_{\mathbb{Q}} \mid z(x) \in \mathrm{Pic}_{\mu}(Y)_{\mathbb{Q}}\},$$

where $z : \widehat{\mathrm{Pic}}(Y)_{\mathbb{Q}} \to \mathrm{Pic}(Y)_{\mathbb{Q}}$ is the homomorphism forgetting Green functions. Then, the push-forward μ_* induces the homomorphism

$$\mu_* : \widehat{\mathrm{Pic}}_{\mu}(Y)_{\mathbb{Q}} \to \widehat{\mathrm{Pic}}_{\mathbf{Br}}(X)_{\mathbb{Q}}.$$

Let us fix an arithmetically ample Hermitian line bundle $\overline{H} = (H, k)$ on X. Here we define the symmetric bi-linear map

$$\Delta_{\mu} : \widehat{\mathrm{Pic}}_{\mu}(Y)_{\mathbb{Q}} \times \widehat{\mathrm{Pic}}_{\mu}(Y)_{\mathbb{Q}} \to \mathbb{R}$$

to be

$$\Delta_{\mu}(x, y) = \widehat{\deg}\left(\widehat{c}_1(\mu^*(\overline{H}))^{d-1} \cdot x \cdot y\right) - \widehat{\deg}\left(\widehat{c}_1(\overline{H})^{d-1} \cdot \mu_*(x) \cdot \mu_*(y)\right).$$

In this section, we will investigate basic properties of Δ_{μ} (cf. Proposition 5.1 and 5.2) and give its application (cf. Corollary 5.5).

Here let us introduce the homomorphism $\delta_{\mu} : \mathrm{Pic}_{\mu}(Y)_{\mathbb{Q}} \to \mathrm{Pic}_{\mu}(Y)_{\mathbb{Q}}$ given by

$$\delta_{\mu}(x) = x - \mu^*(\mu_*(x)).$$

By abuse of notation, the composition

$$\delta_{\mu} \cdot z : \widehat{\mathrm{Pic}}_{\mu}(Y)_{\mathbb{Q}} \xrightarrow{z} \mathrm{Pic}_{\mu}(Y)_{\mathbb{Q}} \xrightarrow{\delta_{\mu}} \mathrm{Pic}_{\mu}(Y)_{\mathbb{Q}}$$

is also denoted by δ_{μ}. First, let us consider the following proposition.

Proposition 5.1. *If $\delta_{\mu}(x) = \delta_{\mu}(x')$ and $\delta_{\mu}(y) = \delta_{\mu}(y')$ for $x, x', y, y' \in \widehat{\mathrm{Pic}}_{\mu}(Y)_{\mathbb{Q}}$, then $\Delta_{\mu}(x, y) = \Delta_{\mu}(x', y')$.*

Proof. First, let us see that if $z(x) = z(x')$ and $z(y) = z(y')$ for $x, x', y, y' \in \widehat{\mathrm{Pic}}_{\mu}(Y)_{\mathbb{Q}}$, then $\Delta_{\mu}(x, y) = \Delta_{\mu}(x', y')$. For this purpose, it is sufficient to see that $\Delta_{\mu}(x, a(\phi)) = 0$ for all C^{∞}-functions ϕ on $Y(\mathbb{C})$. First of all,

$$\widehat{\deg}\left(\widehat{c}_1(\mu^*(\overline{H}))^{d-1} \cdot x \cdot a(\phi)\right) = \frac{1}{2}\int_Y \phi\omega(x) \wedge \mu^*(\Omega^{d-1}).$$

On the other hand, we set $x = (D, g)$. Let g' be a Green function for $\mu_*(D)$. If we set $\psi = \mu_*(g) - g'$, then $\psi \in \mathbf{Br}(X(\mathbb{C}))$. By the definition of the star product and Remark 3.3.4,

$$\begin{aligned}(\mu_*(D), \mu_*(g)) \cdot (0, \mu_*(\phi)) &= (0, \omega(g')\mu_*(\phi) + dd^c(\psi)\mu_*(\phi)) \\ &= (0, \omega(\mu_*(g))\mu_*(\phi)) = (0, \mu_*(\omega(g)\phi)).\end{aligned}$$

Thus, we have

$$\widehat{\deg}\left(\widehat{c}_1(\overline{H})^{d-1} \cdot \mu_*(x) \cdot \mu_*(a(\phi))\right) = \frac{1}{2}\int_Y \phi\omega(x) \wedge \mu^*(\Omega^{d-1}).$$

Thus, using (1) of Proposition 3.2.4, we have

$$\begin{aligned}\deg_K(h)y &= f_*\left(f^*(y)\cdot(\widehat{c}_1(H,k)^d\cdot h)\right) = f_*\left(\widehat{c}_1(H,k)^d\cdot(f^*(y)\cdot h)\right)\\ &= f_*\left(\widehat{c}_1(H,k)^d\cdot(h\cdot f^*(y))\right) = 0.\end{aligned}$$

Therefore, $y=0$. Hence, $x=0$ in $\widehat{\mathrm{CH}}^1(X)_{\mathbb{Q}}$. □

Further, we can give a generalization of [15, Theorem A].

Corollary 4.3. *Let us consider a homomorphism*

$$L : \widehat{\mathrm{CH}}^p_D(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^{p+1}_D(X)_{\mathbb{Q}}$$

given by $L(x)=\widehat{c}_1(\overline{H})\cdot x$. *By abuse of notation, the composition of homomorphisms*

$$\widehat{\mathrm{Pic}}_D(X)_{\mathbb{Q}} \longrightarrow \widehat{\mathrm{CH}}^1_D(X)_{\mathbb{Q}} \xrightarrow{L^{d-1}} \widehat{\mathrm{CH}}^d_D(X)_{\mathbb{Q}}$$

is also denoted by L^{d-1}. *Then, we have the following.*

(1) $\mathrm{Ker}\left(L^{d-1} : \widehat{\mathrm{Pic}}_D(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^d_D(X)_{\mathbb{Q}}\right) = \mathrm{Ker}\left(\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^1(X)_{\mathbb{Q}}\right)$. *In particular, if* X *is regular, then* $L^{d-1} : \widehat{\mathrm{CH}}^1_D(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^d_D(X)_{\mathbb{Q}}$ *is injective.*

(2) *If* $x \in \widehat{\mathrm{Pic}}_{L_1^2}(X)_{\mathbb{Q}}$, $x\neq 0$ *in* $\widehat{\mathrm{CH}}^1_{L_1^2}(X)_{\mathbb{Q}}$, *and* $L^d(x)=0$, *then* $\widehat{\deg}(L^{d-1}(x)\cdot x)<0$.

Proof. First, let us see (2). By virtue of (2) of Proposition 3.2.4,

$$L^{d-1}(x)\cdot x = (\widehat{c}_1(\overline{H})^{d-1}\cdot x)\cdot x = \widehat{c}_1(\overline{H})^{d-1}\cdot x^2.$$

Thus, applying Corollary 4.2 to the case where $h=\widehat{c}_1(\overline{H})$, we have (2).

Next, let us see (1). It is sufficient to show that if $x\in\widehat{\mathrm{Pic}}_D(X)_{\mathbb{Q}}$ and $L^{d-1}(x)=0$, then $x\in\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ and $x=0$ in $\widehat{\mathrm{CH}}^1(X)_{\mathbb{Q}}$. Let us choose $x_0\in\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ such that $z(x)=z(x_0)$ and $\omega(x_0)$ is harmonic with respect to $c_1(\overline{H})$. Then, there is a distribution T on $X(\mathbb{C})$ with $x=x_0+a(T)$. Here, $z(x)\cdot c_1(H)^{d-1}=z(L^{d-1}(x))=0$. Thus, $\omega(x_0)\wedge c_1(\overline{H})^{d-1}=0$. Therefore,

$$\begin{aligned}\widehat{c}_1(\overline{H})^{d-1}\cdot x_0^2 &= \widehat{c}_1(\overline{H})^{d-1}\cdot x_0^2 + a(\omega(x_0)\wedge c_1(\overline{H})^{d-1}T)\\ &= \widehat{c}_1(\overline{H})^{d-1}\cdot x_0\cdot x = x_0\cdot L^{d-1}(x) = 0.\end{aligned}$$

Thus, by virtue of Theorem 4.1, there is $y\in\widehat{\mathrm{CH}}^1(\mathrm{Spec}(O_K))_{\mathbb{Q}}$ with $x_0=f^*(y)$ in $\widehat{\mathrm{CH}}^1(X)_{\mathbb{Q}}$. In particular, $\omega(x_0)=0$. Therefore,

$$\begin{aligned}c_1(\overline{H})^{d-1}\wedge dd^c(T) &= \omega(\widehat{c}_1(\overline{H})^{d-1})\wedge\omega(x_0) + c_1(\overline{H})^{d-1}\wedge dd^c(T)\\ &= \omega(L^{d-1}(x_0)+a(c_1(\overline{H})^{d-1}T)) = \omega(L^{d-1}(x)) = 0.\end{aligned}$$

This implies that $\Delta(T)=0$, where Δ is the Laplacian with respect to $c_1(\overline{H})$. Hence, using the regularity of solutions of the elliptic operator Δ, T is represented by the C^∞-function c which is constant on each connected component of $X(\mathbb{C})$. In particular, $x\in\widehat{\mathrm{Pic}}^1(X)_{\mathbb{Q}}$. Thus, by virtue of (2), we have $x=0$ in $\widehat{\mathrm{CH}}^1(X)_{\mathbb{Q}}$. □

Thus, by the equality condition of (ii) of Lemma 1.1, there is a function c on $X(\mathbb{C})$ such that $\phi = c$ (a. e.) and c is constant on each connected component of $X(\mathbb{C})$. Therefore,

$$x = x_0 + a(c) \in \widehat{\mathrm{Pic}}(X).$$

Hence, by [15, Theorem 1.1], then there are $(D, g) \in \widehat{\mathrm{Div}}(X)$ and a positive integer n such that $nx \sim (D, g)$, D is vertical with respect to $X \to \mathrm{Spec}(O_K)$ and g is constant on each connected component of $X(\mathbb{C})$. Then,

$$\widehat{\deg}\left(\widehat{c}_1(H,k)^{d-1} \cdot (D,g)^2\right) = \sum_{P \in \mathrm{Spec}(O_K) \setminus \{0\}} \deg(H_P^{d-1} \cdot D_P^2).$$

Thus, by Zariski's lemma for integral scheme (cf. Lemma B.1), there is a $\mathbb{Q}$-divisor T on $\mathrm{Spec}(O_K)$ with $f^*(T) = D$ in $Z^1(X)_{\mathbb{Q}}$. Let $K(\mathbb{C})$ be the set of all embeddings of K into $\mathbb{C}$, and, for each $\sigma \in K(\mathbb{C})$, let $X_\sigma = X \otimes_K^\sigma \mathbb{C}$ be the base extension in terms of σ. Then, $X(\mathbb{C}) = \coprod_\sigma X_\sigma$ is nothing more than the decomposition into connected component. Let g_σ be the value of g on X_σ. Then, $(D, g) = f^*(T, \{g_\sigma\}_\sigma)$ in $\widehat{Z}^1(X)_{\mathbb{Q}}$. Thus, if we set $y = (1/n)(T, \{g_\sigma\}_\sigma)$, then $x = f^*(y)$ in $\widehat{\mathrm{CH}}^1(X)_{\mathbb{Q}}$.

Finally, we assume that there is $y \in \widehat{\mathrm{CH}}^1(\mathrm{Spec}(O_K))_{\mathbb{Q}}$ with $x = f^*(y)$ in $\widehat{\mathrm{CH}}^1(X)_{\mathbb{Q}}$. Then, since $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} \otimes \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^2(X)_{\mathbb{Q}}$ passes through $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} \otimes \widehat{\mathrm{CH}}^1(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^2(X)_{\mathbb{Q}}$, and the pairing

$$\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} \otimes \cdots \otimes \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^1(\mathrm{Spec}(O_K))_{\mathbb{Q}}$$

is symmetric, we can see

$$\begin{aligned}
\widehat{\deg}\left(\widehat{c}_1(\overline{H})^{d-1} \cdot x^2\right) &= \widehat{\deg}\left(\widehat{c}_1(\overline{H})^{d-1} \cdot (x \cdot f^*(y))\right) \\
&= \widehat{\deg}\left(\widehat{c}_1(\overline{H})^{d-1} \cdot (f^*(y) \cdot x)\right) \\
&= \widehat{\deg}\left(\widehat{c}_1(\overline{H})^{d-1} \cdot (f^*(y) \cdot f^*(y))\right) = 0.
\end{aligned}$$

Thus, we get all assertions of Theorem 4.1. □

Corollary 4.2. *Let $h \in \widehat{\mathrm{Pic}}_{L_1^2}(X)_{\mathbb{Q}}$ with $\deg_K(h) > 0$ and $(h \cdot h)_{\overline{H}} > 0$. If $x \in \widehat{\mathrm{Pic}}_{L_1^2}(X)_{\mathbb{Q}}$ and $(h \cdot x)_{\overline{H}} = 0$, then $(x \cdot x)_{\overline{H}} \leq 0$. Moreover, if $(x \cdot x)_{\overline{H}} = 0$ and $f_*(\widehat{c}_1(H,k)^{d-1} \cdot (h \cdot x)) = 0$, then $x = 0$ in $\widehat{\mathrm{CH}}^1(X)_{\mathbb{Q}}$.*

Proof. Let us choose a rational number t with $\deg_K(x + th) = 0$. Then, by Theorem 4.1,

$$0 \geq (x + th \cdot x + th)_{\overline{H}} = (x \cdot x)_{\overline{H}} + t^2 (h \cdot h)_{\overline{H}}.$$

Thus, $(x \cdot x)_{\overline{H}} \leq 0$.

Next we assume that $(x \cdot x)_{\overline{H}} = 0$ and $f_*(\widehat{c}_1(H,k)^{d-1} \cdot (h \cdot x)) = 0$. Then, in the above inequality, we can see that $t = 0$. Thus, $\deg_K(x) = 0$ and $(x \cdot x)_{\overline{H}} = 0$. Therefore, by Theorem 4.1, there is $y \in \widehat{\mathrm{CH}}^1(\mathrm{Spec}(O_K))_{\mathbb{Q}}$ with $x = f^*(y)$ in $\widehat{\mathrm{CH}}^1(X)_{\mathbb{Q}}$. Then, by virtue of Proposition A.1,

$$f_*\left(f^*(y) \cdot (\widehat{c}_1(H,k)^d \cdot h)\right) = y \cdot f_*(\widehat{c}_1(H,k)^d \cdot h) = \deg_K(h) y.$$

Remark 3.3.4. Let $x_0 \in \widehat{\mathrm{Pic}}(X)$, $y_0 \in \widehat{\mathrm{Pic}}(X)$, $\phi \in \mathbf{Br}(X(\mathbb{C}))$, and $\psi \in \mathbf{Br}(X(\mathbb{C}))$. Then,

$$(x_0 + a(\phi)) \cdot (y_0 + a(\psi)) = x_0 \cdot y_0 + a(\omega(x_0)\psi + \phi\omega(y_0) + dd^c(\phi)\psi).$$

Indeed, by Proposition 1.2 and (ii) of Lemma 1.1, we have

$$[\partial\bar{\partial}(\phi)\psi] = -[\partial(\phi) \wedge \bar{\partial}(\psi)] = -[\partial(\phi) \wedge \bar{\partial}(\psi)]$$

modulo $\mathrm{Im}(\partial) + \mathrm{Im}(\bar{\partial})$.

4. Hodge index theorem

Let K be a number field, and O_K the ring of integers of K. Let $f : X \to \mathrm{Spec}(O_K)$ be a projective arithmetic variety with the geometrically irreducible generic fiber. Let $\overline{H} = (H, k)$ be an arithmetically ample Hermitian line bundle on X, i.e., (1) H is f-ample, (2) the Chern form $c_1(H, k)$ is positive definite on the infinite fiber $X(\mathbb{C})$, and (3) there is a positive integer m_0 such that, for any integer $m \geq m_0$, $H^0(X, H^m)$ is generated by the set $\{s \in H^0(X, H^m) \mid \|s\|_{\sup} < 1\}$. Let us consider the pairing

$$(\ \cdot\ )_{\overline{H}} : \widehat{\mathrm{Pic}}_{L^2_1}(X)_{\mathbb{Q}} \times \widehat{\mathrm{Pic}}_{L^2_1}(X)_{\mathbb{Q}} \to \mathbb{R}$$

given by $(x \cdot y)_{\overline{H}} = \widehat{\deg}\left(\widehat{c}_1(H, k)^{d-1} \cdot (x \cdot y)\right)$, where $d = \dim X_K$. Further, we have the homomorphism

$$\deg_K : \widehat{\mathrm{Pic}}_{L^2_1}(X)_{\mathbb{Q}} \to \mathbb{Q}$$

given by $\deg_K((D, g)) = (D_K \cdot c_1(H_K)^{d-1})$.

Theorem 4.1. *If $x \in \widehat{\mathrm{Pic}}_{L^2_1}(X)_{\mathbb{Q}}$ and $\deg_K(x) = 0$, then $(x \cdot x)_{\overline{H}} \leq 0$. Moreover, the equality holds if and only if there is $y \in \widehat{\mathrm{CH}}^1(\mathrm{Spec}(O_K))_{\mathbb{Q}}$ with $x = f^*(y)$ in $\widehat{\mathrm{CH}}^1(X)_{\mathbb{Q}}$.*

Proof. Clearly, we may assume that $x \in \widehat{\mathrm{Pic}}_{L^2_1}(X)$. There is $x_0 \in \widehat{\mathrm{Pic}}(X)$ such that $z(x) = z(x_0)$ and $\omega(x_0)$ is harmonic with respect to $c_1(H, k)$. Consequently, $\omega(x_0) \wedge c_1(H, k)^{d-1} = 0$ because $\deg_K(x) = 0$. Then, we can find $\phi \in L^2_{1,loc}(X, \mathbb{R})$ with $x = x_0 + a(\phi)$. Thus, since $x_0 \cdot a(\phi) = a(\phi\omega(x_0))$ and $a(\phi) \cdot a(\phi) = a\left(\frac{-\sqrt{-1}}{2\pi}\partial(\phi) \wedge \bar{\partial}(\phi)\right)$, we have

$$\begin{aligned}(x \cdot x)_{\overline{H}} &= (x_0 \cdot x_0)_{\overline{H}} + 2(x_0 \cdot a(\phi))_{\overline{H}} + (a(\phi) \cdot a(\phi))_{\overline{H}} \\ &= (x_0 \cdot x_0)_{\overline{H}} - \frac{\sqrt{-1}}{4\pi}\int_{X(\mathbb{C})} \partial(\phi) \wedge \bar{\partial}(\phi) \wedge c_1(H, k)^{d-1}.\end{aligned}$$

On the other hand, by (ii) of Lemma 1.1,

$$\sqrt{-1}\int_{X(\mathbb{C})} \partial(\phi) \wedge \bar{\partial}(\phi) \wedge c_1(H, k)^{d-1} \geq 0.$$

Thus, $(x \cdot x)_{\overline{H}} \leq (x_0 \cdot x_0)_{\overline{H}}$. Moreover, by [15, Theorem 1.1], $(x_0 \cdot x_0)_{\overline{H}} \leq 0$. Therefore, we get the first assertion.

Next we assume that the equality holds. Then,

$$\sqrt{-1}\int_{X(\mathbb{C})} \partial(\phi) \wedge \bar{\partial}(\phi) \wedge c_1(H, k)^{d-1} = 0.$$

3.3. **Arithmetic B-Cartier divisor.** Let X be an arithmetic variety. Let Rat_X be the sheaf of rational functions on X. We denote $H^0(X, \mathrm{Rat}_X^\times/\mathcal{O}_X^\times)$ by $\mathrm{Div}(X)$. An element of $\mathrm{Div}(X)$ is called *a Cartier divisor on* X. For a Cartier divisor D on X, we can assign the divisor $[D] \in Z^1(X)$ in the natural way. This gives rise to a homomorphism

$$c_X : \mathrm{Div}(X) \to Z^1(X).$$

Note that c_X is neither injective nor surjective in general. The exact sequence

$$1 \to \mathcal{O}_X^\times \to \mathrm{Rat}_X^\times \to \mathrm{Rat}_X^\times/\mathcal{O}_X^\times \to 1$$

induces to a homomorphism $\mathrm{Div}(X) \to H^1(X, \mathcal{O}_X^\times)$. For a Cartier divisor D on X, the image of D by the above homomorphism induces a line bundle on X. We denote this line bundle by $\mathcal{O}_X(D)$. *An arithmetic Cartier divisor on* X is a pair (D, g) such that $D \in \mathrm{Div}(X)$ and g is a Green function for $D(\mathbb{C})$ on $X(\mathbb{C})$. The set of all arithmetic Cartier divisors on X is denoted by $\widehat{\mathrm{Div}}(X)$, and $\widehat{\mathrm{Pic}}(X)$ is defined by $\widehat{\mathrm{Div}}(X)/\hat{c}_X^{-1}(\widehat{R}^1(X))$, where $\hat{c}_X$ is a natural homomorphism $\widehat{\mathrm{Div}}(X) \to \widehat{Z}^1(X)$.

Let us fix a class B of degeneration of Green functions on $X(\mathbb{C})$, namely, an abelian group with $C^\infty(X, \mathbb{R}) \subseteq B \subseteq L^r_{1,loc}(X, \mathbb{R})$. A pair (D, g) is called *an arithmetic B-Cartier divisor on* X if $D \in \mathrm{Div}(X)$ and g is a B-Green function for $D(\mathbb{C})$ on $X(\mathbb{C})$. We denote the set of all arithmetic B-Cartier divisors on X by $\widehat{\mathrm{Div}}_B(X)$, and define

$$\widehat{\mathrm{Pic}}_B(X) = \widehat{\mathrm{Div}}_B(X)/\hat{c}_X^{-1}(\widehat{R}^1(X)).$$

Note that if X is regular, then $\widehat{\mathrm{Div}}_B(X) = \widehat{Z}^1_B(X)$ and $\widehat{\mathrm{Pic}}_B(X) = \widehat{\mathrm{CH}}^1_B(X)$.

If $B = L^r_{k,loc}(\Omega^{p-1,p-1}_{X(\mathbb{C})}, \mathbb{R})$, then $\widehat{\mathrm{Div}}_B(X)$ and $\widehat{\mathrm{Pic}}_B(X)$ are denoted by $\widehat{\mathrm{Div}}_{L^r_k}(X)$ and $\widehat{\mathrm{Pic}}_{L^r_k}(X)$. Moreover, If $B = \mathbf{Br}(X(\mathbb{C}))$, then $\widehat{Z}^1_B(X)$, $\widehat{\mathrm{Div}}_B(X)$, $\widehat{\mathrm{CH}}^1_B(X)$, and $\widehat{\mathrm{Pic}}_B(X)$ are denoted by $\widehat{Z}^1_{\mathbf{Br}}(X)$, $\widehat{\mathrm{Div}}_{\mathbf{Br}}(X)$, $\widehat{\mathrm{CH}}^1_{\mathbf{Br}}(X)$, and $\widehat{\mathrm{Pic}}_{\mathbf{Br}}(X)$. An element of $\widehat{Z}^1_{\mathbf{Br}}(X)$ (resp. $\widehat{\mathrm{Div}}_{\mathbf{Br}}(X)$) is called a *birational arithmetic divisor* (resp. a *birational arithmetic Cartier divisor*).

We can easily to see that (3.2.2) and (3.2.3) induce

$$\widehat{\mathrm{Pic}}(X) \otimes \widehat{\mathrm{CH}}^p_D(X) \to \widehat{\mathrm{CH}}^{p+1}_D(X). \tag{3.3.1}$$

and

$$\widehat{\mathrm{Pic}}_{L^r_1}(X) \otimes \widehat{\mathrm{CH}}^p_{L^{r'}_1}(X) \to \widehat{\mathrm{CH}}^{p+1}_D(X) \tag{3.3.2}$$

respectively, where $1/r + 1/r' = 1$. Note that if D and Z intersect properly, then (3.3.2) is given by $(D, g_D) \cdot (Z, g_Z) = (D \cdot Z, g_D * g_Z)$, where $D \cdot Z$ is defined as follows. Let s be a rational section of $\mathcal{O}_X(D)$ with $\mathrm{div}(s) = D$, and $Z = a_1 Z_1 + \cdots + a_n Z_n$ the irreducible decomposition as cycles. Then, $s|_{Z_i}$ gives rise to a rational section of $\mathcal{O}_{Z_i}(D)$, so that we define $D \cdot Z$ to be

$$a_1 \,\mathrm{div}(s|_{Z_1}) + \cdots + a_n \,\mathrm{div}(s|_{Z_n}).$$

In particular, if $B \subseteq L^2_{1,loc}(X, \mathbb{R})$, then we have a commutative pairing:

$$\widehat{\mathrm{Pic}}_B(X) \otimes \widehat{\mathrm{Pic}}_B(X) \to \widehat{\mathrm{CH}}^2_D(X). \tag{3.3.3}$$

On the other hand,

$$\begin{aligned}(x \cdot y) \cdot z &= (x \cdot y_0 + a(\omega(x) \wedge \phi)) \cdot z \\ &= (x \cdot y_0) \cdot z_0 + \\ &\qquad a\left([\omega(x \cdot y_0) \wedge \psi] + [\omega(x) \wedge \phi \wedge \omega(z_0)] - \frac{\sqrt{-1}}{2\pi}[\partial(\omega(x) \wedge \phi) \wedge \bar{\partial}(\psi)]\right).\end{aligned}$$

Here $\omega(x \cdot y_0) = \omega(x) \wedge \omega(y_0)$ and $\partial(\omega(x) \wedge \phi) = \omega(x) \wedge \partial(\phi)$. Thus, we have our proposition. □

3.2. **Intersection on singular varieties.** Let X be an arithmetic variety. Let $\overline{E} = (E, h)$ be a Hermitian vector bundle on X. In [5, Theorem 4], the operator

$$\widehat{\mathrm{ch}}(\overline{E}) : \widehat{\mathrm{CH}}^*(X) \to \widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}} \tag{3.2.1}$$

is defined. Here we would like to extend the above operator in two ways.

The first one is

$$\widehat{\mathrm{ch}}(\overline{E}) : \widehat{\mathrm{CH}}^*_D(X) \to \widehat{\mathrm{CH}}^*_D(X)_{\mathbb{Q}}. \tag{3.2.2}$$

This is defined by

$$\widehat{\mathrm{ch}}(\overline{E}) \cdot (x_0 + a(T)) = \widehat{\mathrm{ch}}(\overline{E}) \cdot x_0 + a(\mathrm{ch}(\overline{E}) \wedge T),$$

where $x_0 \in \widehat{\mathrm{CH}}^*(X)$ and $T \in \bigoplus_{p \geq 1} D^{p-1,p-1}(X(\mathbb{C}))$.

To give the second one, let us fix an element ϕ of $\bigoplus_{p \geq 1} L^r_{1,loc}(\Omega^{p-1,p-1}_{X(\mathbb{C})}, \mathbb{R})$. We define

$$\widehat{\mathrm{ch}}(\overline{E}) + a(\phi) : \widehat{\mathrm{CH}}^*_{L^{r'}_1}(X) \to \widehat{\mathrm{CH}}^*_D(X)_{\mathbb{Q}} \tag{3.2.3}$$

as follows, where $1/r + 1/r' = 1$. Let $x \in \widehat{\mathrm{CH}}^*_{L^{r'}_1}(X)$. We choose $x_0 \in \widehat{\mathrm{CH}}^*(X)$ and $\psi \in \bigoplus_{p \geq 1} L^{r'}_{1,loc}(\Omega^{p-1,p-1}_{X(\mathbb{C})}, \mathbb{R})$ with $x = x_0 + a(\psi)$. Then,

$$(\widehat{\mathrm{ch}}(\overline{E}) + a(\phi)) \cdot x = \widehat{\mathrm{ch}}(\overline{E}) \cdot x_0 + a\left([\mathrm{ch}(\overline{E}) \wedge \psi] + [\phi \wedge \omega(x_0)] - \frac{\sqrt{-1}}{2\pi}[\partial(\phi) \wedge \bar{\partial}(\psi)]\right).$$

In the same way as in Proposition 3.1.4, using [5, 4 of Theorem 4], we have the following.

Proposition 3.2.4. *Let $\overline{E}$ and $\overline{F}$ be Hermitian vector bundles on X. Then, we have the following.*

1. $\widehat{\mathrm{ch}}(\overline{E}) \cdot (\widehat{\mathrm{ch}}(\overline{F}) \cdot x) = \widehat{\mathrm{ch}}(\overline{E} \otimes \overline{F}) \cdot x$ *for* $x \in \widehat{\mathrm{CH}}^*_D(X)$.
2. *Let r and r' be real numbers with $1 \leq r, r' < \infty$ and $1/r + 1/r' = 1$. Then, for any $\phi \in \bigoplus_{p \geq 1} L^r_{1,loc}(\Omega^{p-1,p-1}_{X(\mathbb{C})}, \mathbb{R})$ and $x \in \widehat{\mathrm{CH}}^*_{L^{r'}_1}(X)$,*

$$\widehat{\mathrm{ch}}(\overline{E}) \cdot \left((\widehat{\mathrm{ch}}(\overline{F}) + a(\phi)) \cdot x\right) = \left(\widehat{\mathrm{ch}}(\overline{E} \otimes \overline{F}) + a(\mathrm{ch}(\overline{E}) \wedge \phi)\right) \cdot x.$$

From now, we assume that X is regular. We would like to construct a pairing

$$\widehat{\mathrm{CH}}^p_{L^r_1}(X) \otimes \widehat{\mathrm{CH}}^q_{L^{r'}_1}(X) \to \widehat{\mathrm{CH}}^{p+q}_D(X)_{\mathbb{Q}}, \tag{3.1.1}$$

where $1/r + 1/r' = 1$. Roughly speaking, for $(Z_1, g_1) \in \widehat{Z}^p_{L^r_1}(X)$ and $(Z_2, g_2) \in \widehat{Z}^q_{L^{r'}_1}(X)$, $(Z_1, g_1) \cdot (Z_2, g_2)$ is defined by $(Z_1 \cdot Z_2, g_1 * g_2)$ if Z_1 and Z_2 intersect properly.

To define it exactly, for $x \in \widehat{\mathrm{CH}}^p_{L^r_1}(X)$ and $y \in \widehat{\mathrm{CH}}^q_{L^{r'}_1}(X)$, we choose $x_0 \in \widehat{\mathrm{CH}}^p(X)$, $y_0 \in \widehat{\mathrm{CH}}^q(Y)$, $\phi \in L^r_{k,loc}(\Omega^{p-1,p-1}_{X(\mathbb{C})}, \mathbb{R})$, and $\psi \in L^{r'}_{k,loc}(\Omega^{q-1,q-1}_{X(\mathbb{C})}, \mathbb{R})$ with $x = x_0 + a(\phi)$ and $y = y_0 + a(\psi)$. Then, we define $x \cdot y$ to be

$$x \cdot y = x_0 \cdot y_0 + a\left([\omega(x_0) \wedge \psi] + [\phi \wedge \omega(y_0)] - \frac{\sqrt{-1}}{2\pi}[\partial(\phi) \wedge \bar{\partial}(\psi)]\right),$$

where $x_0 \cdot y_0$ is the usual arithmetic intersection. In the same way as in the proof of Proposition 2.1.1, we can see that the above definition does not depend on the choices of x_0, y_0, ϕ and ψ. Moreover, the pairing is commutative by virtue of (ii) of Lemma 1.1.

Here we recall the scalar product

$$\widehat{\mathrm{CH}}^p(X) \otimes \widehat{\mathrm{CH}}^q_D(X) \to \widehat{\mathrm{CH}}^{p+q}_D(X)_{\mathbb{Q}} \tag{3.1.2}$$

introduced in [10, §2.3]. Let $x \in \widehat{\mathrm{CH}}^p(X)$ and $y \in \widehat{\mathrm{CH}}^q_D(X)$. We choose $y_0 \in \widehat{\mathrm{CH}}^q(X)$ and $T \in D^{p-1,p-1}(X(\mathbb{C}))$ with $y = y_0 + a(T)$. Then, $x \cdot y$ is defined by

$$x \cdot y = x \cdot y_0 + a(\omega(x) \wedge T).$$

If $y \in \widehat{\mathrm{CH}}^q_{L^r_k}(X)$, then T can be represented by $\phi \in L^r_{k,loc}(\Omega^{q-1,q-1}_{X(\mathbb{C})}, \mathbb{R})$. Thus, $\omega(x) \wedge T \in L^r_{k,loc}(\Omega^{p+q-1,p+q-1}_{X(\mathbb{C})}, \mathbb{R})$. This observation shows us that (3.1.2) induces

$$\widehat{\mathrm{CH}}^p(X) \otimes \widehat{\mathrm{CH}}^q_{L^r_k}(X) \to \widehat{\mathrm{CH}}^{p+q}_{L^r_k}(X)_{\mathbb{Q}} \tag{3.1.3}$$

Then, we have the following proposition.

Proposition 3.1.4. *Let r and r' be real numbers with $1 \leq r, r' < \infty$ and $1/r + 1/r' = 1$. For $x \in \widehat{\mathrm{CH}}^p(X)$, $y \in \widehat{\mathrm{CH}}^q_{L^r_1}(X)$, and $z \in \widehat{\mathrm{CH}}^s_{L^{r'}_1}(X)$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.*

Proof. We set $y = y_0 + a(\phi)$ and $z = z_0 + a(\psi)$, where $y_0 \in \widehat{\mathrm{CH}}^q(X)$, $z_0 \in \widehat{\mathrm{CH}}^s(Y)$, $\phi \in L^r_{k,loc}(\Omega^{q-1,q-1}_{X(\mathbb{C})}, \mathbb{R})$, and $\psi \in L^{r'}_{k,loc}(\Omega^{s-1,s-1}_{X(\mathbb{C})}, \mathbb{R})$. Then,

$$\begin{aligned} x \cdot (y \cdot z) &= x \cdot \left(y_0 \cdot z_0 + a\left([\omega(y_0) \wedge \psi] + [\phi \wedge \omega(z_0)] - \frac{\sqrt{-1}}{2\pi}[\partial(\phi) \wedge \bar{\partial}(\psi)]\right)\right) \\ &= x \cdot (y_0 \cdot z_0) + a\left(\omega(x) \wedge \left([\omega(y_0) \wedge \psi] + [\phi \wedge \omega(z_0)] - \frac{\sqrt{-1}}{2\pi}[\partial(\phi) \wedge \bar{\partial}(\psi)]\right)\right). \end{aligned}$$

for $\mathrm{div}(s')$. Thus, $-\log h(s,s)$ is a birational Green function for $\mathrm{div}(s)$ because $-\log h(s,s) = \mu_*(-\log h'(s',s'))$.

Next, we assume that $g = -\log h(s,s)$ is a birational Green function for $D = \mathrm{div}(s)$. Let (D',g') be a resolution of singularities of (D,g) by $\mu : Y \to X$. Let s' be a non-zero rational section of $\mathcal{O}_Y(D')$ with $D' = \mathrm{div}(s')$. Let U be a non-empty Zariski open set of X with $\mu^{-1}(U) \simeq U$. Since $D'|_{\mu^{-1}(U)} = D|_U$, there is $u \in H^0(U,\mathcal{O}_U)^\times$ with $s' = us$. Thus, there is an isomorphism $\iota : \mathcal{O}_Y(D')|_{\mu^{-1}(U)} \xrightarrow{\sim} \mathcal{O}_X(D)|_U$ with $\iota(s') = s$. Since g' is a Green function for D', there is a C^∞ Hermitian metric h' of $\mathcal{O}_Y(D')$ with $g' = -\log h'(s',s')$. Then, we have $h'(s',s') = h(s,s)$ over $U = \mu^{-1}(U)$ because $g = \mu_*(g')$. Thus, ι gives rise to an isometry $(\mathcal{O}_Y(D'),h')|_{\mu^{-1}(U)} \xrightarrow{\sim} (\mathcal{O}_X(D),h)|_U$. □

Corollary 2.2.2. *Let g be a birational Green function for a divisor D. Then, there is a birationally Hermitian metric h of $\mathcal{O}_X(D)$ such that $g = -\log h(s,s)$, where s is a non-zero rational section of $\mathcal{O}_X(D)$ with $\mathrm{div}(s) = D$.*

Proof. Since g is a locally integrable function, there is a generalized metric h of $\mathcal{O}_X(D)$ with $g = -\log h(s,s)$. Hence, by Proposition 2.2.1, h must be a birationally Hermitian metric. □

3. Arithmetic B-cycles and their pairing

3.1. **Arithmetic B-cycles.** Let X be an arithmetic variety, i.e., a quasi-projective integral scheme over $\mathbb{Z}$ with the smooth generic fiber over $\mathbb{Z}$. For each p, let $\widehat{R}^p(X)$ be the subgroup of $\widehat{Z}^p(X)$ generated by the following elements:

(a) $((f), -[\log|f|^2])$, where f is a rational function on some subvariety Y of codimension $p-1$ and $[\log|f|^2]$ is the current defined by

$$[\log|f|^2](\gamma) = \int_{Y(\mathbb{C})} (\log|f|^2)\gamma.$$

(b) $(0, \partial(\alpha) + \bar{\partial}(\beta))$, where $\alpha \in D^{p-2,p-1}(X(\mathbb{C}))$, $\beta \in D^{p-1,p-2}(X(\mathbb{C}))$.

A pair (Z,g) is called *an arithmetic D-cycle on X of codimension p* if Z is a cycle of codimension p on X and $g \in D^{p-1,p-1}(X(\mathbb{C}))$. We denote the set of all arithmetic D-cycles of codimension p on X by $\widehat{Z}^p_D(X)$. Moreover, $\widehat{\mathrm{CH}}^p_D(X)$ is defined by $\widehat{Z}^p_D(X)/\widehat{R}^p(X)$.

Let B be a class of degeneration of Green currents for codimension p cycles on $X(\mathbb{C})$. A pair (Z,g) is called *an arithmetic B-cycle of codimension p on X* if $Z \in Z^p(X)$ and g is a B-Green current for $Z(\mathbb{C})$ on $X(\mathbb{C})$. We denote the set of all arithmetic B-cycles of codimension p on X by $\widehat{Z}^p_B(X)$, and define

$$\widehat{\mathrm{CH}}^p_B(X) = \widehat{Z}^p_B(X)/\widehat{R}^p(X).$$

If $B = L^r_{k,loc}(\Omega^{p-1,p-1}_{X(\mathbb{C})},\mathbb{R})$, then $\widehat{Z}^p_B(X)$ and $\widehat{\mathrm{CH}}^p_B(X)$ are denoted by $\widehat{Z}^p_{L^r_k}(X)$ and $\widehat{\mathrm{CH}}^p_{L^r_k}(X)$ respectively. Note that for $x \in \widehat{\mathrm{CH}}^p_B(X)$, there are $x_0 \in \widehat{\mathrm{CH}}^p(X)$ and $\phi \in B$ with $x = x_0 + a(\phi)$.

is equal to

$$\frac{\sqrt{-1}}{2\pi}\left([\phi_1\wedge\partial\bar\partial(\eta_2)]+[\partial(\phi_1)\wedge\bar\partial(\eta_2)]\right)+(\eta_1*f_2-f_2*\eta_1)+$$
$$\frac{\sqrt{-1}}{2\pi}\left([\partial\bar\partial(\eta_1)\wedge\phi_2]+[\partial(\eta_1)\wedge\bar\partial(\phi_2)]\right)-\frac{\sqrt{-1}}{2\pi}\left([\partial(\eta_1)\wedge\bar\partial(\eta_2)]+[\eta_1\wedge\partial\bar\partial\eta_2]\right),$$

which is elements of $\mathrm{Im}(\partial)+\mathrm{Im}(\bar\partial)$ by (i) of Lemma 1.1 and [4, Corollary 2.2.9]. Thus, we get (1).

(2) It is well known that $f_1*f_2=f_2*f_1$ modulo $\mathrm{Im}(\partial)+\mathrm{Im}(\bar\partial)$ (cf. [4, Corollary 2.2.9]). Moreover, by (ii) of Lemma 1.1, $[\partial(\phi_1)\wedge\bar\partial(\phi_2)]=[\partial(\phi_2)\wedge\bar\partial(\phi_1)]$ modulo $\mathrm{Im}(\partial)+\mathrm{Im}(\bar\partial)$. Thus, we have (2). □

2.2. **Birational Green function.** Let X be a complex manifold, and D a divisor on X. Let $\mathbf{Br}(X)$ be the class introduced in §1. A $\mathbf{Br}(X)$-Green function for a divisor D is specially called *a birational Green function for* D. It is easy to see that g is a birational Green function for D if and only if there are a proper bimeromorphic morphism $\mu:Y\to X$ of complex manifolds, a divisor D' on Y, and a Green function g' for D' such that $\mu_*(D')=D$ and $\mu_*(g')=g$. A point at which $\omega(g)$ is not C^∞ is called *a singular point of* g. In this sense, the pair (D',g') is called *a resolution of singularities of* (D,g) *by* μ.

An idea of the birational Green functions arises from the following observation. Let (E,h) a Hermitian vector bundle on X, L a rank-1 saturated subsheaf of E, and s a rational section of L. If $\dim X\geq 2$, then $g=-\log h(s,s)$ is not necessarily a Green function for $D=\mathrm{div}(s)$ because E/L is not locally free in general. However, we can see that g is a birational Green function for D. For, there are a proper bimeromorphic morphism $\mu:Y\to X$ of complex manifolds, and a rank-1 saturated subsheaf L' of $\mu^*(E)$ such that $\mu^*(E)/L'$ is locally free and $\mu_*(L')=L$. Since L' is a sub-line bundle of $\mu^*(E)$, L' possesses a Hermitian metric h' induced from $\mu^*(h)$. Let s' be the rational section of L' corresponding with s. Then, by virtue of Poincaré-Lelong formula, $g'=-\log h'(s',s')$ is a Green function for $D'=\mathrm{div}(s')$. Further, $D=\mu_*(D')$ and $g=\mu_*(g')$ by our construction.

By the above observation, we have the following definition. Let X be a complex manifold, and E a torsion free sheaf on X. We say h is *a birationally Hermitian metric of* E if there is a proper bimeromorphic morphism $\mu:Y\to X$ of complex manifolds, a Hermitian vector bundle (E',h') on Y, and a Zariski open set U of X such that $\mathrm{codim}(X\setminus U)\geq 2$, $\mu^{-1}(U)\simeq U$, (E,h) is a Hermitian vector bundle on U, and $(E',h')|_{\mu^{-1}(U)}\simeq(E,h)|_U$. We say (E',h') is a resolution of singularities of (E,h) by μ. Then, we have the following proposition.

Proposition 2.2.1. *Let L be a line bundle on X and h a generalized metric of L over X. Let s be a non-zero rational section of L. Then, h is a birationally Hermitian metric if and only if $-\log h(s,s)$ is a birational Green function for* $\mathrm{div}(s)$.

Proof. First, we assume that h is a birationally Hermitian metric. Let (L',h') be a resolution of singularities of (L,h) by $\mu:Y\to X$. Let U be a Zariski open set of X as the above. Here, we can find a rational section s' of L' corresponding with s via isomorphism $(L',h')|_{\mu^{-1}(U)}\simeq(L,h)|_U$. Then, $\mu_*(\mathrm{div}(s'))=\mathrm{div}(s)$, and $-\log h'(s',s')$ is a Green function

Proposition 2.1.1. (1) *$g_1 * g_2$ modulo $\mathrm{Im}(\partial) + \mathrm{Im}(\bar{\partial})$ is well defined, namely, $f_1 * f_2 + [\omega(f_1) \wedge \phi_2] + [\phi_1 \wedge \omega(f_2)] - \frac{\sqrt{-1}}{2\pi}[\partial(\phi_1) \wedge \bar{\partial}(\phi_2)]$ modulo $\mathrm{Im}(\partial) + \mathrm{Im}(\bar{\partial})$ does not depend on the choices of f_1, f_2, ϕ_1 and ϕ_2.*

(2) *$g_1 * g_2 = g_2 * g_1$ modulo $\mathrm{Im}(\partial) + \mathrm{Im}(\bar{\partial})$.*

Proof. (1) Let $g_1 = f_1' + \phi_1'$ and $g_2 = f_2' + \phi_2'$ be another expressions of g_1 and g_2, where f_1' is a Green current for Z_1, f_2' is a Green current for Z_2, $\phi_1' \in B$ and $\phi_2' \in B'$. Then, there are smooth forms η_1 and η_2, and currents S_1, T_1, S_2 and T_2 such that

$$f_1' = f_1 + \eta_1 + \partial(S_1) + \bar{\partial}(T_1) \quad \text{and} \quad f_2' = f_2 + \eta_2 + \partial(S_2) + \bar{\partial}(T_2).$$

Thus, we have

$$\phi_1' = \phi_1 - \eta_1 - \partial(S_1) - \bar{\partial}(T_1) \quad \text{and} \quad \phi_2' = \phi_2 - \eta_2 - \partial(S_2) - \bar{\partial}(T_2).$$

First of all, it is well known that

$$f_1' * f_2' = (f_1 + \eta_1) * (f_2 + \eta_2) \quad \text{modulo } \mathrm{Im}(\partial) + \mathrm{Im}(\bar{\partial}). \tag{2.1.1.1}$$

Moreover, since $\omega(f_1') = \omega(f_1 + \eta_1)$ and $\omega(f_1')$ is ∂ and $\bar{\partial}$-closed, we can see that

$$\omega(f_1') \wedge \phi_2' = \omega(f_1 + \eta_1) \wedge (\phi_2 - \eta_2) - \partial(\omega(f_1 + \eta_1) \wedge S_2) - \bar{\partial}(\omega(f_1 + \eta_1) \wedge T_2),$$

which shows us that

$$\omega(f_1') \wedge \phi_2' = \omega(f_1 + \eta_1) \wedge (\phi_2 - \eta_2) \quad \text{modulo } \mathrm{Im}(\partial) + \mathrm{Im}(\bar{\partial}). \tag{2.1.1.2}$$

In the same way,

$$\phi_1' \wedge \omega(f_2') = (\phi_1 - \eta_1) \wedge \omega(f_2 + \eta_2) \quad \text{modulo } \mathrm{Im}(\partial) + \mathrm{Im}(\bar{\partial}). \tag{2.1.1.3}$$

Further, since

$$\partial(\phi_1') = \partial(\phi_1 - \eta_1) - \partial\bar{\partial}(T_1) \quad \text{and} \quad \bar{\partial}(\phi_2') = \bar{\partial}(\phi_2 - \eta_2) - \bar{\partial}\partial(S_2),$$

$\partial\bar{\partial}(T_1)$ (resp. $\bar{\partial}\partial(S_2)$) is a ∂ and $\bar{\partial}$-closed locally L_0^r-form (resp. $L_0^{r'}$-form). Thus, we can see that

$$\partial(\phi_1') \wedge \bar{\partial}(\phi_2') = \partial(\phi_1 - \eta_1) \wedge \bar{\partial}(\phi_2 - \eta_2) + \bar{\partial}(\partial\bar{\partial}(T_1) \wedge (\phi_2 - \eta_2)) + \partial(\phi_1' \wedge \partial\bar{\partial}(S_2)),$$

which says us that

$$\partial(\phi_1') \wedge \bar{\partial}(\phi_2') = \partial(\phi_1 - \eta_1) \wedge \bar{\partial}(\phi_2 - \eta_2) \quad \text{modulo } \mathrm{Im}(\partial) + \mathrm{Im}(\bar{\partial}). \tag{2.1.1.4}$$

Thus, gathering (2.1.1.1), (2.1.1.2), (2.1.1.3), and (2.1.1.4), we obtain that

$$f_1' * f_2' + [\omega(f_1') \wedge \phi_2'] + [\phi_1' \wedge \omega(f_2')] - \frac{\sqrt{-1}}{2\pi}[\partial(\phi_1') \wedge \bar{\partial}(\phi_2')]$$

is equal to

$$\Delta = (f_1 + \eta_1) * (f_2 + \eta_2) + [\omega(f_1 + \eta_1) \wedge (\phi_2 - \eta_2)] + [(\phi_1 - \eta_1) \wedge \omega(f_2 + \eta_2)] - \frac{\sqrt{-1}}{2\pi}[\partial(\phi_1 - \eta_1) \wedge \bar{\partial}(\phi_2 - \eta_2)]$$

modulo $\mathrm{Im}(\partial) + \mathrm{Im}(\bar{\partial})$. Moreover, by easy calculations, we can see that

$$\Delta - \left(f_1 * f_2 + [\omega(f_1) \wedge \phi_2] + [\phi_1 \wedge \omega(f_2)] - \frac{\sqrt{-1}}{2\pi}[\partial(\phi_1) \wedge \bar{\partial}(\phi_2)] \right)$$

and $\bar{\partial}(f) \wedge \mu^*(\lambda)$ is a C^∞-form by Lemma 1.4, we can see that $\partial(\phi) \wedge \bar{\partial}(\psi) \wedge \lambda$ is integrable. In the same way as above, using Lemma 1.4, we can check that $\phi\bar{\partial}(\psi)$ are locally integrable.

Claim 1.7. $\big[\phi\partial\bar{\partial}(\psi)\big] + \big[\partial(\phi) \wedge \bar{\partial}(\psi)\big] = \partial[\phi\bar{\partial}(\psi)]$.

The above equation means that

$$\int_X \phi\partial\bar{\partial}(\psi) \wedge \lambda + \int_X \partial(\phi) \wedge \bar{\partial}(\psi) \wedge \lambda = \int_X \phi\bar{\partial}(\psi) \wedge \partial(\lambda)$$

for all $\lambda \in C_c^\infty(\Omega_X^{n-1,n-1})$. This is equivalent to say that

$$-2\pi\sqrt{-1}\int_Y g\omega \wedge \mu^*(\lambda) + \int_X \partial(g) \wedge \bar{\partial}(f) \wedge \mu^*(\lambda) = \int_X g\bar{\partial}(f) \wedge \partial(\mu^*(\lambda))$$

for all $\lambda \in C_c^\infty(\Omega_X^{n-1,n-1})$. We set $\eta = \bar{\partial}(f) \wedge \mu^*(\lambda)$. Then, η is C^∞ by Lemma 1.4 and

$$d(\eta) = -2\pi\sqrt{-1}\omega \wedge \mu^*(\lambda) - \bar{\partial}(f) \wedge \mu^*(\partial(\lambda)).$$

Thus, using Lemma 1.3,

$$\int_Y \partial(g) \wedge \eta = \int_Y d(g) \wedge \eta = -\int_Y g \wedge d(\eta) = 2\pi\sqrt{-1}\int_Y g\omega \wedge \mu^*(\lambda) + \int_Y g\bar{\partial}(f) \wedge \mu^*(\partial(\lambda)).$$

Hence we get our claim.

Gathering Claim 1.5, Claim 1.6 and Claim 1.7, we can complete the proof of Proposition 1.2. □

2. Degenerate Green currents

2.1. **B-Green currents and their star product.** Let X be a complex manifold, and B a class of degeneration of Green currents for codimension p cycles. Let Z be a cycle of codimension p on X. A current (resp. locally integrable form) g of type $(p-1,p-1)$ on X is called a *B-Green current for Z* (resp. *B-Green form for Z*) if there are a Green current (resp. Green form) f for Z and $\phi \in B$ with $g = f + [\phi]$ (resp. $g = f + \phi$ (a. e.)). We denote $dd^c(g) + \delta_Z$ by $\omega(g)$. For example, if $B = C^\infty(\Omega_X^{p-1,p-1}, \mathbb{R})$, then a B-Green current is nothing more than an usual Green current.

We also fix a class B' of degeneration of Green currents for codimension q cycles. We assume that $B \subseteq L^r_{1,loc}(\Omega_X^{p-1,p-1}, \mathbb{R})$, $B' \subseteq L^{r'}_{1,loc}(\Omega_X^{q-1,q-1}, \mathbb{R})$, and $1/r + 1/r' = 1$. Let Z_1 be a cycle of codimension p on X, and Z_2 a cycle of codimension q on X. Let g_1 be a B-Green current for Z_1, and g_2 a B'-Green current for Z_2. Let us choose a Green current f_1 for Z_1, a Green current f_2 for Z_2, $\phi_1 \in B$, and $\phi_2 \in B'$ such that $g_1 = f_1 + \phi_1$ and $g_2 = f_2 + \phi_2$. We suppose that Z_1 and Z_2 intersect properly. We would like to define the star product $g_1 * g_2$ of g_1 and g_2 to be

$$g_1 * g_2 = f_1 * f_2 + [\omega(f_1) \wedge \phi_2] + [\phi_1 \wedge \omega(f_2)] - \frac{\sqrt{-1}}{2\pi}[\partial(\phi_1) \wedge \bar{\partial}(\phi_2)]$$

as an element of $D^{p+q-1,p+q-1}(X)/(\mathrm{Im}(\partial) + \mathrm{Im}(\bar{\partial}))$. The following proposition says us that the above product is well defined and it is commutative modulo $\mathrm{Im}(\partial) + \mathrm{Im}(\bar{\partial})$.

around $\mu_\alpha(y)$. Then,

$$\pi^*(\omega) = \sum_{s,t} \mu_\alpha^*(\omega_{st}) \left(d\phi_1 \wedge \cdots \wedge \widehat{d\phi_s} \wedge \cdots \wedge d\phi_n \right) \wedge \left(d\bar{\phi}_1 \wedge \cdots \wedge \widehat{d\bar{\phi}_t} \wedge \cdots \wedge d\bar{\phi}_n \right)$$

Since $b \geq 2$, for each s, there is s' with $1 \leq s' \leq b$ and $s' \neq s$. Then,

$$\frac{dz_i}{z_i} \wedge d\phi_{s'} = \frac{dz_i}{z_i} \wedge ((dz_i) f_{s'} + z_i df_{s'}) = dz_i \wedge df_{s'},$$

which shows us that

$$\frac{dz_i}{z_i} \wedge d\phi_1 \wedge \cdots \wedge \widehat{d\phi_s} \wedge \cdots \wedge d\phi_n$$

is a holomorphic form for all s. In the same way, we can see that

$$\frac{d\bar{z}_i}{\bar{z}_i} \wedge d\bar{\phi}_1 \wedge \cdots \wedge \widehat{d\bar{\phi}_t} \wedge \cdots \wedge d\bar{\phi}_n$$

is an anti-holomorphic form for each t. Thus, we get our lemma. □

Let us go back to the proof of Proposition 1.2. Let us pick up arbitrary $\phi, \psi \in \mathbf{Br}(X)$. Choose a proper bimeromorphic morphism $\mu : Y \to X$ of complex manifolds, divisors D and E on Y, and Green functions g for D and f for E such that $\mu_*(D) = \mu_*(E) = 0$, $\phi = \mu_*(g)$ (a. e.), and $\psi = \mu_*(f)$ (a. e.). Changing a model, if necessarily, we may assume that $\mu : Y \to X$ can be obtained by a succession of blowing-ups along smooth and irreducible subvarieties of codimension at least 2, and that the exceptional set is a divisor with only normal crossings. Let ω be a C^∞-form on Y with

$$dd^c([f]) + \delta_E = \frac{-1}{2\pi\sqrt{-1}} \partial\bar{\partial}([f]) + \delta_E = [\omega].$$

Claim 1.5. *$d([\phi]) = [\mu_*(d(g))]$ and $\partial\bar{\partial}([\psi]) = -2\pi\sqrt{-1}[\mu_*(\omega)]$. In particular, $d([\phi])$ and $\partial\bar{\partial}([\psi])$ are represented by locally integrable forms.*

By virtue of Lemma 1.3, $d([g]) = [d(g)]$. Thus,

$$d([\phi]) = d(\mu_*[g]) = \mu_* d([g]) = \mu_*[d(g)] = [\mu_*(d(g))].$$

Further, $\mu_*\partial\bar{\partial}([f]) = -2\pi\sqrt{-1}\mu_*[\omega]$ because $\mu_*(\delta_E) = 0$. Thus,

$$\partial\bar{\partial}([\psi]) = \partial\bar{\partial}(\mu_*[f]) = \mu_*\partial\bar{\partial}([f]) = -2\pi\sqrt{-1}\mu_*[\omega] = -2\pi\sqrt{-1}[\mu_*(\omega)].$$

Claim 1.6. *$\phi\partial\bar{\partial}(\psi)$, $\partial(\phi) \wedge \bar{\partial}(\psi)$, and $\phi\bar{\partial}(\psi)$ are locally integrable. Note that the local integrability of $\partial(\phi) \wedge \bar{\partial}(\phi)$ implies that $\phi \in L^2_{1,loc}(X, \mathbb{R})$.*

Using the equation $\partial\bar{\partial}(\psi) = -2\pi\sqrt{-1}\mu_*(\omega)$, we have

$$\phi\partial\bar{\partial}(\psi) = -2\pi\sqrt{-1}\mu_*(g)\mu_*(\omega) = -2\pi\sqrt{-1}\mu_*(g\omega).$$

Here, since $g\omega$ is locally integrable, so is $\mu_*(g\omega)$. In order to see that $\partial(\phi) \wedge \bar{\partial}(\psi)$ is locally integrable, it is sufficient to see that $\partial(\phi) \wedge \bar{\partial}(\psi) \wedge \lambda$ is integrable for all $\lambda \in C_c^\infty(\Omega_X^{n-1,n-1})$. Since

$$\int_X \partial(\phi) \wedge \bar{\partial}(\psi) \wedge \lambda = \int_Y \partial(g) \wedge \bar{\partial}(f) \wedge \mu^*(\lambda)$$

where $z_i = x_i + \sqrt{-1}y_i$. Then, since $dx_1 \wedge dy_1 = 0$ on S^1_ϵ,

$$\begin{aligned}\int_{S^1_\epsilon \times \Delta_M^{n-1}} \log|z_1|\omega &= \log(\epsilon) \int_{S^1_\epsilon \times \Delta_M^{n-1}} (\alpha_1 dy_1 + \beta_1 dx_1) \wedge (dx_2 \wedge dy_2) \wedge \cdots \wedge (dx_n \wedge dy_n) \\ &= \epsilon \log(\epsilon) \int_0^{2\pi} \int_{\Delta_M^{n-1}} (\alpha_1^\epsilon \cos(\theta) - \beta_1^\epsilon \sin(\theta)) d\theta dx_2 dy_2 \cdots dx_n dy_n,\end{aligned}$$

where $\alpha_1^\epsilon(\theta, z_2, \cdots, z_n) = \alpha_1(\epsilon e^{i\theta}, z_2, \cdots, z_n)$ and $\beta_1^\epsilon(\theta, z_2, \cdots, z_n) = \beta_1(\epsilon e^{i\theta}, z_2, \cdots, z_n)$. Thus, we get our lemma because

$$\lim_{\epsilon \downarrow 0} \epsilon \log(\epsilon) = 0 \quad \text{and} \quad |\alpha_1^\epsilon \cos(\theta) - \beta_1^\epsilon \sin(\theta)| \leq \|\alpha_1\|_{\sup} + \|\beta_1\|_{\sup}.$$

□

Next, let us consider the following lemma.

Lemma 1.4. *Let X be an n-dimensional complex manifold, and*

$$Y = X_N \xrightarrow{\pi_{N-1}} X_{N-1} \xrightarrow{\pi_{N-2}} \cdots \xrightarrow{\pi_0} X_0 = X$$

a succession of blowing-ups along smooth and irreducible subvarieties of codimension at least 2, i.e., for each $0 \leq \alpha < N$, there is a smooth and irreducible subvariety C_α on X_α such that $\operatorname{codim} C_\alpha \geq 2$ *and $\pi_\alpha : X_{\alpha+1} \to X_\alpha$ is the blowing-up along C_α. Let Σ be the exceptional set of $\pi = \pi_0 \cdots \pi_{N-1} : Y \to X$. Let D be a divisor on Y with* $\operatorname{Supp}(D) \subseteq \Sigma$, *and g a Green function for D. If Σ is a normal crossing divisor, then $d(g) \wedge \pi^*(\omega)$ is a C^∞ form for any $\omega \in C^\infty(X, \Omega^{n-1,n-1})$.*

Proof. Let y be an arbitrary point of Y, and $(z_1, \ldots, z_n)$ a local coordinate of Y at y such that $z_1(y) = \cdots = z_n(y) = 0$ and Σ is given by $\{z_1 \cdots z_a = 0\}$ around y. Then, g can be written by a form

$$g = e_1 \log|z_1|^2 + \cdots + e_a \log|z_a|^2 + (C^\infty \text{ function}).$$

Then,

$$d(g) = \sum_{i=1}^{a} e_i \left(\frac{dz_i}{z_i} + \frac{d\bar{z}_i}{\bar{z}_i} \right) + (C^\infty \text{ form}).$$

Thus, it is sufficient to show that

$$\frac{dz_i}{z_i} \wedge \pi^*(\omega) \quad \text{and} \quad \frac{d\bar{z}_i}{\bar{z}_i} \wedge \pi^*(\omega)$$

are C^∞ forms around y for every $1 \leq i \leq a$.

We choose $0 \leq \alpha < N$ such that $\{z_i = 0\}$ is an irreducible component of $\mu_\alpha^{-1}(C_\alpha)$, where $\mu_\alpha = \pi_\alpha \cdots \pi_{N-1} : Y \to X_\alpha$. Moreover, we choose a local coordinate $(w_1, \ldots, w_n)$ of X_α at $\mu_\alpha(y)$ such that $w_1(\mu_\alpha(y)) = \cdots = w_n(\mu_\alpha(y)) = 0$ and C_α is given by an equation $w_1 = \cdots = w_b = 0$. Then, $b \geq 2$ because $\operatorname{codim} C_\alpha \geq 2$. We set $\phi_i = \mu_\alpha^*(w_i)$ for $i = 1, \ldots, n$. By our choice of x_i's and w_j's, the ideal generated by $\phi_1, \ldots, \phi_b$ is contained in the ideal generated by z_i. Thus, there are holomorphic functions $f_1, \ldots, f_b$ around y with $\phi_1 = z_i f_1, \ldots, \phi_b = z_i f_b$. Here we set

$$(\pi_0 \cdots \pi_{\alpha-1})^*(\omega) = \sum_{s,t} \omega_{st} \left(dw_1 \wedge \cdots \wedge \widehat{dw_s} \wedge \cdots \wedge dw_n \right) \wedge \left(d\bar{w}_1 \wedge \cdots \wedge \widehat{d\bar{w}_t} \wedge \cdots \wedge d\bar{w}_n \right)$$

Thus, in order to complete our lemma, it is sufficient to show the following sublemma. □

Sublemma 1.3.1. *Let $(z_1, \ldots, z_n)$ be a coordinate of $\mathbb{C}^n$, and $a_1, \ldots, a_n$ real numbers. Then, for any $\omega \in C_c^\infty(\mathbb{C}^n, \Omega^{2n-1})$,*

$$\int_{\mathbb{C}^n} \log\left(|z_1|^{a_1} \cdots |z_n|^{a_n}\right) d(\omega) = -\int_{\mathbb{C}^n} d\left(\log\left(|z_1|^{a_1} \cdots |z_n|^{a_n}\right)\right) \wedge \omega.$$

Note that

$$d\left(\log\left(|z_1|^{a_1} \cdots |z_n|^{a_n}\right)\right) = \sum_{i=1}^{n} \frac{a_i}{2}\left(\frac{dz_i}{z_i} + \frac{d\bar{z}_i}{\bar{z}_i}\right)$$

and it is a L^1-form.

Proof. Since $\log\left(|z_1|^{a_1} \cdots |z_n|^{a_n}\right) = a_1 \log|z_1| + \cdots + a_n \log|z_n|$, it is sufficient to see that

$$\int_{\mathbb{C}^n} \log|z_1| d(\omega) = -\int_{\mathbb{C}^n} d\left(\log|z_1|\right) \wedge \omega.$$

For $\epsilon > 0$, we set

$$U_\epsilon = \{(z_1, \ldots, z_n) \mid |z_1| \geq \epsilon\} \quad \text{and} \quad D_\epsilon = \{(z_1, \ldots, z_n) \mid |z_1| = \epsilon\}.$$

Then, since $d(\log|z_1|\omega) = d(\log|z_1|)\omega + \log|z_1| d(\omega)$ over U_ϵ, by Stokes' formula,

$$-\int_{D_\epsilon} \log|z_1|\omega = \int_{U_\epsilon} d(\log|z_1|) \wedge \omega + \int_{U_\epsilon} \log|z_1| d(\omega).$$

Moreover,

$$\lim_{\epsilon \downarrow 0} \int_{U_\epsilon} d(\log|z_1|) \wedge \omega = \int_{\mathbb{C}^n} d(\log|z_1|) \wedge \omega \quad \text{and} \quad \lim_{\epsilon \downarrow 0} \int_{U_\epsilon} \log|z_1| d(\omega) = \int_{\mathbb{C}^n} \log|z_1| d(\omega).$$

Thus, it is sufficient to show that

$$\lim_{\epsilon \downarrow 0} \int_{D_\epsilon} \log|z_1|\omega = 0.$$

Let us choose a sufficiently large number M such that $\mathrm{supp}(\omega) \subset \Delta_M^n$, where $\Delta_M = \{z \in \mathbb{C} \mid |z| \leq M\}$. Then, if we set $S_\epsilon^1 = \{z \in \mathbb{C} \mid |z| = \epsilon\}$, we have

$$\int_{D_\epsilon} \log|z_1|\omega = \int_{S_\epsilon^1 \times \Delta_M^{n-1}} \log|z_1|\omega.$$

Here we set

$$\omega = \sum_{i=1}^{n} \Big\{ \alpha_i (dx_1 \wedge dy_1) \wedge \cdots \wedge (\widehat{dx_i} \wedge dy_i) \wedge \cdots \wedge (dx_n \wedge dy_n) \\ + \beta_i (dx_1 \wedge dy_1) \wedge \cdots \wedge (dx_i \wedge \widehat{dy_i}) \wedge \cdots \wedge (dx_n \wedge dy_n) \Big\},$$

(2) $\phi\wedge\partial\bar{\partial}(\psi)$ *is locally integrable forms for any* $\phi,\psi\in\mathbf{Br}(X)$*, and, as currents,* $\bigl[\phi\wedge\partial\bar{\partial}(\psi)\bigr]+\bigl[\partial(\phi)\wedge\bar{\partial}(\psi)\bigr]\in\mathrm{Im}(\partial)$.

Proof. Obviously, $C^\infty(X,\mathbb{R})\subseteq\mathbf{Br}(X)\subseteq L^2_{loc}(X,\mathbb{R})$ because a C^∞-function is a Green function for the zero divisor, and a Green function is locally square integrable. First, let us check that $\mathbf{Br}(X)$ is an abelian group. Choose arbitrary elements $\phi_1,\phi_2\in\mathbf{Br}(X)$. Then, we can easily find a proper bimeromorphic morphism $\mu:Y\to X$ of complex manifolds, divisors D_1,D_2 on Y, and Green functions g_1 for D_1 and g_2 for D_2 such that $\mu_*(D_1)=\mu_*(D_2)=0$, $\phi_1=\mu_*(g_1)$ (a. e.), and $\phi_2=\mu_*(g_2)$ (a. e.). Thus, g_1-g_2 is a Green function for D_1-D_2, and $\mu_*(g_1-g_2)=\phi_1-\phi_2$ (a. e.). Hence, $\phi_1-\phi_2\in\mathbf{Br}(X)$. This shows us that $\mathbf{Br}(X)$ is an abelian group.

In order to check another properties, we need to prepare two lemmas.

Lemma 1.3. *Let X be an n-dimensional complex manifold, D a divisor on X, and g a Green function for D. Then,*

$$\int_X gd(\omega)=-\int_X d(g)\wedge\omega$$

for all $\omega\in C^\infty_c(X,\Omega^{2n-1})$, i.e., ω is a compactly supported $(2n-1)$-form on X. In other words, $d([g])=[d(g)]$. Note that $d(g)$ is a logarithmic form on X, so that $d(g)$ is locally integrable.

Proof. Let $\mu:Y\to X$ be a proper bimeromorphic morphism such that $\mu^{-1}(\mathrm{Supp}(D))$ is a normal crossing divisor. Then,

$$\int_X gd(\omega)=\int_Y\mu^*(g)d(\mu^*(\omega))\quad\text{and}\quad\int_X d(g)\wedge\omega=\int_Y d(\mu^*(g))\wedge\mu^*(\omega).$$

Thus, we may assume that $\mathrm{Supp}(D)$ is a normal crossing divisor.

Let $\{U_\alpha\}_{\alpha\in A}$ be a locally finite open covering of X such that each U_α is isomorphic to a bounded open set of $\mathbb{C}^n$. Let $\sum_{\alpha\in A}\phi_\alpha=1$ be a partition of unity subordinate to $\{U_\alpha\}_{\alpha\in A}$. If

$$\int_X gd(\phi_\alpha\omega)=-\int_X d(g)\wedge\phi_\alpha\omega$$

for all $\alpha\in A$, then

$$\begin{aligned}
-\int_X d(g)\wedge\omega&=\sum_{\alpha\in A}-\int_X d(g)\wedge\phi_\alpha\omega\\
&=\sum_{\alpha\in A}\int_X gd(\phi_\alpha\omega)\\
&=\sum_{\alpha\in A}\int_X g\left(d(\phi_\alpha)\omega+\phi_\alpha d(\omega)\right)\\
&=\int_X g\left(d\left(\sum_{\alpha\in A}\phi_\alpha\right)\omega+\sum_{\alpha\in A}\phi_\alpha d(\omega)\right)\\
&=\int_X gd(\omega).
\end{aligned}$$

other hand, by virtue of the proof of (i),

$$\partial(\phi \wedge \bar{\partial}(\psi_n)) + \bar{\partial}(\phi \wedge \partial(\psi_n)) = (\partial(\phi) \wedge \bar{\partial}(\psi_n)) - (\partial(\psi_n) \wedge \bar{\partial}(\phi))$$

for all n. Thus, we get (ii).

(iii) Let x be an arbitrary point of X, and $\theta_1, \ldots, \theta_n$ a local orthogonal frame of the holomorphic cotangent bundle Ω^1_X around x with respect to Φ such that $\Phi = \sqrt{-1}\sum_i \theta_i \wedge \theta_i$. If we set $\partial(\phi) = \sum_i a_i\theta_i$ around x, then $\bar{\partial}(\phi) = \sum_i \bar{a}_i\bar{\theta}_i$. Thus,

$$\sqrt{-1}\partial(\phi) \wedge \bar{\partial}(\phi) \wedge \Phi^{n-1} = \sum_{i=1}^{n} |a_i|^2 \Phi^n$$

around x. This means that

$$\sqrt{-1}\partial(\phi) \wedge \bar{\partial}(\phi) \wedge \Phi^{n-1}$$

is non-negative on X. Therefore, we get the first assertion.

Next we assume the equality. Then, by the proof of the inequality, we can see that $\partial(\phi) = \bar{\partial}(\phi) = 0$ (a. e.), i.e., $d(\phi) = 0$ (a. e.). Thus, ϕ is a constant almost everywhere. □

An abelian group B is called *a class of degeneration of Green currents for codimension p cycles in L^r* (or simply a class of degeneration) if $C^\infty(\Omega_X^{p-1,p-1}, \mathbb{R}) \subseteq B \subseteq L^r_{1,loc}(\Omega_X^{p-1,p-1}, \mathbb{R})$. For example, $C^\infty(\Omega_X^{p-1,p-1}, \mathbb{R})$ and $L^r_{k,loc}(\Omega_X^{p-1,p-1}, \mathbb{R})$ $(k \geq 1)$ are classes of degeneration in L^r.

Let us consider a non-trivial example of class of degeneration of Green functions. Let $\mu : Y \to X$ be a proper bimeromorphic morphism of complex manifolds, U the maximal open set of X with $\mu^{-1}(U) \xrightarrow{\sim} U$, and ω a form on Y. We define the form $\mu_*(\omega)$ on X to be

$$\mu_*(\omega)(x) = \begin{cases} \omega(\mu^{-1}(x)) & \text{if } x \in U \\ 0 & \text{if } x \notin U. \end{cases}$$

Note that if ω is locally integrable, then $\mu_*(\omega)$ is also locally integrable and $\mu_*([\omega]) = [\mu_*(\omega)]$ as currents. It is easy to see that $\mu_*(\omega_1 \wedge \omega_2) = \mu_*(\omega_1) \wedge \mu_*(\omega_2)$.

Let D be a divisor on X. A locally integrable function g on X is called *a Green function for D* if g is C^∞ over $X \setminus \mathrm{Supp}(D)$ and $dd^c([g]) + \delta_D$ is represented by a C^∞-form. It is easy to see that if g is a Green function for some divisor, then for any points $x \in X$, there are a meromorphic function f around x and a C^∞-function ψ around x with $g = \log|f| + \psi$.

Here we consider the following space $\mathbf{Br}(X)$. A locally integrable function ϕ on X belongs to $\mathbf{Br}(X)$ if and only if there are a proper bimeromorphic morphism $\mu : Y \to X$ of complex manifolds, a divisor D on Y, and a Green function g for D such that $\mu_*(D) = 0$ and $\phi = \mu_*(g)$ (a. e.).

Proposition 1.2. *The space $\mathbf{Br}(X)$ is a class of degeneration of Green functions in L^2. Moreover, the following properties are satisfied.*

(1) *For all $\phi \in \mathbf{Br}(X)$, the differentials $\partial\bar{\partial}([\phi])$ in the sense of currents are represented by locally integrable forms. (By abuse of notation, representatives of $\partial\bar{\partial}([\phi])$ are denoted by $\partial\bar{\partial}(\phi)$.)*

1. Class of degeneration of Green currents

Let X be a complex manifold. We denote by $C^\infty(X)$ (resp. $C^\infty(X,\mathbb{R})$) the set of all complex (resp. real) valued C^∞ functions. For a non-negative integer k and a real number r with $1 \leq r < \infty$, we denote by $L^r_{k,loc}(X)$ (resp. $L^r_{k,loc}(X,\mathbb{R})$) the set of all complex (resp. real) valued functions on X which locally have all weak partial derivatives up to order k in L^r. Let α be a form of type (p,q) on X. We say α is a locally L^r_k-form if all coefficients of α in terms of local coordinates belong to $L^r_{k,loc}$. We denote the set of all (resp. real) locally L^r_k-forms of type (p,q) on X by $L^r_{k,loc}(\Omega^{p,q}_X)$ (resp. $L^r_{k,loc}(\Omega^{p,q}_X,\mathbb{R})$). Let us begin with the following lemma.

Lemma 1.1. (i) *If $\phi \in L^r_{1,loc}(\Omega^{p,p}_X)$ and $\psi \in C^\infty(\Omega^{q,q}_X)$, then $\big[\phi \wedge \partial\bar\partial(\psi)\big] + \big[\partial(\phi) \wedge \bar\partial(\psi)\big] \in \mathrm{Im}(\partial)$ and $\big[\phi \wedge \partial\bar\partial(\psi)\big] + \big[\partial(\psi) \wedge \bar\partial(\phi)\big] \in \mathrm{Im}(\bar\partial)$ as currents.*

(ii) *Let r and r' be real numbers with $1 \leq r, r' < \infty$ and $1/r + 1/r' = 1$. If $\phi \in L^r_{1,loc}(\Omega^{p,p}_X)$ and $\psi \in L^{r'}_{1,loc}(\Omega^{q,q}_X)$, then $[\partial(\phi) \wedge \bar\partial(\psi)] = [\partial(\psi) \wedge \bar\partial(\phi)]$ modulo $\mathrm{Im}(\partial) + \mathrm{Im}(\bar\partial)$.*

(iii) *If X is an n-dimensional connected compact complex manifold with a fundamental form Φ, then*

$$\sqrt{-1}\int_X \partial(\phi) \wedge \bar\partial(\phi) \wedge \Phi^{d-1} \geq 0$$

for all $\phi \in L^2_1(X,\mathbb{R})$. Moreover, the equality holds if and only if ϕ is a constant almost everywhere.

Proof. (i) First of all, $\phi \wedge \bar\partial(\psi) \in L^r_{1,loc}(\Omega^{p+q,p+q+1}_X)$. Thus,

$$\partial(\phi \wedge \bar\partial(\psi)) = \phi \wedge \partial\bar\partial(\psi) + \partial(\phi) \wedge \bar\partial(\psi).$$

Hence, we get the first assertion. In the same way,

$$\bar\partial(\phi \wedge \partial(\psi)) = \phi \wedge \bar\partial\partial(\psi) + \bar\partial(\phi) \wedge \partial(\psi),$$

which shows us the second assertion.

(ii) It is sufficient to see that

$$\partial[\phi \wedge \bar\partial(\psi)] + \bar\partial[\phi \wedge \partial(\psi)] = [\partial(\phi) \wedge \bar\partial(\psi)] - [\partial(\psi) \wedge \bar\partial(\phi)]$$

as currents. This is a local question, so that we may assume that ϕ and ψ can be written in terms of a local coordinate and that all coefficients of ϕ and ψ belong to $L^r_1(X)$ and $L^{r'}_1(X)$ respectively. Since $C^\infty(X) \cap L^{r'}_1(X)$ is dense in $L^{r'}_1(X)$, there is a sequence $\{\psi_n\}$ in $C^\infty(\Omega^{q,q}_X)$ such that all coefficients of ψ_n belong to $L^{r'}_1(X)$ and they converge to the coefficients of ψ in $L^{r'}_1(X)$. Then,

$$\begin{cases} \lim_{n\to\infty}[\phi\partial(\psi_n)] = [\phi\partial(\psi)], \\ \lim_{n\to\infty}[\phi\bar\partial(\psi_n)] = [\phi\bar\partial(\psi)], \\ \lim_{n\to\infty}[\partial(\phi) \wedge \bar\partial(\psi_n)] = [\partial(\phi) \wedge \bar\partial(\psi)], \\ \lim_{n\to\infty}[\partial(\psi_n) \wedge \bar\partial(\phi)] = [\partial(\psi) \wedge \bar\partial(\phi)] \end{cases}$$

as currents. Here note that if T is a current, $\{T_n\}$ is a sequence of currents, and $\lim_{n\to\infty} T_n = T$ as currents, then $\lim_{n\to\infty} \partial(T_n) = \partial(T)$ and $\lim_{n\to\infty} \bar\partial(T_n) = \bar\partial(T)$ as currents. On the

Assuming X is projective over $\mathbb{Z}$, let (H, k) be an arithmetically ample Hermitian line bundle on X, namely, (1) H is ample, (2) the Chern form $c_1(H, k)$ is positive definite on the infinite fiber $X(\mathbb{C})$, and (3) there is a positive integer m_0 such that, for any integer $m \geq m_0$, $H^0(X, H^m)$ is generated by the set $\{s \in H^0(X, H^m) \mid \|s\|_{\sup} < 1\}$. Then, the following is one of main results of this note, which is a generalization of [2], [8] and [15].

Theorem A (cf. Corollary 4.3)**.** *Let us consider a homomorphism*

$$L : \widehat{\mathrm{CH}}^p_D(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^{p+1}_D(X)_{\mathbb{Q}}$$

given by $L(x) = \widehat{c}_1(\overline{H}) \cdot x$*. Then, we have the following.*

(1) $L^{d-1} : \widehat{\mathrm{CH}}^1_D(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^d_D(X)_{\mathbb{Q}}$ *is injective.*

(2) *If* $x \in \widehat{\mathrm{CH}}^1_B(X)_{\mathbb{Q}}$*,* $x \neq 0$*, and* $L^d(x) = 0$*, then* $\widehat{\deg}(L^{d-1}(x) \cdot x) < 0$*.*

On an arithmetic surface, Bost [1] also constructed the same intersection pairing independently for an arithmetic analogue of the Lefschetz theorem. Our motivation is, however, different. In this note, we introduce the special class **Br**, which is called the birational class of degeneration, arising from birational geometry. Namely, a locally integrable function ϕ on $X(\mathbb{C})$ belongs to the class **Br** if and only if there are a proper birational morphism $\mu : Y \to X(\mathbb{C})$ of smooth algebraic schemes over $\mathbb{C}$, a divisor D' on Y, and a Green function g for D' such that $\mu_*(D') = 0$ and $\mu_*(g) = \phi$ (a. e.). In §1, we will check that the birational class **Br** is a class of degeneration (cf. Proposition 1.2). Note that if $\dim X = 2$, then **Br** is nothing more than $C^\infty(X(\mathbb{C}), \mathbb{R})$.

The reason why we need to introduce the birational class comes from the following observation. Let (E, h) be a Hermitian vector bundle of rank 2 on X, and L a rank 1 saturated subsheaf of E. Then, $L_{\mathbb{C}}$ is not necessarily line subbundle of $E_{\mathbb{C}}$ if $\dim X_{\mathbb{Q}} \geq 2$, so that the metric h_L of L induced by h is not necessarily C^∞ over $X(\mathbb{C})$. We can however see that $\widehat{c}_1(L, h_L)$ gives rise to an element of $\widehat{\mathrm{CH}}^1_{\mathbf{Br}}(X)$. Thus, we may consider

$$\widehat{\deg}\left(\widehat{c}_1(H, k)^{d-1} \cdot \widehat{c}_1(L, h_L) \cdot (\widehat{c}_1(E, h) - \widehat{c}_1(L, h_L))\right).$$

We would like to compare the above with $\widehat{\deg}\left(\widehat{c}_1(H, k)^{d-1} \cdot \widehat{c}_2(E, h)\right)$. If we take the geometric case into account, one can guess

$$\widehat{\deg}\left(\widehat{c}_1(H, k)^{d-1} \cdot \widehat{c}_2(E, h)\right) \geq \widehat{\deg}\left(\widehat{c}_1(H, k)^{d-1} \cdot \widehat{c}_1(L, h_L) \cdot (\widehat{c}_1(E, h) - \widehat{c}_1(L, h_L))\right).$$

Actually, we will prove the above inequality in §5. Using this together with the result of [14], we have the following Bogomolov's instability theorem for rank 2 vector bundles in the arithmetic case.

Theorem B (cf. Theorem 6.2)**.** *If* $\widehat{\deg}\left(\widehat{c}_1(H, k)^{d-1} \cdot (4\widehat{c}_2(E, h) - \widehat{c}_1(E, h)^2)\right) < 0$*, then there is a rank* 1 *saturated subsheaf* L *of* E *such that* $L^{\otimes 2} \otimes \det(E)^{-1}$ *has positive degree on each connected component of* $X(\mathbb{C})$ *with respect to* $H_{\mathbb{C}}$*, and that*

$$\widehat{\deg}\left(\widehat{c}_1(H, k)^{d-1} \cdot (2\widehat{c}_1(L, h_L) - \widehat{c}_1(E, h))^2\right) > 0.$$

Finally, we would like to thank Prof. Bost who kindly sent his paper [1] after writing the first version of this note. Inspired with his paper, we could simplify the description of this note.

INTERSECTION PAIRING FOR ARITHMETIC CYCLES WITH DEGENERATE GREEN CURRENTS

ATSUSHI MORIWAKI

Introduction

As we indicated in our paper [10], the standard arithmetic Chow groups introduced by Gillet-Soulé [4] are rather restricted to consider arithmetic analogues of geometric problems. In this note, we would like to propose a suitable extension of the arithmetic Chow group of codimension one, in which the Hodge index theorem still holds as in papers [2], [8] and [15].

Let $X \to \mathrm{Spec}(\mathbb{Z})$ be a regular arithmetic variety with $d = \dim X_{\mathbb{Q}}$. As we defined in [10], $\widehat{\mathrm{CH}}^p_D(X)$ is a group, consisting of pairs (Z, g) with cycles Z of codimension p on X and currents g of type $(p-1, p-1)$ on $X(\mathbb{C})$, modulo arithmetical rational equivalence. It seems to be impossible to give a natural ring structure on the graded module $\widehat{\mathrm{CH}}^*_D(X)_{\mathbb{Q}}$. In [10, §2.3], we showed that $\widehat{\mathrm{CH}}^*_D(X)_{\mathbb{Q}}$ has, however, a natural scalar product of the arithmetic Chow ring $\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}}$, namely, a module structure of $\widehat{\mathrm{CH}}^*_D(X)_{\mathbb{Q}}$ over $\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}}$ as a generalization of [4, Theorem 4.2.3]. In this note, we will introduce suitable subgroups $\widehat{\mathrm{CH}}^p_B(X)$ of $\widehat{\mathrm{CH}}^p_D(X)$ and $\widehat{\mathrm{CH}}^q_{B'}(X)$ of $\widehat{\mathrm{CH}}^q_D(X)$ such that $\widehat{\mathrm{CH}}^p_B(X)$ and $\widehat{\mathrm{CH}}^q_{B'}(X)$ have a natural paring

$$\widehat{\mathrm{CH}}^p_B(X) \otimes \widehat{\mathrm{CH}}^q_{B'}(X) \to \widehat{\mathrm{CH}}^{p+q}_D(X)_{\mathbb{Q}}.$$

In the following, we would like to explain how to construct $\widehat{\mathrm{CH}}^1_B(X)$, for example.

We denote by $C^\infty(X(\mathbb{C}), \mathbb{R})$ (resp. $L^2_{1,loc}(X(\mathbb{C}), \mathbb{R})$) the set of all real valued C^∞-functions (resp. locally square integrable functions with all weak partial derivatives in L^2_{loc}) on $X(\mathbb{C})$. A key point to get $\widehat{\mathrm{CH}}^1_B(X)$ is to fix an abelian group B with $C^\infty(X(\mathbb{C}), \mathbb{R}) \subseteq B \subseteq L^2_{1,loc}(X(\mathbb{C}), \mathbb{R})$. This abelian group B is called a class of degeneration of Green functions. Fixing the class B of degeneration, an arithmetic B-divisor on X is defined to be a pair (D, g) such that D is a divisor on X, and that there are a Green function f for $D(\mathbb{C})$ and $\phi \in B$ with $g = f + \phi$. We denote by $\widehat{Z}^1_B(X)$ the set of all arithmetic B-divisors on X, and define $\widehat{\mathrm{CH}}^1_B(X)$ to be $\widehat{Z}^1_B(X)$ modulo arithmetic linear equivalence. Using the Dirichlet form of $L^2_{1,loc}(X(\mathbb{C}), \mathbb{R})$ together with the usual star product, if $(D_1, g_1), (D_2, g_2) \in \widehat{Z}^1_B(X)$, and D_1 and D_2 have no common component, then we can define the star product $g_1 * g_2$ as well as the usual one (cf. §2.1). In this way, we have a homomorphism

$$\widehat{\mathrm{CH}}^1_B(X) \otimes \widehat{\mathrm{CH}}^1_B(X) \to \widehat{\mathrm{CH}}^2_D(X).$$

as desired.

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