Proof. From the inequality (16), we have that for $|\beta| < \pi/2$,

$$\begin{aligned}\log|a_n| \leq & \frac{1}{2}\sum_{k=1}^{n-1}\log\left(1+\frac{2\cos 2\beta}{k}+\frac{1}{k^2}\right)\\ = & \frac{1}{2}\sum_{k=1}^{n-1}\left(\frac{2\cos 2\beta}{k}\right)+O(1)\\ = & \cos 2\beta\log n+O(1)\end{aligned}$$

as $n \to +\infty$. Therefore we obtain the estimate (17). □

Remark. In the case $|\beta| < \pi/4$, this is shown by Basgöze and Keogh in [3](1970).

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On the other hand, since $\overline{\omega(\mathbb{D}\setminus\mathcal{M})} \not\ni -e^{-2i\beta}$, we have

$$\sup_{z\in\mathbb{D}\setminus\mathcal{M}}(1-|z|^2)|T_f(z)| < \|T_{f_\beta}\|_1.$$

Combining both estimates, we can conclude $\|T_f\|_1 < \|T_{f_\beta}\|_1$.

Now the proof of Theorem 1 is completed.

3. Proof of Theorem 2

From (5), it follows that

$$\begin{aligned} S_{f_\beta} =& (T_{f_\beta})' - \frac{1}{2}(T_{f_\beta})^2 \\ =& -c\frac{e^{2i\beta}\{e^{2i\beta}(e^{2i\beta}-1)z^2+4(e^{2i\beta}-1)z+6\}}{2(1-z)^2(1+ze^{2i\beta})^2}. \end{aligned}$$

So we also have

$$(1-|z|^2)^2|S_{f_\beta}(z)| = |c|\frac{(1-|z|^2)^2|e^{2i\beta}(e^{2i\beta}-1)z^2+4(e^{2i\beta}-1)z+6|}{2|1-z|^2|1+ze^{2i\beta}|^2}.$$

It follows easily that $(1-|z|^2)^2|S_{f_\beta}(z)| \to 6$ as $z\to -e^{-2i\beta}$ radially. By the Kraus-Nehari theorem, $\sup_{z\in\mathbb{D}}|S_{f_\beta}(z)|(1-|z|^2)^2 \le 6$. Therefore we obtain $\|S_{f_\beta}\|_2 = 6$ for any $|\beta| < \pi/2$.

4. Order estimate of the coefficients

Knowing of the norm $\|T_f\|_1$ enables us to estimate the growth of coefficients of f (cf. [5]). However the sharp estimate of coefficients of $f\in SP(\beta)$ has been already obtained by Zamorski [10] in 1960. We would like to remark that we can derive the sharp growth estimate of coefficients of $f\in SP(\beta)$ from this.

Theorem 4.1 (Zamorski)**.** *If $f(z) = z+a_2z^2+a_3z^3+\cdots$ is in $SP(\beta)$ and $|\beta|<\pi/2$, then*

$$|a_n| \le \prod_{k=1}^{n-1}\left|1+\frac{e^{2i\beta}}{k}\right| \tag{16}$$

for any $n\ge 2$. The equality in (16) *holds for some $n\ge 2$ if and only if f is a rotation of the β-spiral Koebe function f_β.*

Remark. This is also shown in terms of generalized spiral-like functions by C. Burniak, J. Stankiewicz and Z. Stankiewicz [4](1980).

Corollary 4.1. *Let $|\beta|<\pi/2$ and $f(z) = z+a_2z^2+a_3z^3+\cdots$ be a β-spiral-like function. Then it holds that*

$$a_n = O(n^{\cos 2\beta}) \quad (n\to+\infty). \tag{17}$$

This order estimate is sharp.

(i) If $\sin^2\beta < \frac{1}{8}$, (15) holds since $-\frac{1-8\sin^2\beta}{4\sin\beta} \leq 0$ and $g(0) = 4\sin\beta \geq 0$.

(ii) If $\frac{1}{8} \leq \sin^2\beta < \frac{3}{8}$, (15) holds since $0 \leq -\frac{1-8\sin^2\beta}{4\sin\beta} < \frac{4}{3}\sin\beta$ and $-\frac{64\sin^4\beta - 48\sin^2\beta + 1}{8\sin\beta} \geq 0$.

(iii) If $\frac{3}{8} \leq \sin^2\beta \leq \frac{3}{4}$, (15) holds since $\frac{4}{3}\sin\beta \leq -\frac{1-8\sin^2\beta}{4\sin\beta}$ and

$$g(\frac{4}{3}\sin\beta) = -\frac{64}{9}\sin\beta(\sin^2\beta - \frac{3}{4}) \geq 0.$$

Consequently we have for $|\beta| \leq \pi/3$, $(1-|z|^2)|T_{f_\beta}(z)| \leq 2|2+e^{2i\beta}|$ on $\mathbb{D}$. Noting that $(1-|z|^2)|T_{f_\beta}(z)|$ tends to $2|2+e^{2i\beta}|$ as z tends to $1-0$ along the real axis, we have $\|T_{f_\beta}\|_1 = 2|2+e^{2i\beta}|$.

Next we consider the case $\beta > \pi/3$. In this case, $\|T_{f_\beta}\|_1$ is strictly larger than $2|2+e^{2i\beta}|$. In fact, we have $0 \leq 1/\sin\beta \leq m_0$ and $E(1/\sin\beta) > 2|2+e^{2i\beta}|$. Thus we can see that there exists a point in $\{|z+2e^{-2i\beta}| > |1+2e^{-2i\beta}|\} \cap \mathbb{D}$ where $(1-|z|^2)|T_{f_\beta}(z)|$ attains the maximum on $\mathbb{D}$ and no such one in $\{|z+2e^{-2i\beta}| > |1+2e^{-2i\beta}|\} \cap \partial\mathbb{D}$. Therefore we can conclude that a rotation of f_β is a unique extremal function from Lemma 2.2.

Moreover, for $0 \leq m \leq \frac{4}{3}\sin\beta$, we have a uniform estimate:

$$\begin{aligned} E(m) &= 2m\cos\beta\left(1 + \left|1 - \frac{ie^{-i\beta}}{m - ie^{-i\beta}}\right|\right) \\ &\leq 2m\cos\beta\left(2 + \frac{1}{|m - ie^{-i\beta}|}\right) \\ &\leq \frac{8}{3}\sin 2\beta + 2\cos\beta\frac{m}{|m - ie^{-i\beta}|} \\ &\leq \frac{8}{3}\sin 2\beta + 2\sqrt{1+\cos^2\beta}. \end{aligned}$$

Thus we obtain $\|T_{f_\beta}\|_1 \to 2$ as $\beta \to \pi/2$.

Finally, we will show that for $|\beta| \leq \pi/3$, f_β is also the essentially unique extremal function in $SP(\beta)$.

Let $f \in SP(\beta)$ not be a rotation of f_β. Noting Corollary 2.1, we consider the only case that $\overline{\omega(\mathbb{D})} \ni -e^{-2i\beta}$. Put $\varepsilon := 2|2+e^{2i\beta}| - 2 > 0$. Noting that

$$\limsup_{\mathbb{D}\ni z \to -e^{-2i\beta}} (1-|z|^2)|T_{f_\beta}(z)| = 2,$$

we take $r > 0$ such that

$$\mathcal{N} := \{z \in \mathbb{D}; |z+e^{-2i\beta}| < r \text{ and } (1-|z|^2)|T_{f_\beta}(z)| < 2+\varepsilon\}.$$

We note that $2+\varepsilon < 2|2+e^{2i\beta}| = \|T_{f_\beta}\|_1$. Next put $\mathcal{M} := \omega^{-1}(\mathcal{N})$. From (11) in Lemma 2.2, we obtain the following:

$$\sup_{z\in\mathcal{M}}(1-|z|^2)|T_f(z)| \leq \sup_{z\in\mathcal{N}}(1-|z|^2)|T_{f_\beta}(z)| \leq 2+\varepsilon < \|T_{f_\beta}\|_1.$$

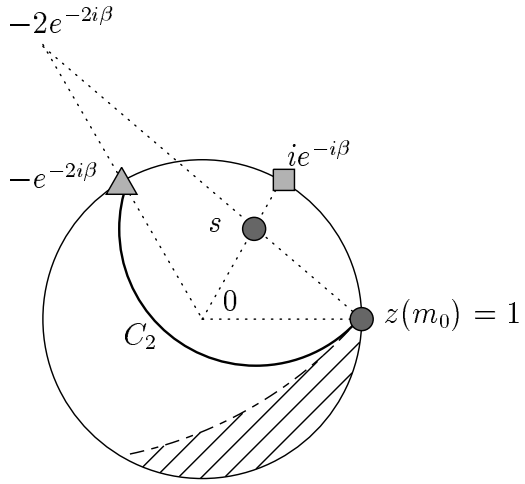

FIGURE 2. the level curve C_2 in the case $m = m_0$.

Noting that $2k\cos\beta + (k^2-1)\sin\beta < 0$ since $m \geq 0$ and $k^2 \leq 1$, we also have $r = \frac{1+k^2}{1-k^2} m\cos\beta$. It follows that for $z \in C_2$,

$$|s + 2e^{-2i\beta}| = \sqrt{m^2 + 4 - 4m\sin\beta},$$
$$\frac{1-k^2}{1+k^2} = \frac{m\cos\beta}{\sqrt{m^2+1-2m\sin\beta}} \quad \text{and}$$
$$(1-|z|^2)|T_{f_\beta}(z)| \leq E(m) := 2m\cos\beta\left(1 + \sqrt{\frac{m^2+4-4m\sin\beta}{m^2+1-2m\sin\beta}}\right).$$

We also note that $z(m_0) = 1 \in \partial\mathbb{D}$ and that $z(m) \in \mathbb{D}$ for $0 \leq m < m_0$.

We consider the case $0 \leq \beta \leq \pi/3$. Noting that $2|2 + e^{2i\beta}| = 2\sqrt{5 + 4\cos 2\beta}$ and $E(m_0) = 4\sin 2\beta$, we can see that $E(m_0) \leq 2|2 + e^{2i\beta}|$. So it is sufficient to prove that $E(m) \leq E(m_0)$ for $0 \leq m \leq m_0$. We can see that $E(m) \leq E(m_0)$ for $0 \leq m \leq m_0$ if and only if

(15)
$$g(m) := 2m^2\sin\beta + (1 - 8\sin^2\beta)m + 4\sin\beta \geq 0 \quad (0 \leq m \leq m_0).$$

This holds trivially if $\beta = 0$. We can see that

$$g(m) = 2\sin\beta\left(m + \frac{1-8\sin^2\beta}{4\sin\beta}\right)^2 - \frac{64\sin^4\beta - 48\sin^2\beta + 1}{8\sin\beta} \text{ and}$$
$$\frac{4}{3}\sin\beta - \left(-\frac{1-8\sin^2\beta}{4\sin\beta}\right) = \frac{3-8\sin^2\beta}{12\sin\beta},$$

so we have the following:

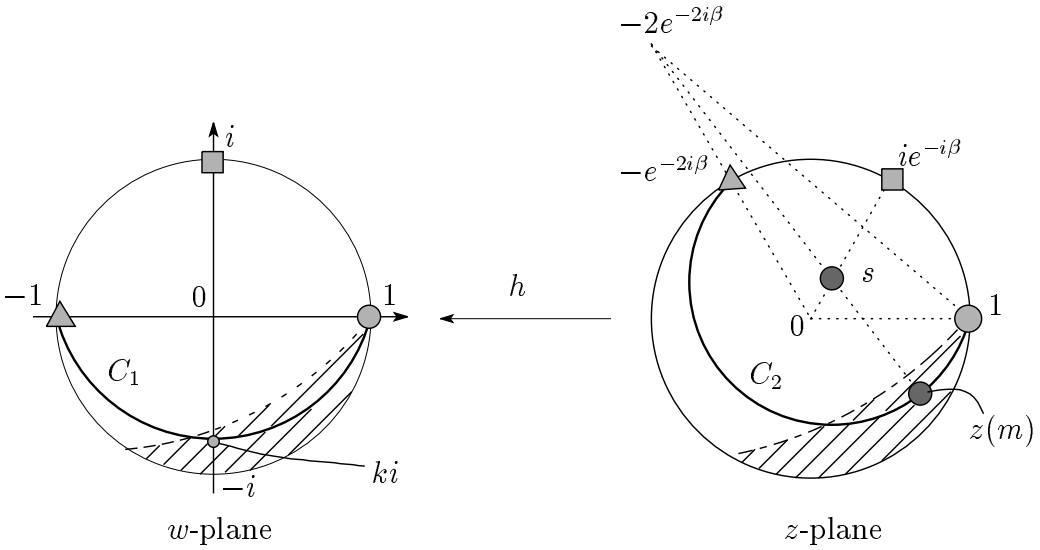

FIGURE 1. the level curve C_1 and C_2

we can calculate the center s and the radius r of C_2 :

$$s = \frac{ie^{-i\beta}(k^2-1)}{2k\cos\beta + (k^2-1)\sin\beta}, \tag{13}$$

$$r = \frac{(k^2+1)\cos\beta}{|2k\cos\beta + (k^2-1)\sin\beta|}. \tag{14}$$

Putting $m := \frac{k^2-1}{2k\cos\beta+(k^2-1)\sin\beta}$, we have $s = m \cdot ie^{-i\beta}$. On the level curve C_2, $|z + 2e^{-2i\beta}|$ takes the maximum at the point $z(m)$ in Figure 1, which is the intersection of the circular arc C_2 and the straight line passing through $-2e^{-2i\beta}$ and $s = m \cdot ie^{-i\beta}$. So we obtain that

$$(1-|z|^2)|T_{f_\beta}(z)| \leq 2\frac{1-k^2}{1+k^2}(|s+2e^{-2i\beta}| + r).$$

on C_2.

Since we are considering the case $|z + 2e^{-2i\beta}| > |1 + 2e^{-2i\beta}|$, we can assume $0 \leq m \leq m_0 := \frac{4}{3}\sin\beta$. We note that if $m = m_0$, then C_2 is tangential to the circle $\{|z + 2e^{-2i\beta}| = |1 + 2e^{-2i\beta}|\}$ and the tangent point between them is $z(m_0) = 1$, and that if m moves from 0 to m_0, the level curve C_2 sweeps out $\{|z + 2e^{-2i\beta}| > |1 + 2e^{-2i\beta}|\} \cap \mathbb{D}$ (see Figure 2).

Lemma 2.2. *Let $f \in SP(\beta)$ not be a rotation of f_β. For $z \in \mathbb{D}$,*

$$(1-|z|^2)|T_f(z)| < (1-|\omega|^2)|T_{f_\beta}(\omega)|. \tag{11}$$

In particular, $\|T_f\|_1 \le \|T_{f_\beta}\|_1$. If $(1-|z|^2)|T_{f_\beta}(z)|$ attains the supremum only in $\mathbb{D}$, this inequality is strict.

From now on, we turn to our attention to the norm of T_{f_β}. If $f \in SP(-\beta)$, then $g(z) := \overline{f(\overline{z})} \in SP(\beta)$ and $\|T_f\|_1 = \|T_g\|_1$, so we can assume $\beta \ge 0$ without any loss of generality.

We consider the conformal automorphism $w \mapsto z = h(w)$ of $\mathbb{D}$ with $h(1) = 1$, $h(-e^{-2i\beta}) = -1$ and $h(ie^{-i\beta}) = i$. This is given by the relation

$$\frac{w-1}{w+1} = e^{-i\beta}\frac{z-1}{z+e^{-2i\beta}}. \tag{12}$$

Differentiating (12), we have

$$1-|z|^2 = (1-|w|^2)\frac{2|z+e^{-2i\beta}|^2}{|c||w+1|^2}.$$

From (12), we also have

$$|1-z| = |z+e^{-2i\beta}|\left|\frac{w-1}{w+1}\right|.$$

Thus we obtain

$$|c|\frac{1-|z|^2}{|1-z||z+e^{-2i\beta}|} = 2\frac{1-|w|^2}{|w^2-1|} \quad \text{and}$$

$$(1-|z|^2)|T_{f_\beta}(z)| = |c|\frac{(1-|z|^2)|z+2e^{-2i\beta}|}{|1-z||1+ze^{2i\beta}|} = 2\frac{1-|w|^2}{|w^2-1|}\cdot|z+2e^{-2i\beta}|.$$

Since $(1-|w|^2)/|w^2-1| \le 1$, we have

$$(1-|z|^2)|T_{f_\beta}(z)| \le 2|2+e^{2i\beta}|$$

on $|z+2e^{-2i\beta}| \le |1+2e^{-2i\beta}|$.

To obtain the estimate of $(1-|z|^2)|T_{f_\beta}(z)|$ on $|z+2e^{-2i\beta}| > |1+2e^{-2i\beta}|$, we use some geometric argument.

For $|k|<1$, the circular arc C_1 passing through the three points ± 1 and ki in the w-plane is the following:

$$\frac{1-|w|^2}{|w^2-1|} = \frac{1-k^2}{1+k^2}.$$

So C_1 is the level curve of $(1-|w|^2)/|w^2-1|$. Put $C_2 = h^{-1}(C_1)$. Since C_2 is the circle passing through the three points 1, $h^{-1}(ki)$ and $-e^{-2i\beta}$,

Proof of Lemma 2.1. We take such U as above. Put $c_1 := \min_{z\in U}(|2+\omega(z)e^{2i\beta}|-1)$ and $c_2 := \max_{z\in U}(|2+\omega(z)e^{2i\beta}|-1)$. We note that $c_1 > 0$ and $0 < c_2 \leq 2$. It follows that

$$\begin{aligned}
&|2+\omega e^{2i\beta}||z|(1-|\omega|^2) - \{|2+\omega e^{2i\beta}||\omega|(1-|z|^2) + (|z|^2-|\omega|^2)\}\\
\geq&(|z|-|\omega|)\{(1-|z|)(1-|\omega|) + (|2+\omega e^{2i\beta}|-1)|z||\omega|\}\\
\geq&(|z|-|\omega|)\{(1-|z|)(1-|\omega|) + c_1|z||\omega|\}.
\end{aligned}$$

From this, putting $c_3 := 1/2(c_2+1)$ and $c_4 = c_1/2(c_2+1) > 0$, we have

$$\begin{aligned}
D &\leq 1 - \Big(\frac{(|z|-|\omega|)(1-|z|)(1-|\omega|)}{|2+\omega e^{2i\beta}||z|(1-|\omega|^2)} + \frac{c_1(|z|-|\omega|)|\omega|}{|2+\omega e^{2i\beta}|(1-|\omega|^2)}\Big)\\
&\leq 1 - \{c_3(1-|\omega/z|)(1-|z|)(1-|\omega|) + c_4|\omega|(1-\frac{1-|z|}{1-|\omega|})\}.
\end{aligned}$$

In Yamashita [8] (p. 313, (6.8**a)), it is shown that for a holomorphic map $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ which is not a rotation at the origin,

$$|\omega(z)| \leq |z|Q(|z|) \leq |z| \quad (z \in \mathbb{D}), \tag{7}$$

where

$$\begin{aligned}
Q(x) &= \frac{x^2+Bx+A}{Ax^2+Bx+1} \quad (0 \leq x \leq 1),\\
&A = |\omega'(0)| < 1 \quad \text{and}\\
B &= \frac{|\omega''(0)|}{2(1-|\omega'(0)|)} \leq 1 + A < 2.
\end{aligned}$$

It follows from this that

$$D \leq 1 - \{c_3(1-Q(|z|))(1-|z|)(1-|z|Q(|z|)) + c_4|\omega|(1-\frac{1-|z|}{1-|z|Q(|z|)})\}. \tag{8}$$

It is easy to see that

$$1 - \frac{1-x}{1-xQ(x)} \geq 0 \quad (0 \leq x \leq 1) \tag{9}$$

and that the equality in (9) holds if and only if $x = 0$. Therefore we can conclude that

$$\sup_{z\in\mathbb{D}}\{c_3(1-Q(|z|))(1-|z|)(1-|z|Q(|z|)) + c_4|\omega|(1-\frac{1-|z|}{1-|z|Q(|z|)})\} > 0. \tag{10}$$

Now the proof is completed. □

We consider the case $\overline{\omega(\mathbb{D})} \ni -e^{-2i\beta}$. Then $c_4 = c_1/2(c_2+1) = 0$ but $0 < c_3$, so the inequality similar to (10):

$$c_3(1-Q(|z|))(1-|z|)(1-|z|Q(|z|)) > 0$$

holds for $z \in \mathbb{D}$. Thus we obtain the following.

$\beta\} =: \mathbb{H}_\beta$. The univalent map $q(z) = (1 + ze^{2i\beta})/(1 - z)$ on $\mathbb{D}$ satisfies $q(0) = 1$ and $q(\mathbb{D}) = \mathbb{H}_\beta$. Then p is subordinate to q, i.e. there exists a holomorphic function $\omega = \omega_f : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ such that

$$p = q \circ \omega = \frac{1 + \omega e^{2i\beta}}{1 - \omega}. \tag{4}$$

In particular, the β-spiral Koebe function f_β satisfies $P_{f_\beta} = q$, i.e. $\omega = id_{\mathbb{D}}$. By the logarithmic differentiation of (4), we have

$$\begin{aligned} T_f(z) =& \frac{f''(z)}{f'(z)} = c\frac{\frac{\omega}{z}(1 + \omega e^{2i\beta}) + \omega'}{(1 - \omega)(1 + \omega e^{2i\beta})}, \text{ thus} \\ (1 - |z|^2)T_f(z) =& c\frac{(1 - |z|^2)\{\frac{\omega}{z}(2 + \omega e^{2i\beta}) + (\omega' - \frac{\omega}{z})\}}{(1 - \omega)(1 + \omega e^{2i\beta})}. \end{aligned}$$

Here we set $c := e^{2i\beta} + 1$. Putting $\omega = id_{\mathbb{D}}$, we also have

$$T_{f_\beta}(z) = c\frac{2 + ze^{2i\beta}}{(1 - z)(1 + ze^{2i\beta})} \text{ and} \tag{5}$$

$$(1 - |z|^2)T_{f_\beta}(z) = c\frac{(1 - |z|^2)(2 + ze^{2i\beta})}{(1 - z)(1 + ze^{2i\beta})}. \tag{6}$$

By the Schwarz-Pick lemma for ω/z, we have $(1 - |z|^2)|z\omega' - \omega| \leq |z|^2 - |\omega|^2$. So we can estimate as

$$\begin{aligned} (1 - |z|^2)|T_f(z)| \leq& \frac{|\omega|(1 - |z|^2)}{|z|(1 - |\omega|^2)} \cdot |c|\frac{(1 - |\omega|^2)|2 + \omega e^{2i\beta}|}{|1 - \omega||1 + \omega e^{2i\beta}|} \\ &+ \frac{|z|^2 - |\omega|^2}{|z|(1 - |\omega|^2)} \cdot |c|\frac{1 - |\omega|^2}{|1 - \omega||1 + \omega e^{2i\beta}|} \\ \leq& \frac{|2 + \omega e^{2i\beta}||\omega|(1 - |z|^2) + (|z|^2 - |\omega|^2)}{|2 + \omega e^{2i\beta}||z|(1 - |\omega|^2)}\|T_{f_\beta}\|_1. \end{aligned}$$

To show $\|T_f\|_1 < \|T_{f_\beta}\|_1$ for $SP(\beta) \ni f$ such that $|\omega'| < 1$, we show the following.

Lemma 2.1. *Let $f \in SP(\beta)$ not be a rotation of f_β. For any domain $U \subset \mathbb{D}$ with $\overline{\omega(U)} \not\ni -e^{-2i\beta}$, there exists a positive constant $C < 1$ such that*

$$D := \frac{|2 + \omega e^{2i\beta}||\omega|(1 - |z|^2) + (|z|^2 - |\omega|^2)}{|2 + \omega e^{2i\beta}||z|(1 - |\omega|^2)} \leq C \quad (z \in U).$$

Thus we have $\sup_{z\in U}(1 - |z|^2)|T_f(z)| \leq C\|T_{f_\beta}\|_1 < \|T_{f_\beta}\|_1$.

Corollary 2.1. *Let $f \in SP(\beta)$ not be a rotation of f_β. If $\overline{\omega(\mathbb{D})} \not\ni -e^{-2i\beta}$, then $\|T_f\|_1 < \|T_{f_\beta}\|_1$.*

Proof. We can take $\mathbb{D}$ itself as such U in Lemma 2.1. □

[11]. If the reader would like to know the related works about norm estimates of pre-Schwarzian derivatives for other classes of A, see [9], [7] and [5].

In the present paper, we shall give the best possible estimate of the norms of pre-Schwarzian derivatives for the class $SP(\beta)$.

Theorem 1. *For any $f \in SP(\beta)$, where $\beta \in (-\pi/2, \pi/2)$, we have the following.*

I) *In the case $|\beta| \le \pi/3$, we have*

$$\|T_f\|_1 \le \|T_{f_\beta}\|_1 = 2|2 + e^{2i\beta}|. \tag{1}$$

II) *In the case $|\beta| > \pi/3$, we have $\|T_f\|_1 \le \|T_{f_\beta}\|_1$, where*

$$\|T_{f_\beta}\|_1 = \max_{0 \le m \le \frac{4}{3}\sin\beta} 2m\cos\beta \left(1 + \sqrt{\frac{m^2 + 4 - 4m\sin\beta}{m^2 + 1 - 2m\sin\beta}}\right) \quad \textit{and} \tag{2}$$

$$2|2 + e^{2i\beta}| < \|T_{f_\beta}\|_1 < \frac{8}{3}\sin 2\beta + 2\sqrt{1 + \cos^2\beta}. \tag{3}$$

In particular, $\|T_{f_\beta}\|_1 \to 2$ as $|\beta| \to \pi/2$.

In both cases, the equality $\|T_f\|_1 = \|T_{f_\beta}\|_1$ holds if and only if f is a rotation of the β-spiral Koebe function.

If $|\beta| \le \pi/3$, the function $|T_f(z)|(1 - |z|^2)$ does not attain its supremum in $\mathbb{D}$. However if $|\beta| > \pi/3$, it does. This phenomenon of phase transition seems to be quite interesting.

On the other hand, we also obtain the estimate of the norms of Schwarzian derivatives of β-spiral-like functions.

Theorem 2. *Assume $|\beta| < \pi/2$. For any $f \in SP(\beta)$, $\|S_f\|_2 \le \|S_{f_\beta}\|_2 = 6$.*

In Theorem 2, the extremality of f_β is trivial since the Kraus–Nehari theorem says that $\|S_f\|_2 \le 6$ for any univalent $f \in A$.

The proof of Theorem 1 and 2 will be given in Section 2 and 3. Knowing of the norm $\|T_f\|_1$ of $f \in A$ enables us to estimate the growth of coefficients of f. For example, the following holds.

Theorem 1.1 ([5])**.** *For $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in A$ such that $\|T_f\|_1 \le 2\lambda$, we have*

$$a_n = O(n^{\lambda - 1}) \quad (n \to +\infty).$$

We also remark on the sharp order estimate of coefficients of $f \in SP(\beta)$ in Section 4.

2. Proof of Theorem 1

Let $f \in A$ be a β-spiral-like function. We set $p(z) = P_f(z) = zf'(z)/f(z)$. Then, by assumption, p is a holomorphic function on $\mathbb{D}$ satisfying $p(0) = 1$ and $p(\mathbb{D}) \subset \{w \in \mathbb{C}; -\frac{\pi}{2} + \beta < \arg w < \frac{\pi}{2} +$

THE NORM ESTIMATES OF PRE-SCHWARZIAN DERIVATIVES OF SPIRAL-LIKE FUNCTIONS

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Abstract. For a constant $\beta \in (-\pi/2, \pi/2)$, a normalized analytic function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ on the unit disk is said to be β-spiral-like if $\Re(zf'(z)/e^{i\beta} f(z)) > 0$ for any point z in the unit disk. In this paper, for such a function f, we shall present the optimal estimate of the norm of f''/f'.

1. Introduction

Let A denote the set of analytic functions f on the unit disk $\mathbb{D}$ normalized so that $f(0) = f'(0) - 1 = 0$. For a constant $\beta \in (-\pi/2, \pi/2)$, a function $f \in A$ is called *β-spiral-like* if f is univalent on $\mathbb{D}$ and for any $z \in \mathbb{D}$, the β-logarithmic spiral $\{f(z)\exp(-e^{-i\beta}t); t \geq 0\}$ is contained in $f(\mathbb{D})$. It is equivalent to the condition that $\Re(zf'(z)/e^{i\beta}f(z)) > 0$ in $\mathbb{D}$. We set $SP(\beta)$ is the set of β-spiral-like functions. We call $f_\beta(z) := z(1-z)^{-2e^{-i\beta}\cos\beta} \in SP(\beta)$ the β-spiral Koebe function. Note that $f_0(z) = z(1-z)^{-2}$ is the Koebe function K, and that the β-spiral Koebe function conformally maps the unit disk onto the complement of the β-logarithmic spiral $\{f_\beta(-e^{-2i\beta})\exp(-e^{-i\beta}t); t \leq 0\}$ in $\mathbb{C}$. If the reader would like to know the known results about these classes of the functions, see, for example, [1].

For a locally univalent holomorphic function f, we define

$$T_f = \frac{f''}{f'} \quad \text{and} \quad S_f = (T_f)' - \frac{1}{2}(T_f)^2,$$

which are said to be the *pre-Schwarzian derivative* (or nonlinearity) and the *Schwarzian derivative* of f, respectively. For a locally univalent function f in the unit disk, we define the norms of T_f and S_f by

$$\|T_f\|_1 = \sup_{z\in\mathbb{D}} |T_f(z)|(1-|z|^2) \quad \text{and} \quad \|S_f\|_2 = \sup_{z\in\mathbb{D}} |S_f(z)|(1-|z|^2)^2,$$

respectively. As well as $\|S_f\|_2$, the norm $\|T_f\|_1$ has a significant meaning in the theory of Teichmüller spaces. For example, see [6], [2] and

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