Homotopy Type of Map(S^4, BSU(2)) and Classifying Spaces of Gauge Groups

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Abstract

The homotopy type of classifying spaces of gauge groups of principal SU(2) bundle over S^4 are classified by the absolute value of their instanton numbers.

1 Introduction

Let P_k be the principal SU(2) bundle over S^4 with $c_2(P_k) = k \in \mathbb{Z}$, $\mathcal{G}_k$ its gauge group. It is known ([1]) that $B\mathcal{G}_k \simeq \mathrm{Map}_k(S^4, B\mathrm{SU}(2))$, where $\mathrm{Map}_k(S^4, B\mathrm{SU}(2))$ is the connected component of $\mathrm{Map}(S^4, B\mathrm{SU}(2))$ including the map inducing P_k. In this paper we will show the following result. Let $B\mathcal{G}_{k(p)}$ denote the $B\mathcal{G}_k$ localized at a prime p.

Theorem 1.1. *The following four conditions are equivalent.*

1. $|k| = |l|$
2. $\mathcal{G}_k \cong \mathcal{G}_l$
3. $B\mathcal{G}_k \simeq B\mathcal{G}_l$
4. $B\mathcal{G}_{k(p)} \simeq B\mathcal{G}_{l(p)}$ *for any prime* p.

Remark 1.2. In the case of based gauge groups, $\mathcal{G}_k^* \cong \mathcal{G}_0^*$ and $B\mathcal{G}_k^* \simeq B\mathcal{G}_0^*$.

Remark 1.3. The conditions (3) and (4) are not always equivalent for general spaces.

Remark 1.4. It is known that two compact Lie groups G and H are isomorphic as Lie groups if and only if BG and BH are homotopy equivalent ([4]), but this does not always hold for non compact groups. For example, $B\mathbb{C}^\times \simeq B\mathrm{SL}(2,\mathbb{R})$ but $\mathbb{C}^\times \ncong \mathrm{SL}(2,\mathbb{R})$.

Moreover if $|k| \neq |l|$, then $\tilde{\mathcal{G}}_k \ncong \tilde{\mathcal{G}}_l$, where $\tilde{\mathcal{G}}_k$ is the universal covering group of the identity component of $\mathcal{G}_k$. More precisely we have the following result.

Theorem 1.5. *The following four conditions are equivalent.*

1. $|k| = |l|$
2. $\tilde{\mathcal{G}}_k \cong \tilde{\mathcal{G}}_l$
3. $B\tilde{\mathcal{G}}_k \simeq B\tilde{\mathcal{G}}_l$
4. $B\tilde{\mathcal{G}}_{k(p)} \simeq B\tilde{\mathcal{G}}_{l(p)}$ *for any prime* p.

In [7], we showed that if $B\mathcal{G}_{k(p)} \simeq B\mathcal{G}_{l(p)}$ then $(k,p) = (l,p)$ for any prime p and by the result of [3], $(k,4) = (l,4)$. Therefore we must detect the maximal power of p dividing k and l. We improve the method used in [7]. For a fixed prime p, put (cf. [2])

$$M_k = \mathrm{Map}_k(S^4, B\mathrm{SU}(2))_{(p)} \simeq \mathrm{Map}_k(S^4, \mathbb{H}P^\infty_{(p)}).$$

Of course $B\mathcal{G}_{k(p)} \simeq M_k$. We have the following fibration

$$\Omega^4\mathbb{H}P^\infty_{(p)} \to M_k \to \mathbb{H}P^\infty_{(p)} \tag{1}$$

and we study the section of this fibration.

2 Proof of Theorem

Since S^4 admits a degree -1 map, *1* implies *2*. It is clear that *2* implies *3* and *3* implies *4*.

The next proposition shows that *4* implies *1*.

Proposition 2.1. *Let p be a prime. If $M_k \simeq M_l$ then $(k,p^r) = (l,p^r)$ for any $r \in \mathbb{N}$.*

In the rest of the paper, we prove this proposition.

Lemma 2.2. *If $(p,q)=1$, then $M_k \simeq M_{qk}$.*

Proof . Let $q\colon S^4 \to S^4$ be a map of degree q. This map induces a homotopy equivalence $\Omega^4\mathbb{H}P^\infty_{(p)} \xrightarrow{\simeq} \Omega^4\mathbb{H}P^\infty_{(p)}$ and a map $M_k \to M_{qk}$. The result follows from the following commutative diagram

$$\begin{array}{ccccc} \Omega^4\mathbb{H}P^\infty_{(p)} & \longrightarrow & M_k & \longrightarrow & \mathbb{H}P^\infty_{(p)} \\ {\scriptstyle\simeq}\downarrow & & \downarrow & & \| \\ \Omega^4\mathbb{H}P^\infty_{(p)} & \longrightarrow & M_{qk} & \longrightarrow & \mathbb{H}P^\infty_{(p)}. \end{array}$$

□

By [3], $\pi_3(B\mathcal{G}_k) \cong \mathbb{Z}/(12,k)$. Therefore for $p=2,3$, if $M_k \simeq M_l$ and $(k,p)=1$, then $(l,p)=1$ and $(k,p^r)=1=(l,p^r)$ for any $r\in\mathbb{N}$. If $(k,4)=2$, then $(l,4)=2$ and $(k,2^r)=(l,2^r)$.

Hereafter we assume $4|k$ if $p=2$ and $3|k$ if $p=3$ otherwise stated.

Let $d_p(k)$ be the maximal number n such that there exists a map $f\colon \mathbb{H}P^n \to M_k$ and $(ev_k\circ f)^*\colon H^4(\mathbb{H}P^\infty_{(p)};\mathbb{Z}_{(p)}) \to H^4(\mathbb{H}P^n;\mathbb{Z}_{(p)})$ is an isomorphism. If there exists a map $f\colon \mathbb{H}P^\infty \to M_k$ satisfying this condition, we define $d_p(k)=\infty$. Note that if $p\geq 5$, we have

$$\pi_4(M_k) \xrightarrow[\cong]{(ev_k)_*} \pi_4(\mathbb{H}P^\infty_{(p)})$$

by the homotopy exact sequence for the fibration (1) because $\pi_3(\Omega^4\mathbb{H}P^\infty_{(p)}) = \pi_4(\Omega^{\mathbb{H}}P^\infty_{(p)}) = 0$. This homomorphism is epic for $p=2,3$ if $4|k$, $3|k$ respectively by the result of [3]. Therefore there exists a map $f\colon \mathbb{H}P^1 = S^4 \to M_k$ satisfying the condition and $d_p(k)$ is well defined. Note that $d_p(0)=\infty$.

Lemma 2.3. *If there exists a map $f\colon \mathbb{H}P^n \to M_k$ such that*

$$(ev_k\circ f)^*\colon H^4(\mathbb{H}P^\infty_{(p)};\mathbb{Z}_{(p)}) \to H^4(\mathbb{H}P^n;\mathbb{Z}_{(p)})$$

is an isomorphism, then there exists a map $g\colon \mathbb{H}P^n \to M_k$ such that $ev_k\circ g$ is homotopic to the map $\mathbb{H}P^n \xrightarrow{l} \mathbb{H}P^n_{(p)} \xrightarrow{i} \mathbb{H}P^\infty_{(p)}$, where l is a localization and i is the inclusion. (We need not assume that $4|k$ nor $3|k$ in this lemma.)

Proof . By the cellular approximation theorem we may assume $ev_k\circ f$ maps into $\mathbb{H}P^n_{(p)}$ and by the definition of the localization $ev_k\circ f$ factors through $\mathbb{H}P^n_{(p)}$ thus $ev_k\circ f$ decomposes as follows

$$ev_k\circ f\colon \mathbb{H}P^n \xrightarrow{l} \mathbb{H}P^n_{(p)} \xrightarrow{h} \mathbb{H}P^n_{(p)} \xrightarrow{i} \mathbb{H}P^\infty_{(p)}.$$

By the assumption h induces an isomorphism of the $\mathbb{Z}_{(p)}$ cohomology. Since $\mathbb{H}P^n_{(p)}$ is p local and simply connected h is a homotopy equivalence. Since M_k is p local, f factors through $\mathbb{H}P^n_{(p)}$,

$$f\colon \mathbb{H}P^n \xrightarrow{l} \mathbb{H}P^n_{(p)} \xrightarrow{\tilde{f}} M_k.$$

Put $g = \tilde{f} \circ h^{-1} \circ l\colon \mathbb{H}P^n \to M_k$. Since $ev_k \circ \tilde{f} \circ l = ev_k \circ f = i \circ h \circ l$, we have $ev_k \circ \tilde{f} \simeq i \circ h$ hence $ev_k \circ g = ev_k \circ \tilde{f} \circ h^{-1} \circ l \simeq i \circ l$. □

Lemma 2.4. *If $k \neq 0$, then $d_p(k) < \infty$.*

Proof . Assume $p^r \nmid k$. If there exists a map $\mathbb{H}P^n \xrightarrow{f} M_k$ we obtain a map $\Phi\colon \mathbb{H}P^n \times S^4 \to \mathbb{H}P^\infty_{(p)}$ such that the following commutes.

$$\begin{array}{ccc} \mathbb{H}P^n \times S^4 & \xrightarrow{\Phi} & \mathbb{H}P^\infty_{(p)} \\ \uparrow & & \uparrow \nabla \\ \mathbb{H}P^n \vee S^4 & \xrightarrow{(i\circ l)\vee k} & \mathbb{H}P^\infty_{(p)} \vee \mathbb{H}P^\infty_{(p)} \end{array}$$

where ∇ is the folding map. There exists an element $x \in K(\mathbb{H}P^\infty_{(p)}; \mathbb{Z}_{(p)})$ such that $ch(x) = \sum_{j\geq 1} \lambda_j u^j$, where $\lambda_j = 2/(2j)!$ and $u \in H^4(\mathbb{H}P^\infty_{(p)}; \mathbb{Z}_{(p)})$ is a generator. Let a, b be generators of $K(S^4), H^4(S^4; \mathbb{Z})$ respectively such that $ch(a) = -b$. Then in the $\mathbb{Z}_{(p)}$ coefficient K theory we have

$$\Phi^*(x) = x - ka + \sum_{j\geq 1} \varepsilon_j x^j a,$$

where $\varepsilon_j \in \mathbb{Z}_{(p)}$ and in the $\mathbb{Z}_{(p)}$ coefficient cohomology

$$\Phi^*(u) = u + kb.$$

We have

$$\begin{aligned} \Phi^* ch(x) &= \Phi^*(\sum \lambda_j u^j) \\ &= \sum \lambda_j (u + kb)^j \\ &= \sum \lambda_j (u^j + jku^{j-1}b) \end{aligned}$$

and for $l \leq n$ the coefficient of $u^l b$ is

$$(l+1)k\lambda_{l+1} = \frac{k}{(2l+1)!}.$$

Since $ch(x^j) = ch(x)^j = (\sum \lambda_i u^i)^j$,the coefficient $a_{j,l}$ of u^l of $ch(x^j)$ is

$$a_{j,l} = \sum_{\sum n_k i_k = l} \lambda_{i_1}^{n_i} \cdots \lambda_{i_q}^{n_q} = \sum_{\sum n_k i_k = l} \frac{2^l}{(2i_1)!^{n_1} \cdots (2i_q)!^{n_q}}$$

and hence $(2l)!a_{j,l} \in \mathbb{Z}$. Therefore the coefficient of $u^l b$ of $ch\Phi^*(x)$ is $-\sum \varepsilon_j a_{j,l}$ and $-(2l)!\sum \varepsilon_j a_{j,l} \in \mathbb{Z}_{(p)}$. Thus we have

$$(2l)!\frac{k}{(2l+1)!} = \frac{k}{2l+1} \in \mathbb{Z}_{(p)}.$$

Since $p^r \nmid k$ we have $n < (p^r - 1)/2$. □

Lemma 2.5. *If $M_k \simeq M_l$, then $d_p(k) = d_p(l)$.*

Proof . Consider a map $f\colon \mathbb{H}P^{d_p(k)} \to M_k$ realizing the maximal number $d_p(k)$. Since $\pi_i(\mathbb{H}P^n) = \pi_i(\mathbb{H}P^\infty_{(p)}) = 0\,(i = 1, 2, 3)$, by the definition of $d_p(k)$ we have

$$(ev_k \circ f)_*\colon \pi_4(\mathbb{H}P^{d_p(k)}) \otimes \mathbb{Z}_{(p)} \xrightarrow{\cong} \pi_4(\mathbb{H}P^\infty_{(p)})$$

and hence

$$f_*\colon \pi_4(\mathbb{H}P^{d_p(k)}) \otimes \mathbb{Z}_{(p)} \xrightarrow{\cong} \pi_4(M_k)$$

if $p \geq 5$. Therefore the composite map

$$\mathbb{H}P^{d_p(k)} \xrightarrow{f} M_k \xrightarrow{\simeq} M_l \xrightarrow{ev_l} \mathbb{H}P^\infty_{(p)}$$

induces an isomorphism

$$\pi_4(\mathbb{H}P^{d_p(k)}) \otimes \mathbb{Z}_{(p)} \xrightarrow{\cong} \pi_4(\mathbb{H}P^\infty_{(p)}),$$

we have $d_p(k) \leq d_p(l)$ and $d_p(k) = d_p(l)$. If $p = 2, 3$, we can show this lemma similarly using the fact that the map

$$\pi_4(M_k) \xrightarrow{(ev_k)_*} \pi_4(\mathbb{H}P^\infty_{(p)})$$

is epic and the kernel is a finite group. □

Lemma 2.6. *If $k \neq 0$ and $p \geq 3$, then $d_p(k) < d_p(pk)$.*

Proof . By the definition of $d_p(k)$ we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{H}P^{d_p(k)} \times S^4 & \xrightarrow{\Phi} & \mathbb{H}P^{\infty}_{(p)} \\ \uparrow & & \uparrow \nabla \\ \mathbb{H}P^{d_p(k)} \vee S^4 & \xrightarrow{(i\circ l)\vee k} & \mathbb{H}P^{\infty}_{(p)} \vee \mathbb{H}P^{\infty}_{(p)}. \end{array}$$

The only obstruction to extend the map

$$\mathbb{H}P^{d_p(k)+1} \vee S^4 \xrightarrow{(i\circ l)\vee pk} \mathbb{H}P^{\infty}_{(p)} \vee \mathbb{H}P^{\infty}_{(p)} \longrightarrow \mathbb{H}P^{\infty}_{(p)}$$

to $\mathbb{H}P^{d_p(k)+1} \times S^4$ lives in the image of

$$\begin{array}{c} H^{4(d_p(k)+2)}\big(\mathbb{H}P^{d_p(k)+1} \times S^4, \mathbb{H}P^{d_p(k)+1} \times \{*\} \cup \mathbb{H}P^{d_p(k)} \times S^4; \pi\big) \\ \downarrow{\scriptstyle (1\times p)^* = p\times} \\ H^{4(d_p(k)+2)}\big(\mathbb{H}P^{d_p(k)+1} \times S^4, \mathbb{H}P^{d_p(k)+1} \times \{*\} \cup \mathbb{H}P^{d_p(k)} \times S^4; \pi\big), \end{array}$$

where $\pi = \pi_{4d_p(k)+7}(\mathbb{H}P^{\infty}_{(p)})$. If $p \geq 3$ and $i > 3$, by [5], the order of elements of $\pi_{i+1}(\mathbb{H}P^{\infty}_{(p)}) \cong \pi_i(S^3_{(p)})$ is p, this obstruction vanishes. Therefore $d_p(pk) \geq d_p(k) + 1$. □

By these lemmas we can prove Proposition 2.1 for odd primes. Note that $M_0 \not\simeq M_k$ for any $k \neq 0$. Let s, t be integers prime to p such that $sk = p^n tl$. We may assume $n \geq 0$. If $M_k \simeq M_l$, by Lemma 2.2, $M_{sk} \simeq M_{tl}$. By Lemma 2.5, $d_p(p^n tl) = d_p(sk) = d_p(tl)$ and by Lemma 2.6, n must be 0. Therefore $(k, p^r) = (sk, p^r) = (tl, p^r) = (l, p^r)$ for any $r \in \mathbb{N}$.

If $p = 2$, it may happen that $d_2(k) = d_2(2k)$. Hereafter we assume that $p = 2$. Let $\varphi_n \colon S^{4n+3} \to \mathbb{H}P^n$ be the attaching map of the top cell of $\mathbb{H}P^{n+1}$. Put $X^n_r = \mathbb{H}P^n \cup_{2^r\varphi_n} e^{4n+4}$. Then we have the following commutative diagram

$$\begin{array}{ccc} X^n_{r+1} & \longrightarrow & X^n_r \\ {\scriptstyle i}\uparrow & & \downarrow \\ \mathbb{H}P^n & \longrightarrow & \mathbb{H}P^{n+1}. \end{array}$$

For each nonzero integer k divisible by 4 and $n \leq d_2(k)$, define $r_n(k)$ to be the minimal number r such that there exists a map f,

$$\mathbb{H}P^n \xrightarrow{i} X^n_r \xrightarrow{f} M_k \xrightarrow{ev_k} \mathbb{H}P^{\infty}_{(2)},$$

which satisfies that $ev_k \circ f \circ i$ induces an isomorphism $H^4(\mathbb{H}P^\infty_{(2)}; \mathbb{Z}_{(2)}) \to H^4(\mathbb{H}P^n; \mathbb{Z}_{(2)})$. Since $\pi_{4n+3}(\mathbb{H}P^\infty_{(2)})$ is finite, $r_n(k) < \infty$. Of course if $n < d_2(k)$, $r_n(k) = 0$ and if $n = d_2(k)$, $r_n(k) > 0$. We can prove the following lemma quite similarly to the case of $d_p(k)$.

Lemma 2.7. *If $M_k \simeq M_l$, then for all $n \leq d_2(k) = d_2(l)$, $r_n(k) = r_n(l)$.*

Lemma 2.8. *If $n = d_2(k)$, $r_n(k) > r_n(2k)$.*

Proof . Put $r = r_n(k)$. We have a map $\Phi\colon X^n_r \times S^4 \to \mathbb{H}P^\infty_{(2)}$ such that $\Phi \circ (i \times *)\colon \mathbb{H}P^n \to \mathbb{H}P^\infty_{(2)}$ is the inclusion. When restricted to $X^n_r \times \{*\} \cup \mathbb{H}P^n \times S^4$, Φ naturally extends to $X^n_{r-1} \times \{*\} \cup \mathbb{H}P^n \times S^4$ and we also write this map Φ. Let o be the obstruction class to extend Φ to $X^n_{r-1} \times S^4$. Consider the following commutative diagram.

$$\begin{array}{ccc} H^{4n+8}(X^n_{r-1} \times S^4, X^n_{r-1} \times \{*\} \cup \mathbb{H}P^n \times S^4; \pi_{4n+7}(\mathbb{H}P^\infty_{(2)})) & = & \pi_{4n+7}(\mathbb{H}P^\infty_{(2)}) \\ \downarrow{\scriptstyle (i\times 1)^*} & & \downarrow{\scriptstyle 2\times} \\ H^{4n+8}(X^n_r \times S^4, X^n_r \times \{*\} \cup \mathbb{H}P^n \times S^4; \pi_{4n+7}(\mathbb{H}P^\infty_{(2)})) & = & \pi_{4n+7}(\mathbb{H}P^\infty_{(2)}). \end{array}$$

Since $(i \times 1)^*(o) = 0$, the order of o is 2 therefore the composite map

$$\Phi \circ (1 \times 2)\colon X^n_{r-1} \times \{*\} \cup \mathbb{H}P^n \times S^4 \to \mathbb{H}P^\infty_{(2)}$$

extends to $X^n_{r-1} \times S^4$ and $r_n(2k) \leq r - 1$. □

Using these lemmas, we can prove Proposition 2.1 for $p = 2$ quite similarly to the odd prime case.

Proof of Theorem 1.5. Let $\tilde{M}_k$ be the two connected cover of M_k, then $\tilde{M}_k \simeq B\tilde{\mathcal{G}}_k$. Using $\tilde{M}_k$, we can define $\tilde{d}_p(k)$ and $\tilde{r}_n(k)$ similarly to $d_p(k)$ and $r_n(k)$. It is easy to verify that these are homotopy invariant. Moreover, since $\mathbb{H}P^n$ is 3-connected, $\tilde{d}_p(k) = d_p(k)$ and $\tilde{r}_n(k) = r_n(k)$. Then we can prove this theorem similarly to Theorem 1.1. □

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