A natural horseshoe-breaking family which has a period doubling bifurcation as the first bifurcation

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Abstract

In this paper, we construct a horseshoe-breaking family in which a period doubling bifurcation of a periodic point with period 2 occurs before the first tangency, and in fact it is the first bifurcation.

1 Introduction

Let $F : M \to M$ be a diffeomorphism on a manifold M. We say that F is *C^r-structurally stable* if there is a homeomorphism $h : M \to M$ which satisfies $h^{-1} \circ G \circ h = F$ for any diffeomorphism G near F in C^r-topology.

F is called *hyperbolic* on an F-invariant compact set Λ if there exist a splitting of the tangent bundle $TM|_\Lambda = E^u \oplus E^s$ and constants $C > 0$ and $\lambda > 1$ which satisfy that

$$\|DF^n(v_u)\| \geq C\lambda^n\|v_u\|,\ \|DF^{-n}(v_s)\| \geq C\lambda^n\|v_s\|$$

for any $n \in \mathbf{N}$ and any $v_u \in E^u$, $v_s \in E^s$. We call F *a hyperbolic diffeomorphism* if the non-wandering set of F is hyperbolic.

The hyperbolicity and the stability are very important concepts in the research of dynamical systems and they are mutually related. For example, it is known that F is C^1-structurally stable, if and only if F is hyperbolic and F has some additional conditions (*e.g.* [7] and [3]).

Smale's horseshoe[8] is a typical dynamical system which has the stability and the hyperbolicity. A horseshoe is defined as a planer diffeomorphism which maps D to $F(D)$ as in Figure 1, and which is hyperbolic on $\Lambda = \bigcap_{n\in\mathbf{Z}} F^n(D)$. We call a horseshoe diffeomorphism *an affine horseshoe* if it maps the rectangles A and B to A' and B' affinely.

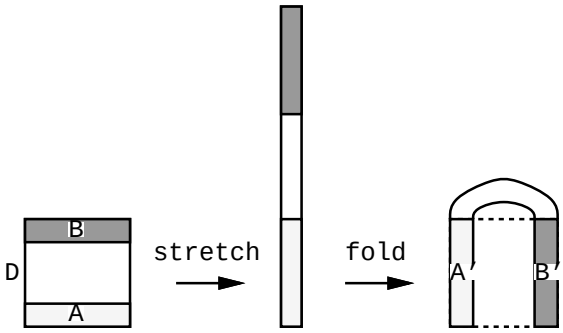

Figure 1: Smale's horseshoe

How does a horseshoe lose its stability? One of the typical deformation F_t of a horseshoe is that $F_t(D)$ is pushed down as a parameter varies until $F_t(D)$ becomes contained in D, like in Figure 2. We call such a 1-parameter family *a horseshoe-breaking family*. We do not attempt to give a precise definition for it. One may, however, take the family $\{F_t = y_t \circ F\}_{0<t\leq 1}$ as a natural horseshoe-breaking family, where $y_t(x, y) = (x, ty)$ and where F is an affine horseshoe that expands along the y-axis and contracts along the x-axis.

We are interested in the bifurcation set of such a horseshoe-breaking family.

Let $\{F_t\}_{T_0\leq t\leq T_1}$ be a horseshoe-breaking family where F_{T_1} is an affine horseshoe and $F_{T_0}(D)$ is assumed to lie in D.

We say that the family $\{F_t\}$ undergoes a bifurcation at t_0 when F_{t_0} loses its stability. The first bifurcation is the bifurcation that occurs at the largest parameter value, namely, at $t_{bif} = \inf\{t | F_{t'}$ is stable for all $t' > t\}$.

Since F_{T_1} is a horseshoe, one finds a hyperbolic fixed point p, so that some segments in its stable and unstable manifolds $W^s(p; F_{T_1})$ and $W^u(p; F_{T_1})$ give the boundary of D. We assume that the hyperbolic fixed point p persists for all t and denote it by p_t. The first tangency of $\{F_t\}$ is a tangency of $W^s(p_t; F_t)$ and $W^u(p_t; F_t)$ that occurs at the largest parameter value, namely, at $t_{tan} = \inf\{t | W^s(p_{t'})$ intersects $W^u(p_{t'})$ transversely for all $t' > t\}$.

It is known that a tangency of the stable and the unstable manifold of a hyperbolic periodic point generically gives rise to a complicated bifurcation (see [5]). Therefore, t_{bif} is larger than or equal to t_{tan}.

Palis and Takens [5] posed the following question :

"Is the first tangency generically the first bifurcation for horseshoe-breaking families ?"

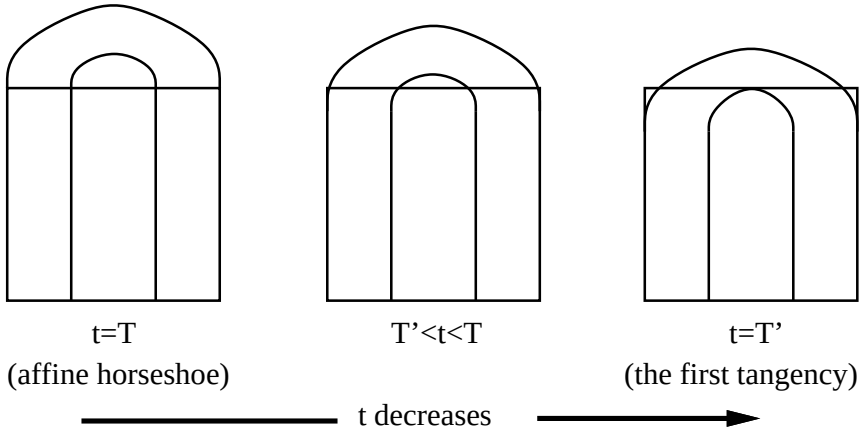

Figure 2: A horseshoe-breaking family

Kiriki [1] formulated a class of horseshoe-breaking families and constructed an example of a horseshoe-breaking family in which the first tangency is the first bifurcation. The basic set of the horseshoe in his family has large fractal dimension. In [2], he also constructed a horseshoe-breaking family in which the first tangency is the first bifurcation, with an additional property that the basic set has an arbitrarily small fractal dimension. Notice that the latter family has more than one folded parts.

One can consider a similar problem for the Hénon-like family :

$$F(x,y) = (af(x) - by, x).$$

Smillie [9] showed that the first tangency is the first bifurcation in the Hénon family (*i.e.* $f(x) = x(1-x)$). Moreover, Yang [10] obtained the same conclusion for any f which is C^2-near to $x(1-x)$.

In this paper, we construct a natural C^1 horseshoe-breaking family in which, as the first bifurcation, a period doubling bifurcation occurs before the first tangency. The basic set of the horseshoe in this family has a small fractal dimension.

Main Theorem *There is a C^1 diffeomorphism F such that the 1-parameter family $F_t = y_t \circ F$, (where $y_t(x,y) = (x, ty)$) satisfies the following properties:*

1. *F_t is an affine horseshoe on $[0,1] \times [0,1]$ for sufficiently large $t > 1$.*

2. *A periodic point with period 2 undergoes a period doubling bifurcation and it is the first bifurcation.*

We divide the proof in two parts. The proof of each part is outlined as follows :

Proof of the 1-dimension part

1. We construct a piecewise-linear map $\hat{f}_{a,K,K'}$ as in Figure 3. We prove that $\hat{f} = \hat{f}_{a,K,K'}$ has a 2-periodic sink and the critical value of $\hat{f}$ is equal to 1 for some a, K, K'. Obviously, the map $\hat{f}_t(x) = t \cdot \hat{f}(x)$ is affine and uniformly expanding on $[0,1] \cap \hat{f}_t^{-1}[0,1]$ for sufficiently large t. Therefore the family $\{\hat{f}_t\}$ satisfies the desired properties in the sense that:

   (a) $\hat{f}_t$ is a 1-dimensional affine 'horseshoe' for sufficiently large $t > 1$.

   (b) The 2-periodic point in the horseshoe invariant set has to change its stability before $t = 1$ and hence before the tangency.

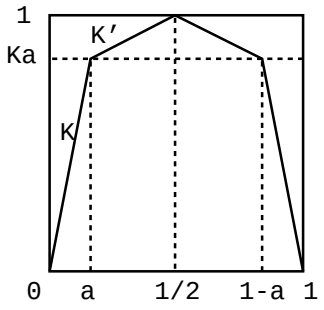

Figure 3: Graph of $\hat{f}_{a,K,K'}$

2. We modify $\hat{f}$ into a C^1 function f in neighborhoods of a and $1-a$ by using the logarithmic function. We show that in the family $\{f_t(x) = t \cdot f(x)\}$ a 2-periodic sink appears after a sub-critical period doubling bifurcation.

3. We check the hyperbolicity of f_t before the period doubling bifurcation. Once that is done, we can conclude that f_t is an affine horseshoe for sufficiently large $t > 1$ and that the period doubling bifurcation is the first bifurcation in this family.

Proof of the 2-dimension part

1. We extend the 1-dimensional map f to a 2-dimensional diffeomorphism $F : \mathbf{R}^2 \to \mathbf{R}^2$ as in Figure 4. Notice that F is semi-conjugate to f on a dynamically important set by the projection to y-axis. The family F_t is given by $F_t = y_t \circ F$.

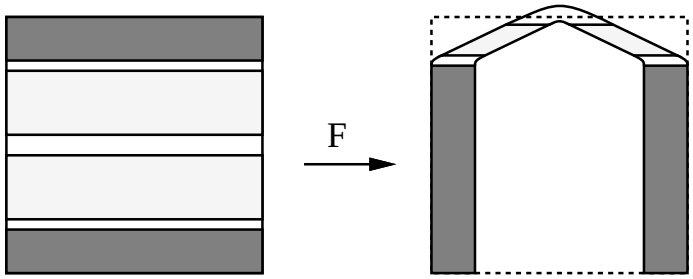

Figure 4: $F : \mathbf{R}^2 \to \mathbf{R}^2$

2. It can be shown that the non-wandering set of F_t is contained in $[0,1] \times [0,1]$. We use a sequence of filtrations to prove it.

3. We show that a 2-periodic point of F_t undergoes a period doubling bifurcation at some $t > 1$.

4. We verify the hyperbolicity of F_t on the non-wandering set before the bifurcation of the 2-periodic point occurs. This completes the proof of the main theorem.

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2 A family of 1-dimension maps

In this section, we construct a family of unimodal maps whose first bifurcation is a period doubling bifurcation.

2.1 A piecewise linear map

Let $\hat{f}_{a,K,K'} : \mathbf{R} \to \mathbf{R}$ be a piecewise linear map given by

$$\hat{f}_{a,K,K'}(x) = \begin{cases} Kx & \text{for } x \in (-\infty, a) \\ Ka + K'(x-a) & \text{for } x \in [a, \frac{1}{2}) \\ Ka + K'(1-x-a) & \text{for } x \in [\frac{1}{2}, 1-a] \\ K(1-x) & \text{for } x \in (1-a, \infty), \end{cases} \tag{1}$$

where $0 < a < \frac{1}{2}$ and $0 < K' < K$.

The map $\hat{f} = \hat{f}_{a,K,K'}$ is continuous on $\mathbf{R}$ and C^1 on $\mathbf{R} - \{a, 1-a, \frac{1}{2}\}$, which is unimodal and convex, and satisfies $\hat{f}(0) = 0$ and $\hat{f}(1-x) = \hat{f}(x)$.

Lemma 1 *For any $0 < a < \frac{1}{6}$, there exist $K > K' > 0$ such that $\hat{f}_{a,K,K'}(\frac{1}{2}) = 1$ and that $\hat{f}_{a,K,K'}$ has a 2-periodic sink.*

(Proof) Let K and K' be the numbers as follows.

$$K = \frac{1 + \sqrt{1 - \frac{4}{5}a}}{2a}, \ K' = \frac{1 - Ka}{\frac{1}{2} - a}.$$

It is easy to see that $0 < K' < K$ and $\hat{f}_{a,K,K'}(\frac{1}{2}) = 1$.

If we show the existence of a 2-periodic sink, the lemma is proven. Since $K(1 - Ka) = \frac{1}{5}$ and $K > \frac{1}{2a}$, we have $1 - Ka < \frac{2}{5}a < a$, and therefore,

$$\hat{f}^2(a) = \hat{f}(Ka) = K(1 - Ka) = \frac{1}{5} > a.$$

Since $\hat{f}^2(a) > a$ and $\hat{f}^2(\frac{1}{2}) = 0 < \frac{1}{2}$, there exists a 2-periodic point p in $(a, \frac{1}{2})$. It holds that

$$|(\hat{f}^2)'(p)| = |(\hat{f}^2(\tfrac{1}{2}) - \hat{f}^2(a))/(\tfrac{1}{2} - a)| < 1,$$

since $\hat{f}^2$ is linear on $(a, \frac{1}{2})$. Therefore, p is a 2-periodic sink. □

Remark 1 K, K' chosen in the proof of Lemma 1 satisfy that

$$K^2K' > 1, \ KK' < \frac{3}{5}, \ K > 5 \text{ and } Ka > 1 - a.$$

In the rest of this paper, we assume that $0 < a < \frac{1}{6}$ and K, K' are as in the proof of Lemma 1.

2.2 Smoothing the piecewise linear map

In this section, we modify the map $\hat{f}$ to f on neighborhoods of $a, 1-a$ so that f becomes C^1 on $\mathbf{R} - \{\frac{1}{2}\}$.

We can choose a small positive number δ such that $K(a-\delta) > 1-(a-\delta)$, $K(1-K(a-\delta)) < \frac{1}{4}$, and $p \in (a+\delta, \frac{1}{2})$, where p is the 2-periodic sink of $\hat{f}$ obtained in Lemma 1.

We define f as follows:

$$f(x) = \begin{cases} g(x) & \text{for } x \in [a-\delta_-, a+\delta_+] \\ g(1-x) & \text{for } x \in [1-(a+\delta_+), 1-(a-\delta_-)] \\ \hat{f}(x) & \text{otherwise,} \end{cases}$$

where $g(x) = c_1 \log(x+c_2) + c_3$, for some real numbers c_1, c_2, c_3 and positive numbers δ_-, δ_+.

Lemma 2 *We can choose $c_1, c_2, c_3, \delta_-, \delta_+$ so that f is C^1 on $\mathbf{R} - \{\frac{1}{2}\}$ and that δ_-,δ_+ are less than δ.*

(Proof) Let $c_1 = K'\delta$, then

$$(c_1 \log x)'|_{x=\delta} = \frac{c_1}{\delta} = K', \ (c_1 \log x)'|_{x=\frac{K'}{K}\delta} = \frac{c_1}{\left(\frac{K'}{K}\right)\delta} = K.$$

One can find suitable $c_2, c_3, \delta_-, \delta_+$ such that $g(x) = c_1 \log(x+c_2)+c_3$ satisfies

$$g(a-\delta_-) = \hat{f}(a-\delta_-), \ g(a+\delta_+) = \hat{f}(a+\delta_+),$$

$$g'(a-\delta_-) = K, \ g'(a+\delta_+) = K'.$$

These conditions clearly imply that $f(x)$ is C^1 except at $x = \frac{1}{2}$.

Finally, one has $\delta_-, \delta_+ < \delta$ since $0 < \delta_- + \delta_+ = (a+\delta_+) - (a-\delta_-) = \delta - \frac{K'}{K}\delta < \delta$.□

In what follows, we fix $\delta_-, \delta_+, c_1, c_2, c_3$ as above.

2.3 A family of 1-dimensional maps

A 1-parameter family $\{f_t\}_{1<t\le t_0}$ is defined as

$$f_t(x) = t \cdot f(x),$$

where $t_0 := \frac{1}{K(a-\delta)}$.

Remark 2 It holds that $t_0 < \frac{6}{5}$ since $K(a-\delta) > 1-a > \frac{5}{6}$. As commented in Remark 1, one have $KK' < \frac{3}{5}$, and therefore, $t_0 K \cdot t_0 K' < t_0^2 KK' < \frac{36}{25} \cdot \frac{3}{5} < 1$.

Let Λ_t and $\Omega(f_t)$ be the maximal invariant set of f_t in $[0,1]$ and the non-wandering set of f_t, respectively.

Obviously, for $t = t_0$, f_{t_0} equals to the tent map with slope $t_0 K, -t_0 K$ on $f_{t_0}^{-1}((-\infty, 1])$, and hence, $f_t|_{\Lambda_t}$ is conjugate to the one-sided 2-shift.

Proposition 3 *For $t > 1$, $\Omega(f_t) \subset \Lambda_t$.*

(Proof) Firstly, we show that $\Omega(f_t)$ lies in $\mathbf{R}_+$. Choose any $x < 0$ and let $U = (\frac{tK}{2}x, \frac{1}{2}x)$. Since $tK > \frac{5}{2}$, one has $x \in U$. For arbitrary $n \geq 1$,

$$f_t^n(U) \cap U = \left(\frac{(tK)^{n+1}}{2} x, \frac{(tK)^n}{2} x \right) \cap U = \emptyset.$$

and hence, $x \notin \Omega(f_t)$.

If $x > 1$, then $f_t(x) < 0$, and hence, $f_t(x) \notin \Omega(f_t)$. Since the non-wandering set is f_t-invariant, $x \notin \Omega(f_t)$ if $f_t^n(x) \notin [0,1]$ for some n. Therefore $x \in f_t^{-n}([0,1])$ for arbitrary $n \geq 1$ if $x \in \Omega(f_t)$. □

Lemma 4 *For $t > 1$, $f_t^2|_{I_{01}(t)}$ has a unique fixed point p_t and $(f_t^2)'(p_t)$ is continuous, where $I_{01}(t) = f_t^{-1}((\frac{1}{2}, 1)) \cap (0, \frac{1}{2})$.*

(Proof) $I_{01}(t)$ is not empty since $f_t(\frac{1}{2}) = t > 1$ and f_t maps $(0, \frac{1}{2})$ to $(0, f_t(\frac{1}{2}))$ diffeomorphically.

Since $(f_t)'|_{(0,\frac{1}{2})} > 0$ and $(f_t)'|_{(\frac{1}{2},1)} < 0$,

$$f_t^2|_{I_{01}(t)} : I_{01}(t) \to (0, f_t(\tfrac{1}{2})) \supset (0,1)$$

is an orientation reversing diffeomorphism. This implies that $f_t^2|_{I_{01}(t)}$ has a unique 2-periodic point p_t.

The continuity of $(f_t^2)'(p_t)$ follows, because

$$(f_t^2 - id)'|_{I_{01}(t)}(x) = (f_t^2)'|_{I_{01}(t)}(x) - 1 < -((tK')^2 + 1) < 0,$$

and hence, by the implicit function theorem, p_t is a C^1 function. □

2.4 The bifurcation of the 2-periodic point p_t

In this section, we prove that p_t undergoes a period doubling bifurcation which is characterized by the following proposition.

Proposition 5 (See [6, pp. 220]) *Let $\{G_t : \mathbf{R} \to \mathbf{R}\}$ be a 1-parameter family of functions, and assume that $G : (x,t) \mapsto G_t(x)$ is C^r $(r \geq 3)$ and satisfies the following conditions for some $(x_0, t_0) \in \mathbf{R}^2$.*

1. $G(x_0, t_0) = x_0$,
2. $\dfrac{\partial G}{\partial x}(x_0, t_0) = -1$,
3. $\alpha = \left[\dfrac{\partial^2 G}{\partial t \partial x} + \dfrac{1}{2}\left(\dfrac{\partial G}{\partial t}\right)\left(\dfrac{\partial^2 G}{\partial x^2}\right)\right](x_0, t_0) \neq 0$,
4. $\beta = \left[\dfrac{1}{3!}\dfrac{\partial^3 G}{\partial x^3} + \left(\dfrac{1}{2!}\dfrac{\partial^2 G}{\partial x^2}\right)^2\right](x_0, t_0) \neq 0$.

Then the 1-parameter family $\{G_t\}$ undergoes a period doubling bifurcation at $t = t_0$. More specifically, there is a differentiable curve of fixed points, $x(t)$, passing through x_0 at t_0, and the stability of the fixed point changes at t_0. There is also a differentiable curve γ passing through (x_0, t_0) so that $\gamma - \{(x_0, t_0)\}$ is the union of hyperbolic 2-periodic orbits. The curve γ is tangent to the line $\mathbf{R} \times \{t_0\}$ at (x_0, t_0), so γ is the graph of a function of x, $t = m(x)$ with $m'(x_0) = 0$ and $m''(x_0) = -2\beta/\alpha \neq 0$. □

In order to verify the assumption of the proposition, we estimate the orbits of p_t and $a - \delta_-$.

Lemma 6 *For $t > 1$, $f_t(p_t) > 1 - (a - \delta_-)$.*

(Proof) $f_t(p_t) > \frac{1}{2}$ since $p_t \in I_{01}(t)$.

Suppose $f_t(p_t) \leq 1 - (a - \delta_-)$. $f_t|_{(\frac{1}{2},1)}$ is orientation reversing, and hence,

$$p_t = f_t^2(p_t) \geq f_t(1 - (a - \delta_-)) = f_t(a - \delta_-) = tK(a - \delta_-) > 1 - (a - \delta_-) > \frac{1}{2},$$

which contradicts $p_t \in I_{01} \subset (0, \frac{1}{2})$. □

Lemma 7 *For $t > 1$, $f_t(x) > 1 - (a - \delta_-)$ and $f_t'(f_t(x)) = -tK$, if $a - \delta_- < x < \frac{1}{2}$.*

(Proof) Since $f_t|_{(0,\frac{1}{2})}$ is increasing, $f_t(x) > f_t(a - \delta_-) > 1 - (a - \delta_-)$, and hence, $f_t'(f_t(x)) = -tK$. □

Lemma 8 *For $t > 1$, $f_t^2(a - \delta_-) < \frac{1}{2}$.*

(Proof) Since $tK(a-\delta_-) > K(a-\delta_-) > 1-(a-\delta_-)$,

$$\begin{aligned} f_t^2(a-\delta_-) &= tK(1-tK(a-\delta_-)) \\ &= tK(1-t)+t^2K(1-K(a-\delta_-)) \end{aligned}$$

The last term of the inequality is less than $\frac{1}{2}$, since $1 < t < \frac{6}{5}$ and $K(1-K(a-\delta_-)) < \frac{1}{4}$. □

Let $t_1 = \min_{t \le t_0}\{|(f_{t'}^2)'(p_{t'})| > 1 \text{ for all } t' > t\}$. Notice that $t_1 > 1$ since p_t is a sink if t is near 1.

Lemma 9 *(1) For any $t_1 < t < t_0$, $p_t \in [0, a+\delta_+]$. (2) $p_{t_1} \in (a-\delta_-, a+\delta_+)$.*

(Proof) Notice that $|f_t'(f_t(p_t))| = tK$ from Lemma 6. If $p_t \in (a+\delta_+, \frac{1}{2})$, then

$$|(f_t^2)'(p_t)| = |f_t'(f_t(p_t))||f_t'(p_t)| = tK \cdot tK' < 1.$$

Therefore $t > t_1$ implies $p_t \in [0, a+\delta_+]$.

If $p_t \in [0, a-\delta_-]$, then

$$|(f_t^2)'(p_t)| = |f_t'(f_t(p_t))||f_t'(p_t)| = tK \cdot tK > 1.$$

$p_{t_1} \in (a-\delta_-, a+\delta_+)$ since $|(f_t^2)'(p_{t_1})| = 1$. □

Proposition 10 *$\{f_t^2\}$ has a period doubling bifurcation of p_t at $t = t_1$.*

(Proof) Let $G(x,t) = f_t^2(x)$. From Lemma 6,

$$G(x,t) = tK(1-tf(x))$$

for (x,t) near (p_{t_1}, t_1). G is C^3 in a neighborhood of (p_{t_1}, t_1).

We calculate α and β in Proposition 5.

$$\begin{aligned} \alpha &= \left[\frac{\partial^2 G}{\partial t \partial x} + \frac{1}{2}\left(\frac{\partial G}{\partial t}\right)\left(\frac{\partial^2 G}{\partial x^2}\right)\right](p_{t_1}, t_1) \\ &= -2t_1 f'(p_{t_1}) - \frac{1}{2}t^2K^2(1-2t_1 f(p_{t_1}))f''(p_{t_1}). \end{aligned}$$

$f'(p_{t_1}) > 0$ and $f''(p_{t_1}) < 0$ since f is a logarithmic and increasing function in $[a-\delta_-, a+\delta_+]$ and p_{t_1} is contained there by the previous lemma. These estimates and the inequality $f_{t_1}(p_{t_1}) > \frac{1}{2}$ imply $\alpha < 0$.

Since f_{t_1} is logarithmic near p_{t_1}, so is $G(\cdot, t_1)$, and hence has positive Schwarzian derivative. Since -6β is the Schwarzian derivative of $G(\cdot, t_1)$ at p_{t_1}, one has $\beta < 0$.

Therefore, we have shown that $G(x) = f_t^2(x)$ and (p_{t_1}, t_1) satisfy the conditions in Proposition 5. □

2.5 Hyperbolicity before the period doubling bifurcation

Proposition 11 *If* $|(f_t^2)'(p_t)| > 1$, *then there exists* $\alpha > 1$ *such that for any* $x \in \Lambda_t$, $|(f_t^n)'(x)| \geq \alpha$ *for some* $n = 1, 2, 3, 4$.

This proposition implies that f_t is hyperbolic for $t > t_1$, and hence, the period doubling bifurcation at $t = t_1$ is the first bifurcation of $\{f_t\}$.

(Proof) Fix a point x in $\Lambda_t - \{p_t\}$. We have only to consider the case $x < \frac{1}{2}$ since $f_t(x) = f_t(1-x)$. Notice that $f_t'(f_t(x)) = -tK$ for $p_t \leq x < \frac{1}{2}$ from Lemma 6 and that $|f_t'(x)| \geq tK'$ for all $x \neq \frac{1}{2}$.

Suppose $p_t \in [a + \delta_+, \frac{1}{2})$, then

$$|(f_t^2)'(p_t)| = |f_t'(p_t)||f_t'(f_t(p_t))| = tK' \cdot tK = t^2 KK' < 1.$$

This inequality contradicts the assumption of the proposition. Therefore, we may assume $p_t \in (0, a + \delta_+)$.

Firstly, $f_t'(x) = tK > 1$ for $x \in [0, a - \delta_-]$. Secondly, suppose that $p_t \in [0, a - \delta_-]$ and that $x \in (a - \delta_-, \frac{1}{2})$. Then, $f_t'(f_t(x)) = -tK$ and $f_t^2(x) < p_t$ because $p_t < x$, $f_t((p_t, \frac{1}{2})) \subset (\frac{1}{2}, \infty)$ and f_t is monotone on the both sides of $\frac{1}{2}$. The second inequality $f_t^2(x) < p_t$ implies $f_t'(f_t^2(x)) = tK$. Therefore,

$$|(f_t^3)'(x)| = |f_t'(x)||f_t'(f_t(x))||f_t'(f_t^2(x))| \geq tK' \cdot tK \cdot tK = t^3 K^2 K' > 1.$$

Since the proposition holds if $x \in [0, a - \delta_-]$ or $p_t \in [0, a - \delta_-]$, let $p_t \in (a - \delta_-, a + \delta_+)$ and $x \in (a - \delta_-, \frac{1}{2})$ in the rest of proof.

Since $x \in (a - \delta_-, \frac{1}{2})$, f_t is increasing on $(0, \frac{1}{2})$ and $f_t(a - \delta_-) > 1 - (a - \delta_-)$, $f_t(x) > 1 - (a - \delta_-)$. This implies $f_t'(f_t(x)) = -tK$.

If $f_t^2(x) \leq a - \delta_-$, then $f_t'(f_t^2(x)) = tK$, and hence,

$$|(f_t^3)'(x)| = |f_t'(x)||f_t'(f_t(x))||f_t'(f_t^2(x))| \geq tK' \cdot tK \cdot tK = t^3 K^2 K' > 1.$$

Now we consider the case $f_t^2(x) > a - \delta_-$. Notice that $f_t^2|_{[a-\delta_-, a+\delta_+]} = -C_1 \log(x + C_2) + C_3$ for some $C_1, C_2, C_3 \in \mathbf{R}$, since $f_t|_{[a-\delta_-, a+\delta_+]}$ is a logarithmic function and $f_t|_{f_t([a-\delta_-, a+\delta_+])}$ is linear. Since $f_t^2|_{[a-\delta_-, a+\delta_+]}$ is decreasing, $C_1 > 0$. Since f_t^2 is defined at $x = a - \delta_-$, $C_2 > -(a - \delta_-)$. We denote $G_t(x)$ as $-C_1 \log(x + C_2) + C_3$. Since p_t is a fixed point of f_t^2 and is contained in $[a - \delta_-, a + \delta_+]$, p_t is also a fixed point of G_t.

Since G_t' is negative and increasing, $0 > G_t'(y) > G_t'(a + \delta_+) = (f_t^2)'(y)$ for any $y > a + \delta_+$, and hence, $(f_t^2)'(x) \leq G_t'(x)$ and $(f_t^2)(x) \leq G_t(x)$ for any $x > a - \delta_-$. Therefore, if $x, f_t^2(x) > a - \delta_-$, then $(f_t^2)'(f_t^2(x)) \leq G_t'(f_t^2(x)) \leq G_t'(G_t(x))$, because G_t' is increasing. These inequalities imply

$$|(f_t^4)'(x)| = |(f_t)'^2(x)||(f_t^2)'(f_t^2(x))| \geq |G_t'(x)||G_t'(G_t(x))| = |(G_t^2)'(x)|,$$

if $x, (f_t^2)(x) > a - \delta_-$ and $p_t \in [a - \delta_-, a + \delta_+]$.

Proof of the proposition is completed if we show $\inf |(G_t^2)'|$ is bigger than 1, which follows from the following lemma. □

Lemma 12 *For a positive number C_1 and real numbers C_2, C_3, we assume that $\phi(x) = -C_1 \log(x + C_2) + C_3$ has a fixed point p.*

If $|\phi'(p)| > 1$, then $\inf_{x,\phi(x)>-C_2} |(\phi^2)'(x)| > \alpha$ for some $\alpha > 1$.

(Proof) By translating the coordinate, we may assume that $p = 0$. Then, $\phi(x)$ is given by $\phi(x) = -C_1 \log \frac{x+C_2}{C_2}$ for some $C_1, C_2 > 0$.

Choose x so that $\phi^2(x)$ is defined, then

$$(\phi^2)'(x) = C_1^2 \frac{1}{h(x)} \text{ and } (\phi^2)''(x) = C_1^2 \frac{-h'(x)}{(h(x))^2},$$

where $h(x) = (x + C_2)(\phi(x) + C_2)$.

Since ϕ is a decreasing function, so is h'. In fact, $h'(x) = \phi(x) + C_2 - C_1$.

Let $x_0 = C_2 \exp(\frac{C_2}{C_1} - 1) - C_2$. Since $x_0 + C_2 > 0$ and $\phi(x_0) + C_2 = C_1 > 0$, $\phi^2(x_0)$ is well-defined. Since $h'(x_0) = 0$ and h' is decreasing,

$$\begin{array}{lllll} x < x_0 & \Longrightarrow & h'(x) > 0 & \Longrightarrow & (\phi^2)''(x) < 0 \\ x > x_0 & \Longrightarrow & h'(x) < 0 & \Longrightarrow & (\phi^2)''(x) > 0, \end{array}$$

This means that $x = x_0$ is the minimal point of $(\phi^2)'$.

We estimate the value of $(\phi^2)'(x_0)$.

$$(\phi^2)'(x_0) = C_1 \frac{1}{C_2(\exp(\frac{C_1}{C_2} - 1))} = \frac{\exp(1 - \frac{C_2}{C_1})}{(\frac{C_2}{C_1})},$$

Since $|\phi'(0)| = |-C_1 \frac{1}{0+C_2}| = \frac{C_1}{C_2}$,

$$(\phi^2)'(x_0) = |\phi'(0)| \exp(1 - |\phi'(0)|^{-1}).$$

Therefore, $|\phi'(0)| > 1$ implies that $\inf(\phi^2)' = (\phi^2)'(x_0) > 1$. □

3 A family of 2-dimensional maps

In this section, we define a family of 2-dimensional maps $\{F_t\}$ and prove that the assertion in the main theorem holds for the family.

3.1 Constructing a horseshoe-breaking family

Similarly as in the case of 1-dimensional maps, we first construct a 2-dimensional diffeomorphism F and define a family $\{F_t\}$ as $F_t(x, y) = y_t \circ F(x, y)$, where $y_t(x, y) = (x, ty)$.

Fix t_2 such that $1 < t_2 < t_1$. Since $f_{t_2}(x) = (\frac{t_2}{t}) f_t(x) \geq f_t(x)$ for any $t \geq t_2$, we can take a small positive number ϵ such that $f_t(\frac{1}{2} - \epsilon) > 1$ for any $t \geq t_2$.

We define a smooth function h as follows:

$$h(x) = \begin{cases} 0 & (x \in (-\infty, a-\eta)) \\ x-a & (x \in (a+\eta, \frac{1}{2}-\epsilon), \end{cases}$$

and

$$h(1-x) = h(x),$$

where η is a small positive number.

Let $0 < b < \min\{a, \frac{K'}{3}\}$. The following lemma is necessary to verify that the diffeomorphism F given below is well-defined.

Lemma 13 *There exists a positive number γ which satisfies*

$$b(1+\gamma) + h(y) < \frac{1}{2} - \epsilon \text{ for any } y \leq \frac{1}{2} - \epsilon.$$

(Proof) Since $b + h(y) < a + (y-a) = y \leq \frac{1}{2} - \epsilon$, the assertion follows for a sufficiently small γ. □

Now we define a diffeomorphism F. $F : \mathbf{R}^2 \to \mathbf{R}^2$ is given by a diffeomorphism of the form

$$F(x,y) = \begin{cases} (bx + h(y), f(y)) & ((x,y) \in (-\infty, 1+\gamma) \times (-\infty, \frac{1}{2}-\epsilon)) \\ (1-bx-h(y), f(y)) & ((x,y) \in (-\infty, 1+\gamma) \times (\frac{1}{2}+\epsilon, \infty)), \end{cases} \tag{2}$$

which satisfies **the filtration conditions**:

(F1)' $F([1+\gamma, \infty] \times \mathbf{R}) \subset (0,1) \times (-\infty, 2)$,

(F2)' $F([-\gamma, 1+\gamma] \times [\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon]) \subset (0,1) \times [f(\frac{1}{2}-\epsilon), \infty)$,

(F3)' $F((-\infty, -\gamma] \times [\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon]) \subset \mathbf{R} \times [f(\frac{1}{2}-\epsilon), \infty)$.

From Lemma 13, a map of the form (2) is a diffeomorphism on $(-\infty, 1+\gamma) \times (-\infty, \frac{1}{2}-\epsilon) \cup (-\infty, 1+\gamma) \times (\frac{1}{2}+\epsilon, \infty)$, and hence, is extended to a diffeomorphism F on the whole plane $\mathbf{R}^2$.

The diffeomorphism F_t, defined by $F_t(x,y) = y_t \circ F(x,y)$, has similar properties, namely, it has the form

$$F_t(x,y) = \begin{cases} (bx + h(y), f_t(y)) & ((x,y) \in (-\infty, 1+\gamma) \times (-\infty, \frac{1}{2}-\epsilon)) \\ (1-bx-h(y), f_t(y)) & ((x,y) \in (-\infty, 1+\gamma) \times (\frac{1}{2}+\epsilon, \infty)), \end{cases} \tag{3}$$

and satisfies **the filtration conditions** for any $t \geq t_2$:

(F1) $F_t([1+\gamma, \infty] \times \mathbf{R}) \subset (0,1) \times (-\infty, 2)$,

(F2) $F_t([-\gamma, 1+\gamma] \times [\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon]) \subset (0,1) \times (1, \infty)$,

(F3) $F_t((-\infty, -\gamma] \times [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]) \subset \mathbf{R} \times (1, \infty)$.

We note that F_t is semi-conjugate to f_t on $[0,1] \times ([0, \frac{1}{2} - \epsilon] \cup [\frac{1}{2} + \epsilon, 1])$,

$$\pi_y \circ F_t = f_t \circ \pi_y,$$

where π_y is the projection to the y-axis.

For $t = t_0$, F_{t_0} is an affine horseshoe diffeomorphism, since f_{t_0} is conjugate to a tent map on $f_{t_0}^{-1}([0,1])$.

3.2 Non-wandering set of F_t

Let Λ_t and $\Omega(F_t)$ be the maximal invariant set of F_t in $[0,1] \times [0,1]$ and the non-wandering set of F_t, respectively. In this subsection, we prove that $\Omega(F_t)$ is contained in Λ_t. For this purpose, we use a sequence of filtrations associated to Λ_t.

For a small positive number μ which satisfies $0 < \mu < \gamma$, we define the filtration $\mathbf{M}(\mu)$ as follows.

$$\begin{aligned}
\mathbf{M}(\mu)_0 &= \emptyset, \\
\mathbf{M}(\mu)_1 &= I_{\frac{1}{2}\mu} \times (-\infty, -\mu], \\
\mathbf{M}(\mu)_2 &= I_\mu \times \mathbf{R}, \\
\mathbf{M}(\mu)_3 &= \mathbf{R}^2,
\end{aligned}$$

where $I_\mu = (-\mu, 1 + \mu)$.

Lemma 14 *For any μ $(0 < \mu < \gamma)$, $\mathbf{M}(\mu)$ is a filtration. The invariant set associated to the filtration, $\bigcup_{i=1}^{3} \bigcap_{n \in \mathbf{Z}} F_t^n(\mathbf{M}(\mu)_i)$, is contained in $I_\mu \times [0,1]$.*

(Proof) We check that $F_t(\mathbf{M}(\mu)_i) \subset \mathrm{Int}\mathbf{M}(\mu)_i$ and calculate the maximal invariant set of $\mathbf{M}(\mu)_i - \mathbf{M}(\mu)_{i-1}$ for $i = 1, 2, 3$.

If $(x, y) \in \mathbf{M}(\mu)_1$, then $F_t(x, y) \in \mathrm{Int}\mathbf{M}(\mu)_1$, since $bx \in \mathrm{Int}(I_{\frac{1}{2}\mu})$ and $f_t(y) = tKy < y$. Since $f_t((-\infty, -\mu]) = (\infty, -tK\mu]$,

$$\bigcap_{n \in \mathbf{Z}} \mathbf{M}(\mu)_1 = \emptyset.$$

From (3) and (F2), $F_t(\mathbf{M}(\mu)_2)$ is contained in $\mathrm{Int}(\mathbf{M}(\mu)_2)$. For $(x, y) \in I_\mu \times (-\infty, 0)$, $F_t^n(x, y) = (b^n x, (tK)^n y)$, and hence, $F_t^n(x, y) \in \mathbf{M}(\mu)_1$ for sufficiently large n. For $(x, y) \in I_\mu \times (1, \infty)$, $F_t(x, y) = (bx, tK(1 - y)) \in I_\mu \times (-\infty, 0)$. Therefore, $F_t^n(x, y) \in \mathbf{M}(\mu)_1$ for sufficiently large n if $(x, y) \in I_\mu \times ((-\infty, 0) \cup (1, \infty))$. This implies

$$\bigcap_{n \in \mathbf{Z}} F_t^n(\mathbf{M}(\mu)_2 - \mathbf{M}(\mu)_1) \subset I_\mu \times [0,1].$$

From (3), (F1) and (F2), we have $F_t((1,\infty)\times\mathbf{R}) \subset [0,1]\times\mathbf{R} \subset \mathbf{M}(\mu)_2$ and $F_t((-\infty,-\mu)\times(1,\infty)) \subset (1,\infty)\times\mathbf{R}$, which imply $F_t^2((-\infty,-\mu)\times(1,\infty)) \subset \mathbf{M}(\mu)_2$. Therefore, the maximal invariant set of $\mathbf{R}^2 - \mathbf{M}(\mu)_2$ is contained in $(-\infty,-\mu)\times(-\infty,1]$.

Since $F_t^{-1}((-\infty,-x)\times(-\infty,1]) \subset (-\infty,-xb^{-1})\times(-\infty,1]$ for any $x>0$, $F_t^{-n}((-\infty,-\mu)\times(-\infty,1]) \subset (-\infty,-\mu b^{-n})\times(-\infty,1]$ for any $n \geq 0$. This implies

$$\bigcap_{n\in\mathbf{Z}} F_t^n(\mathbf{R}^2 - \mathbf{M}(\mu)_2) = \bigcap_{n\in\mathbf{Z}} F_t^n((-\infty,-\mu)\times(-\infty,1]) = \emptyset.$$

From the above arguments, the invariant set associated to the filtration, $\bigcup_{i=1}^3 \bigcap_{n\in\mathbf{Z}} F_t^n(\mathbf{M}(\mu)_i)$, is contained in $I_\mu \times [0,1]$. □

Proposition 15

$$\Omega(F_t) \subset \Lambda_t.$$

(Proof) From the above lemma,

$$\Omega(F_t) \subset \bigcap_{n\in\mathbf{Z}} F_t^n(I_\mu \times [0,1]).$$

Since μ is an arbitrary small positive number less than γ,

$$\Omega(F_t) \subset \bigcap_{n\in\mathbf{Z}} F_t^n([0,1]\times[0,1]) = \Lambda_t. \square$$

3.3 Bifurcation of the 2-periodic point

Recall that p_t is the 2-periodic point of f_t given in Section 2.3. $F_t^2|_{[0,1]\times\{p_t\}} : [0,1]\times\{p_t\} \to [0,1]\times\{p_t\}$ is a contraction map, and hence, there exists a unique 2-periodic point of F_t, q_t, in $[0,1]\times\{p_t\}$ with

$$\begin{aligned} DF_t^2|_{q_t} &= DF_t|_{F_t(q_t)} \cdot DF_t|_{q_t} \\ &= \begin{pmatrix} -b & * \\ 0 & f_t'(f_t(p_t)) \end{pmatrix} \begin{pmatrix} b & * \\ 0 & f_t'(p_t) \end{pmatrix} \\ &= \begin{pmatrix} -b^2 & * \\ 0 & (f_t^2)'(p_t) \end{pmatrix}. \end{aligned}$$

Since $0 < b^2 < 1$ and the period doubling bifurcation of p_t occurs at $t = t_1$ in the family $\{f_t\}$, the period doubling bifurcation of q_t occurs in the family $\{F_t\}$.

3.4 Transversality of the stable and unstable manifolds

For any $t \in [t_2, t_0]$, the local stable manifold $W^s_{loc}(0; F_t)$ and the local unstable manifold $W^u_{loc}(0; F_t)$ are respectively contained in the x-axis and the y-axis.

The following lemma shows the transversality of $W^s(0)$ and $W^u(0)$.

Lemma 16 *For $t_2 < t < t_0$, there exist closed cone fields C^u and C^s on $T\mathbf{R}^2|_{\Lambda_t}$ such that for any $r \in \Lambda_t$,*

1. $(DF_t)_r(C^u(r)) \subset C^u(F_t(r))$,
   $(DF_t)_r^{-1}(C^s(r)) \subset C^s(F_t(r))$,
2. $C^u(r) \cap C^s(r) = \{0_r\}$,
3. $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathrm{Int}C^u(r)$ *and* $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathrm{Int}C^s(r)$.

(Proof) Let

$$C^u(r) = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in T_r\mathbf{R}^2 \;\middle|\; |v_1| \le \frac{3}{K'}|v_2| \right\}$$

and

$$C^s(r) = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in T_r\mathbf{R}^2 \;\middle|\; |v_1| \ge \frac{4}{K'}|v_2| \right\}.$$

Obviously, they satisfy the second and third assertions of the lemma. We check the first assertion.

For $r \in \Lambda_t$ and $v = \begin{pmatrix} \alpha \\ 1 \end{pmatrix} \in C^u(r)$, we have

$$(DF_t)_r(v) = \begin{pmatrix} \pm b & h' \\ 0 & {f_t}' \end{pmatrix} \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \begin{pmatrix} \pm b\alpha + h' \\ {f_t}' \end{pmatrix}.$$

Since $|f_t'| \ge K'$ and

$$|\pm b\alpha + h'| \le b|\alpha| + |h'| \le b \cdot \frac{3}{K'} + 1 \le 1 + 1 < 3,$$

C^u satisfies the first assertion.

Similarly, for $r \in \Lambda_t$ and $v = \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \in C^s(r)$, we have

$$(DF_t)_r^{-1}(v) = \frac{1}{|b{f_t}'|} \begin{pmatrix} {f_t}' & \pm h' \\ 0 & \pm b \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \frac{1}{|b{f_t}'|} \begin{pmatrix} {f_t}' \pm \alpha h' \\ \pm b\alpha \end{pmatrix}.$$

Since

$$|{f_t}' \pm \alpha h'| \ge |f_t'| - |\alpha||h'| \ge K' - \frac{K'}{4} = \frac{3K'}{4} > b,$$

C^s also satisfies the first assertion. □

3.5 Hyperbolicity before the period doubling bifurcation

Let $E^u(r) = \bigcap_{n\in\mathbf{Z}} F_t^n(C^u(F_t^{-n}(r)))$ and $E^s(r) = \bigcap_{n\in\mathbf{Z}} F_t^n(C^s(F_t^{-n}(r)))$.

We prove that $E^u \oplus E^s$ is the hyperbolic splitting for F_t for $t_1 < t < t_0$. Once it is done, the period doubling bifurcation at $t = t_1$ is the first bifurcation of the family $\{F_t\}$, and hence, the proof of the main theorem is completed.

Lemma 17 *For any $r \in \Lambda_t$, $E^u(r)$ and $E^s(r)$ are 1-dimensional vector spaces.*

(Proof) We prove the lemma by the contraction principle (*c.f.* [4]).

We define two spaces of line fields $L^u = \left\{ r \mapsto \left\langle \begin{pmatrix} 1 \\ \alpha(r) \end{pmatrix} \right\rangle \middle| \alpha \in C^0(\Lambda_t, \mathbf{R}) \right\}$ and $L^s = \left\{ r \mapsto \left\langle \begin{pmatrix} \beta(r) \\ 1 \end{pmatrix} \right\rangle \middle| \beta \in C^0(\Lambda_t, \mathbf{R}) \right\}$. For line fields $l : r \mapsto \left\langle \begin{pmatrix} v_1(r) \\ v_2(r) \end{pmatrix} \right\rangle$ and $l' : r \mapsto \left\langle \begin{pmatrix} v_1'(r) \\ v_2'(r) \end{pmatrix} \right\rangle$, let $d(l,l')_u = \sup_{r\in\Lambda_t} \left| \frac{v_2}{v_1} - \frac{v_2'}{v_1'} \right|$ if $l, l' \in L^u$ and $d(l,l')_s = \sup_{r\in\Lambda_t} \left| \frac{v_1}{v_2} - \frac{v_1'}{v_2'} \right|$ if $l, l' \in L^s$. (L^u, d_u) and (L^s, d_s) become complete metric spaces.

Let $L_c^\sigma = \{ l \in L^\sigma | l(r) \subset C^\sigma(r) \text{ for any } r \in \Lambda_t \}$ for $\sigma = u, s$, which is a closed subset of L^σ.

If we prove that DF (*resp.* DF^{-1}) is a contraction on L_c^u (*resp.* L_c^s), then E^u (*resp.* E^s) is the image of a unique DF-invariant line field in L_c^u (*resp.* L_c^s).

For $v = \left\langle \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \right\rangle$ and $v' = \left\langle \begin{pmatrix} 1 \\ \beta \end{pmatrix} \right\rangle \in L^u$,

$$\begin{aligned} d(DF_t(v), DF_t(v'))_u &= \sup_{r\in\Lambda_t} \left\{ \left| \frac{\pm b\alpha(r) \pm h'}{f_t'} - \frac{\pm b\beta(r) \pm h'}{f_t'} \right| \right\} \\ &= \sup_{r\in\Lambda_t} \left\{ \left| \frac{b}{f_t'} \right| |\alpha(r) - \beta(r)| \right\} \le \frac{1}{3} d(v, v')_u, \end{aligned}$$

and hence, DF is a contraction of L_c^u.

For $v = \left\langle \begin{pmatrix} \alpha \\ 1 \end{pmatrix} \right\rangle$ and $v' = \left\langle \begin{pmatrix} \beta \\ 1 \end{pmatrix} \right\rangle \in L^s$,

$$\begin{aligned} d(DF_t^{-1}(v), DF_t^{-1}(v'))_s &= \sup_{r\in\Lambda_t} \left\{ \left| \frac{b\alpha(r)}{f_t' \pm \alpha(r)h'} - \frac{b\beta(r)}{f_t' \pm \beta(r)h'} \right| \right\} \\ &= \sup_{r\in\Lambda_t} \left\{ \frac{b}{\left(1 \pm \frac{\alpha(r)h'}{f_t'}\right)\left(1 \pm \frac{\beta(r)h'}{f_t'}\right)} |\alpha(r) - \beta(r)| \right\}. \end{aligned}$$

Since $|\alpha|, |\beta| \le \frac{K'}{4}$, $|h'| \le 1$, $|f_t'| \ge K'$ and $b < a < \frac{1}{6}$,

$$d(DF_t^{-1}(v), DF_t^{-1}(v'))_s \le b \left(\frac{4}{3}\right)^2 d(v, v')_s < \frac{8}{27} d(v, v')_s,$$

and hence, DF^{-1} is a contraction of L^s_c.□

Lemma 18 *$E^u(r) \oplus E^s(r)$ is the hyperbolic splitting of $T\mathbf{R}^2|_{\Lambda_t}$ for $t_1 < t < t_0$.*

(Proof) If there exist $C > 0$ and $\lambda > 1$ such that $\|(DF^n)_r v_u\| > C\lambda^n \|v_u\|$ and $\|(DF^{-n})_r v_s\| > C\lambda^n \|v_s\|$ for any $r \in \Lambda_t$, $v_u \in C^u(r)$, $v_s \in C^s(r)$ and $n \in \mathbf{N}$, then $E^u \oplus E^s$ is the hyperbolic splitting.

Fix a point $r = (x, y)$ in Λ_t. Since it is in Λ_t, y is in the maximal f_t-invariant set of [0,1].

$$(DF_t^n)_r = \prod_{k=0}^{n-1} (DF_t)_{(F_t^k(r))} = \prod_{k=0}^{n-1} \begin{pmatrix} \pm b & * \\ 0 & f_t'(f_t^k(y)) \end{pmatrix} = \begin{pmatrix} \pm b^n & * \\ 0 & (f_t^n)'(y) \end{pmatrix}.$$

From the hyperbolicity of f_t, there exist positive constants C and $\lambda > 1$ such that $|(f_t^n)'| > C\lambda^n$, and hence, we can choose n_0 such that $|(f_t^{n_0})'| > 2(1 + \left(\frac{3}{K'}\right)^2)$. For any $v_u = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in C^u(r)$,

$$\|(DF_t^{n_0})_r v_u\| = \left\| \begin{pmatrix} * \\ (f_t^{n_0})'(r) \cdot v_2 \end{pmatrix} \right\| \geq |(f_t^{n_0})'(r) \cdot v_2| > 2(1 + \left(\frac{3}{K'}\right)^2)|v_2| \geq \|v_u\|.$$

For any $v_s = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in C^s(r)$,

$$\left\|(DF_t^{-1})_r v_s\right\| = \left\| \begin{pmatrix} \pm\frac{1}{b} \cdot v_1 \pm \frac{h'}{b f_t'} \cdot v_2 \\ * \end{pmatrix} \right\| \geq \frac{|v_1| - |\frac{h'}{f_t'} \cdot v_2|}{b} > \frac{1 - \frac{1}{4}}{b}|v_1|.$$

Since $\|v\| \leq \sqrt{(1 + (\frac{K'}{4})^2)} \cdot |v_1| < \frac{5}{4}|v_1|$ and $|b| < \frac{1}{6}$,

$$\left\|(DF_t^{-1})_r v_s\right\| > \frac{3}{4} \cdot 6 \cdot \frac{4}{5}\|v_s\| = \frac{18}{5}\|v_s\|.\square$$

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