Order of decay of solutions of certain second order differential equations at infinity

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Introduction

The differential equation

$$u''(t) + f(t)u(t) = 0 \tag{0.1}$$

for unknown function u with a given coefficient function f was extensively studied in the middle of this century by A.Wintner and others, and we have many results on the condition for the equation to be oscillatory, on (successive) approximations of solutions of (0.1) and on the distribution of zeros of a solution and so on.

The present author encountered the above type of equation in his study of embeddings of discrete series into generalized Gel'fand-Graev representations. To determine unitary embeddings of discrete series, we need to estimate the order of solutions of certain differential equations. In this paper, an estimate of decay of solutions at infinity is studied.

In §1, we prepare some elementary lemmas used in the estimation.

In §§2-4, we consider the following type of equation among the equations written as (0.1):

$$u''(t) + f(t)^2 u(t) = 0\,. \tag{0.2}$$

Some lemmas related to zeros of a solutions of (0.2) are given in §2.

In §3 the following theorem is proved.

THEOREM A (Theorem 3.1). *Let f be a smooth function on $\mathbb{R}$ and assume that f is in $\mathcal{F}_{a,p,A}$ with $0 < p < \frac{1}{2}$. Then, for each real-valued solution u of (0.2),*

$$u(t) = O(f(t)^{-\frac{1}{2}}) \text{ as } t \to \infty\,.$$

Here the family $\mathcal{F}_{a,p,A}$ is defined in §2.

In this theorem, an upper bound for $u(t)f(t)^{\frac{1}{2}}$ is explicitly given.

1. Comparison lemmas

Throughout this paper, we assume that the coefficient function f in (0.1) is real-valued and of C^2-class. The equation (0.1) and its solutions are considered on a half-line $[a_0, \infty)$ with $a_0 \in \mathbb{R}$. The following lemma was proved in Wintner's paper [7].

Lemma 1.1. *Let a_0 be a real constant and assume that the function f satisfies the condition $\int_{a_0}^{\infty} f(t)dt = \infty$. Then, (0.1) is oscillatory, that is, every real-valued solution of (0.1) has infinitely many zeros on the interval (a_0, ∞).*

Remark 1.1. A necessary and sufficient condition of (0.1) to be oscillatory is given in Wintner's paper [6].

The Sturm comparison theorem implies the following lemma.

Lemma 1.2. *Let f and g be real-valued smooth functions on $\mathbb{R}$ and suppose that real-valued smooth functions u and v are solutions of differential equations*

$$u''(t) + f(t)u(t) = 0\,,$$
$$\text{and} \quad v''(t) + g(t)v(t) = 0\,,$$

respectively. Assume that for two real numbers α and β $(\alpha < \beta)$,

$$u(\alpha) = v(\alpha) = 0\,, \quad 0 < u'(\alpha) \le v'(\alpha)\,,$$
$$u(t) > 0 \quad \text{and} \quad f(t) \ge g(t) \quad \text{for } \alpha < t < \beta\,.$$

Then $u(t) \le v(t)$ for $\alpha \le t \le \beta$.

For determining the order of solutions of (0.1) at infinity, we prove a lemma.

Lemma 1.3. *Let f be a real-valued smooth function on $\mathbb{R}$ and u a real-valued solution of the differential equation*

$$u''(t) + f(t)^2 u(t) = 0\,.$$

Assume that for two real numbers α and β $(\alpha < \beta)$,

$$u(\alpha) = 0 \quad \text{and} \quad u(t) \ne 0 \quad \text{for } \alpha < t < \beta\,.$$

Further suppose that f is positive on the interval $[\alpha, \beta]$ and that $f(t)^{-\frac{1}{2}}$ is convex on (α, β). Then

$$0 \le u'(\alpha)^{-1}u(t) \le f(\alpha)^{-\frac{1}{2}} f(t)^{-\frac{1}{2}} \sin\Big(\int_{\alpha}^{t} f(s)ds\Big)$$

for $\alpha \le t \le \beta$.

Remark: Since u is not identically zero, $u'(\alpha) \neq 0$.

Proof of Lemma 1.3. Put $u_0(t) = u'(\alpha)^{-1}u(t)$ and $v(t) = f(\alpha)^{-\frac{1}{2}}f(t)^{-\frac{1}{2}}\sin(\int_\alpha^t f(s)\,ds)$. Then it is easily seen that

$$u_0(\alpha) = v(\alpha) = 0 \quad \text{and} \quad u_0'(\alpha) = v'(\alpha) = 1\,.$$

A straightforword calculation gives that

$$v''(t) + \left[f(t)^2 - f(t)^{\frac{1}{2}}(f_{\frac{1}{2}})''(t)\right] v(t) = 0\,,$$

where $f_{\frac{1}{2}}(t) = f(t)^{-\frac{1}{2}}$.

Since $f_{\frac{1}{2}}$ is convex, $f(t)^2 \geq f(t)^2 - f(t)^{\frac{1}{2}}(f_{\frac{1}{2}})''(t)$. So Lemma 1.2 implies that

$$0 < u_0(t) \leq v(t) \quad \text{for } \alpha \leq t \leq \beta\,.$$

This inequality yields the assertion of the lemma. ∎

For two real numbers a and p ($p > 0$), let $\mathfrak{F}_{a,p}$ be the totality of real-valued smooth functions f on $\mathbb{R}$ satisfying the following two conditions:

(i) Put $I = [a, \infty)$, then f is positive-valued, strictly increasing and convex on I.

(ii) Define a smooth function f_p on I by $f_p(t) = f(t)^{-p}$, then f_p is also convex on I.

For a given real-valued function f on $\mathbb{R}$, let $\mathcal{S}_f$ be the totality of real-valued solutions u of the differential equation $u''(t) + f(t)^2u(t) = 0$.

Remark 1.2. (i) If $0 < p_1 < p_2$, then $\mathfrak{F}_{a,p_1} \subset \mathfrak{F}_{a,p_2}$ for all a.

(ii) Let f be a smooth function in $\mathfrak{F}_{a,p}$. Then, since f_p is convex on the interval $I = [a, \infty)$, $-f_p'(t) = pf(t)^{-(p+1)}f'(t)$ is non-increasing on I. So, we see that $f'(t) \leq C{\cdot}f(t)^{p+1}$ for $t \geq a$, where $C = f'(a)f(a)^{-(p+1)}$.

2. Properties about zeros of a solution of equation (0.2)

Let f and u be real-valued smooth functions on $\mathbb{R}$ and take a number α_1 such that $u(\alpha_1) = 0$. Suppose that $f \in \mathfrak{F}_{\alpha_1,p}$ and $u \in \mathcal{S}_f$, then Lemma 1.1 implies that u has infinitely many zeros on the interval $[\alpha_1, \infty)$. Since the set of all zeros of u has no accumlation point, we enumerate the totality of zeros of u in $[\alpha_1, \infty)$ in increasing order by $\{\alpha_n \,|\, n = 1, 2, \ldots\}$. Then the following lemma shows the distribution of α_n's.

Lemma 2.1. *Let f, u and $\{\alpha_n\}$ be as above and assume that $f \in \mathfrak{F}_{\alpha_1,\frac{1}{2}}$, then*

$$\int_{\alpha_n}^{\alpha_{n+1}} f(t)dt \leq \pi \quad (\forall\, n) \quad \textit{and} \quad \lim_{n\to\infty}\int_{\alpha_n}^{\alpha_{n+1}} f(t)dt = \pi\,.$$

Proof. By Remark 1.2(ii) above, we see that

$$\text{there is a positive constant } C \text{ such that } f'(t) \le C\cdot f(t)^{\frac{3}{2}} \quad (\forall\, t \ge \alpha_1)\,. \tag{2.1}$$

Since f is convex and strictly increasing, $\lim_{t\to\infty} f(t) = \infty$. Therefore, for each number $a > 1$, there is a constant $T > \alpha_1$ such that

$$f(t) \ge \frac{3C^2}{4(a^2-1)} \qquad (\forall\, t \ge T)\,. \tag{2.2}$$

Now consider zeros of u, then it is seen that $\lim_{n\to\infty} \alpha_n = \infty$, for the set $\{\alpha_n\}$ has no accumlation point. So $\alpha_n \ge T$ for all sufficiently large n. For such an α_n, put $v(t) = a^{-1} f(\alpha_n)^{-\frac{1}{2}} f(t)^{-\frac{1}{2}} \sin(a\int_{\alpha_n}^t f(s)ds)$. Then, a straightforward calculation gives that

$$\begin{gathered} v''(t) + \left[a^2 f(t)^2 - f(t)^{\frac{1}{2}} (f_{\frac{1}{2}})''(t)\right] v(t) = 0\,, \\ v(\alpha_n) = 0\,, \quad v'(\alpha_n) = 1\,. \end{gathered}$$

Moreover, by the convexity of f, (2.1) and (2.2), we see that for $t \ge T$,

$$\begin{aligned} \left[a^2 f(t)^2 - f(t)^{\frac{1}{2}} (f_{\frac{1}{2}})''(t)\right] - f(t)^2 &= (a^2-1) f(t)^2 - \tfrac{1}{4} f(t)^{-2} \left[3f'(t)^2 - 2f(t)f''(t)\right] \\ &\ge \left[(a^2-1) f(t)^4 - \tfrac{3}{4} f'(t)^2\right] f(t)^{-2} \\ &\ge \left[(a^2-1)\cdot \frac{3C^2}{4(a^2-1)} \cdot C^{-2}\cdot f'(t)^2 - \frac{3}{4} f'(t)^2\right] f(t)^{-2} \\ &= 0\,. \end{aligned}$$

Let β be the constant given by $a\int_{\alpha_n}^{\beta} f(s)ds = \pi$, then Lemma 1.2 implies that $0 \le v(t) \le u'(\alpha_n)^{-1} u(t)$ for $\alpha_n \le t \le \beta$. By this inequality, we see that $\beta \le \alpha_{n+1}$, or

$$\int_{\alpha_n}^{\alpha_{n+1}} f(s)ds \ge \frac{\pi}{a} \quad \text{if} \quad \alpha_n > T\,. \tag{2.3}$$

Similarly, Lemma 1.3 gives that $\int_{\alpha_n}^{\alpha_{n+1}} f(s)ds \le \pi$ for all n. This fact and (2.3) yield the assertion of the lemma. ∎

Remark 2.1. Let C be the constant in (2.1), then the above proof shows the following facts for any function $f \in \mathfrak{F}_{\alpha,\frac{1}{2}}$ and $u \in \mathcal{S}_f$.

(i) Let $a > 1$ be a constant and assume that $f(t) \ge 3C^2[4(a^2-1)]^{-1}$ for $t \ge t_0$. Then $\int_{\alpha_1}^{\alpha_2} f(t)dt \ge a^{-1}\pi$ for any two adjacent zeros α_1 and α_2 of u such that $t_0 \le \alpha_1 < \alpha_2$.

(ii) Since $\lim_{t\to\infty} f(t) = \infty$, we have $\lim_{t\to\infty} f(t)^4 f'(t)^{-2} \ge \lim_{t\to\infty} C^{-2}\cdot f(t) = \infty$. So, there is a constant T such that $f(t)^4 f'(t)^{-2} \ge \frac{3}{8}$ for $t \ge T$.

For a positive number A, define a subfamily $\mathcal{F}_{a,p,A}$ of $\mathfrak{F}_{a,p}$ by

$$\mathcal{F}_{a,p,A} = \{ f \in \mathfrak{F}_{a,p} \,|\, f'(t) \geq A \cdot f(t) \text{ for } t \geq a \} .$$

Let f and u be smooth functions on $\mathbb{R}$ and α_1 be a zero of u. Assume that $f \in \mathcal{F}_{a,p,A}$ $(0 < p < \frac{1}{2})$ and $u \in \mathcal{S}_f$. Since $\mathcal{F}_{a,p,A} \subset \mathfrak{F}_{a,\frac{1}{2}}$, we can take a constant $T (> a)$ such that $f(t)^4 f'(t)^{-2} \geq \frac{3}{8}$ for $t \geq T$, by Remark 2.1(ii). Let $\{\alpha_n\}$, $\alpha_n < \alpha_{n+1} (\forall n)$ be the totality of zeros of u in the interval $[T, \infty)$.

Using these numbers, define constants C_0, j_f and t_j $(j = 0, 1, \ldots)$ as follows:

$$\begin{aligned} C_0 &= f'(T) f(T)^{-\frac{3}{2}} , \\ j_f &= \left\lfloor \tfrac{8}{3} C_0^{-2} f(\alpha_1) \right\rfloor , \\ f(t_j) &= \begin{cases} \frac{3}{8} C_0^2 (j + j_f) & \text{if } j > 0 \\ \max\{ f(T), \frac{3}{8} C_0^2 j_f \} & \text{if } j = 0 , \end{cases} \end{aligned}$$

where $\lfloor x \rfloor$ denotes the largest integer which does not exceed x.

Since f is strictly increasing,

$$f(t) \geq C_0^2 f'(T)^{-2} f(T)^3 \cdot f(T) \geq \tfrac{3}{8} C_0^2 \quad \text{for } t \geq T .$$

This implies that $j_f > 0$. Note that t_j's are uniquely determined. It is eaily seen that $t_j < t_{j+1}$ for all j and $\lim_{j \to \infty} t_j = \infty$. Now define a set of integers I_j by $I_j = \{ n \,|\, t_j \leq \alpha_n < t_{j+1} \}$, then the number of elements of I_j is estimated as in the following lemma.

Lemma 2.2. *For a function f in $\mathcal{F}_{a,p,A}$ $(0 < p < \frac{1}{2})$, define I_j as above, then*

$$|I_j| \leq \left\lfloor \frac{3\sqrt{3}\, C_0^2}{4\pi A} \right\rfloor + 1$$

for all $j = 0, 1, \ldots$.

Proof. Suppose that n and $n+1$ are both in I_j. Then, since $\pi / \sqrt{3} \leq \int_{\alpha_n}^{\alpha_{n+1}} f(s) ds = (\alpha_{n+1} - \alpha_n) f(\xi)$ for some $\xi \in (\alpha_n, \alpha_{n+1})$ and f is an increasing function,

$$\begin{aligned} \alpha_{n+1} - \alpha_n &\geq \pi (\sqrt{3}\, f(\xi))^{-1} \\ &\geq \pi (\sqrt{3}\, f(\alpha_{n+1}))^{-1} \\ &\geq \pi (\sqrt{3}\, f(t_{j+1}))^{-1} , \\ \text{so,} \qquad \alpha_{n+1} - \alpha_n &\geq 8\sqrt{3}\, \pi C_0^{-2} \left[9(j + j_f + 1) \right]^{-1} . \end{aligned} \tag{2.4}$$

By definition of t_j's, we see that $\frac{3}{8} C_0^2 = f(t_{j+1}) - f(t_j) = (t_{j+1} - t_j) f'(\xi')$ for some $\xi' \in (t_j, t_{j+1})$. Since $f'(t) \geq A f(t)$ for all $t \geq T$,

$$\begin{aligned} t_{j+1} - t_j &= \tfrac{3}{8} C_0^2 f'(\xi')^{-1} \\ &\leq \tfrac{3}{8} C_0^2 A^{-1} f(\xi')^{-1} \\ &\leq \tfrac{3}{8} C_0^2 A^{-1} f(t_j)^{-1} \end{aligned}$$

so, $$t_{j+1} - t_j \le A^{-1}(j + j_f)^{-1}\,. \tag{2.5}$$

Put $k = |I_j|$ and let n be the least element in I_j, then an upper bound for k is given by (2.4) and (2.5) as follows:

$$\begin{aligned}(k-1)\cdot 8\sqrt{3}\,\pi C_0^{-2}\,[9(j+j_f+1)]^{-1} &\le \sum_{i=1}^{k-1}(\alpha_{n+i} - \alpha_{n+i-1}) \\ &\le t_{j+1} - t_j \\ &\le A^{-1}(j+j_f)^{-1},\end{aligned}$$

that is,

$$k \le 1 + 3\sqrt{3}\,(8\pi)^{-1}C_0^2(j+j_f+1)\cdot A^{-1}(j+j_f)^{-1} \le 1 + 3\sqrt{3}\,C_0^2(4\pi A)^{-1}\,.$$

This implies the assertion of the lemma. ∎

Lemma 2.3. *Let f, u, T and $\{\alpha_n\}$ be as mentioned before the previous lemma. Then the sequence $\{|u'(\alpha_n)|f(\alpha_n)^{-\frac{1}{2}}\}$ is bounded.*

Proof. By Remark 1.2(ii), we see that

$$f'(t) \le C_0\cdot f(t)^{\frac{3}{2}} \quad \text{and} \quad f'(t) \le C_1\cdot f(t)^{p+1} \quad \text{for } t \ge T\,, \tag{2.6}$$

where $C_0 = f'(T)f(T)^{-\frac{3}{2}}$ and $C_1 = f'(T)f(T)^{-(p+1)}$. By Remark 2.1(i), we see that

$$\int_{\alpha_n}^{\alpha_{n+1}} f(s)ds \ge \frac{\pi}{\sqrt{3}} \quad \text{for } n = 1, 2, \ldots\,. \tag{2.7}$$

Now we define j_f, t_j $(j = 0, 1, \ldots)$ and I_j as mentioned before Lemma 2.2. Since α_n and α_{n+1} are adjacent zeros of u, the signatures of $u'(\alpha_n)$ and $u'(\alpha_{n+1})$ are mutually distinct. If $t_j \le \alpha_n < t_{j+1}$, then we obtain the following estimate by using Lemma 1.3:

$$\begin{aligned}0 &> u'(\alpha_n)^{-1}u'(\alpha_{n+1}) \\ &= u'(\alpha_n)^{-1}\left[u'(\alpha_n) + \int_{\alpha_n}^{\alpha_{n+1}} u''(t)dt\right] \\ &= 1 - \int_{\alpha_n}^{\alpha_{n+1}} u'(\alpha_n)^{-1}f(t)^2u(t)dt \\ &\ge 1 - f(\alpha_n)^{-\frac{1}{2}}\int_{\alpha_n}^{\alpha_{n+1}} f(t)^{\frac{3}{2}}\sin(\int_{\alpha_n}^{t} f(s)ds)dt \qquad \text{(by Lemma 1.3)} \\ &= 1 + f(\alpha_n)^{-\frac{1}{2}}\left[f(t)^{\frac{1}{2}}\cos(\int_{\alpha_n}^{t} f(s)ds)\right]_{\alpha_n}^{\alpha_{n+1}}\end{aligned}$$

$$
\begin{aligned}
&\quad -\tfrac{1}{2}f(\alpha_n)^{-\frac{1}{2}}\int_{\alpha_n}^{\alpha_{n+1}} f'(t)f(t)^{-\frac{1}{2}}\cos(\int_{\alpha_n}^{t} f(s)ds)dt \\
&= f(\alpha_n)^{-\frac{1}{2}}f(\alpha_{n+1})^{\frac{1}{2}}\cos(\int_{\alpha_n}^{\alpha_{n+1}} f(s)ds) \\
&\quad -\tfrac{1}{2}f(\alpha_n)^{-\frac{1}{2}}\int_{\alpha_n}^{\alpha_{n+1}} \left(f'(t)f(t)^{-\frac{3}{2}}\right) f(t)\cos(\int_{\alpha_n}^{t} f(s)ds)dt \\
&\geq -f(\alpha_n)^{-\frac{1}{2}}f(\alpha_{n+1})^{\frac{1}{2}} - \tfrac{1}{2}f(\alpha_n)^{-\frac{1}{2}}\left[f'(t)f(t)^{-\frac{3}{2}}\sin(\int_{\alpha_n}^{t} f(s)ds)\right]_{\alpha_n}^{\alpha_{n+1}} \\
&\quad +\tfrac{1}{2}f(\alpha_n)^{-\frac{1}{2}}\int_{\alpha_n}^{\alpha_{n+1}} \left[f''(t)f(t)^{-\frac{3}{2}} - \tfrac{3}{2}f'(t)^2 f(t)^{-\frac{5}{2}}\right]\sin(\int_{\alpha_n}^{t} f(s)ds)dt \\
&\geq -f(\alpha_n)^{-\frac{1}{2}}f(\alpha_{n+1})^{\frac{1}{2}} - \tfrac{1}{2}f(\alpha_n)^{-\frac{1}{2}}f'(\alpha_{n+1})f(\alpha_{n+1})^{-\frac{3}{2}}\sin(\int_{\alpha_n}^{\alpha_{n+1}} f(s)ds) \\
&\quad -\tfrac{3}{4}f(\alpha_n)^{-\frac{1}{2}}\int_{\alpha_n}^{\alpha_{n+1}} f'(t)^2 f(t)^{-\frac{5}{2}}\sin(\int_{\alpha_n}^{t} f(s)ds)dt \qquad (\text{for } f''(t)\geq 0) \\
&\geq -f(\alpha_n)^{-\frac{1}{2}}f(\alpha_{n+1})^{\frac{1}{2}} - \tfrac{1}{2}C_0 f(\alpha_n)^{-\frac{1}{2}}\sin(\int_{\alpha_n}^{\alpha_{n+1}} f(s)ds) \\
&\quad -\tfrac{3}{4}C_1^2 f(\alpha_n)^{-\frac{1}{2}}\int_{\alpha_n}^{\alpha_{n+1}} f(t)^{2p-\frac{3}{2}}f(t)\sin(\int_{\alpha_n}^{t} f(s)ds)dt \qquad (\text{by (2.6)}) \\
&\geq -f(\alpha_n)^{-\frac{1}{2}}f(\alpha_{n+1})^{\frac{1}{2}} - \tfrac{1}{2}C_0 f(\alpha_n)^{-\frac{1}{2}}\sin(\int_{\alpha_n}^{\alpha_{n+1}} f(s)ds) \\
&\quad -\tfrac{3}{4}C_1^2 f(\alpha_n)^{2p-2}\int_{\alpha_n}^{\alpha_{n+1}} f(t)\sin(\int_{\alpha_n}^{t} f(s)ds)dt \qquad (\text{for } 2p-\tfrac{3}{2}<0) \\
&\geq -f(\alpha_n)^{-\frac{1}{2}}f(\alpha_{n+1})^{\frac{1}{2}} - \tfrac{1}{2}C_0 f(\alpha_n)^{-\frac{1}{2}}\sin(\int_{\alpha_n}^{\alpha_{n+1}} f(s)ds) - \tfrac{3}{2}C_1^2 f(\alpha_n)^{2p-2}
\end{aligned}
$$

Put $r_n = u'(\alpha_n)f(\alpha_n)^{-\frac{1}{2}}$, then the above inequality implies that

$$
|r_{n+1}r_n^{-1}| \leq 1 + \tfrac{1}{2}f(\alpha_{n+1})^{-\frac{1}{2}}C_0\sin(\int_{\alpha_n}^{\alpha_{n+1}} f(s)ds) + \tfrac{3}{2}C_1^2 f(\alpha_n)^{2p-\frac{3}{2}}f(\alpha_{n+1})^{-\frac{1}{2}}. \qquad (2.8)
$$

Since $f(t_j) > 3C_0^2\left\{4\left[(1+(j+j_f)^{-1})^2-1\right]\right\}^{-1}$, Remark 2.1(i) gives that

$$
\pi \geq \int_{\alpha_n}^{\alpha_{n+1}} f(s)ds \geq \pi\left[1+(j+j_f)^{-1}\right]^{-1} > \pi\left[1-(j+j_f)^{-1}\right].
$$

Combined with this estimate, (2.6) and (2.8) give that

$$
\begin{aligned}
|r_{n+1}r_n^{-1}| &\leq 1 + \tfrac{1}{2}f(\alpha_{n+1})^{-\frac{1}{2}}C_0\cdot\pi(j+j_f)^{-1} + \tfrac{3}{2}C_1^2 f(\alpha_n)^{2p-\frac{3}{2}}f(\alpha_{n+1})^{-\frac{1}{2}} \\
&\leq 1 + \tfrac{1}{2}f(t_j)^{-\frac{1}{2}}C_0\pi(j+j_f)^{-1} + \tfrac{3}{2}C_1^2 f(t_j)^{2p-2} \\
&= 1 + \tfrac{1}{3}\sqrt{6}\,\pi(j+j_f)^{-\frac{3}{2}} + \tfrac{3}{2}C_1^2(\tfrac{3}{8}C_0^2)^{2p-2}(j+j_f)^{2p-2}.
\end{aligned}
$$

Put $C_2 = \frac{3}{2}C_1^2(\frac{3}{8}C_0^2)^{2p-2} = \frac{3}{2}(\frac{3}{8})^{2p-2}[f'(T)f(T)^{-2}]^{4p-2}$ and

$$p_j = 1 + \tfrac{1}{3}\sqrt{6}\,\pi(j+j_f)^{-\frac{3}{2}} + C_2(j+j_f)^{2p-2}\,, \tag{2.9}$$

then we obtain the following estimate:

$$t_j \le \alpha_n < t_{j+1} \implies |r_{n+1}r_n^{-1}| \le p_j\,. \tag{2.10}$$

For the number of elements of the set I_j, Lemma 2.2 gives the following inequality:

$$|I_j| \le \left\lfloor \frac{3\sqrt{3}\,C_0^2}{4\pi A} \right\rfloor + 1\,.$$

Put $N = \lfloor 3\sqrt{3}\,C_0^2(4\pi A)^{-1}\rfloor + 1$, then for an ineger $n > 1$ in I_k,

$$\begin{aligned}
|r_n r_2^{-1}| &\le \prod_{i=2}^{n-1} |r_{i+1}r_i^{-1}| \\
&\le \left(\prod_{j=0}^{k-1}\prod_{i\in I_j} |r_{i+1}r_i^{-1}|\right)\cdot\left(\prod_{i\in I_k,\, i<n} |r_{i+1}r_i^{-1}|\right) \\
&\le \left(\prod_{j=0}^{k-1} p_j^{|I_j|}\right)\cdot\left(\prod_{i\in I_k,\, i<n} p_k\right) && \text{(by (2.10))} \\
&\le \prod_{j=0}^{k} p_j^N \\
&\le \prod_{j=0}^{\infty} p_j^N && \text{(for } p_j > 1\ (\forall j)) \\
&\le \exp\left(N\cdot\sum_{j=0}^{\infty}\left(\tfrac{1}{3}\sqrt{6}\,\pi(j+j_f)^{-\frac{3}{2}} + C_2(j+j_f)^{2p-2}\right)\right) && \text{(see (2.9))} \\
&\le \exp\left(\left(\sqrt{6}\,\pi + C_2(2-2p)(1-2p)^{-1}\right)N\right)\,.
\end{aligned}$$

So we see that

$$|r_n| \le |r_2|\exp\left(\left(\sqrt{6}\,\pi + C_2(2-2p)(1-2p)^{-1}\right)N\right) \quad \text{for } n > 1\,. \tag{2.11}$$

This inequality (2.11) gives the assertion of the lemma. ∎

Corollary. *Let f, u, $\{\alpha_n\}$, T, N and r_n be as in the previous lemma and define three constants C, T' and M by*

$$\begin{aligned}
C &= \sqrt{6}\,\pi + \frac{3-3p}{1-2p}\cdot\left(\frac{3}{8}\right)^{2p-2}\left[f'(T)f(T)^{-2}\right]^{4p-2}\,, \\
T' &= T + 2\pi f(T)^{-1}\,, \\
M &= \max_{T\le t\le T'} |u(t)|\,.
\end{aligned}$$

Then, $|r_n| \le 2\pi M \exp(CN) f(T')^2 f(T)^{-\frac{3}{2}}$ *for* $n > 1$.

Proof. Since $f(t) \ge f(T) > 0$ for $t \ge T$, it is easily seen that $\alpha_1 - T \le \pi f(T)^{-1}$ and that $\alpha_2 - \alpha_1 \le \pi f(T)^{-1}$. So,

$$T \le \alpha_1 < \alpha_2 \le T' .$$

We can take a constant β in (α_1, α_2) such that $u'(\beta) = 0$, for $u(\alpha_1) = u(\alpha_2) = 0$. For such a β, we see that

$$\begin{aligned} |u'(\alpha_2)| &= \left| \int_\beta^{\alpha_2} u''(t) dt \right| \\ &\le \int_\beta^{\alpha_2} |u''(t)| dt \\ &= \int_\beta^{\alpha_2} f(t)^2 |u(t)| dt \\ &\le \int_T^{T'} f(t)^2 |u(t)| dt \\ &\le \int_T^{T'} M f(T')^2 dt \\ &\le 2\pi M f(T')^2 f(T)^{-1} . \end{aligned}$$

Thus we obtain the following estimate for r_2:

$$\begin{aligned} |r_2| &= |u'(\alpha_2) f(\alpha_2)^{-\frac{1}{2}}| \\ &\le 2\pi M f(T')^2 f(T)^{-1} \cdot f(T)^{-\frac{1}{2}} \\ &= 2\pi M f(T')^2 f(T)^{-\frac{3}{2}} . \end{aligned} \tag{2.12}$$

Together with (2.11) in the proof of Lemma 2.3, inequality (2.12) gives that

$$\begin{aligned} |r_n| &\le |r_2| \exp(CN) \\ &\le 2\pi M f(T')^2 f(T)^{-\frac{3}{2}} \exp(CN) . \end{aligned}$$

This inequality is exactly the assertion of the corollary. ∎

3. Evaluation of decay of solutions of equation (0.2)

Using lemmas in previous two sections, we can determine the order of a solution of differential equation $u''(t) + f(t)^2 u(t) = 0$. The following theorem gives an upper bound of $u(t) f(t)^{\frac{1}{2}}$ explicitly.

Theorem 3.1. *Let f be a smooth function on $\mathbb{R}$ and assume that f is in $\mathcal{F}_{a,p,A}$ with $0 < p < \frac{1}{2}$. Then, for each real-valued solution u of the equation $u''(t) + f(t)^2 u(t) = 0$,*

$$u(t) = O(f(t)^{-\frac{1}{2}}) \text{ as } t \to \infty .$$

More precisely, take $T(> a)$, T', N, C and M as in Lemma 2.3 and in its corollary, then we see that

$$|u(t)| \le MC_f f(t)^{-\frac{1}{2}} \qquad \text{for } t \ge T' ,$$

where $C_f = 2\pi \exp(CN) f(T')^2 f(T)^{-\frac{3}{2}}$.

Proof. By Corollary of Lemma 2.3, it is seen that

$$|u'(\alpha)| f(\alpha)^{-\frac{1}{2}} < 2\pi \exp(CN) M f(T')^2 f(T)^{-\frac{3}{2}} = MC_f ,$$

for any zero $\alpha \ge T'$ of u. Let α be an arbitrary zero of u such that $\alpha \ge T'$ and β the least zero in the interval (α, ∞). Since u has infinitely many zeros on the interval $[T', \infty)$ and the set of zeros of u has no accumlation point, there exists such a β and Lemma 1.3 implies that

$$|u(t)| \le |u'(\alpha)| f(\alpha)^{-\frac{1}{2}} f(t)^{-\frac{1}{2}} \sin(\int_\alpha^t f(s) ds) ,$$

for $\alpha \le t \le \beta$. So, we see that

$$|u(t)| \le MC_f f(t)^{-\frac{1}{2}} \quad \text{for } \alpha \le t \le \beta .$$

Since α is arbitrary, this inequality implies the assertion of the theorem. ∎

Remark 3.1. Put $\mathcal{P} = \{P \in \mathbb{R}[t] \mid P(t) = \sum_{j=0}^n a_j t^j \text{ with } n > 0 \text{ and } a_n > 0\}$, then we have the following facts for the family of functions $\mathcal{F}_{a,p,A}$.

(i) For any $P = \sum_{j=0}^n a_j t^j \in \mathcal{P}$ and $p > 0$, $\exp(P(\cdot)) \in \mathcal{F}_{a,p,a_n}$ for sufficiently large a.

(ii) Let φ be a polynomial in $\mathcal{P}$ with $\deg \varphi \ge 2$ and f be a smooth function in $\mathcal{F}_{a,p,A}$. Then, $\varphi \circ f \in \mathcal{F}_{a',p,A}$ for sufficiently large a'.

(iii) Assume that $f \in \mathcal{F}_{a,p,A}$ and that $\varphi \in \mathcal{F}_{b,q,B}$. Then, for every number $r > q$, $\varphi \circ f \in \mathcal{F}_{c,r,1}$ for sufficiently large c.

4. Examples

Here, two examples are given. The former is concerned with author's work [4], and the latter gives a well-known estimate for Bessel functions but the method used here seems to be quite different from ordinal one. For some notions in representation theory of semisimple Lie groups, see, for example, [5].

Example 1: Relation to embeddings of discrete series representations

Let G be a normal real form of a connected, simply connected, complex simple Lie group of type G_2, $G = KAN$ an Iwasawa decomposition of G and η a character of N. The copmlexified Lie algebra of G is denoted by $\mathfrak{g}$.

Let τ be a finite-dimensional representation of K with representation space V. We define two function spaces $C^\infty(G;\eta)$ and $C^\infty_\tau(G;\eta)$ respectively by

$$C^\infty(G;\eta) = \{f : G \xrightarrow{C^\infty} \mathbb{C} \,|\, f(gn) = \eta(n)^{-1} f(g)\ (\forall g \in G, n \in N)\}\,,$$
$$C^\infty_\tau(G;\eta) = \{f : G \xrightarrow{C^\infty} V \,|\, f(kgn) = \eta(n)^{-1}\tau(k) f(g)\ (\forall g \in G, k \in K, n \in N)\}\,.$$

Define a G-action on $C^\infty(G;\eta)$ by $g{\cdot}f(x) = f(g^{-1}x)$ for $g, x \in G$ and $f \in C^\infty(G;\eta)$, then the space $C^\infty(G;\eta)$ has a $(\mathfrak{g}, K)$-module structure. When considered as a $(\mathfrak{g}, K)$-module, $C^\infty(G;\eta)$ is denoted by $\mathrm{Ind}_N^G(\eta)$.

Let π_Λ be the discrete series representation of G with Harish-Chandra parameter Λ. We denote the Blattner parameter of π_Λ by λ and the irreducible representation of K with heighest weight λ is denoted by τ_λ. Define a differential operator $\mathcal{D}_{\lambda,\eta}$ on $C^\infty_{\tau_\lambda}(G;\eta)$ as in [3, §1]. Then, Theorem 2.4 of H.Yamashita's paper [3] reduces the determination of embedding to a study of solution space $\mathrm{Ker}\,\mathcal{D}_{\lambda,\eta}$. Since discrete series representations are unitary, we are especially interested in their unitary embeddings into (unitarily) induced representations. To determine such embeddings, we consider some conditions, for example square integrability of fuctions in $\mathrm{Ker}\,\mathcal{D}_{\lambda,\eta}$. So, estimate of decay of solutions of the equation $\mathcal{D}_{\lambda,\eta} f = 0$ is of importance in this problem.

In his computation of $\mathrm{Ker}\,\mathcal{D}_{\lambda,\eta}$, the author has encountered a differential equation reduced to the following one (see (4.11) in [4]):

$$u''(t) + e^t u(t) = 0\,. \tag{4.1}$$

Let ψ_j $(j = 1, 2)$ be analytic functions defined by

$$\psi_1(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} t^n\,,$$
$$\psi_2(t) = \sum_{n=0}^{\infty} 2\Big(\sum_{m=1}^{n} \frac{1}{m}\Big) \frac{(-1)^n}{(n!)^2} t^n\,.$$

Using these functions ψ_1 and ψ_2, we define two functions φ_1 and φ_2 by

$$\varphi_1(t) = \psi_1(e^t)\,,$$
$$\varphi_2(t) = \psi_2(e^t) - t\psi_1(e^t)\,.$$

The function ψ_1 is a solution for the differential equation

$$tu''(t) + u'(t) + u(t) = 0\,,$$

and two functions ψ_1 and ψ_2 are related as

$$t\psi_2''(t) + \psi_2'(t) + \psi_2(t) = 2\psi_1'(t)\,.$$

So, functions φ_1 and φ_2 are linearly independent solutions of (4.1).

By Remark 3.1, we see that Theorem 3.1 is applicable for functions φ_1 and φ_2. For these two functions, Theorem 3.1 shows that $\varphi_j = O(e^{-t/4})$ for $j = 1, 2$. Using this estimate, we obtain

$$\begin{aligned}\psi_1(t) &= O(t^{-\frac{1}{4}})\,,\\ \psi_2(t) &= O(t^{-\frac{1}{4}}\log t)\,,\end{aligned}$$

as $t \to \infty$.

Example 2: Estimate of Bessel functions

Denote Bessel function of degree n by J_n as usual, then ψ_1 in Example 1 and J_0 are related as $J_0(x) = \psi_1((\frac{1}{2}x)^2)$. So, our Theorem 3.1 gives that $J_0(x) = O(x^{-\frac{1}{2}})$ as $x \to \infty$. Similarly, by using Thorem 3.1 and the fact that $J_n(e^x)$ is a solution for the equation

$$u''(x) + (e^{2x} - n^2)u(x) = 0\,,$$

we obtain the estimate

$$J_n(x) = O(x^{-\frac{1}{2}}) \text{ as } x \to \infty\,.$$

This estimate for J_n are usually derived from asymptotic expansions of J_n's and it seems that Theorem 3.1 gives a somewhat easier way to obtain those estimates.

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