ASYMPTOTIC DISTRIBUTION OF EIGENVALUES FOR PAULI OPERATORS WITH NONCONSTANT MAGNETIC FIELDS

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Introduction

In the present work we study the asymptotic distribution of discrete eigenvalues near the bottom of essential spectrum for two and three dimensional Pauli operators perturbed by electric potentials falling off at infinity. The special emphasis is placed on the case that the Pauli operators have nonconstant magnetic fields.

The Pauli operator describes the motion of a particle with spin in a magnetic field and it acts on the space $L^2(R^3) \otimes C^2$. The unperturbed Pauli operator is given by

$$H_P = (-i\nabla - A)^2 - \sigma \cdot B$$

under a suitable normalization of units, where $A : R^3 \to R^3$ is a magnetic potential, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ with components

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the vector of 2×2 Pauli matrices and $B = \nabla \times A$ is a magnetic field. We write $(x, z) = (x_1, x_2, z)$ for the coordinates over the three dimensional space $R^3 = R^2_x \times R_z$. Throughout the entire discussion, we suppose that the magnetic field B has a constant direction. For notational brevity, the field is assumed to be directed along the positive z axis, so that B takes the form

$$B(x) = (0, 0, b(x)).$$

Since the magnetic field B is a closed two form, it is easily seen that B is independent of the z variable. We identify $B(x)$ with the function $b(x)$. Let $A(x) = (a_1(x), a_2(x), 0)$ with real function $a_j \in C^1(R^2_x)$ be a magnetic potential

associated with $b(x)$. Then $b(x) = \partial_1 a_2 - \partial_2 a_1$, $\partial_j = \partial/\partial x_j$, and the Pauli operator takes the simple form

$$H_P = \begin{pmatrix} H_+ - \partial_z^2 & 0 \\ 0 & H_- - \partial_z^2 \end{pmatrix},$$

where

$$H_\pm = H_0 \mp b, \quad H_0 = \Pi_1^2 + \Pi_2^2, \quad \Pi_j = -i\partial_j - a_j. \tag{0.1}$$

The magnetic field b is represented as the commutator $b = i[\Pi_2, \Pi_1]$ and hence $H_\pm$ can be rewritten as

$$H_\pm = (\Pi_1 \pm i\Pi_2)^*(\Pi_1 \pm i\Pi_2). \tag{0.2}$$

This implies that $H_\pm \geq 0$ is nonnegative. If, in particular, $b(x) > c > 0$ is positive, then $H_- \geq c$ becomes strictly positive. On the other hand, it is known ([1, 7]) that H_+ has zero as an eigenvalue with infinite multiplicities. We further know (see [3] for example) that the non–zero spectra of operators H_+ and H_- coincide with each other. Thus H_+ has zero as the bottom of its essential spectrum and the bottom is an isolated eigenvalue with inifinite multiplicities.

We first discuss the two dimensional case. We consider the Pauli operator

$$H(V) = H_+ - V = \Pi_1^2 + \Pi_2^2 - b - V \tag{0.3}$$

perturbed by an electric potential $V(x)$. As stated above, the unperturbed operator H_+ has zero as an isolated eigenvalue with infinite multiplicities. If the perturbation $V(x)$ falling off at infinity is added to this operator, then the infinite multipicities are resolved and the above operator $H(V)$ has discrete (positive or negative) eigenvalues accumulating at the origin. We are concerned with how the inifinite multiplicities of zero eigenvalue are resolved. The aim is to study the asymptotic distribution near the origin of such discrete eigenvalues.

We shall formulate the obtained result more precisely. We assume that the magnetic field $b(x) \in C^1(R_x^2)$ fulfills the following assumption : There exists β, $0 < \beta \leq 1$, such that

$$(b) \qquad 1/C \leq b(x) \leq C, \qquad |\nabla b(x)| \leq C \langle x \rangle^{-\beta}$$

for some $C > 1$, where $\langle x \rangle = (1 + |x|^2)^{1/2}$. The constant β is used with the meaning ascribed above throughout the entire discussion. If $V(x)$ is a real bounded function, then the operator $H(V)$ formally defined by (0.3) admits a unique self–adjoint realization in the space $L^2 = L^2(R_x^2)$ with natural domain $\{u \in L^2 : H(V)u \in L^2\}$. We denote by the same notation $H(V)$ this self–adjoint realization. We now mention the first main theorem.

Theorem 1. *Suppose that the above assumption (b) is fulfilled. If a real function $V(x) \in C^1(R^2_x)$ satisfies*

$$|V(x)| \leq C \langle x \rangle^{-m}, \qquad |\nabla V(x)| \leq C \langle x \rangle^{-m-1}, \quad C > 0,$$

for some $m > 0$, then one has :

(i) Let $N(H(V) < -\lambda)$, $\lambda > 0$, denote the number of negative eigenvalues less than $-\lambda$ of operator $H(V)$. Then, for any δ, $0 < \delta \ll 1$, small enough, there exists K_δ such that

$$\begin{aligned} |N(H(V) < -\lambda) - (2\pi)^{-1} \int_{V(x)>\lambda} b(x)\, dx| \\ \leq (2\pi)^{-1} \int_{(1-\delta)\lambda<|V(x)|<(1+\delta)\lambda} b(x)\, dx + \delta\lambda^{-2/m} + K_\delta. \end{aligned}$$

(ii) Let $0 < c < b_0/3$, $b_0 = \inf b(x)$, be fixed and let $N(\lambda < H(V) < c)$, $0 < \lambda < c$, be the number of positive eigenvalues lying in the interval (λ, c) of operator $H(V)$. Then, for any $\delta > 0$ small enough, there exists K_δ such that

$$\begin{aligned} |N(\lambda < H(V) < c) - (2\pi)^{-1} \int_{V(x)<-\lambda} b(x)\, dx| \\ \leq (2\pi)^{-1} \int_{(1-\delta)\lambda<|V(x)|<(1+\delta)\lambda} b(x)\, dx + \delta\lambda^{-2/m} + K_\delta. \end{aligned}$$

As an immediate consequence of Theorem 1, we obtain the folllowing

Corollary. *Let the notations be as above. Assume that $b(x)$ and $V(x)$ fulfill the same assumptions as in Theorem 1. If, in addition, $V(x)$ satisfies the assumption*

$$\limsup_{\lambda \to 0} \lambda^{2/m} \int_{(1-\delta)\lambda<|V(x)|<(1+\delta)\lambda} dx = o(1), \quad \delta \to 0,$$

then one has :

$$N(H(V) < -\lambda) = (2\pi)^{-1} \int_{V(x)>\lambda} b(x)\, dx + o(\lambda^{-2/m}),$$

$$N(\lambda < H(V) < c) = (2\pi)^{-1} \int_{V(x)<-\lambda} b(x)\, dx + o(\lambda^{-2/m})$$

as $\lambda \to 0$.

Next we proceed to the three dimensional case. Let $b(x) \in C^1(R^2_x)$ be again the magnetic field satisfying the assumption (b). We consider the three dimensional perturbed Pauli operator

$$H = H(V) = \Pi_1^2 + \Pi_2^2 - \partial_z^2 - b - V,$$

which acts on the space $L^2(R^3) = L^2(R^2_x \times R_z)$, where $V = V(x, z)$ is a real function decaying at infinity. The essential spectrum of the unperturbed three dimensional Pauli operator without potential V begins at the origin and occupies the whole positive axis. On the other hand, the perturbed operator H has an infinite number of negative eigenvalues accumulating the origin. The second main theorem is formulated as follows.

Theorem 2. *Let $H = H(V)$ be as above. Suppose that the magnetic field $b(x)$ fulfills the assumption (b). If a real function $V(x,z) \in C^1(R^3)$ satisfies*

$$\langle x,z\rangle^{-m}/C \le V(x,z) \le C\,\langle x,z\rangle^{-m}, \quad |\nabla V(x,z)| \le C\,\langle x,z\rangle^{-m-1}, \quad C > 1,$$

for some $m > 0$, where $\langle x,z\rangle = (1+|x|^2+|z|^2)^{1/2}$, then one has :

(i) If $0 < m < 2$, then

$$N(H < -\lambda) = 2(2\pi)^{-2} \int_{V(x,z)>\lambda} b(x)(V(x,z)-\lambda)^{1/2}\,dxdz\,(1+o(1))$$

as $\lambda \to 0$.

(ii) Assume that $m > 2$. Let $w(x)$ be defined as

$$w(x) = \int V(x,z)\,dz,$$

where the integration without domain attached is taken over the whole space. If $w(x)$ fulfills

$$\limsup_{\lambda\to 0} \lambda^{2/(m-1)} \int_{(1+\delta)\lambda > w(x) > (1-\delta)\lambda} dx = o(1), \quad \delta \to 0,$$

then

$$N(H < -\lambda) = (2\pi)^{-1} \int_{w(x)>2\lambda^{1/2}} b(x)\,dx\,(1+o(1)), \quad \lambda \to 0.$$

Remark. The above theorem can be extended to a certain class of potentials with indefinite sign or weak local singularities, although the detailed matters are not discussed here. Such a class of potentials includes the negative Coulomb potential as one of typical examples.

The problem on the asymptotic distribution of eigenvalues for Pauli operators perturbed by electric potentials has been already studied by [8, 11] when $b(x) = b > 0$ is a uniform magnetic field. Both the works make an essential use of the uniformity of magnetic fields and the methods developed there do not seem to apply directly to the case of nonconstant magnetic fields. Roughly speaking, the difficulty arises from the fact that magnetic potentials which actually appear in Pauli operators undergo nonlocal changes even under local changes of magnetic fields. This makes it difficult to control nonconstant magnetic fields by a local approximation of uniform magnetic fields. To prove the two main theorems, some new devices are required in many states of the proof. We also note that the present method may extend to the case of periodic magnetic fields for which the second assumption in (b) is in general violated. We will discuss the matter in detail elsewhere.

Recently several works have been done on the spectral problems of Pauli operators with nonconstant magnetic fields. For example, the Lieb–Thirring inequality for negative eigenvalues has been discussed in [5, 9] and the asymptotic behavior of ground state densities in the strong field limit has been studied in [4]. The present work is motivated by these works.

1. Preliminaries

This is a preparatory section. In this section, we prove simple but useful (upper and lower) bounds on the number of eigenvalues near the bottom of essential spectrum and we briefly explain the perturbation theory for singular numbers of compact operators. These results are used as important tools to prove the main theorems. We also fix several basic notations used throughout the future discussion.

1.1. Bounds on the number of eigenvalues. Let $H_{\pm}$ be as in (0.1) and let $H(V)$ be the two dimensional perturbed operator defined by (0.3). We first establish several useful bounds on the number of eigenvalues near the origin of operator $H(V)$. As previously stated, H_+ has the remarkable spectral property that H_+ has zero, bottom of its spectrum, as an isolated eigenvalue with infinite multiplicities and also the non–zero spectra of operators H_+ and H_- coincide with each other. Let $P : L^2(R^2_x) \to L^2(R^2_x)$ be the eigenprojection associated with the zero eigenspace of H_+. We write Q for $Id - P$, Id being the identity operator. These two notations P and Q are used as basic notations throughout the entire discussion. By assumption (b), $H_- \geq b_0 > 0$, $b_0 = \inf\, b(x)$, is strictly positive and hence we have

$$QH_+Q \geq b_0\, Q \tag{1.1}$$

in the form sense. Let T be a given self–adjoint operator. We also use the notations $N(T < \lambda)$ and $N(T > \lambda)$ to denote the number of eigenvalues less and more than λ of operator T, respectively.

Lemma 1.1. *Let $V(x)$ be a real bounded function decaying at infinity and let $H(V)$ denote the Pauli operator perturbed by potential V. Then $PVP : L^2(R^2_x) \to L^2(R^2_x)$ is a compact operator, and one has :*

$$N(H(V) < -\lambda) \geq N(PVP > \lambda), \quad \lambda > 0,$$

and

$$N(H(V) < -\lambda) \leq N(P(V + V^2/c)P > \lambda) + N(Q(H_+ - V - c)Q < -\lambda)$$

for any $c > 0$. If, in particular, $0 < c < b_0/2$, $b_0 = \inf\, b(x)$, then

$$N(Q(H_+ - V - c)Q < -\lambda) < \infty \tag{1.2}$$

is bounded uniformly in λ, $0 < \lambda \ll 1$, small enough.

Proof. The lemma is easy to prove. The compactness and the first inequality are obvious. To prove the second one, we use the form inequality

$$-PVQ - QVP \geq -c\, Q - PV^2P/c$$

for any $c > 0$. This implies that

$$H(V) \geq Q(H_+ - V - c)Q - P(V + V^2/c)P$$

and hence the second inequality follows at once. If $c < b_0/2$, it follows from (1.1) that $Q(H_+ - c)Q \geq (b_0/2)Q$. The multiplication operator V is relatively compact with respect to H_+. Hence the last statement also follows immediately from the famous Weyl theorem on the invariance of essential spectum under perturbation of compact operators. $\square$

Lemma 1.2. *Let $V(x)$ be as in Lemma 1.1 and let $0 < c < b_0/3$ be as in Theorem 1 (ii). Then, for any $d > 0$ small enough, there exists $g = g_d > 0$ such that*

$$N(\lambda < H(V) < c) \lessgtr N(P(2cV \mp gV^2)P < -2c\lambda + \lambda^2) + N_{\pm d}(\lambda),$$

and the quantities $N_{\pm d}(\lambda)$ on the right side are bounded uniformly in λ, $0 < \lambda \ll 1$, small enough, where $N_{\pm d} = N_{\pm d}(\lambda)$ are given by

$$N_{\pm d} = N(Q((1 \mp d)(H_+ - c)^2 \mp gV^2)Q < (c - \lambda)^2) - N(c \leq H(V) < 2c - \lambda).$$

Proof. Let c be as in the lemma. We define

$$K(V) = (H(V) - c)^2 = H_c^2 - VH_c - H_cV + V^2$$

with $H_c = H_+ - c$. Then

$$N(\lambda < H(V) < c) = N(K(V) < (c - \lambda)^2) - N(c \leq H(V) < 2c - \lambda).$$

By a simple calculation, $PK(V)P = P(c^2 + 2cV + V^2)P$, and also we have the following form inequalities :

$$PK(V)Q + QK(V)P \lessgtr \pm Q(d\,H_c^2 + V^2)Q \pm g\,PV^2P$$
$$QK(V)Q \lessgtr Q((1 \pm d)H_c^2 \pm g\,V^2)Q$$

for some $g = g_d > 0$. These relations imply that

$$K(V) \lessgtr P(c^2 + 2cV \pm g\,V^2)P + Q((1 \pm d)H_c^2 \pm g\,V^2)Q$$

with another $g > 0$, which yields the desired upper and lower estimates for $N(\lambda < H(V) < c)$. The second statement can be verified by the same reasoning as used for proving (1.2). $\square$

1.2. Perturbation theory for singular numbers. We use the perturbation theory for singular numbers of compact operators as another basic tool to

prove the main theorems. We here summarize briefly several important properties of singular numbers. We refer to [6] for details.

Let $T : X \to X$ be a compact (not necessarily self–adjoint) operator acting on a separable Hilbert space X. We write $|T|$ for $\sqrt{TT^*}$. The singular number $\{s_n(T)\}$, $n \in N$, of compact operator T is defined as the non–increasing sequence of eigenvalues of $|T|$ and it has the following properties : $s_n(T) = s_n(T^*)$ and

$$s_{n+m-1}(T_1 + T_2) \leq s_n(T_1) + s_m(T_2) \tag{1.3}$$

for two compact operators T_1 and T_2. We now write

$$N(|T| > \lambda) = \#\{n \in N : s_n(T) > \lambda\}, \quad \lambda > 0,$$

according to the previous notation. If $T : X \to X$ is a compact self–adjoint operator, then it immediately follows that

$$N(|T| > \lambda) = N(T > \lambda) + N(T < -\lambda), \quad \lambda > 0.$$

If, in particular, $T \geq 0$ is nonnegative, then $N(|T| > \lambda) = N(T > \lambda)$. The lemma below is repeatedly used for proving the main theorems.

Lemma 1.3. *Let $\lambda_1, \lambda_2 > 0$ be such that $\lambda_1 + \lambda_2 = \lambda$. Then*

$$N(|T_1 + T_2| > \lambda) \leq N(|T_1| > \lambda_1) + N(|T_2| > \lambda_2).$$

If, in particular, $T_1,\ T_2 \geq 0$ are nonnegative, then one has :

$$\begin{aligned} &N(T_1 + T_2 > \lambda) \leq N(T_1 > (1-\delta)\lambda) + N(T_2 > \delta\lambda), \\ &N(T_1 - T_2 > \lambda) \geq N(T_1 > (1+\delta)\lambda) - N(T_2 > \lambda) - N(T_2 > \delta\lambda) \end{aligned}$$

for any $\delta > 0$ small enough.

Proof. We prove only the third inequality. The other inequalities are an immediate consequence of (1.3). To prove this, we decompose T_1 into $T_1 = (T_1 - T_2) + T_2$ and use (1.3) again. Then

$$\begin{aligned} N(T_1 > (1+\delta)\lambda) &\leq N(|T_1 - T_2| > \lambda) + N(T_2 > \delta\lambda) \\ &= N(T_1 - T_2 > \lambda) + N(T_1 - T_2 < -\lambda) + N(T_2 > \delta\lambda). \end{aligned}$$

Since $N(T_1 - T_2 < -\lambda) \leq N(T_2 > \lambda)$, the inequality in question follows at once. $\square$

1.3. Commutator estimate. For later reference, we further prepare the following

Lemma 1.4. *Let P again denote the eigenprojection associated with the zero eigenspace of H_+. Assume that $V(x) \in C^1(R^2_x)$ satisfies*

$$|V(x)| \le C\,\langle x\rangle^{-m}, \qquad |\nabla V(x)| \le C\,\langle x\rangle^{-m-1}, \quad C > 0,$$

for some $m > 0$. Then the commutator $[P, V] = PV - VP$ takes the form

$$[P, V] = [V, Q] = B\,\langle x\rangle^{-m-1}$$

for some bounded operator B on $L^2 = L^2(R^2_x)$.

Proof. The lemma is easy to prove. We give only a sketch for a proof. Recall that zero is the isolated eigenvalue of H_+. Let Γ be a small circle around the origin in the complex plane. Then the eigenprojection P is represented by the line integral

$$P = \frac{1}{2\pi i}\oint_\Gamma (\zeta - H_+)^{-1}\,d\zeta$$

along the closed curve Γ. Let Π_j, $j = 1, 2$, be as in (0.1) and we write V_j for $\partial_j V = \partial V/\partial x_j$. Then we have

$$[(H_+ - \zeta)^{-1}, V] = (H_+ - \zeta)^{-1}[V, H_+](H_+ - \zeta)^{-1}$$

with

$$[V, H_+] = i\sum_{j=1}^{2}(V_j\Pi_j + \Pi_j V_j).$$

As is easily seen,

$$\Pi_j(H_+ - \zeta)^{-1}, \;\; (H_+ - \zeta)^{-1}\Pi_j : L^2 \to L^2$$

are bounded. By use of a simple inductive argument, we can also show that the operators

$$\langle x\rangle^{-l}(H_+ - \zeta)^{-1}\langle x\rangle^l, \;\; \langle x\rangle^{-l}\Pi_j(H_+ - \zeta)^{-1}\langle x\rangle^l, \;\; \langle x\rangle^{-l}(H_+ - \zeta)^{-1}\Pi_j\langle x\rangle^l$$

are all bounded for nonnegative integer $l \ge 0$. Hence, by interpolation, it follows from assumption that $[P, V]\langle x\rangle^{m+1}$ is bounded. This completes the proof. □

2. Two dimensional case : proof of Theorem 1

In the present section we prove the first main theorem (Theorem 1). The proof is long and it is divided into three steps.

2.1. First step : class of potentials decaying slowly. The first step toward the proof is to prove Theorem 1 for a special class of potentials decaying slowly at infinity.

Proposition 2.1. *Let β, $0 < \beta \leq 1$, be as in assumption (b). Assume that $V(x) \in C^1(R^2_x)$ fulfills*

$$0 \leq V(x) \leq C \langle x \rangle^{-m}, \quad |\nabla V(x)| \leq C \langle x \rangle^{-m-1}, \quad C > 0,$$

for some $m > 0$. If $m < 2\beta/3$, then, for any $\delta > 0$ small enough, there exists $K_\delta > 0$ such that

$$N(H(V) < -\lambda) \lesseqgtr (2\pi)^{-1} \int_{V(x) > (1 \mp \delta)\lambda} b(x)\, dx \pm \delta \lambda^{-2/m} \pm K_\delta.$$

Remark. If a family of potentials $V_z(x)$ fulfills the assumption in the lemma for C independent of parameter z, then the constant K_δ can be chosen uniformly in z. This is used in the later discussion.

The proof of the proposition uses the min–max principle and it is based on the following lemma due to [2].

Lemma 2.2. *Let Q_R be a cube with side R. Let H_B be the Schrödinger operator with constant magnetic field $B > 0$. If we consider H_B under zero Dirichlet boundary conditions over the domain Q_R and denote by $N_D(H_B < \lambda; Q_R)$, $\lambda > 0$, the number of eigenvalues less that λ, then there exists $c > 0$ independent of λ, R and Λ, $0 < \Lambda < R/2$, such that :*

(1) $N_D(H_B < \lambda; Q_R) \leq (2\pi)^{-1} B\, |Q_R|\, f(\lambda/B)$

(2) $N_D(H_B < \lambda; Q_R) \geq (2\pi)^{-1} (1 - \Lambda/R)^2 B\, |Q_R|\, f((\lambda - c\Lambda^{-2})/B),$

where $|Q_R| = R^2$ is the measure of cube Q_R and

$$f(\lambda) = \#\{n \in N_* = N \cup \{0\} : 2n + 1 \leq \lambda\}.$$

Proof of Proposition 2.1. The proof is rather lengthy and is divided into several steps.

(1) We begin by fixing several notations used in the proof. We may assume that $\lambda > 0$ is sufficiently small. By assumption, $m < 2\beta/3$. Hence we can take ρ as

$$1/2 < \rho < \beta/2m - 1/4 < 1/m. \tag{2.1}$$

Let $\{Q_k\}$, $k = (k_1, k_2) \in Z \times Z$, be a family of disjoint open cubes with side $\lambda^{-\rho}$, which covers the plane $R^2 = \cup_{k \in Z \times Z} \overline{Q}_k$, $\overline{Q}_k$ being the closure of Q_k. We denote by $y_k \in R^2$ the center of cube Q_k and by $\tilde{Q}_k \supset Q_k$ another open cube with the same center y_k and side $(1 + \delta)\lambda^{-\rho}$ for $\delta > 0$ small enough. We further introduce a nonnegative smooth partition $\{\varphi_k\}$ subject to the covering above. We can construct the partition in such a way that : (a) $\sum_{k \in Z \times Z} \varphi_k(x)^2 = 1$ over R^2_x ; (b) $\operatorname{supp} \varphi_k \subset \tilde{Q}_k$; (c) $|\nabla \varphi_k(x)| \leq C\, \delta^{-1} \lambda^\rho$ for $C > 0$ independent of k, δ and λ. Then we obtain

$$H(V) = \sum_{k \in Z \times Z} \varphi_k (H(V) - \Phi) \varphi_k \tag{2.2}$$

in the form sense, where $\Phi(x) = \sum_{k\in Z\times Z} |\nabla\varphi_k(x)|^2$. This relation is often called the IMS localization formula ([3]).

(2) We first show the upper estimate. Since $2\,\rho > 1$ by (2.1), we have

$$\Phi(x) = O(\delta^{-2})\lambda^{2\rho} \le \delta\lambda$$

for $\lambda < \lambda_\delta \ll 1$ small enough. Let $H_k(V)$ be the self–adjoint realization of operator $H(V)$ in the space $L^2(\tilde{Q}_k)$ under zero Dirichlet boundary conditions and denote by

$$N_k(\lambda) = N_D(H_k(V) < -\lambda; \tilde{Q}_k)$$

the number of negative eigenvalues less than $-\lambda$ of operator $H_k(V)$. Then it follows from (2.2) that

$$N(H(V) < -\lambda) \le \sum_{k\in Z\times Z} N_k((1-\delta)\lambda). \tag{2.3}$$

(3) We denote by $b_k = b(y_k)$ the value of $b(x)$ at the center $y_k = (y_{1k}, y_{2k})$ of cube Q_k. Recall that $A(x) = (a_1(x), a_2(x))$ is the magnetic potential associated with the field $b(x)$. Let

$$r_{jk}(x) = a_j(x) - A_{jk}(x), \quad A_{jk}(x) = a_j(y_k) + \nabla a_j(y_k)\cdot(x-y_k),$$

for $1 \le j \le 2$ and $k \in Z\times Z$. Then it follows that

$$\partial_1 r_{2k} - \partial_2 r_{1k} = b(x) - b_k$$

and hence we can represent $r_k = (r_{1k}, r_{2k})$ as $r_k(x) = e_k(x) + \nabla g_k(x)$ with some $g_k \in C^1(R^2_x)$, where $e_k = (e_{1k}(x), e_{2k}(x))$ is given by

$$\begin{aligned} e_{1k}(x) &= -\int_0^1 s(x_2 - y_{2k})(b(y_k + s(x-y_k)) - b_k)\,ds, \\ e_{2k}(x) &= \int_0^1 s(x_1 - y_{1k})(b(y_k + s(x-y_k)) - b_k)\,ds. \end{aligned} \tag{2.4}$$

We now define the two operators

$$\begin{aligned} H_{0k} &= (-i\partial_1 - A_{1k})^2 + (-i\partial_2 - A_{2k})^2, \\ T_k(V) &= \exp(-ig_k)H_k(V)\exp(ig_k) \end{aligned}$$

acting on $L^2(\tilde{Q}_k)$ under zero Dirichlet boundary conditions. By definition, H_{0k} has the constant magnetic field b_k, while $T_k(V)$ has $(A_{1k} + e_{1k}, A_{2k} + e_{2k})$ as a magnetic potential and it is unitarily equivalent to $H_k(V)$, so that

$$N_k(\lambda) = N_D(H_k(V) < -\lambda; \tilde{Q}_k) = N_D(T_k(V) < -\lambda; \tilde{Q}_k).$$

(4) We evaluate the quantity $N_k((1-\delta)\lambda)$ on the right side of (2.3). By (2.1), we can choose ν as $(4\rho+1)/2\beta < \nu < 1/m$. Obviously, ν obeys the inequality

$$\rho < (\rho+1)/(m+1) < (\rho+1)/\beta < (4\rho+1)/2\beta < \nu < 1/m. \tag{2.5}$$

Let $X_0 = \{k \in Z \times Z : |y_k| \le \lambda^{-\nu}\}$. We first evaluate $N_k((1-\delta)\lambda)$ with $k \in X_0$. To do this, we further divide $\tilde{Q}_k$ into a family of disjoint open cubues Q_{kl}, $l \in Z \times Z$, with center y_{kl} and constant side d, $1 \le d \le 2$. The total number L of such cubes is bounded by $O(\lambda^{-2\rho})$ uniformly in $k \in X_0$. We denote by $\tilde{Q}_{kl} \supset Q_{kl}$ the cube with the same center y_{kl} and side $3d/2$. For the cube $\tilde{Q}_{kl}$, we define the magnetic potential $e_{kl}(x)$ and the two operators H_{0kl} and $T_{kl}(V)$ as in the previous step. Then we have

$$N_k((1-\delta)\lambda) \le \sum_l N_D(T_{kl}(V) < \gamma; \tilde{Q}_{kl})$$

for some $\gamma > 0$ independent of k, l and λ. This is derived by making use of the same argument as used for proving (2.3). As is easily seen, the magnetic potential $e_{kl}(x)$ is uniformly bounded and hence it follows that

$$T_{kl}(V) \ge H_{0kl}/2 - c$$

in the form sense over the space $L^2(\tilde{Q}_{kl})$ for some $c > 0$ independent of k and l. This, together with Lemma 2.2, yields that

$$N_D(T_{kl}(V) < \gamma; \tilde{Q}_{kl}) = O(1)$$

is uniformly bounded. Thus we have

$$N_k((1-\delta)\lambda) = O(\lambda^{-2\rho}), \quad \lambda \to 0,$$

uniformly in δ. Since the number $\#(X_0)$ of elements in the set X_0 is bounded by $O(\lambda^{-2\nu+2\rho})$, we obtain by (2.5) that

$$\sum_{k \in X_0} N_k((1-\delta)\lambda) = O(\lambda^{-2\nu}) = o(\lambda^{-2/m}). \tag{2.6}$$

If k belongs to the set $X_\infty = \{k \in Z \times Z : |y_k| \ge M\,\lambda^{-1/m}\}$, then we can take $M \gg 1$ so large that $|V(x)| \le \lambda/2$ for $x \in Q_k$ with $k \in X_\infty$. Hence

$$\sum_{k \in X_\infty} N_k((1-\delta)\lambda) = 0. \tag{2.7}$$

(5) Let $M \gg 1$ be as above. We deal with the case

$$k \in X = \{k \in Z \times Z : \lambda^{-\nu} < |y_k| < M\,\lambda^{-1/m}\}.$$

Set $V_k = V(y_k)$. If $k \in X$, then it follows from (2.4) and (2.5) that :

$$|e_k(x)|^2 = O(|y_k|^{-2\beta})|x - y_k|^4 = O(\lambda^{2\nu\beta - 4\rho}) = o(\lambda),$$
$$|b(x) - b_k| = O(|y_k|^{-\beta})|x - y_k| = O(\lambda^{\nu\beta - \rho}) = o(\lambda),$$
$$|V(x) - V_k| = O(|y_k|^{-(m+1)})|x - y_k| = O(\lambda^{\nu(m+1) - \rho}) = o(\lambda)$$

for $x \in \tilde{Q}_k$. We further divide X into two sets

$$X_1 = \{k \in X : V_k > (1 - 2\delta)\lambda\}, \quad X_2 = \{k \in X : V_k \leq (1 - 2\delta)\lambda\}.$$

If $k \in X_2$, then $V(x) < (1 - \delta)\lambda$ for $x \in \tilde{Q}_k$ and hence $N_k((1 - \delta)\lambda) = 0$. On the other hand, if $k \in X_1$, then it follows that

$$T_k(V) \geq (1 - \delta)H_{0k} - b_k - V_k - \delta\lambda$$

for $\lambda < \lambda_\delta \ll 1$ small enough. Set

$$\sigma_k(\lambda) = (b_k + V_k - \lambda + 2\delta\lambda)/(1 - \delta), \quad k \in X_1,$$

for λ as above. Then we have

$$N_k((1 - \delta)\lambda) \leq N(H_{0k} < \sigma_k(\lambda)).$$

Since $\sigma_k(\lambda) < 2\, b_k$, $f(\sigma_k(\lambda)/b_k) \leq 1$, f being as in Lemma 2.2. Hence

$$N_k((1 - \delta)\lambda) \leq (2\pi)^{-1}\, b_k\, |\tilde{Q}_k| = (2\pi)^{-1}\, b_k\, |Q_k|(1 + O(\delta))$$

by Lemma 2.2. The number $\#(X_1)$ of elements in the set X_1 is bounded by $O(\lambda^{-2/m+2\rho})$ and hence

$$(1 + O(\delta)) \sum_{k \in X_1} b_k\, |Q_k| = \sum_{k \in X_1} \int_{Q_k} b(x)\, dx + \delta\, O(\lambda^{-2/m}).$$

As is easily seen,

$$Q_k \subset \{x \in R^2_x : V(x) > (1 - 3\delta)\lambda\}$$

for $k \in X_1$. Thus we obtain that

$$\sum_{k \in X} N_k((1 - \delta)\lambda) \leq (2\pi)^{-1} \int_{V(x) > (1 - 3\delta)\lambda} b(x)\, dx + \delta\, O(\lambda^{-2/m})$$

for λ small enough. This, together with (2.3), (2.6) and (2.7), yields the desired upper bound.

(6) The lower bound can be derived in a similar way. We give only a sketch for it. Let Q_k be as above and let

$$\tilde{N}_k(\lambda) = N_D(\tilde{H}_k(V) < -\lambda; Q_k),$$

where $\tilde{H}_k(V)$ denotes the self–adjoint realization of $H(V)$ in the space $L^2(Q_k)$ under zero Dirichlet boundary conditions. Then we have

$$N(H(V) < -\lambda) \geq \sum_{k \in Z \times Z} \tilde{N}_k(\lambda).$$

The quantity $\tilde{N}_k(\lambda)$ on the right side is evaluated by use of Lemma 2.2 (2) with $\Lambda = \delta\lambda^{-\rho}$, $0 < \delta \ll 1$. This is done in almost the same way as used for proving the upper bound. Thus the proof of the proposition is now completed. $\square$

2.2. Second step : extension to a wider class of potentials. The aim of the second step is to extend Proposition 2.1 to a wider class of potentials decaying not necessarily slowly at infinity. The argument here is based on the following

Proposition 2.3. *Let $u_m(x) = \langle x \rangle^{-m}$ for $m > 0$. Then one has*

$$N(Pu_mP > \lambda) = O(\lambda^{-2/m}), \quad \lambda \to 0.$$

We proceed with the argument, accepting this proposition as proved. The proof of Proposition 2.3 is given at the end of this section (subsection 2.4) after completing the proof of Theorem 1.

Lemma 2.4. *Assume that a real function $V(x) \in C^1(R^2_x)$ fulfills*

$$|V(x)| \leq C \langle x \rangle^{-m}, \quad |\nabla V(x)| \leq C \langle x \rangle^{-m-1}, \quad C > 0,$$

for some $m > 0$. Then one has

$$N(PV^2P > \lambda^2) \geq N((PVP)^2 > \lambda^2)$$

and

$$N(PV^2P > \lambda^2) \leq N((PVP)^2 > (1-\delta)\lambda^2) + C_\delta \lambda^{-2/(m+1)}, \quad C_\delta > 0,$$

for any $\delta > 0$ small enough.

Proof. The proof uses Lemmas 1.3, 1.4 and Proposition 2.3. We decompose the operator PV^2P as

$$PV^2P = (PVP)^2 + PVQVP = T_1^2 + T_2.$$

Since $PV^2P \geq T_1^2$, the lower estimate can be easily obtained. By Lemma 1.3, we have

$$N(PV^2P > \lambda^2) \leq N(T_1^2 > (1-\delta)\lambda^2) + N(T_2 > \delta\,\lambda^2).$$

Let $T = QVP$. Then we assert that

$$N(T_2 > \lambda^2) = N(|T| > \lambda) = O(\lambda^{-2/(m+1)}).$$

If this is shown, then the upper estimate follows at once. To prove the assertion above, we use Lemma 1.4. According to this lemma, the operator T takes the form

$$T = [Q, V]P = B\, u_{m+1} P$$

with some bounded operator B. Since the singular number $s_n(T)$ satisfies $s_n(T) \le \|B\|\, s_n(u_{m+1}P)$, we have

$$N(|T| > \lambda) \le N(Pu_{2(m+1)}P > c\lambda^2)$$

for some $c > 0$. This, together with Proposition 2.3, proves the assertion in question and the proof is complete. □

Lemma 2.5. *Assume that $V(x) \in C^1(R^2_x)$ fulfills the same assumption as in Proposition 2.1. If $m < 2\beta/3$ again, then*

$$N(PVP > \lambda) \lesseqgtr (2\pi)^{-1} \int_{V(x)>(1\mp\delta)\lambda} b(x)\, dx \pm \delta\lambda^{-2/m} \pm K_\delta$$

for any $\delta > 0$ small enough.

Proof. The lemma is easy to prove. Since $N(H(V) < -\lambda) \ge N(PVP > \lambda)$ by Lemma 1.1, Proposition 2.1 implies the upper bound. By Lemma 1.1 again, we have

$$N(H(V) > -\lambda) \le N(P(V + V^2/c)P > \lambda) + N(Q(H_+ - V - c)Q < -\lambda).$$

If $0 < c < b_0/2$ for $b_0 = \inf b(x)$, then the second quantity on the right side is bounded uniformly in λ. We apply Lemma 1.3 with $T_1 = PVP$ and $T_2 = PV^2P/c$ to the first quantity. Then Propositions 2.1 and 2.3 yield the lower bound. Thus the proof is completed. □

Lemma 2.6. *Assume that $V(x) \in C^1(R^2_x)$ fulfills*

$$\langle x\rangle^{-m}/C \le V(x) \le C\langle x\rangle^{-m}, \quad |\nabla V(x)| \le C\langle x\rangle^{-m-1}, \quad C > 1,$$

for some $m > 0$. Then, for any $\delta > 0$ small enough, there exists K_δ such that

$$N(PVP > \lambda) \lesseqgtr (2\pi)^{-1} \int_{V(x)>(1\mp\delta)\lambda} b(x)\, dx \pm \delta\lambda^{-2/m} \pm K_\delta.$$

Proof. We first note that if $V(x)$ satisfies the assumption above for $m > 0$, then $V(x)^{1/2}$ also satisfies the same assumption for m replaced by $m/2$. If we make repeated use of Lemma 2.4, the proof is reduced to the case of potentials falling off slowly at infinity. Thus the lemma immediately follows from Lemma 2.5. □

By approximation, the lemma above can be shown to remain true for a slightly wider class of potentials.

Lemma 2.7. *Assume that $V(x) \in C^1(R^2_x)$ fulfills*

$$0 \le V(x) \le C\,\langle x\rangle^{-m}, \quad |\nabla V(x)| \le C\,\langle x\rangle^{-m-1}, \quad C > 0,$$

for some $m > 0$. Then the same result as in Lemma 2.6 is still valid for such a class of potentials.

Remark. (see the remark after Proposition 2.1.) If a family of potentials $V_z(x)$ fulfills the assumption in the above lemma for $C > 0$ independent of parameter z, then the constant K_δ can be chosen uniformly in z. This is used for proving Theorem 2.

Proof. Let $V(x)$ be as in the lemma. We set $V_\epsilon(x) = V(x) + \epsilon\langle x\rangle^{-m}$ for $\epsilon > 0$ small enough. Then $V_\epsilon(x)$ fulfills the assumption in Lemma 2.6 and

$$V(x) \le V_\epsilon(x) \le V(x) + 2\epsilon\,\langle x\rangle^{-m}. \tag{2.8}$$

We first evaluate the upper bound. By (2.8),

$$N(PVP > \lambda) \le N(PV_\epsilon P > \lambda)$$

and hence it follows from Lemma 2.6 that

$$N(PVP > \lambda) \le (2\pi)^{-1} \int_D b(x)\,dx + \delta\lambda^{-2/m} + K_\delta, \quad K_\delta = K_\delta(\epsilon),$$

for any $\delta > 0$ small enough, where

$$D = \{x \in R^2_x : V_\epsilon(x) > (1-\delta)\lambda\}.$$

We now take $\epsilon = \delta^2$ and divide D into two domains

$$D_1 = \{x \in D : V(x) > \delta\,\langle x\rangle^{-m}\}, \quad D_2 = \{x \in D : V(x) \le \delta\,\langle x\rangle^{-m}\}.$$

If $x \in D_2$, then it follows from (2.8) that

$$(1-\delta)\lambda < V_\epsilon(x) \le V(x) + 2\delta^2\langle x\rangle^{-m} \le (\delta + 2\delta^2)\langle x\rangle^{-m}$$

with $\epsilon = \delta^2$, and hence $|x| \le \delta^{1/m}\,O(\lambda^{-1/m})$. Thus

$$\int_{D_2} dx = \delta^{2/m} O(\lambda^{-2/m}).$$

On the other hand, if $x \in D_1$, then

$$(1-\delta)\lambda < V_\epsilon(x) \le V(x) + 2\delta^2\langle x\rangle^{-m} \le (1+2\delta)V(x).$$

This implies that $V(x) > (1-4\delta)\lambda$ for $x \in D_1$, and hence

$$\int_{D_1} b(x)\,dx \le \int_{V(x)>(1-4\delta)\lambda} b(x)\,dx.$$

Thus the desired upper bound is obtained.

The lower bound is also proved in a similar way. Let $u_m(x) = \langle x\rangle^{-m}$ again. If we use Lemma 1.3, then it follows from (2.8) that

$$\begin{aligned} N(PV_\epsilon P > \lambda) \le & N(P(V+2\epsilon u_m)P > \lambda) \\ \le & N(PVP > (1-\delta)\lambda) + N(Pu_mP > \delta\lambda/2\epsilon) \end{aligned}$$

for any $\delta > 0$ small enough. We take $\epsilon = \delta^2$ again. Then we have by Proposition 2.3 that

$$N(Pu_mP > \delta\lambda/2\epsilon) = \delta^{2/m}\,O(\lambda^{-2/m}).$$

We further apply Lemma 2.6 to the quantity $N(PV_\epsilon P > \lambda)$ on the left side. Since $V_\epsilon(x) \ge V(x)$, the desired lower bound can be easily obtained. Thus the proof is complete. □

2.3. Third step : completion of proof. In this step we complete the proof of Theorem 1. We further require the following lemma to complete the proof.

Lemma 2.8. *Let $V(x) \in C^1(R^2_x)$ satisfy the assumption in Theorem 1 for $m > 0$ and let $V_\pm(x) = \max(\pm V(x), 0)$. Then one has :*

$$\begin{aligned} N(PV_+P > \lambda) &\lessgtr (2\pi)^{-1} \int_{V(x)>(1\mp\delta)\lambda} b(x)\,dx \pm \delta\lambda^{-2/m} \pm K_\delta, \\ N(PV_-P > \lambda) &\lessgtr (2\pi)^{-1} \int_{V(x)<-(1\mp\delta)\lambda} b(x)\,dx \pm \delta\lambda^{-2/m} \pm K_\delta \end{aligned}$$

for any $\delta > 0$ small enough.

Proof. The proof is done in almost the same way as in the proof of Lemma 2.7. We approximate $V_+(x)$ by a smooth potential. Let

$$W_\epsilon(x) = \epsilon^2\langle x\rangle^2 \int \varphi((x-y)/\epsilon\langle x\rangle)(V_+(y) + \epsilon\langle y\rangle^{-m})\,dy$$

for $\epsilon > 0$ small enough, where $\varphi \in C_0^\infty(R^2_x)$ is a nonnegative smooth function with support in the unit ball with the origin at center, and $\int \varphi(x)\,dx = 1$. If $x \in R^2$ satisfies $|V(x)| \ge M\epsilon\,\langle x\rangle^{-m}$ for some $M \gg 1$, then $V_+(y)$ always equals $V(y)$ or 0 in the region $\{y : |y-x| < \epsilon\langle x\rangle\}$. Hence we have

$$|V_+(x) - W_\epsilon(x)| \le L\epsilon\,\langle x\rangle^{-m}, \quad L \ge 1,$$

for x as above. On the other hand, if x satisfies $|V(x)| < M\epsilon \langle x\rangle^{-m}$, then it follows that

$$|V_+(x) - W_\epsilon(x)| \le |V(x)| + W_\epsilon(x) \le L\epsilon \langle x\rangle^{-m}$$

with another $L > 1$. In any case, we have $|V_+(x) - W_\epsilon(x)| \le L\epsilon \langle x\rangle^{-m}$ for some $L > 1$. We now set $V_\epsilon(x) = W_\epsilon(x) + L\epsilon \langle x\rangle^{-m}$. Then $V_\epsilon(x) \in C^1(R^2_x)$ satisfies the assumption in Lemma 2.6 and it obeys

$$V_+(x) \le V_\epsilon(x) \le V_+(x) + 2L\epsilon \langle x\rangle^{-m}.$$

This enables us to repeat the same argument as in the proof of Lemma 2.7 and we obtain the desired upper and lower bounds for $N(PV_+P > \lambda)$. The same argumet applies to $N(PV_-P > \lambda)$ also. Thus the proof is complete. $\square$

Proof of Theorem 1. (i) Since $|V(x)|^2 = O(|x|^{-2m})$ as $|x| \to \infty$, we combine Lemmas 1.1, 1.3 and Proposition 2.3 to obtain that

$$N(PVP > \lambda) \le N(H(V) < -\lambda) \le N(PVP > (1-\delta)\lambda) + C_\delta \lambda^{-1/m}$$

for any $\delta > 0$ small enough. Thus the proof is reduced to evaluating the upper and lower bounds on $N(PVP > \lambda)$. Since $N(PVP > \lambda) \le N(PV_+P > \lambda)$, the upper bound follows from Lemma 2.8. To prove the lower bound, we make use of Lemma 2.4 to obtain that

$$\begin{aligned} N(PV^2P > (1+\delta)\lambda^2) &\le N((PVP)^2 > \lambda^2) + C_\delta \lambda^{-2/(m+1)} \\ &= N(PVP > \lambda) + N(P(-V)P > \lambda) + C_\delta \lambda^{-2/(m+1)}. \end{aligned}$$

Hence we have

$$N(PVP > \lambda) \ge N(PV^2P > (1+\delta)\lambda^2) - N(PV_-P > \lambda) - C_\delta \lambda^{-2/(m+1)}.$$

The bounds on the two quantities $N(PV^2P > \lambda^2)$ and $N(PV_-P > \lambda)$ on the right side have already been established in Lemmas 2.7 and 2.8, respectively. Thus the lower bound is also obtained by combining these two lemmas.

(ii) To prove the second statement, we make use of Lemma 1.2 in place of Lemma 1.1. If we again use Lemma 1.3 and Proposition 2.3, then the proof is reduced to evaluating the upper and lower bounds on $N(PVP < -\lambda) = N(P(-V)P > \lambda)$. This is done in exactly the same way as in the proof of the first statement (i). $\square$

2.4. Proof of Proposition 2.3. It remains to prove Proposition 2.3, which has played an important role in the above proof of Theorem 1. We here give the proof of this proposition. We start by preparing two lemmas.

Lemma 2.9. *Let $\alpha,\ K > 0$ and let $f(\lambda)$ be a monotone decreasing nonnegative function over $(0,\infty)$. If $f(\lambda)$ satisfies*

$$f(\lambda) \le C_\epsilon\, \lambda^{-\alpha} + K\, f(\lambda/\epsilon), \quad C_\epsilon > 0,$$

for any $\epsilon > 0$ small enough, then

$$f(\lambda) = O(\lambda^{-\alpha}), \qquad \lambda \to 0.$$

Proof. The lemma is easy to prove. By repeated use of the inequality in the assumption, we have

$$f(\lambda) \le C_\epsilon(1 + K\epsilon^\alpha + \ldots + K^n\epsilon^{n\alpha})\lambda^{-\alpha} + K^{n+1} f(\lambda/\epsilon^{n+1}).$$

We now fix ϵ so small that $K\epsilon^\alpha < 1/2$. If $\epsilon^n \le \lambda < \epsilon^{n-1}$ for some integer $n \gg 1$, then $K^{n+1} < K^2\epsilon^{-\alpha(n-1)} \le K^2\lambda^{-\alpha}$ and also we have

$$f(\lambda) \le (2C_\epsilon + K^2\, f(1/\epsilon))\lambda^{-\alpha}$$

uniformly in $n \gg 1$. This proves the lemma. □

Lemma 2.10. *Let $\chi(x)$ be a bounded nonnegative function with compact support. Then one has*

$$N(P\chi P > \lambda) = O(\lambda^{-\epsilon}), \qquad \lambda \to 0,$$

for any $\epsilon > 0$ small enough.

The lemma above plays a basic role in proving Proposition 2.3. We first complete the proof of the proposition, accepting this lemma as proved.

Proof of Proposition 2.3. The proof is done by induction. We use Proposition 2.1 with $V = u_m$. Then it follows from Lemma 1.1 that

$$N(PVP > \lambda) \le N(H(V) < -\lambda) = O(\lambda^{-2/m}), \quad V = u_m,$$

for $0 < m < 2\beta/3$. Thus the proposition is true for $0 < m \ll 1$ small enough. We now assume the case $m = k$ as an inductive assumption, and we shall prove that

$$N(Pu_{2k}P > \lambda) = O(\lambda^{-2/2k}). \tag{2.9}$$

If this is shown, then the proposition follows by induction.

To prove (2.9), we decompose $Pu_{2k}P$ into

$$Pu_{2k}P = (Pu_kP)^2 + Pu_kQu_kP.$$

Using Lemma 1.4, we repeat the same commutator computation as in the proof of Lemma 2.4. Then we obtain that

$$N(Pu_{2k}P > \lambda) \le N(Pu_kP > (\lambda/2)^{1/2}) + N(Pu_{2(k+1)}P > c\lambda)$$

for some $c > 0$, and hence

$$N(Pu_{2k}P > \lambda) \le O(\lambda^{-1/k}) + N(Pu_{2(k+1)}P > c\lambda)$$

by the inductive assumption. We now denote by $\chi_R(x)$ the characteristic function of the ball with the origin at center and radius R. Take $R = R_\epsilon$ so large that

$$u_{2(k+1)} \le \chi_R + c\,\epsilon\, u_{2k}/2$$

for $\epsilon > 0$ small enough, $c > 0$ being as above. Then it follows from Lemma 1.3 that

$$N(Pu_{2(k+1)}P > c\lambda) \le N(P\chi_R P > c\lambda/2) + N(Pu_{2k}P > \lambda/\epsilon).$$

By Lemma 2.10, the first quantity on the right side is of order $O(\lambda^{-1/k})$, where the order estimate depends on $R = R_\epsilon$. If we set $f(\lambda) = N(Pu_{2k}P > \lambda)$, then we have

$$f(\lambda) \le C_\epsilon \lambda^{-1/k} + f(\lambda/\epsilon)$$

for some $C_\epsilon > 0$. Hence Lemma 2.9 yields (2.9) and the proof is complete. □

We proceed to the proof of Lemma 2.10. This is proved through a series of lemmas.

Lemma 2.11. *Let $b(x) \in C^1(R^2_x)$ be a bounded real function. Then there exists a real solution $\varphi(x) \in C^2(R^2_x)$ to equation $\Delta\,\varphi = b$ with bound*

$$|\varphi(x)| = O(\exp(c|\log r|^2)), \qquad r = |x| \to \infty,$$

for some $c > 0$.

Proof. We work in the complex plane $C \simeq R^2_x$. We write $b(\zeta)$, $\zeta = x_1 + ix_2$, for $b(x)$ and decompose $b(\zeta)$ into the sum $b(\zeta) = \sum_{j=0}^{\infty} b_j(\zeta)$, where b_j is a C^1–smooth function having support in

$$\{\zeta \in C : R_{j-1} < |\zeta| < R_{j+1}\}, \quad R_j = 2^j, \;\; R_{-1} = -\infty.$$

We define $f_j(\zeta)$ by

$$f_j(\zeta) = (1/\pi) \int b_j(\xi)/(\zeta - \xi)\, dm(\xi),$$

where $dm(\xi)$ is the Lebesgue measure over the complex plane. Then f_j is a solution to the $\overline{\partial}$ equation

$$\overline{\partial}\, f_j = (1/2)(\partial_1 + i\partial_2) f_j = b_j$$

and it satisfies $|f_j(\zeta)| \le c\,2^{2j}$. Let $j > L$, $L \gg 1$ being fixed. We expand f_j into $f_j(\zeta) = \sum_{k=0}^{\infty} a_{kj}\,\zeta^k$ for $|\zeta| < R_{j-2}$, where

$$a_{kj} = -(1/\pi) \int b_j(\xi)/\xi^{k+1}\, dm(\xi).$$

These coefficients a_{kj} obey the bound

$$|a_{kj}| \le c\,2^{-j(k-1)+k}$$

for some $c > 0$, because $|\xi|^{-k-1} \le 2^{-(j-1)(k+1)}$ on the support of b_j. We set

$$p_j(\zeta) = \sum_{k=0}^{2j} a_{kj}\zeta^k, \quad j > L.$$

Then $p_j(\zeta)$ is analytic over the whole plane and it satisfies $\overline{\partial}\, p_j = 0$. We make use of the above bounds on a_{kj} to obtain that

$$|f_j(\zeta) - p_j(\zeta)| \le \begin{cases} c\,2^{-j}, & |\zeta| < R_{j-2}, \\ c\,2^{2j(l-j+1)}, & R_{l-2} \le |\zeta| < R_{l-1}, \quad l \ge j, \end{cases}$$

for another $c > 0$. If $0 \le j \le L$, we define $p_j(\zeta)$ as $p_j(\zeta) = 0$. Then the series

$$g(\zeta) = \sum_{j=0}^{\infty} (f_j(\zeta) - p_j(\zeta))$$

is convergent locally uniformly and it satisfies the equation $\overline{\partial}\, g = b$. It can be easily shown that $g(\zeta)$ obeys the bound

$$|g(\zeta)| = O(\exp(c|\log r|^2)), \qquad r = |\zeta| \to \infty.$$

We now write $g(\zeta) = g_1(\zeta) - ig_2(\zeta)$, g_j being a real function. Since $b(\zeta)$ is real valued, it follows by equation $\overline{\partial}\, g = b$ that

$$\partial_1 g_1 + \partial_2 g_2 = 2b, \quad \partial_1 g_2 - \partial_2 g_1 = 0.$$

Hence, if we define

$$\varphi(x) = (1/2)\Big(\int_0^{x_1} g_1(t,0)\,dt + \int_0^{x_2} g_2(x_1,t)\,dt\Big),$$

then $\varphi(x)$ satifies

$$\partial_1 \varphi = g_1/2, \quad \partial_2 \varphi = g_2/2, \quad \Delta\varphi = b.$$

It is also easily seen that $\varphi(x)$ obeys the bound in the lemma. Thus the desired solution is constructed and the proof is complete. $\square$

We here introduce some new notations. Let $\mathcal{A}(C)$ be the class of analytic functions over the complex plane C. For given real function $\psi(x) \in C^2(R_x^2)$, we define the subspace $\mathcal{K}_\psi(R_x^2)$ of $L^2(R_x^2)$ by

$$\mathcal{K}_\psi(R_x^2) = \{u \in L^2(R_x^2) : u = he^{-\psi} \text{ with } h \in \mathcal{A}(C)\}$$

and we denote by P_ψ the orthogonal projection on $\mathcal{K}_\psi(R_x^2)$. The argument below shows that such a subspace is closed.

Let $\varphi(x)$ be as in Lemma 2.11. By construction in the lemma, we can choose a magnetic potential $(a_1(x), a_2(x))$ associated with the field $b(x)$ to be divergenceless

$$a_1(x) = -\partial_2\varphi(x), \qquad a_2(x) = \partial_1\varphi(x).$$

Then we have

$$\Pi_1 + i\Pi_2 = -ie^{-\varphi}(\partial_1 + i\partial_2)e^{\varphi}$$

by a simple calculation. This, together with relation (0.2), implies that the zero eigenspace of H_+ just coincides with $\mathcal{K}_\varphi(R_x^2)$ and hence the eigenprojection P is realized as the projection P_φ on $\mathcal{K}_\varphi(R_x^2)$.

Lemma 2.12. *Let $\psi_j(x) \in C^2(R_x^2)$, $1 \le j \le 2$, be a real function and let $\chi(x)$ be as in Lemma 2.10, χ being of compact support. Write $\mathcal{K}_j$ and P_j for $\mathcal{K}_{\psi_j}(R_x^2)$ and P_{ψ_j}, respectively. If $\psi_1(x) \le \psi_2(x)$, then one has*

$$N(P_1\chi P_1 > \lambda) \le N(P_2\chi P_2 > \lambda/\gamma),$$

where

$$\gamma = \max_{x \in \mathrm{supp}\,\chi} \exp(2\theta(x)), \qquad \theta(x) = \psi_2(x) - \psi_1(x) \ge 0.$$

Proof. The lemma is proved by a simple application of the min–max principle. Let $\mathcal{L}_n$, $n \ge 0$, be the totality of subspaces in $L^2 = L^2(R_x^2)$ with dimension not exceeding n. Each element of $\mathcal{L}_n$ is expressed as a subspace $F_n = S(f_1, \ldots, f_n)$ spanned by the linear hulls of $f_j \in L^2$, $1 \le j \le n$. We define the injective mapping $\sigma : \mathcal{L}_n \to \mathcal{L}_n$ by

$$\sigma(F_n) = G_n = S(g_1, \ldots, g_n)$$

with $g_j = \exp(-\theta)f_j \in L^2$ and we denote the range of such a mapping as $\mathcal{M}_n = \sigma(\mathcal{L}_n)$. Let $w = h\exp(-\psi_1) \in \mathcal{K}_1$ with $h \in \mathcal{A}(C)$. If w belongs to the orthogonal complement $G_n^\perp$ of $G_n = \sigma(F_n)$ with $F_n = S(f_1, \ldots, f_n)$, then $\tilde{w} = h\exp(-\psi_2) \in \mathcal{K}_2$ with the same $h \in \mathcal{A}(C)$ belongs to $F_n^\perp$, and also it follows that $(\tilde{w}, \tilde{w}) \le (w, w)$ and

$$(\chi w, w)/(w, w) \le \gamma\,(\chi\tilde{w}, \tilde{w})/(\tilde{w}, \tilde{w}),$$

where (,) stands for the scalar product in the space L^2. Thus we have

$$\min_{G_n \in \mathcal{M}_n} \max_{u \in \mathcal{K}_1 \cap G_n^\perp} (\chi u, u)/(u, u) \leq \gamma \min_{F_n \in \mathcal{L}_n} \max_{v \in \mathcal{K}_2 \cap F_n^\perp} (\chi v, v)/(v, v). \tag{2.10}$$

We now denote by $\lambda_n(T)$ the n–th eigenvalue of nonnegative compact operator T. By the min–max principle,

$$\lambda_{n+1}(P_j \chi P_j) = \min_{F_n \in \mathcal{L}_n} \max_{u \in \mathcal{K}_j \cap F_n^\perp} (\chi u, u)/(u, u)$$

for $n \geq 0$. Hence, if we take account of the inclusion relation $\mathcal{M}_n \subset \mathcal{L}_n$, the lemma follows from (2.10). □

Lemma 2.13. *Let $\chi(x)$ be again as in Lemma 2.10 and let $\psi = \psi(r) \in C^\infty(R_x^2)$ be a spherically symmetric real function. Define $q(r)$ as*

$$q(r) = \Delta\psi = \psi''(r) + \psi'(r)/r.$$

Assume that $q(r)$ satsifies the following four assumptions :

$$q(r) > 0, \quad \int_0^\infty r q(r)\, dr = \infty, \quad q'(r) > 0, \quad \lim_{r \to \infty} q'(r)/q(r)^{3/2} = 0.$$

Then one has

$$N(P_\psi \chi P_\psi > \lambda) = O(\lambda^{-\epsilon}), \qquad \lambda \to 0,$$

for any $\epsilon > 0$ small enough.

We shall first show that the above lemma readily implies Lemma 2.10 in question.

Proof of Lemma 2.10. To prove the lemma, we apply Lemma 2.13 with $\psi(r) = \exp(c(r^2 + 1))$ for $c \gg 1$ large enough. Let $\varphi(x)$ be as in Lemma 2.11. Then $\varphi(x) \leq \psi(r)$. We can easily check that $q(r) = \Delta\psi$ fulfills all the assumptions in Lemma 2.13. Hence we combine Lemmas 2.12 and 2.13 to obtain the desired conclusion. □

Proof of Lemma 2.13. It suffices to prove the lemma for the characteristic function $\chi_R(r)$ of the ball with the origin at center and radius R. Let $p(r) = r\psi'(r)$. If we take account of relation $p'(r) = rq(r)$, then it follows by assumption that

$$p(r) = r\psi'(r) = \int_0^r s q(s)\, ds \to \infty, \quad r \to \infty,$$

and hence $\psi(r)/\log r \to \infty$. Thus we have

$$\int_0^\infty r^n \exp(-2\psi(r))\, dr < \infty$$

for all nonnegative integers $n \in N_* = N \cup \{0\}$. This implies that $\mathcal{K}_\psi(R_x^2)$ is spanned by a family of orthogonal functions

$$u_n(x) = (x_1 + ix_2)^n \exp(-\psi(r)) = r^n \exp(in\theta) \exp(-\psi(r)), \quad n \in N_*,$$

where (r, θ) denotes the polar coordinates over the plane. We define

$$\begin{aligned} e_n =& (u_n, u_n) = 2\pi \int_0^\infty r^{2n+1} \exp(-2\psi(r))\, dr, \\ \alpha_n =& (\chi_R u_n, u_n) = 2\pi \int_0^\infty \chi_R(r) r^{2n+1} \exp(-2\psi(r))\, dr \end{aligned}$$

for $n \in N_*$. The operator $P_\psi \chi_R P_\psi$ is realized as an infinite diagonal matrix with α_n/e_n as diagonal components, so that this operator has $\lambda_n = \alpha_n/e_n$ as eigenvalues. We analyse the behavior as $n \to \infty$ of λ_n.

First it is easy to see that

$$\alpha_n \le 2\pi \int_0^R r^{2n+1}\, dr = O(e^{cn}), \quad n \to \infty, \tag{2.11}$$

for some $c > 0$. We rewrite e_n as

$$e_n = 2\pi \int_0^\infty \exp(2f_n(r))\, dr,$$

where

$$f_n(r) = (n + 1/2) \log r - \psi(r).$$

We look at the stationary point of $f_n(r)$ with

$$f_n'(r) = ((n + 1/2) - p(r))/r = 0, \quad p(r) = r\psi'(r).$$

Since $p(r)$ is a strictly increasing function, there exists a unique root τ_n to equation

$$p(\tau_n) = \int_0^{\tau_n} sq(s)\, ds = n + 1/2. \tag{2.12}$$

As is easily seen, $\tau_n \to \infty$ as $n \to \infty$.

We estimate the values $q(r)$ and $f_n(r)$ at the stationary point $r = \tau_n$ to study the behavior as $n \to \infty$ of e_n. By assumption,

$$q(r) = o(\int_0^r q(s)^{3/2}\, ds) = q(r)^{1/2} o(\int_0^r sq(s)\, ds), \quad r \to \infty,$$

and hence it follows from (2.12) that

$$q(\tau_n) = o(n^2), \qquad n \to \infty. \tag{2.13}$$

By a simple calculation, we have

$$\psi(r) = p(r)\log r - \int_0^r sq(s)\log s\, ds + \psi(0).$$

Hence it again follows from (2.12) that

$$f_n(\tau_n) = \int_0^{\tau_n} sq(s)\log s\, ds - \psi(0).$$

This implies that

$$\lim_{n\to\infty} f_n(\tau_n)/n = \infty. \tag{2.14}$$

We now evaluate the lower bound on e_n. Since $\psi''(r) = -\psi'(r)/r + q(r)$, $f_n''(r)$ is calculated as

$$f_n''(r) = (p(r) - (n+1/2))/r^2 - q(r).$$

By assumption, $\lim_{r\to\infty} q'(r)/q(r)^{3/2} = 0$ and hence it follows that $1/c \le q(r)/q(\tau_n) \le c$ for some $c > 1$, if $|r - \tau_n| < q(\tau_n)^{-1/2}$. We obtain

$$|p(r) - (n+1/2)| = |p(r) - p(\tau_n)| = |\int_{\tau_n}^r sq(s)\, ds| \le c_1 r^2 q(\tau_n)$$

and $|f_n''(r)| \le c_2 q(\tau_n)$ for r as above. Thus we can take $K \gg 1$, K being independent of $n \gg 1$, so large that

$$f_n(r) = f_n(\tau_n) + q(\tau_n)O((r-\tau_n)^2) \ge f_n(\tau_n) - K$$

again for r as above. This yields the lower bound

$$e_n \ge c\, q(\tau_n)^{-1/2}\exp(2f_n(\tau_n)) \tag{2.15}$$

for some $c > 0$ independent of $n \gg 1$. We combine (2.11) and (2.13) $\sim$ (2.15) to obtain that

$$(\log\lambda_n)/n = (\log\alpha_n - \log e_n)/n \to -\infty, \quad n\to\infty,$$

and hence $\lambda_n = O(n^{-L})$ for any $L \gg 1$. Thus the proof of the lemma is now completed. □

3. Three dimensional case : proof of Theorem 2

In this section we prove the second main theorem (Theorem 2). Let $V = V(x,z)$ be the potential as in the theorem. Throughout the proof, the z variable is often regarded as a parameter and we sometimes write $V_z(x)$ for $V(x,z)$. The next lemma is a restatement of Lemma 2.7 (see also Remark after this lemma) and it plays a basic role in proving Theorem 2.

Lemma 3.1. *Let $V_z(x) = V(x,z)$ be as in Theorem 2. Then, for any $\delta > 0$ small enough, there exists $K_\delta > 0$ independent of z such that*

$$N(PV_zP > \lambda) \lessgtr (2\pi)^{-1} \int_{V_z(x)>(1\mp\delta)\lambda} b(x)\,dx \pm \delta\lambda^{-2/m} \pm K_\delta.$$

3.1. Proof of Theorem 2 (i)**.** We first prove Theorem 2 (i). The proof uses an argument similar to that in the proof of Proposition 2.1 and it is done through several steps.

Proof of Theorem 2 (i). (1) We begin by fixing several notations used throughout the proof. We may assume that $\lambda > 0$ is sufficiently small. Let

$$F(\lambda) = 2(2\pi)^{-2} \int_{V(x,z)>\lambda} b(x)(V(x,z)-\lambda)^{1/2}\,dx\,dz$$

be the leading term of the asymptotic formula to be proved. As is easily seen, $F(\lambda)$ behaves like $F(\lambda) \sim \lambda^{1/2-3/m}$ as $\lambda \to 0$. We fix ρ as

$$1/2 < \rho < 1/m.$$

Let $\{I_k\}$, $k \in Z$, be a family of disjoint open intervals with length $|I_k| = \lambda^{-\rho}$, which covers the whole line $R = \cup_{k\in Z}\overline{I}_k$. We denote by z_k the center of interval I_k and define another open interval J_k as

$$J_k = \{z \in R : |z - z_k| < (1+\delta)\lambda^{-\rho}/2\} \supset I_k$$

for $\delta > 0$ small enough. We further introduce a nonnegative smooth partition $\{\varphi_k\}$ associated with the covering above. We can construct the partition in such a way that : (a) $\sum_{k\in Z}\varphi_k(z)^2 = 1$ over R_z ; (b) $\operatorname{supp}\varphi_k \subset J_k$; (c)

$$|(d/dz)^l\varphi_k(z)| \le C_l\,\delta^{-l}\,\lambda^{\rho l}$$

for C_l independent of k, δ and λ.

(2) We first show the upper bound

$$\limsup_{\lambda\to 0} N(H < -\lambda)/F(\lambda) \le 1. \tag{3.1}$$

If we again write $V_z(x)$ for $V(x,z)$, then we see, using the same argument as in the proof of Lemma 1.1, that

$$N(H < -\lambda) \le N(-\partial_z^2 - P(V_z + cV_z^2)P < -\lambda) + C$$

for some c, $C > 0$, because

$$N(-\partial_z^2 - P(V_z + cV_z^2)P < -\lambda) = N(P(-\partial_z^2 - V_z - cV_z^2)P < -\lambda).$$

Set $W_z(x) = V_z(x) + cV_z(x)^2$ for $c > 0$ as above. Since $2\,\rho > 1$ by the above choice, we have

$$\sum_{k\in Z} |d\varphi_k(z)/dz|^2 = O(\delta^{-2})\lambda^{2\rho} \le \delta\lambda$$

for $\lambda < \lambda_\delta \ll 1$ small enough, and hence it follows from the IMS localization formula that

$$-\partial_z^2 - PW_zP \ge \sum_{k\in Z} \varphi_k(-\partial_z^2 - PW_zP)\varphi_k - \delta\,\lambda$$

in the form sense. Let

$$N_k(\lambda) = N_D(-\partial_z^2 - PW_zP < -\lambda; J_k)$$

denote the number of negative eigenvalues less than $-\lambda$ of operator $-\partial_z^2 - PW_zP$ acting in the space $L^2(R_x \times J_k)$ under zero Dirichlet boundary conditions at the boundary of interval J_k. Then we have

$$N(H < -\lambda) \le \sum_{k\in Z} N_k((1-\delta)\lambda) + C. \tag{3.2}$$

(3) Let $u_m(x) = \langle x\rangle^{-m}$ again and let λ_j denote the j–th eigenvalue of the operator $Pu_mP : L^2(R_x^2) \to L^2(R_x^2)$. Since $W_z(x) \le d\,u_m(x)$ uniformly in z for some $d > 0$, it follows that

$$-\partial_z^2 - PW_zP \ge -\partial_z^2 - d\,Pu_mP.$$

If the operator on the right side is considered as a self–adjoint operator acting in $L^2(R_x^2 \times J_k)$ with zero Dirichlet boundary conditions, it has $l^2\pi^2/|J_k|^2 - d\lambda_j$ as eigenvalues, $l \ge 1$ being an integer. Since λ_j obeys the bound $\lambda_j \le cj^{-m/2}$, $c > 0$, by Proposition 2.3, there exists $\gamma > 0$ such that

$$l^2\pi^2/|J_k|^2 - d\lambda_j > -(1-\delta)\lambda$$

for $j > [\gamma\lambda^{-2/m}]$, [ ] being the Gauss notation.

(4) We evaluate the quantities on the right side of (3.2). We first consider the case in which k is in the set $\Gamma_0 = \{k \in Z : |z_k| \le \lambda^{-\rho}/\delta\}$. By the observation in step (3), we have

$$N_k((1-\delta)\lambda)) \le \sum_{j=1}^{[\gamma\lambda^{-2/m}]} ((|J_k|/\pi)(d\,\lambda_j - (1-\delta)\lambda)_+^{1/2} + 1),$$

where $(s)_+ = \max(s, 0)$. A simple calculation using the above bound $\lambda_j \le cj^{-m/2}$ shows that the sum on the right side obeys the bound $O(\lambda^{1/2-2/m-\rho})$

uniformly in δ. As is easily seen, the number $\#(\Gamma_0)$ of elements in the set Γ_0 is bounded by $O(\delta^{-1})$ uniformly in λ. Hence it follows that

$$\sum_{k\in\Gamma_0} N_k((1-\delta)\lambda) = O(\delta^{-1})\, O(\lambda^{1/2-2/m-\rho}) = o(\lambda^{1/2-3/m}) = o(F(\lambda)) \quad (3.3)$$

as $\lambda \to 0$, because $\rho < 1/m$ by choice.

If k belongs to the set $\Gamma_\infty = \{k \in Z : |z_k| \geq M\,\lambda^{-1/m}\}$, then we can take $M \gg 1$ so large that $PW_zP : L^2(R^2_x) \to L^2(R^2_x)$ is bounded with bound not exceeding $\lambda/2$ for $z \in J_k$. This implies that $N_k((1-\delta)\lambda) = 0$ and hence

$$\sum_{k\in\Gamma_\infty} N_k((1-\delta)\lambda) = 0. \quad (3.4)$$

(5) We deal with the remaining case in which k belongs to the set

$$\Gamma = \{k \in Z : \lambda^{-\rho}/\delta < |z_k| < M\,\lambda^{-1/m}\}.$$

We set $v_k(x) = V(x, z_k)$ for $k \in \Gamma$ and denote by λ_{jk} the j–th eigenvalue of the operator $Pv_kP : L^2(R^2_x) \to L^2(R^2_x)$. By assumption,

$$W_z(x) = v_k(x)(1 + O(\delta)), \quad z \in J_k,$$

uniformly in $k \in \Gamma$ for $\lambda < \lambda_\delta \ll 1$ small enough. Hence we have

$$N_k((1-\delta)\lambda) \leq \sum_{j=1}^{[\gamma\lambda^{-2/m}]} ((|J_k|/\pi)((1+d\delta)\lambda_{jk} - (1-\delta)\lambda)_+^{1/2} + 1)$$

for some $d,\ \gamma > 0$ independent of $k \in \Gamma$. Let $m_k(\nu) = N(Pv_kP > \nu)$. Then $m_k(\nu) = 0$ for $\nu > R \gg 1$ and hence it follows that

$$N_k((1-\delta)\lambda) \leq -(|J_k|/\pi) \int_{\eta_\delta(\lambda)}^{R} ((1+d\delta)\nu - (1-\delta)\lambda)^{1/2}\, dm_k(\nu) + O(\lambda^{-2/m}),$$

where

$$\eta_\delta(\lambda) = (1-\delta)\lambda/(1+d\delta) = \lambda(1 + O(\delta)). \quad (3.5)$$

We now denote by $E_k(\lambda)$ the integral on the right side. The number $\#(\Gamma)$ of elements in the set Γ is bounded by $O(\lambda^{-1/m+\rho})$ uniformly in δ. Hence we have

$$\sum_{k\in\Gamma} N_k((1-\delta)\lambda) \leq \sum_{k\in\Gamma} E_k(\lambda) + O(\lambda^{\rho-3/m}) = \sum_{k\in\Gamma} E_k(\lambda) + o(\lambda^{1/2-3/m}). \quad (3.6)$$

We evaluate the integral $E_k(\lambda)$ by making use of partial integration. If we define

$$g_k(\nu) = (2\pi)^{-1} \int_{v_k(x)>\nu} b(x)\, dx,$$

then it follows from Lemma 3.1 that

$$m_k(\nu) \le g_k((1-\delta)\nu) + \delta\nu^{-2/m} + K_\delta$$

for $k \in \Gamma$. It should be noted that the constant K_δ on the right side can be taken independently of k. Thus the integral $E_k(\lambda)$ under consideration is estimated as

$$E_k(\lambda) \le E_{k0}(\lambda) + E_{k1}(\lambda) + E_{k2}(\lambda),$$

where

$$E_{k0}(\lambda) = -(|J_k|/\pi) \int_{\eta_\delta(\lambda)}^{R} ((1+d\delta)\nu - (1-\delta)\lambda)^{1/2}\, dg_k((1-\delta)\nu),$$
$$E_{k1}(\lambda) = \delta\, O(\lambda^{1/2-2/m-\rho}), \quad E_{k2}(\lambda) = K_\delta\, O(\lambda^{-\rho}) = \delta\, O(\lambda^{1/2-2/m-\rho}).$$

Hence it follows from (3.6) that

$$\sum_{k\in\Gamma} N_k((1-\delta)\lambda) \le \sum_{k\in\Gamma} E_{k0}(\lambda) + \delta\, O(\lambda^{1/2-3/m}) \tag{3.7}$$

for λ small enough. We look at the integral $E_{k0}(\lambda)$. By the definition of Stieltjes integral,

$$\int f(\nu)\, dg_k((1-\delta)\nu) = -(2\pi)^{-1} \int b(x) f(v_k(x)/(1-\delta))\, dx$$

and hence this integral equals

$$E_{k0}(\lambda) = 2(2\pi)^{-2}|J_k|\, c_\delta \int_{v_k(x)>(1-\delta)\eta_\delta(\lambda)} b(x)(v_k(x) - (1-\delta)\eta_\delta(\lambda))^{1/2}\, dx,$$

where $\eta_\delta(\lambda)$ is again in (3.5) and

$$c_\delta = (1+d\delta)^{1/2}/(1-\delta)^{1/2} = 1 + O(\delta).$$

Since $|J_k| = |I_k|(1+O(\delta))$ and since

$$\int_{v_k(x)>\lambda} dx = O(\lambda^{-2/m}), \quad \lambda \to 0,$$

uniformly in $k \in \Gamma$, we can easily see that

$$E_{k0}(\lambda) = 2(2\pi)^{-2}|I_k| \int_{v_k(x)>\lambda} b(x)(v_k(x)-\lambda)^{1/2}\, dx + \delta^{1/2}\, O(\lambda^{1/2-2/m-\rho}).$$

If $z \in I_k$, then $v_k(x) = V(x,z)(1+O(\delta))$ and

$$\int_{V(x,z)>\lambda} V(x,z)^{1/2}\, dx = O(\lambda^{1/2-2/m})$$

uniformly in $z \in I_k$. Hence

$$E_{k0}(\lambda) = 2(2\pi)^{-2} \int_{I_k} \int_{V(x,z)>\lambda} b(x)(V(x,z)-\lambda)^{1/2}\, dx\, dz + \delta^{1/2}\, O(\lambda^{1/2-2/m-\rho}).$$

As is easily seen,

$$\sum_{k\in\Gamma_0} \int_{I_k} \int_{V(x,z)>\lambda} b(x)(V(x,z)-\lambda)^{1/2}\, dx\, dz = o(\lambda^{1/2-3/m}),$$
$$\sum_{k\in\Gamma_\infty} \int_{I_k} \int_{V(x,z)>\lambda} b(x)(V(x,z)-\lambda)^{1/2}\, dx\, dz = 0.$$

Hence it follows from (3.7) that

$$\sum_{k\in\Gamma} N_k((1-\delta)\lambda) \le F(\lambda)(1+O(\delta^{1/2}))$$

for $\lambda < \lambda_\delta \ll 1$ small enough. Thus we combine this estimate with (3.2) $\sim$ (3.4) to obtain that

$$N(H<-\lambda) \le \sum_{k\in Z} N_k((1-\delta)\lambda) + C \le F(\lambda)(1+O(\delta^{1/2})) + C,$$

which yields the desired upper bound (3.1).

(6) We can show the lower bound

$$\liminf_{\lambda\to 0} N(H<-\lambda)/F(\lambda) \ge 1$$

in a similar way. We give only a sketch for it. First we get

$$N(H<-\lambda) \ge N(-\partial_z^2 - PV_zP < -\lambda)$$

by the min–max principle. Let

$$\tilde{N}_k(\lambda) = N_D(-\partial_z^2 - PV_zP < -\lambda; I_k).$$

Then we have

$$N(H<-\lambda) \ge \sum_{k\in Z} \tilde{N}_k(\lambda).$$

The lower estimate for each quantity $\tilde{N}_k(\lambda)$ on the right side is obtained by repeating almost the same argument as used for proving (3.1). The aymptotic formula of Theorem (i) is now established. $\square$

3.2. Rough estimate. We proceed to the proof of Theorem 2 (ii). Before doing it, we derive a somewhat rough upper bound as $\lambda \to 0$ on the quantity $N(H<-\lambda)$ in question. The lemma below is required to prove Theorem 2 (ii).

Lemma 3.2. *Let $U(x,z) = (1+|x|^2+|z|^2)^{-m/2}$ with $m > 2$ and let $H(U)$ be the three dimensional Pauli operator perturbed by potential U. Then one has*

$$N(H(U) < -\lambda) = O(\lambda^{-1/(m-1)-\epsilon}), \quad \lambda \to 0,$$

for any $\epsilon > 0$ small enough.

The proof is based on the following lemma due to [10].

Lemma 3.3. *Let $T = -d^2/dz^2 - v$ act on $L^2(R_z)$, where $v(z) \geq 0$ is assumed to fulfill*

$$\int (1+|z|)v(z)\,dz < \infty.$$

Then one has :

(1) *The number N of negative eigenvalues of operator T obeys the bound*

$$N \leq 1 + \int |z| v(z)\,dz.$$

(2) *Let $E = -\alpha^2$, $\alpha > 0$, be the ground state energy of operator T. If $\int |z|v(z)\,dz < 1$, then*

$$\alpha \geq (1/2) \int v(z)\,dz \times (1 - \int |z|v(z)\,dz).$$

Proof of Lemma 3.2. We write $U_z(x)$ for $U(x,z)$. Then it is easily seen that

$$N(H(U) < -\lambda) \leq N(-\partial_z^2 - cPU_zP < -\lambda) + C$$

for some $c,\ C > 0$. Hence it follows by the Birman–Schwinger principle that

$$N(H(U) < -\lambda) \leq N(T(\lambda) > 1/c) + C,$$

where

$$T(\lambda) = (-\partial_z^2 + \lambda)^{-1/2} PU_z P(-\partial_z^2 + \lambda)^{-1/2} : L^2(R^3) \to L^2(R^3).$$

Thus, to prove the lemma, it is enough to show that

$$N(T(\lambda) > 1) = O(\lambda^{-1/(m-1)-\epsilon}).$$

We may assume that $\lambda \ll 1$ is small enough. We set $\rho = 1/(2(m-1))$ and fix $L \gg 1$ arbitrarily. We further introduce the notations

$$I_1 = \{z : |z| < \lambda^{-\rho/L}\}, \quad I_{L+1} = \{z : |z| \geq \lambda^{-\rho}\},$$

and

$$I_k = \{z : \lambda^{-(k-1)\rho/L} \le |z| < \lambda^{-k\rho/L}\}, \quad 2 \le k \le L.$$

Let $\chi_k(z)$, $1 \le k \le L+1$, denote the characteristic function of I_k. Then we have

$$U_z(x) = \sum_{k=1}^{L+1} \chi_k(z) U_z(x) = \sum_{k=1}^{L+1} U_{kz}(x)$$

and hence it follows from Lemma 1.3 that

$$N(T(\lambda) > 1) \le \sum_{k=1}^{L+1} N(T_k(\lambda) > \eta), \quad \eta = 1/(L+1), \tag{3.8}$$

where

$$T_k(\lambda) = (-\partial_z^2 + \lambda)^{-1/2} P U_{kz} P (-\partial_z^2 + \lambda)^{-1/2}.$$

We below evaluate each quantity $N(T_k(\lambda) > \eta)$ on the right side of (3.8).

We first consider the case $k = L+1$. Take σ as $2 < \sigma < m$. Then we have

$$P U_{(L+1)z} P \le \chi_{(L+1)}(z)(1 + |z|^2)^{-\sigma/2} P u_{m-\sigma} P$$

in the form sense, where $u_m(x) = \langle x \rangle^{-m}$ again. The j–th eigenvalue of operator $P u_{m-\sigma} P : L^2(R_x^2) \to L^2(R_x^2)$ obeys the bound $\lambda_j \le c j^{-(m-\sigma)/2}$ for some $c > 0$ independent of j, so that

$$N(T_{L+1}(\lambda) > \eta) \le \sum_{j=1}^{\infty} N(-\partial_z^2 - v_{j(L+1)} < -\lambda),$$

where the potential $v_{j(L+1)}(z) \ge 0$ satisfies the estimate

$$v_{j(L+1)}(z) \le c_L \, j^{-(m-\sigma)/2} \chi_{L+1}(z)(1 + |z|^2)^{-\sigma/2}.$$

By Lemma 3.3, we have that

$$N(-\partial_z^2 - v_{j(L+1)} < -\lambda) \le 1 + j^{-(m-\sigma)/2} O(\lambda^{\rho(\sigma-2)})$$

and that

$$N(-\partial_z^2 - v_{j(L+1)} < -\lambda) = 0, \quad j > c\,\lambda^{-\nu},$$

for some $c > 0$, where

$$\nu = 2(1/2 - \rho(\sigma - 1))/(m - \sigma) = 1/(m-1).$$

This yields that

$$N(T_{L+1}(\lambda) > \eta) = O(\lambda^{-1/(m-1)}). \tag{3.9}$$

Next we consider the case $1 \le k \le L$. Since

$$PU_{kz}P \le \chi_k(z)(1+|z|^2)^{-1/2} Pu_{m-1}P,$$

we repeat the same argumet as above to obtain that

$$N(T_k(\lambda) > \eta) \le \sum_{j=1}^{\infty} N(-\partial_z^2 - v_{jk} < -\lambda),$$

where the potential $v_{jk}(z) \ge 0$ obeys the bound

$$v_{jk}(z) \le c_L\, j^{-(m-1)/2} \chi_k(z)(1+|z|^2)^{-1/2}.$$

Hence it again follows from Lemma 3.3 that

$$N(-\partial_z^2 - v_{jk} < -\lambda) \le 1 + j^{-(m-1)/2} O(\lambda^{-k\rho/L})$$

and $N(-\partial_z^2 - v_{jk} < -\lambda) = 0$ for $j > c\,\lambda^{-2\rho-\epsilon}$, where $\epsilon = 4\,\rho^2/L \to 0$ as $L \to \infty$. Thus a simple calculation shows that

$$N(T_k(\lambda) > \eta) = O(\lambda^{-2\rho-\epsilon}), \quad 1 \le k \le L.$$

This, together with (3.8) and (3.9), completes the proof. □

3.3. Proof of Theorem 2 (ii)**.** We now prove Theorem 2 (ii). This is also verified through several steps.

Proof of Theorem 2 (ii). (1) Let

$$G(\lambda) = (2\pi)^{-1} \int_{w(x) > 2\,\lambda^{1/2}} b(x)\,dx$$

be the leading term of the asymptotic formula in Theorem 2 (ii). As is easily seen, $G(\lambda)$ behaves like $G(\lambda) \sim \lambda^{-1/(m-1)}$ as $\lambda \to 0$. We prove only the upper bound

$$\limsup_{\lambda \to 0} N(H < -\lambda)/G(\lambda) \le 1. \tag{3.10}$$

A similar argument applies to the lower bound also. In fact, the lower bound is easier to prove than the upper one.

We first fix several notations used in the proof. Let $W_z(x) = V_z(x) + c\,V_z(x)^2$ be again as in the proof of Theorem 2 (i). Let I_K be the interval

$$I_K = \{z : |z| < K\,\lambda^{-1/2}\}$$

for $K \gg 1$ large enough. We often write $\delta = 1/K \ll 1$ and let $\delta = 1/K \to 0$ in the discussion below. If $|z| \ge \lambda^{-1/2}$, then $W_z(x) = O(\lambda^{m/2}) = o(\lambda)$ uniformly

in x and hence we obtain by making use of the same argument as in the proof of Theorem 2 (i) that

$$N(H < -\lambda) \leq N_D(-\partial_z^2 - PW_zP < -(1-\delta)\lambda; I_K) + C \tag{3.11}$$

with $\delta = 1/K$.

(2) We now consider the operator $-\partial_z^2$ with zero Dirichlet boundary conditions in the interval I_K. This operator has the eigenvalue

$$\lambda_j = \lambda\,(\pi/2K)^2 j^2, \qquad j \geq 1,$$

and the corresponding normalized eigenfunction is given by

$$\psi_j(z;\lambda) = (-1)^{[j/2]} K^{-1/2} \lambda^{1/4} \sin\,(j((\pi/2K)\lambda^{1/2} z + \pi/2)). \tag{3.12}$$

Let $0 < \nu \ll 1$ be fixed small enough and set $L = 2K[\lambda^{-\nu}]$. Then we define the projection $P_L : L^2(R_x^2 \times I_K) \to L^2(R_x^2 \times I_K)$ as

$$(P_L u)(x,z) = \sum_{j=1}^{L} \int_{I_K} u(x,z)\psi_j(z;\lambda)\,dz\,\psi_j(z;\lambda)$$

and write Q_L for $Q_L = Id - P_L$. By definition, these propjections commute with P. Since

$$P_L W_z Q_L + Q_L W_z P_L \leq \delta\, P_L W_z P_L + \delta^{-1}\, Q_L W_z Q_L$$

in the form sense, we get

$$N_D(-\partial_z^2 - PW_zP < -(1-\delta)\lambda; I_K) \leq N_\delta(\lambda; P_L) + N_\delta(\lambda; Q_L), \tag{3.13}$$

where

$$\begin{aligned} N_\delta(\lambda; P_L) &= N(P_L(-\partial_z^2 - (1+\delta)PW_zP)P_L < -(1-\delta)\lambda), \\ N_\delta(\lambda; Q_L) &= N(Q_L(-\partial_z^2 - (1+1/\delta)PW_zP)Q_L < -(1-\delta)\lambda). \end{aligned}$$

We assert that

$$\limsup_{\lambda \to 0} N_\delta(\lambda; Q_L)/G(\lambda) = 0. \tag{3.14}$$

We shall prove this assertion. Let $W_{\delta z}(x) = (1+1/\delta)W_z(x)$. By definition, $Q_L(-\partial_z^2)Q_L \geq \pi^2\,\lambda^{1-2\nu} Q_L$ and hence we have

$$Q_L(-\partial_z^2 - PW_{\delta z}P)Q_L \geq Q_L(-\partial_z^2/2 - PW_{\delta z}P + (\pi^2/2)\,\lambda^{1-2\nu})Q_L.$$

This implies that

$$\begin{aligned} N_\delta(\lambda; Q_L) &\leq N(-\partial_z^2/2 - PW_{\delta z}P < -(\pi^2/2)\lambda^{1-2\nu} - (1-\delta)\lambda) \\ &\leq N(H(V_\delta) < -(\pi^2/2)\lambda^{1-2\nu}), \end{aligned}$$

where $H(V_\delta)$ is the three dimensional Pauli operator perturbed by potential $V_\delta(x,z) = 2\,W_{\delta z}(x)$. Thus the assertion (3.14) follows from Lemma 3.2 at once. By (3.11) and (3.13), the proof of the upper bound (3.10) is now reduced to showing that

$$\lim_{\delta\to 0}\limsup_{\lambda\to 0} N_\delta(\lambda; P_L)/G(\lambda) \le 1. \tag{3.15}$$

We here note that Lemma 3.2 is only used for proving the upper bound and this is not required for the lower one.

(3) To prove (3.15), we work in the space $\mathcal{X}_L = \sum \oplus L^2(R_x^2)$, L summands. We represent $-\partial_z^2$ and W_z in the matrix form. The operator $-\partial_z^2$ is represented as the diagonal matrix

$$\lambda\,\Lambda_L = \lambda\,\mathrm{diag}\,(\mu_1, \dots, \mu_L), \quad \mu_j = (\pi/2)^2(j/K)^2, \tag{3.16}$$

and the multiplication operator W_z has the representation

$$\Gamma_L(\lambda) = \Gamma_L(x;\lambda) = (\gamma_{jk}(x:\lambda))_{1\le j,k\le L}$$

with component

$$\gamma_{jk}(x;\lambda) = \int_{I_K} \psi_j(z;\lambda) W_z(x) \psi_k(z;\lambda)\,dz.$$

We analyse the behavior as $\lambda \to 0$ of $\Gamma_L(x;\lambda)$. Let

$$E_0 = (e_0, 0, \dots, e_0, 0), \quad e_0 = {}^t(1, 0, \dots, 1, 0),$$

be the constant matrix of size $L \times L$ generated by the column vector e_0. As is easily seen, E_0 is nonnegative and

$$\mathrm{rank}\, E_0 = 1. \tag{3.17}$$

We further define $u(x)$ as

$$u(x) = \int_{-\infty}^{\infty} W_z(x)\,dz = \int_{-\infty}^{\infty} (V(x,z) + cV(x,z)^2)\,dz,$$

which behaves like

$$u(x) = w(x) + O(|x|^{-2m+1}), \quad |x| \to \infty. \tag{3.18}$$

Lemma 3.4. *Let $\kappa = (m-2)/2m > 0$. Then $\Gamma_L(x;\lambda)$ takes the form*

$$\Gamma_L(x;\lambda) = K^{-1}\lambda^{1/2}\, u(x)(E_0 + O_L(\lambda^{\kappa-\nu})) + O_L(\lambda^{1+\kappa-\nu}),$$

where the notation $O_L(\lambda^\sigma)$ denotes a class of matrices of size $L\times L$, $L = 2K[\lambda^{-\nu}]$, with components bounded by $O(\lambda^\sigma)$ as $\lambda \to 0$ uniformly in x, the bound $O(\lambda^\sigma)$ being dependent on K.

Proof. Let $\gamma_{jk}(x;\lambda)$ be defined above and let $J = (-\lambda^{-1/2+\kappa}, \lambda^{-1/2+\kappa}) \subset I_K$. We divide the integral into two parts

$$\gamma_{jk}(x;\lambda) = \{\int_J + \int_{I_K\setminus J}\}\psi_j(z;\lambda)W_z(x)\psi_k(z;\lambda)\,dz.$$

Since $W_z(x)$ obeys the bound $W_z(x) = O((|x|+|z|)^{-m})$ at infinity, the integral over $I_K \setminus J$ is seen to be bounded by $O(\lambda^{1+\kappa})$ uniformly in x and we have

$$\int_J W_z(x)\,dz = u(x) + O(\lambda^{1/2+\kappa}).$$

The eigenfunction $\psi_j(z;\lambda)$ defined by (3.12) behaves like

$$\psi_j = \begin{cases} K^{-1/2}\,\lambda^{1/4}(1+O(s^2)) & \text{for } j \text{ odd} \\ K^{-1/2}\,\lambda^{1/4}O(s) & \text{for } j \text{ even} \end{cases}$$

as $s = j(\pi/2K)\lambda^{1/2}z \to 0$, so that we have

$$\psi_j = \begin{cases} K^{-1/2}\,\lambda^{1/4}(1+O(\lambda^{2(\kappa-\nu)})) & \text{for } j \text{ odd} \\ K^{-1/2}\,\lambda^{1/4}O(\lambda^{\kappa-\nu}) & \text{for } j \text{ even} \end{cases}$$

on the interval J. If we take account of these relations, the desired relation can be obtained after a simple calculation. □

(4) Let κ be as in Lemma 3.4 and let Λ_L be as in (3.16). We may assume that $\nu/\kappa \ll 1$. We define

$$Y_L = (\Lambda_L + Id)^{-1/2},$$

which satisfies $Y_L \le Id$ in the form sense. If $A_L(\lambda) \in O_L(\lambda^\sigma)$, then $A_L(\lambda) \le O(\lambda^{\sigma-\nu})\,Id$. Hence we obtain by Lemma 3.4 that

$$Y_L\Gamma_L(x;\lambda)Y_L \le K^{-1}\lambda^{1/2}u(x)(Y_LE_0Y_L + d\lambda^{\kappa/2-2\nu}Id) + \delta\lambda\,Id, \quad d>0,$$

for $\lambda < \lambda_\delta \ll 1$ small enough. We should note that $d > 0$ is independent of K. We further define

$$Z_L(x;\lambda) = K^{-1}u(x)(Y_LE_0Y_L + d\,\lambda^{\kappa/2-2\nu}Id).$$

Let $(\ ,\ )_L$ denote the L^2 scalar product in $\mathcal{X}_L = \sum \oplus L^2(R_x^2)$. We regard Λ_L, $P\Gamma_L(\lambda)P$ and $PZ_L(\lambda)P$ as operators acting on $\mathcal{X}_L$. If $v \in \mathcal{X}_L$ obeys

$$((\lambda\,\Lambda_L - (1+\delta)P\Gamma_L(\lambda)P)v, v)_L < -(1-\delta)\lambda\,(v,v)_L,$$

then $\tilde{v} = (\Lambda_L + Id)^{1/2}v \in \mathcal{X}_L$ satisfies

$$(1-3\delta)\lambda^{1/2}(\tilde{v},\tilde{v})_L < (PZ_L(\lambda)P\tilde{v},\tilde{v})_L$$

for $\delta > 0$ small enough. Thus it follows that

$$N_\delta(\lambda; P_L) \le N(PZ_L(\lambda)P > (1-3\delta)\lambda^{1/2})$$

and hence we have

$$N_\delta(\lambda; P_L) \le \sum_{j=1}^{L} N(Pv_j(\lambda)P > (1-3\delta)\lambda^{1/2}), \tag{3.19}$$

where $v_j(x;\lambda)$ is the eigenvalue of $Z_L(x;\lambda)$. Since $\operatorname{rank} Y_L E_0 Y_L = 1$ by (3.17), these eigenvalues are easily calculated as $v_j = dK^{-1}\lambda^{\kappa/2-2\nu}u(x)$ for $2 \le j \le L$ and

$$v_1 = K^{-1}u(x)(\operatorname{tr}(Y_L E_0 Y_L) + d\lambda^{\kappa/2-2\nu}).$$

(5) We evaluate the sum on the right side of (3.19). We have

$$N(Pv_j(\lambda)P > (1-3\delta)\lambda^{1/2}) \le N(PuP > (K/d)(1-3\delta)\lambda^{1/2-\kappa/2+2\nu})$$

for $2 \le j \le L$. Since $u(x) = O(|x|^{-(m-1)})$ as $|x| \to \infty$, it follows from Proposition 2.3 that

$$N(Pv_j(\lambda)P > (1-3\delta)\lambda^{1/2}) = O(\lambda^{-2(1/2-\kappa/2+2\nu)/(m-1)})$$

for j as above. Hence we can take ν so small that

$$\sum_{j=2}^{L} N(Pv_j(\lambda)P > (1-3\delta)\lambda^{1/2}) = o(\lambda^{-1/(m-1)}) = o(G(\lambda)).$$

Next we consider the quantity with $j = 1$. Let μ_j be as in (3.16). We can calculate the trace $\operatorname{tr}(Y_L E_0 Y_L)$ as

$$\operatorname{tr}(Y_L E_0 Y_L) = \sum_{j=1}^{L/2}(\mu_{2j-1}+1)^{-1} = \sum_{j=1}^{L/2}(\pi^2((j-1/2)/K)^2+1)^{-1}$$

with $L = 2K[\lambda^{-\nu}]$, so that

$$\lim_{K=1/\delta\to\infty} K^{-1}\operatorname{tr}(Y_L E_0 Y_L) = (1/\pi)\int_0^\infty (s^2+1)^{-1}\,ds + o(1) = 1/2 + o(1)$$

as $\lambda \to 0$. Thus

$$Pv_1(\lambda)P \le (1/2 + c\delta)PuP, \quad c > 0,$$

for $\lambda < \lambda_\delta \ll 1$ small enough. If we take account of bahavior (3.18), then the upper bound (3.10) follows from Theorem 1 (i). $\square$

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