J such that $J \subset \bigcap_{k=2}^{\infty} J_k$ and $|J| > \epsilon$. Since I.M.$\{f^m(J_k)\}_{m=0}^{n_k} \leq 2$, we have I.M.$\{f^n(J)\}_{n=0}^{\infty} \leq 2$. This implies $\lim_{n\to\infty} |f^n(J)| = 0$. By inequality 6, we have $\lim_{k\to\infty} |J_k| = \infty$. It contradicts that $|J_k| \leq |I|$. □

Now, we prove Theorem B. Assume that $\{p_i\}_{i=1}^{\infty}$ is the set of periodic attractors of a C^2 diffeomorphism F, and that the period n_k of p_k goes to infinity. Let Λ be the set of accumulation points of $\{p_i\}$. We also assume that Λ has a dominated splitting with a contracting direction.

Lemma 19 *There is an integer i_0 such that $\Lambda_0 = \Lambda \cup \{F^n(p_i) | i \geq i_0, n = 0, 1, \cdots, n_i - 1\}$ has a dominated splitting with a contracting direction.*

(Proof) By the arguments similar to that for hyperbolic case, one can show that there is a neighborhood U of Λ such that $\bigcap_{n\in\mathbf{Z}} F^n(U)$ has an F-invariant continuous splitting with a contracting direction. □

By Theorem A, we can take a finite union of periodic orbits S and a Markov cover of $\Lambda_0 \setminus S$. For infinitely many points of $\{p_i\}$, the orbits of them are contained in Λ_0. Hence, the reduced map associated to the cover has infinitely many periodic attractors with unbounded period. It contradicts Proposition 18. We complete the proof of Theorem B.

References

[1] M. Asaoka A C^1 example with infinitely many periodic attractors which cannot be approximated by diffeomorphisms with homoclinic tangencies, preprint.

[2] R. Mañé, An ergodic closing lemma, Ann. of Math., **100** (1982), 503-540.

[3] R. Mañé, Hyperbolicity, sinks and measure in one dimensional dynamics, Commun. Math. Phys., **100** (1985), 495-524.

[4] S. Newhouse, Diffeomorphisms with infinitely many sinks, Topology, **13** (1974), 9-18.

[5] J. Palis and F. Takens, *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations*, Cambridge Univ. Press, 1993.

[6] A. Schwartz, A generalization of a Poincaré-Bendixson theorem to closed two-dimensional manifolds, Amer. J. Math., **85** (1963), 453-458.

[7] M. Shub, *Global Stability of Dynamical Systems*, Springer-Verlag, 1986.

and that if $p \in T_i$ connects R_i and R_j then $F(p) \in T_i$. Let $\{P_k\}_{k=1}^s$ be the collection of all connected components of $\{P_{i,j} \setminus (T_i \cup S_i)\}$. We define $\alpha(k), \beta(k)$ by $\alpha(k) = i$ and $\beta(k) = j$ if P_k is the component of $P_{i,j} \setminus (T_i \cup S_i)$ for some i and j. From the construction of $P_k, \alpha(k), \beta(k)$, we have $\Lambda_1 \setminus R_i' \subset \bigcup_{\alpha(k)=i} \overline{P_k}$ and $F(\partial_h P_k) \subset T_i \sup S_i \subset R_i' \setminus \bigcup_{\alpha(k')=\beta(k)} \mathrm{int} P_k$. □

Now, we prove Theorem A. Take $\{P_k\}_{k=1}^s, \{\alpha(k), \beta(k)\}_{k=1}^s, S$ as in Lemma 16. Choosing an affine transformation $\mu_k(x, y) = (2x, a_k y + b_k)$, we get $\bar{\varphi}_k = \varphi_{\alpha(k)} \circ \mu_k$ so that $\bar{\varphi}_k([-1, 1]^2) = P_k$. Let $Z_k = \{l = 1, \cdots, s | \beta(k) = \alpha(l)\}$. Then, $\{(\overline{P_k}, \bar{\varphi}_k, Z_k)\}_{k=1}^s$ is a Markov cover of $\Lambda_1 \setminus S$. Indeed, since $\Lambda_1 \setminus S \subset \bigcup_{i=1}^s R_i'$ and $\Lambda_1 \setminus S \cap R_i' \subset \bigcup_{\alpha(k)=i} \overline{P_k}$, we have $\Lambda_1 \setminus S \subset \bigcup_{k=1}^s \overline{P_k}$ and $F(\Lambda_1 \setminus S \cap \overline{P_k}) \subset \bigcup_{l \in Z_k} \overline{P_l}$. One can see that other conditions in the definition of a Markov cover also follows from Lemma 16. The estimate of distortion of the Markov map induced by the cover follows from Lemma 8. We complete the proof of Theorem A. □

3 Finiteness of periodic attractors

We prove the following theorem.

Theorem B *Assume a C^2 diffeomorphism $F : M \to M$ has infinitely many periodic attractors $\{p_i\}$ with unbounded periods. Then, the set of accumulation points of $\{p_i\}$ does not have a dominated splitting with a contracting direction.*

First we define topological Markov maps in general and show Mañé's theorem for topological Markov maps.

Definition 17 Let I be a finite union of compact intervals and I_* a finite union of open subintervals of I. We call $f : I_* \to I$ *a topological Markov map* if $\overline{f(J)}$ is a connected component of I and f is a homeomorphism on J for any connected component J of I_*.

Proposition 18 *Let $f : I_* \to I$ be a topological Markov map. Assume that there exists a function $D : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\mathrm{dist}(f^n, J) \le D(\sum_{m=0}^{n-1} |f^m(J)|)$ for any $n \in \mathbf{N}$ and for any interval $J \subset \bigcap_{m=0}^{n-1} f^{-m}(I_*)$. Then, there exists an integer N such that any periodic orbit in $\bigcup_k I_k$ with minimal period $n \ge N$ is not attractive.*

(Proof) Assume that $\{p_i\}_{i=1}^\infty$ is the set of periodic attractors and that the period n_k of p_k goes to infinity. Without loss of generality, we can assume that p_k lies in a connected component I_1 of I_*. Moreover, replacing p_k by an element in its orbit, we can assume $(p_k, p_1) \cap \{F^n(p_k) | n = 0, \cdots, n_k - 1\} = \emptyset$. Let J_k be the maximal interval which satisfies that $f^m(J_k) \subset I_*$ and $f^m(J_k) \cap \{F^n(p_k) | n = 0, \cdots, n_k - 1\} = \{F^m(p_k)\}$ for any $m = 0, \cdots, n$. Then, it is easy to see that $\mathrm{I.M.}\{f^m(J_k)\}_{m=0}^{n-1} \le 2$. This implies that $\mathrm{dist}(f^{n_k}(J_k), J_k) \le D(\sum_{m=0}^{n-1} |f^m(J_k)|) \le D(2|I|)$, where $|I|$ is the sum of the length of the components of I. Since f is a topological Markov map, one can also see $f^{n_k}(J_k) \supset \{p_k, p_1\}$. There exists a positive number $\delta > 0$ such that $|f^{n_k}(J_k)| > \delta$ for any $k \ge 2$. Therefore, we have

$$|f^{n_k}(J)| \ge e^{D(2|I|)} \frac{\delta}{|J_k|} |J| \tag{6}$$

for any $k \ge 2$ and any interval $J \subset J_k$. Since J_k contains an attractive periodic point, there is an interval $J \subset J_k$ such that $|f^{n_k}(J)| < |J|$. Therefore, there exists a positive number ϵ such that $|J_k| > 2\epsilon$ for any k. Without loss of generality, we can assume that $p_k \to p$ for some $p \in \overline{I_1}$ and that there exists an interval

Let $B_0 = B$ and B_{n+1} be the connected component of $B_n \cap F^{-(n+1)} \circ \varphi_{i_{(n+1)}}([-1,1] \times [-2,2])$ which contains p. By Corollary 12, $[B_n]_i = B_n$ for any n. First, we claim there exists an integer N such that if $n \geq N$ then $B_n = B_N$. If not, there exist two integers n_1, n_2, $j = 1, \cdots, r$ and $q \in \partial_h R_j$ such that $F^{n_k}(B_{n_k}) \subset R_j$ and $F^{n_k}(\mathrm{int} B_{n_k - 1}) \cap [q]_j \neq \emptyset$. Without loss of generality, we can assume that $[F^{n_1}(B_{n_1})]_j \subset [F^{n_2}(B_{n_1})]_j$. Since $F^{n_1}(p)$ is an accumulation point of boundary points and $F^{n_2}(B_{n_2})$ has no boundary points, we have $F_{n_1}(p) \in [F^{n_2}(p)]_j$. Since $p \in B$ is regular, $|F^{l+1}([F^{n_2}(p)]_j)| < \lambda |F^l([F^{n_2}(p)]_j)|$. This implies that $\lim_{l \to \infty} F^{l|n_1 - n_2|}(p) = q$ for some $q \in \mathrm{Per}(F)$ and hence $p \in W^s(q)$. It contradicts the fact that p is not a boundary point. Note that $\mathrm{int} B_N \neq emptyset$.

Next, we claim $\mathrm{I.M.}\{[B_n]_j | n \geq N, i_n = j\} \leq 2$ for any j. If not, there exist integers $n_1, n_2, n_3 \geq N$ and $j = 1, \cdots, r$ such that $i_{n_k} = j$ for any $k = 1, 2, 3$ and $\mathrm{I.M.}\{[B_{n_k}]_j\}_{k=1,2,3} = 3$. Without loss of generality, we can assume that $[B_{n_1}] \subset [B_{n_2}] \cup [B_{n_3}]$. Indeed, it follows from the fact that if three intervals have a common point then one of them must be contained in the union of other two intervals. Since B_{n_k} has no boundary point, $F^{n_k}(p) \in \partial_h([B_{n_2}] \cup [B_{n_3}])$ for any $k = 1, 2, 3$. Therefore, we have $[F^{n_k}(p)] = [F^{n_k'}(p)]$ for some $k, k' = 1, 2, 3$. This implies $p \in W^s(p)$ for some q. It contradicts to that p is not a boundary point.

By the second claim, there exists a constant $C > 0$ such that $\sum_{m=0}^{\infty} |F^m(l)| < C$ for any C^1-vertical curve $l \subset B_N$. By Lemma 9, we can take a C^1-vertical curve l' and $q \in \mathrm{Per}(F)$ such that $l' \subset \bigcap_{n=0}^{\infty} (\mathrm{int} W_0)$, $[l' \cap B_N]_j = B_N$, $q \in l'$ and $\lim_{n \to \infty} |F^n(l')| = 0$ since $p \in B_N \cap \overline{\mathrm{Per}(F)}$ and $\inf_{n \in \mathbf{N}} d(F^n(B_N), M \setminus W_0) \geq d(\bigcap_{n=0}^{\infty} F^{-n}(R), M \setminus W_0) > 0$. Thus, we have $l \subset W^s(p)$ and which contradicts the fact that B_N has no boundary points. □

Lemma 16 *There exist connected sets $R_i' \subset R_i$ for any $i = 1, \cdots, r$, a family of connected sets $\{P_k\}_{k=1}^s$, integers $\{\alpha(k), \beta(k)\}_{k=1}^s$ and a finite set $S \subset \Lambda_1$ such that*

1. *$P_k \subset R'_{\alpha(k)} \cap F^{-1}(R'_{\beta(k)})$ for any $k = 1, \cdots, s$,*
2. *$\mathrm{int} P_k \cap \mathrm{int} P_l = \emptyset$ if $\alpha(k) = \alpha(l)$,*
3. *$[R_i']_i = R_i'$ and $[P_k]_{\alpha(k)} = P_k$,*
4. *S consists of orbits of some periodic attractors,*
5. *$\Lambda_1 \setminus S \subset \bigcup_{i=1}^r R_i'$,*
6. *$\Lambda_1 \cap R_i' \subset \bigcup_{\alpha(k)=i} \overline{P_k}$,*
7. *$F(\partial_h P_k) \subset R'_{\beta(k)} \setminus \left(\bigcup_{\alpha(k')=\beta(k)} \mathrm{int} P_k' \right)$.*

(Proof) By Lemma 15, we take a family $\{P_{i,j}\}_{i,j=1,\cdots,r}$ of closed sets with finitely many components so that $[P_{i,j}]_i = P_{i,j}$, $P_{i,j} \subset R_i$, $F(P_{i,j}) \subset \varphi_j([-1,1]^2)$, $\partial_h P_{i,j}$ consists of boundary points and $\Lambda_1 \cap \varphi_i([-2,2] \times [-1,1]) \subset \bigcup_{l=1}^r P_{i,l}$ for any $i, j = 1, \cdots, r$. Let R_i' be the minimal closed set which satisfies that $R_i' \supset \varphi_i([-2,2] \times [-1,1]) \cup \bigcup_{j=1}^r P_{i,j}$ and $[R_i']_i = R_i'$. Then, we have $\Lambda_1 \cap R_i' \subset \bigcup_{j=1}^r P_{i,j}$.

Let $S' = \{q \in \mathrm{Per}(F) | \ p \in W^s(q') \text{ for some } i = 1, \cdots, r, \text{ a regular point } p \in \partial_h R_i \text{ and } q \in O(q')\}$ and $S = \{q \in S' | \ q \in \mathrm{int} W^s(q)\}$. Then, S' and S are finite sets and F-invariant. Fix a small positive number $\epsilon > 0$ so that $W^s_\epsilon(q) = W^s(q) \cap \{q' | d(q', q) < \epsilon\}$ is connected for any $q \in S'$. Let $S_i = \left[\left(\bigcup_{q \in S'} W^s_\epsilon(q) \right) \cap R_i' \right]_i$ and T_i be the set of points $q \in [F^n(p)]_i$ for some $p \in \bigcup_{i',j'=1}^r (\partial_h P_{i',j'} \cup \partial_h S_i')$ and some sequence $\{i_m\}_{m=0,\cdots,n}$ such that $F^m(p)$ connects R_{i_m} and $R_{i_{(m+1)}}$ for any $m = 0, \cdots, n-1$. It is easy to see that $T_i \cup S_i$ has finitely many components

To simplify notations we replace F^{n_0} by F and $\bigcap_{m=0}^{n_0-1} F^{-m}(W_0)$ by W_0.

We define $\pi_2 : \mathbf{R}^2 \to \mathbf{R}$ by the projection to the second component. Let $[A] = [A]_i = \varphi_i \circ \pi_2^{-1} \circ \pi_2(\varphi_i^{-1}(A))$ for a set $A \subset R_i$. For any set $A \subset R_i$ with $[A]_i = A$, we define $\partial_h A = \varphi_i([-2,2] \times \partial A')$, where $A' = \pi_2 \circ \varphi_i^{-1}(A) \subset [-2,2]$ and $\partial A'$ is the boundary of A'.

We say a point $p \in R$ connects R_i and R_j for some $i, j = 1, \cdots, r$ if $p \in R_i$ and $F([p]_i) \subset \varphi_j([-1,1] \times [-2,2])$.

Lemma 13 *If p connects R_i and R_j, then the connected component P of $R_i \cap F^{-1}(R_j)$ which contains p satisfies that $[P]_i = P$.*

(Proof) Since $F(\{\pm 2\} \times [-2,2])$ is a C^1-vertical curve and $\varphi_j^{-1}(F(\{\pm 2\} \times [-2,2])) \cap [-1,1] \times [-2,2] \neq \emptyset$, there exist $\frac{1}{9}$-Lipschitz functions $x_1, x_2 : [-2,2] \to [-2,2]$ and real numbers $y_1, y_2 \in [-2,2]$ such that $F(P) = \varphi_i(\{(x,y) \in [-2,2]^2 |\ x_1(y) \leq x \leq x_2(y), y_1 \leq y \leq y_2\})$. □

We say a point $p \in R$ is regular when there exists a sequence $\{i_n\}_{n=0}^{\infty}$ such that $F^n(p)$ connects R_{i_n} and $R_{i_{(n+1)}}$ for any integer n. Note that any $p \in \Lambda_1$ is regular and $F(q)$ is regular if $q \in R$ is regular.

We say a point $p \in R$ is a boundary point when p is not regular or $p \in W^s(q)$ for some $q \in \mathrm{Per}(F)$. Note that any $p \in \mathrm{Per}(F) \cap R$ is a boundary point. It is easy to see that $F^{-1}(p)$ is a boundary point if $p \in R$ is a boundary point and $F^{-1}(p) \in R$.

Lemma 14 *If $p \in R_i$ is a boundary point, then any point $q \in [p]_i$ is also a boundary point.*

(Proof) Assume $q \in [p]_i$ is regular and there exists a sequence $\{i_n\}$ such that $F^n(q)$ connects R_{i_n} and $R_{i_{(n+1)}}$. Since $F(q) \in \varphi_{i_1}([-1,1] \times [-2,2])$, we have $F([p]_i) \subset R_{i_1}$. This implies that p is regular. There exists $p' \in \mathrm{Per}(F)$ such that $p \in W^s(p')$. Since $F^n([p]_i) \subset R_{i_n}$ for any $n \geq 1$, $\lim_{n\to\infty} |F^n([p]_i)| = 0$. Therefore, we have $q \in [p]_i \subset W^s(p')$. □

Let A be a set and $\{A_\alpha\}_\alpha$ be a family of subsets of A. Then, we define the intersection multiplicity by

$$\mathrm{I.M.}(\{A_\alpha\}) = \sup_{x \in A} \#\{\alpha | x \in A_\alpha\}.$$

Since (R_i, φ_i) is a canonical coordinate, there exists a constant C_0 such that

$$C_0^{-1} \leq \frac{|\pi_2 \circ \varphi_i^{-1}(l)|}{|l|} \leq C_0$$

for any C^1 vertical curve $l \subset R_i$.

Lemma 15 *For any $p \in \Lambda_1$, there exist a connected set $P \subset R_i$ with $[P]_i = P$ such that $(\Lambda_1 \cap U) \cap R_i = (\Lambda_1 \cap U) \cap P$ for some neighborhood U of p, $F(P) \subset \varphi_{i_1}([-1,1]^2)$ for some i_1 and any point in $\partial_h P$ is a boundary point.*

(Proof) Let $F(p) \in \varphi_{i_n}((-\frac{1}{2}, \frac{1}{2})^2)$ for any $n \geq 1$ and P' be the component of $R_i \cap F^{-1} \circ \varphi_{i_1}([-1,1]^2)$ which contains p. Notice that $p \in [\mathrm{int}P']_i$ and $[P']_i = P'$ by Corollary 12.

To lead contradiction, we assume that there exists a connected set $B \subset P'$ such that $p \in B$, $[B]_i = B$ and B contains no boundary points. Since $F(p) \in \overline{\mathrm{Per}(F) \cap \mathrm{int}R_{i_1}}$ and any periodic point in R is a boundary point, we have $F(p) \notin F([\mathrm{int}B]_i)$. This implies that $p \in \partial B$.

(Proof) Let $\alpha = \inf d(F^n(l), M \setminus W_0) > 0$, $\beta = \sum_{n=0}^{\infty} |F^n(l)| < \infty$, $\gamma = \alpha / \sup |F^n(l)|$ and $0 < \delta < \min\{\gamma \exp(-D(1+\gamma)\beta), \gamma\}$. We fix l' and prove the claim by induction. For $n = 0$, the lemma is trivial. Assume that $|F^m(l')| \leq (1+\gamma)|F^m(l)|$ for any $m \leq n-1$. We have $|F^m(l')| \leq |F^m(l)| + \alpha$. Hence, $F^m(l') \subset W_0$ for any $m \leq n-1$. This implies that $\mathrm{dist}(F^n, l') \leq D(\sum_{m=0}^{n-1} |F^m(l')|) \leq D((1+\gamma)\beta)$. Since $F^n|_l$ is diffeomorphic, we apply the mean value theorem and have $\|D_z F^n|_l\| = |F^n(l)|/|l|$ for some z. For any $z' \in l'$,

$$\|D_{z'} F^n|_l\| \leq \exp D((1+\gamma)\beta) \cdot \frac{|F^n(l)|}{|l|} < \frac{\gamma}{\delta} \cdot \frac{|F^n(l)|}{|l|}.$$

Therefore, we have $|F^n(l')| < |F^n(l)| + \frac{\gamma}{\delta} \cdot \frac{|F^n(l)|}{|l|} |l' \setminus l| \leq (1+\gamma)|F^n(l)|$. □

2.3 A Markov cover

Definition 10 Let R_p be a compact neighborhood of a point $p \in \Lambda_0$ and $\varphi_p : [-2,2]^2 \to R_p$ a diffeomorphism. We call (φ_p, R_p) *a canonical coordinate* centered at p if

1. $\varphi_p(0,0) = p$,
2. $\varphi_p([-2,2] \times \{y\})$ and $\varphi_p(\{x\} \times [-2,2])$ are C^1 curves and are tangent to E_1 and E_2, respectively, for any x and y,
3. for any C^1 vertical curve $l \subset R_p$, there exists a $\frac{1}{9}$-Lipschitz function $\tilde{l} : [-2,2] \to [-2,2]$ such that $\varphi_p^{-1}(l) \subset \{(\tilde{l}(y), y) | y \in [-2,2]\}$,
4. $\sup\{|\varphi_p(\{x\} \times [-2,2])|, |\varphi_p([-2,2] \times \{y\})|\} < \frac{1}{3} d(\Lambda_0, M \setminus W_0)$.

It is easy to prove that any $p \in \Lambda_0$ has a canonical coordinate centered at p.

Let $\mathrm{Per}(F)_*$ be the set consisting of all accumulation points of $\mathrm{Per}(F)$ and $\Lambda_1 = \Lambda_0 \cap \mathrm{Per}(F)_*$. Since Λ_1 is compact, we can take canonical coordinates $(\varphi_i, R_i)_{i=1}^r$ centered at some points in Λ_1 so that $\Lambda_1 \subset \bigcup_{i=1}^r \varphi_i((-\frac{1}{2}, \frac{1}{2})^2)$. Notice that $\Lambda_0 \setminus \bigcup_{i=1}^r \varphi_i((-\frac{1}{2}, \frac{1}{2})^2)$ consists of finitely many periodic points and each of them are isolated in Λ_0. Let $R = \bigcup_{i=1}^r R_i$.

Lemma 11 *There exists an integer n_0 such that for any $p \in \Lambda_1$ with $p \in R_i$ and $F^{n_0}(p) \in \varphi_i'((-\frac{1}{2}, \frac{1}{2})^2)$ for some $i, i' = 1, \cdots, r$, $F^{n_0} \circ \varphi_i([-2,2] \times \{y\}) \subset \varphi_{i'}([-\frac{3}{4}, \frac{3}{4}] \times \{\bar{y}\})$, where $p = \varphi_i(x,y) = F^{-n_0} \circ \varphi_{i'}(\bar{x}, \bar{y})$.*

(Proof) Let a constant K be the maximum of the Lipschitz constants of $\{\varphi_j, \varphi_j^{-1}\}_{j=1}^r$ and n_0 be an integer with $K^2 \lambda^{n_0} < \frac{1}{8}$. Since $p \in \Lambda_1$, we have $F^m \circ \varphi_i([-2,2] \times \{y\}) \subset \mathrm{int} W_0$. Hence, $|F^m \circ \varphi_i([-2,2] \times \{y\})| \leq \lambda^m |\varphi_i([-2,2] \times \{y\})|$ for any $m \geq 0$. Since one of the components of $\varphi_{i'}^{-1} \circ F^{n_0} \circ \varphi_i([-2,2] \times \{y\})$ is contained in $[-2,2] \times \{\bar{y}\}$, we have

$$\varphi_{i'}^{-1} \circ F^{n_0} \circ \varphi_i([-2,2] \times \{y\}) \subset [\bar{x} - K^2 \lambda^{n_0}, \bar{x} + K^2 \lambda^{n_0}] \times \{\bar{y}\} \subset [-\frac{3}{4}, \frac{3}{4}] \times \{\bar{y}\}. \square$$

By the definition of canonical coordinates, we have the following corollary.

Corollary 12 *Under the same notation as before, the component of $\varphi_{i'}^{-1} \circ F^{n_0}(R_i)$ containing $\varphi_{i'}^{-1}(p)$ has the form $\{(x,y) \in [-2,2]^2 \mid x_1(y) \leq x \leq x_2(y), y_1 \leq y \leq y_2\}$, where $x_1, x_2 : [-2,2] \to [-1,1]$ are $\frac{1}{9}$-Lipschitz function and $|y_1|, |y_2| \leq 1$.* □

(Proof) Denote $(E_1(z))^\perp$ by $E_\perp(z)$. Let $P_1 : T_zM|_{W_0} \to E_1(z)$ and $P_2 : T_zM|_{W_0} \to E_\perp(z)$ be orthogonal projections, $\Theta : W_0 \ni z \mapsto |\det D_zF|/\|D_zF|_{E_1(z)}\|$ and $\Theta_n : l \ni z \mapsto \prod_{m=0}^{n-1} \Theta(F^m(z))$. Notice that

$$\Theta_n(z) = \frac{|\det D_zF^n|}{\|D_zF^n|_{E_1(z)}\|} = \frac{\|P_2 \circ DF^n \circ P_2(v(z))\|}{\|P_2(v(z))\|}$$

for any $v(z) \in C_2(z)$. Since $E_1(z), E_\perp(z)$ and $|\det D_zF|$ are Lipschitz, $\Theta(z)$ is also Lipschitz. This implies that there exists a constant $K_0 > 0$ such that

$$|\log \Theta_n(z) - \log \Theta_n(z')| \le K_0 \sum_{m=0}^{n-1} d(F^m(z), F^m(z'))$$

for any integer n and $z, z' \in l$.

Since $E_1 \subset C_1(z)$ and $C_2(z)$ intersects with the angle away from 0, there is a constant $K_1 > 0$ such that

$$K_1^{-1} < \frac{\|P_2 v\|}{\|v\|} < K_1$$

for any non-zero vector $v \in C_2(W_0)$. Applying this inequality to $v \in C_2(l), DF^n(v)$ and $DF \circ P_2 v$, we have

$$K_1^{-3} < \frac{\|DF^n(v)\|}{\|v\|} \cdot \left(\frac{\|P_2 \circ DF^n \circ P_2 v\|}{\|P_2 v\|}\right)^{-1} \cdot \frac{\|DF^n \circ P_2 v\|}{\|P_2 \circ DF^n v\|} < K_1^3. \tag{5}$$

Since $DF^n v = DF^n \circ P_2 v + DF^n \circ P_1 v$, then we have

$$\begin{aligned} \left|\frac{\|DF^n \circ P_2 v\|}{\|DF^n v\|} - 1\right| &= \left|\frac{\|DF^n \circ P_2 v\| - \|DF^n v\|}{\|DF^n v\|}\right| \\ &\le \frac{\|DF^n \circ P_1 v\|}{\|DF^n v\|} \le \lambda^n \frac{\|P_1 v\|}{\|v\|} < 1 \end{aligned}$$

for any $v \in C_2(\bigcap_{m=0}^n F^{-1}(W_0))$. This inequality and (5) imply that there exist a constant K_2 such that

$$K_2^{-1} < \frac{(\|DF^n v\|/\|v\|)}{\Theta_n(z)} < K_2$$

for any $n \in \mathbf{N}$, $z \in l$ and non-zero vector $v \in C_2(z)$. Therefore, we have

$$\begin{aligned} \left|\log \frac{\|DF^n v\|}{\|v\|} - \log \frac{\|DF^n v'\|}{\|v'\|}\right| &\le \log K_2 + |\log \Theta_n(z) - \log \Theta_n(z')| \\ &\le \log K_2 + K_0 \sum_{m=0}^{n-1} d(F^m(z), F^m(z')) \end{aligned}$$

for any $n \in \mathbf{N}$, $z, z' \in l$, $v \in C_2(z)$ and $v' \in C_2(v')$. □

Next lemma is a variation of that given by Schwartz [6].

Lemma 9 *Let $l \subset W_0$ be a C^1-vertical curve with $\inf d(F^n(l), M \setminus W_0) > 0$. Then, there exist $\delta, \gamma > 0$ such that $|F^n(l')| \le (1+\gamma)|F^n(l)|$ and $F^n(l') \subset W_0$ for any $n \in \mathbf{N}$ and any C^1-vertical curve $l' \supset l$ with $|l'| \le (1+\gamma)|l|$.*

Take $\xi \in \Gamma_L(W,K)$ and $z, z' \in U_{z_i} \cap F^{-1}(U_{z_j})$. Let $\bar{x} = \phi_{z_i}(z)$, $\bar{x'} = \phi_{z_i}(z')$, $x = \phi_{z_j}(F(z))$, $x' = \phi_{z_j}(F(z'))$. Denote $y_j(\xi)$ and $y_i(F^*(\xi))$ by y and $\bar{y}$ respectively.

We calculate the Lipschitz constant of $F^*(\xi)$:

$$\begin{aligned}
&|\bar{y}(\bar{x}) - \bar{y}(\bar{x}')| \\
&= \left| \frac{-C_z + A_z y(x)}{D_z - B_z y(x)} - \frac{-C_{z'} + A_{z'} y(x')}{D_{z'} - B_{z'} y(x')} \right| \\
&\leq \frac{(1+\epsilon)^2}{|D_z|\,|D_{\bar{z}}|} \{(1+7\epsilon)K'|D_z| \cdot |\bar{x} - \bar{x}'| + (1+\epsilon)\,|A_{z'}|\,|D_z| \cdot K|x - x'|\} \\
&\leq (1+\epsilon)^2 \left\{ (1+7\epsilon)\frac{K'}{|D_z|} + (1+\epsilon)^3 |A_z| K \right\} |\bar{x} - \bar{x}'|,
\end{aligned}$$

where K' is the Lipschitz constant of $z \to \begin{pmatrix} A_z & B_z \\ C_z & D_z \end{pmatrix}$.

Since $\sup_{z \in W_0} |A_z| < 1$, $\inf_{z \in W_0} |D_z| > 0$ and ϵ is very small,

$$K_0 = \frac{(1+\epsilon)^3(1+7\epsilon)K'}{(1 - (1+\epsilon)^6 \sup|A_z|) \inf|D_z|}.$$

is a positive number. If $K > K_0$, then $|\bar{y}(\bar{x}) - \bar{y}(\bar{x}')| < \frac{K}{1+\epsilon}|\bar{x} - \bar{x}'|$ for any $\bar{x}, \bar{x}' \in \phi_{z_i}(W \cap F^{-1}(W))$. This implies $F^*(\Gamma_L(W,K)) \subset \Gamma_L(W \cap F^{-1}(W), K)$. □

Since $\bar{W} \cap F^{-1}(\bar{W})$ is a neighborhood of Λ_0, we can take a compact neighborhood W_0 of Λ_0 with $W_0 \subset \mathrm{int}(\bar{W} \cap F^{-1}(\bar{W}))$.

Corollary 7 *There exist $E_1 \in \Gamma_1(W_0)$ and $L_0 > K_0$ such that $E_1 \in \Gamma_L(W_0, L_0)$ and $E_1(z) = F^*(E_1)(z)$ if $z \in W_0 \cap F^{-1}(W_0)$.*

(Proof) Take a Lipschitz function $h : \bar{W} \to [0,1]$ such that $h = 1$ on W_0 and $h = 0$ on $\bar{W} \setminus W_0$. We define $G : \Gamma_1(\bar{W}) \to \Gamma_1(\bar{W})$ by $G(\xi)(z) = \langle e_1(z) + \{h(z)y(z)\}e_2(z) \rangle$, where $F^*(\xi) = \langle e_1(z) + y(z)e_2(z) \rangle$. Note that $G(\xi)(z) = F^*(\xi)(z)$ for any $z \in W_0$.

If a constant L_0 is sufficiently larger than the Lipschitz constant of h, then we have $G(\Gamma_L(\bar{W}, L_0)) \subset \Gamma_L(\bar{W}, L_0)$. Since $d(G(\xi), G(\eta)) \leq d(F^*(\xi), F^*(\xi))$ for any $\xi, \eta \in \Gamma_L(\bar{W}, L_0)$, we get a unique fixed point ξ_0 of G and $E_1 = \xi_0|_{W_0}$ satisfies the assertions of the corollary. □

Remark In general, one can show the unique fixed point of F^* on $\Gamma_1(\bigcup_{n=0}^{\infty} F^{-n}(W_0))$ is C^1. However, it is not known whether the analogy of Corollary 7 for C^1 invariant foliation is true or not.

Let $E_2 = \bar{E_2}|_{W_0}$. Then, we have $TM|_{W_0} = E_1 \oplus E_2$. These subbundles define the required Lipschitz foliations.

2.2 Estimate of distortion

We define the distortion of F^n on an C^1 curve $l \subset M$ by

$$\mathrm{dist}(F^n, l) = \sup_{z, z' \in l} \log \frac{\|DF^n|_l(z)\|}{\|DF^n|_l(z')\|}.$$

We call a C^1 curve l in W_0 vertical if $T_z l \subset C_2(z)$ for all $z \in l$. Denote the length of a curve l by $|l|$.

Lemma 8 (distortion estimate) *There exists a function $D : \mathbf{R} \to \mathbf{R}$ such that $\mathrm{dist}(F^n, l) < D(\alpha)$ holds for any C^1 vertical curve l in $\bigcap_{m=0}^{n} F^{-m}(W_0)$ with $\sum_{m=0}^{n} |F^n(l)| = \alpha$.*

(Proof) Let $W' = W \cap F^{-1}(W')$ and

$$D_z F(e_1(z), e_2(z)) = (e_1(F(z)), e_2(F(z))) \begin{pmatrix} A_z & B_z \\ C_z & D_z \end{pmatrix}$$

for $z \in W'$. For $\xi, \eta \in \Gamma_1(W)$ with $\xi(z) = \langle e_1(z) + y(z)e_2(z) \rangle$ and $\eta(z) = \langle e_1(z) + y'(z)e_2(z) \rangle$, we have $d(\xi, \eta) = \sup_{z \in W'} |y(z) - y'(z)|$ and

$$\begin{aligned} & d(F^*(\xi), F^*(\eta)) \\ = & \sup_{z \in W'} \left| \frac{-C_z + y(F(z))A_z}{D_z - y(F(z))B_z} - \frac{-C_z + y'(F(z))A_z}{D_z - y'(F(z))B_z} \right| \\ \leq & \sup_{z \in W'} \frac{|A_z D_z - B_z C_z|}{|D_z - y(F(z))B_z||D_z - y'(F(z))|} |y(F(z)) - y'(F(z))|. \end{aligned} \tag{4}$$

Since $DF^{-1}(e_1(F(z))) \in C_1(z)$ by Lemma 3, we have $|C_z| \leq \epsilon |D_z|$. Similarly, we also have $|B_z| \leq \epsilon |D_z|$ and $|A_z|/|D_z| < (1+\epsilon)\lambda + \epsilon$. These inequalities and (4) implies that

$$d(F^*(\xi), F^*(\eta)) \leq \frac{(1+\epsilon)(\lambda+\epsilon)}{(1-\epsilon)^2} d(\xi, \eta).$$

Since $\lambda < 1$ and ϵ is very small, F^* is a contraction. □

Next, we estimate Lipschitz constants of subbundles. Fix a compact neighborhood $\bar{W}$ of Λ_0 with $\bar{W} \subset U$.

Denote the natural projection $TM \to M$ by π. For each $z \in \bar{W}$, choose a local chart (U_z, ϕ_z) such that U_z is contractive. Then, we have continuous sections $z' \mapsto e_1(z') = e_1(z'; z)$ and $z' \mapsto e_2(z') = e_2(z'; z)$ of $TM|_{U_z}$ such that $e_1(z')$ and $e_2(z')$ are unit vectors in $\bar{E}_1(z')$ and $\bar{E}_2(z')$ respectively. Since E_1 and E_2 are C^1, e_1 and e_2 are C^1. We take a C^1 trivialization $\Phi_z : \pi^{-1}(U_z) \to \phi_z(U_z) \times \mathbf{R}^2$ of TM near z such that $\Phi_z(e_1(z')) = (\phi_z(z'), (\pm 1, 0))$ and $\Phi_z(e_2(z')) = (\phi_z(z'), (0, \pm 1))$ for all $z \in U_z \cap U$ since $TM|_U = \bar{E}_1 \oplus \bar{E}_2$ is a C^1 splitting.

For any $z' \in U_z \cap U$, we define $\psi_{z,z'} : \pi^{-1}(z') \to \mathbf{R}^2$ by $\Phi_z(v) = (\phi_z(\pi(v)), \psi_{z,\pi(v)}(v))$. Since $\bar{W}$ is compact, we choose $z_1, \cdots, z_k$ such that $\bigcup_{i=1}^k U_{z_i} \subset U$.

Definition 5 Let W be a compact subset of $\bar{W}$ and $\xi \in \Gamma_1(W)$. For each $i = 1, \cdots, k$, we have a function $y_i(\xi) : \phi_{z_i}(W \cap U_{z_i}) \subset \mathbf{R}^2 \to \mathbf{R}$ by

$$\psi_{z_i,z}(\xi(z)) = \langle (1, y_i(\xi)(x)) \rangle,$$

for any $x = \phi_{z_i}(z) \in \phi_{z_i}(W \cap U_{z_i})$. We say ξ is K-Lipschitz when all $y_i(\xi)$ are K-Lipschitz.

Let $\Gamma_L(W, K) = \{ \xi \in \Gamma_1(W) |\ \xi \text{ is } K\text{-Lipschitz } \}$.

Lemma 6 *There exists a positive number K_0 such that*

$$F^*(\Gamma_L(W, K)) \subset \Gamma_L(W \cap F^{-1}(W), K)$$

for any compact subset $W \subset \bar{W}$ and $K \leq K_0$.

(Proof) Let $W' = W \cap F^{-1}(W)$ and

$$D_z F(e_1(z), e_2(z)) = (e_1(F(z)), e_2(F(z))) \begin{pmatrix} A_z & B_z \\ C_z & D_z \end{pmatrix}$$

for each $z \in W'$. Notice that $|C_z| \leq \epsilon |D_z|$, $|B_z| \leq \epsilon |D_z|$ and $|A_z|/|D_z| < (1+\epsilon)\lambda + \epsilon$. We can assume $|D_z| < (1+\epsilon)|D_{z'}|$ for any $z, z' \in U_{z_i}$ if we make U_{z_i} sufficiently small.

2 Proof of the Theorem A

Let M be a two-dimensional manifold and $F : M \to M$ be a C^2 diffeomorphism. Let $\mathrm{Per}(F)$ be the set of periodic points and Λ_0 be a compact F-invariant set contained in the closure of $\mathrm{Per}(F)$.

2.1 A contracting foliation

We assume that Λ_0 has a dominated splitting $TM|_{\Lambda_0} = E_- \oplus E_+$ with a contracting direction E_-. Then, there exists a positive number λ such that

$$\frac{\|DF(v)\|}{\|DF(v')\|} < \lambda$$

for any $p \in \Lambda$ and $v \in E_-(p), v' \in E_+(p)$.

Similarly to the case of a hyperbolic set, we can take two invariant cones on a neighborhood of Λ_0. For any splitting $TM|_U = E \oplus E'$ on a open set U and positive number ϵ, we let $C_1(z;\epsilon,E,E') = \{\alpha e(z) + \beta e'(z)|\ |\beta| \le \epsilon|\alpha|\}$ and $C_2(z;\epsilon,E,E') = \{\alpha e(z) + \beta e'(z)|\ |\alpha| \le \epsilon|\beta|\}$, where $e(z), e'(z)$ be unit vectors in $E(z), E'(z)$ respectively.

Lemma 3 *For any small positive number ϵ, a open neighborhood U and a C^1 splitting $TM|_U = \bar{E}_1 \oplus \bar{E}_2$ such that*

$$DF^{-1}(C_1(F(z);\epsilon,\bar{E}_1,\bar{E}_2)) \subset \mathrm{int}C_1(z;\epsilon,\bar{E}_1,\bar{E}_2) \cup \{0\}, \quad (1)$$

$$DF(C_2(z;\epsilon,\bar{E}_1,\bar{E}_2)) \subset \mathrm{int}C_2(F(z);\epsilon,\bar{E}_1,\bar{E}_2) \cup \{0\} \quad (2)$$

and

$$\frac{\|DF(v)\|}{\|DF(v')\|} < \lambda \quad (3)$$

for any $z \in U \cap F^{-1}(U)$, $v \in C_1(z;\epsilon,\bar{E}_1,\bar{E}_2)$ and $v' \in C_2(z;\epsilon,\bar{E}_1,\bar{E}_2)$.

(Proof) We extend the splitting $E_- \oplus E_+$ to an open neighborhood U of Λ_0. If U is sufficiently small, any $z \in U$ satisfies the conditions (1), (2) and (3). Since each subbundle E_- and E_+ can be approximated by C^1 subbundles and any splitting sufficiently near $E_- \oplus E_+$ satisfies (1) and (3), we can take the required C^1 splitting $\bar{E}_1(z) \oplus \bar{E}_2(z)$. □

In this section, we assume ϵ is sufficiently smaller than $1 - \lambda$. We denote $C_1(z;\epsilon,\bar{E}_1,\bar{E}_2)$ and $C_2(z;\epsilon,\bar{E}_1,\bar{E}_2)$ simply by $C_1(z)$ and $C_2(z)$ respectively.

Let $P(V)$ be the projective space of real vector space V. Denote the element in $P(V)$ spanned by a non-zero vector $v \in V$ by $\langle v \rangle$. The set $\bigcup_{z\in M} P(T_zM)$ becomes a fiber bundle naturally. We denote this bundle by LM. For any compact subset W of U, let $\Gamma(W)$ be the set of sections of $LM|_W$ and $\Gamma_1(W) = \{\xi \in \Gamma(W)|\xi(z) \subset C_1(z)$ for any $z \in W\}$. We define distance between two sections $\xi, \eta \in \Gamma_1(W)$ by $d(\xi,\eta) = \sup_{z\in W} |\alpha_\xi(z) - \alpha_\eta(z)|$, where $\xi(z) = \langle e_1(z) + \alpha_\xi(z)e_2(z)\rangle$, $\eta(z) = \langle e_1(z) + \alpha_\eta(z)e_2(z)\rangle$ and $e_1(z)$ and $e_2(z)$ are unit vectors in $E_1(z)$ and $E_2(z)$ respectively. It is easy to see that $(\Gamma_1(W), d)$ is a complete metric space. Define a map $F^* : \Gamma_1(W) \to \Gamma_1(W \cap F^{-1}(W)$ by $F^*(\xi)(z) = DF^{-1}(\xi(F(z))$.

Lemma 4 *For any compact subset W of U, the map $F^* : \Gamma(W) \to \Gamma(W) \cap F^{-1}(W)$ is a contraction. Especially, $F^* : \Gamma(\bigcup_{n=0}^{\infty} F^{-n}(W)) \to \Gamma(\bigcup_{n=0}^{\infty} F^{-n}(W))$ has a unique fixed point.*

Finiteness of periodic attractors For one-dimensional dynamical systems, Mañé [3] showed the following result.

Theorem (Mañé) *Let I be a compact interval and $f : I \to I$ a C^1-map. Assume that the variation of $\log|DF|$ is bounded on some closed subset $J \subset I$. Then, for any closed f-invariant set $\Lambda \subset J$, the periods of periodic attractors in Λ is bounded.*

For two-dimensional dynamical systems, there are two remarkable results on the finiteness of periodic attractors. Newhouse [4] showed that if a C^2 diffeomorphism has a homoclinic tangency, namely, if the stable and unstable manifolds of a hyperbolic periodic point tangentially, then there exists an open set whose closure contains the diffeomorphism and whose generic element has infinitely many periodic attractors. Mañé [2] then showed that, for a generic C^1 diffeomorphism, either the non-wandering set is hyperbolic or it has infinitely many attractors or repellors. The Newhouse's result tells homoclinic tangencies implies coexistence of infinitely many periodic attractors or repellors and Mañé's result tells that coexistence of infinitely many periodic attractors implies lack of hyperbolicity. A natural question which comes out from these results may be:

"Can a C^r diffeomorphism on a two-dimensional manifold with infinitely many periodic attractors be C^r approximated by a diffeomorphism with a homoclinic tangency?"

Related to this question, it has been known that a diffeomorphism with a dominated splitting on the closure of periodic points cannot have a homoclinic tangency. This yields the following question:

"Can a C^r diffeomorphism having a dominated splitting on the closure of periodic points have infinitely many periodic attractors?"

For C^1 diffeomorphisms, the answer is negative [1]. We give in this paper an affirmative answer to this question for C^2 case by using Theorem A and Mañé's theorem applied to the reduced one-dimensional dynamical systems.

Theorem B *Let $F : M \to M$ be a C^2 diffeomorphism on a two-dimensional manifold M. Assume that F has infinitely many periodic attractors with unbounded periods. Then, the set of accumulation points of the attractors cannot have a dominated splitting with a contracting direction.*

Outlined the proof of Theorem A is follows: In subsection 2.1, we construct two Lipschitz foliations E_1 and E_2 on a neighborhood of the invariant set having a dominated splitting. A foliation E_1 is DF-invariant and DF is contracting on it. Another foliation E_2 is not DF-invariant but the angle between E_1 and E_2 is away from zero and satisfies that $\|DF|_{E_1}\|/\|DF_{E_2}\| < 1$. In subsection 2.2, we show a key lemma which gives a distortion estimate. In fact, we bound distortion of F^n on C^1 curves by using the C^2 smoothness of F and Lipschitz condition on E_1. One of the applications given here is a variation of a lemma due to Schwartz [6] for C^2 map on intervals. In subsection 2.3, we construct a Markov cover by using coordinates induced by the foliations E_1 and E_2. Once we take a cover of the invariant set by these coordinates, DF preserves horizontal lines. To get a Markov cover, we divide this cover by those horizontal lines that correspond to "boundary points". Indeed, we show that any points in the invariant set is surrounded by boundary points by using estimate of distortion and make a Markov cover as boxes bounded by boundary points.

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5. $F \circ \varphi_k([-1,1] \times \{y\}) \subset \varphi_l([-1,1] \times \{y'\})$ for any $l \in Z_k$ and $q = \varphi_k(x,y) = F^{-1}(\varphi_l(x',y'))$.

Once a Markov cover $\{(P_k, \varphi_k, Z_k)\}_{k=1}^s$ is given, it allows us to reduce the dynamics of F on $\bigcup_{k=1}^s P_k$ to a one dimensional dynamics. Let $\pi_2 : \mathbf{R}^2 \to \mathbf{R}$ be a natural projection to the second component, $I = [-1,1] \times \{1, \cdots, s\}$, $I_k = \left(\pi_2 \circ \varphi_k(\mathrm{int}P_k \cap F^{-1}(\bigcup_{l \in Z_k} \mathrm{int}P_l))\right) \times \{k\}$ and $I_* = \bigcup_{k=1}^s I_k$. We define a continuous map $f : I_* \to I$ by $f(x,k) = \left((\pi_2 \circ \varphi_l^{-1}) \circ F \circ (\pi_2 \circ \varphi_k^{-1})^{-1}(x), l\right)$ for any $x \in I_k$ with $\varphi_k(0,x) \in \mathrm{int}P_k \cap F^{-1}(\mathrm{int}P_l)$ and $l \in Z_k$. Then, f is well-defined and satisfies that

$$\tilde{\pi}_l \circ F(p) = f \circ \tilde{\pi}_k(p)$$

for any $p \in P_k \cap F^{-1}(P_l)$ with $l \in Z_k$, where $\tilde{\pi}_j(p) = (\pi_2 \circ \varphi_j^{-1}(q), j)$ for any $q \in P_j$. Moreover, f is topologically Markov, namely, if J is a connected component of I_k then f is homeomorphic on J and

$$\overline{f(J)} = [-1,1] \times \{l\}$$

for some $l \in Z_k$. We call $f : I_* \to I$ *the reduced map* of F associated to the Markov cover $\{(P_k, \varphi_k, Z_k)\}$.

One of the aim of this paper is to prove that under some assumptions, an invariant compact set having so called a dominated splitting has a Markov cover and the reduced map associated to it has a good metric property.

To describe the main theorem, we prepare some definitions.

Definition 2 Let F be a diffeomorphism on a manifold and Λ a compact F-invariant set. A splitting $TM|_\Lambda = E_- \oplus E_+$ is called *dominated* if E_- and E_+ are F-invariant subbundles and there exists a positive number $\lambda < 1$ such that

$$\frac{\|DF(v)\|}{\|DF(v')\|} < \lambda$$

for any $p \in \Lambda$ and any unit vectors $v \in E_-(p), v' \in E_+(p)$.

We say a dominated splitting $E_- \oplus E_+$ *has a contracting direction* E_- when $\|DF|_{E_-}\| < \lambda'$ for some $0 < \lambda < 1$.

Let I be an interval and $f : I \to I$ a Lipschitz map. We define distortion $\mathrm{dist}(f, J)$ on a subinterval $J \subset I$ by

$$\mathrm{dist}(f,J) = \sup\left\{\log\frac{|f(J_1)|}{|J_1|} - \log\frac{|f(J_2)|}{|J_2|} \mid J_1, J_2 \subset I \text{ subintervals}\right\}$$

Remark that for any C^1 map f the definition of $\mathrm{dist}(f,J)$ is equivalent to the usual definition of distortion $\mathrm{dist}(f,J) = \sup_{x,y \in J} \log \frac{|Df(x)|}{|Df(y)|}$.

Now, we state one of our main results.

Theorem A *Let $F : M \to M$ be a C^2 diffeomorphism on a two-dimensional manifold M. Assume that a compact F-invariant set $\Lambda_0 \subset \overline{\mathrm{Per}(F)}$ has a dominated splitting $TM|_{\Lambda_0} = E_- \oplus E_+$ with a contracting direction E_-. Then, there exists finite periodic orbits $\{O(p_i)\}_{i=1}^t$ and a Markov cover $\{(P_k, \varphi_k, Z_k)\}_{k=1}^s$ of $\Lambda \setminus \bigcup_{i=1}^t O(p_i)$. Moreover, there exists a function $D : \mathbf{R} \to \mathbf{R}_+$ such that $dist(f^n, J) \le D(\alpha)$ for any interval $J \in \bigcap_{m=0}^n F^{-m}(I_*)$ with $\sum_{m=0}^{n-1} |F^m(J)| \le \alpha$, where the reduced map $f : I_* \to I$ associated to the cover.*

Markov covers and finiteness of periodic attractors for diffeomorphisms with a dominated splitting

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Abstract

In this paper, we show existence of Markov covers for C^2 surface diffeomorphisms with a dominated splitting under some assumptions. Using a Markov cover, we can reduce the dynamics to a one-dimensional dynamical system having a good metric property. As an application, we show finiteness of periodic attractors for the above diffeomorphisms.

1 Introduction

Let F be a diffeomorphism on a manifold. We say an F-invariant compact set Λ is *hyperbolic* when there exist two numbers $K > 0$ and $0 < \lambda < 1$ and a DF-invariant splitting $TM|_\Lambda = E^s \oplus E^u$ such that

$$\|DF^n|_{E^s}\| < K\lambda^n \text{ and } \|DF^{-n}|_{E^u}\| < K\lambda^n$$

for any $n \in \mathbf{N}$. It is known that if a hyperbolic set Λ is locally maximal, *i.e.* $\Lambda = \bigcap_{n\in\mathbf{Z}} F^n(U)$ for some neighborhood U of Λ, then there exists a partition of Λ, which is so called a Markov partition, and it gives complete information in terms of symbolic dynamics associated to the partition. See for example [7]. If the dimension of manifold is two, then we can represent a Markov partition as a family of boxes in the manifold. See for example [5, Appendix 2].

From now on, we assume F is a diffeomorphism on a two-dimensional manifold. We consider an analogous concept to a Markov partition as a family of boxes, which permits interiors of the boxes to intersect and has only one invariant foliation.

Definition 1 Let Λ be a compact F-invariant set. For each $k = 1, \cdots s$, let P_k a compact subset of M, $\varphi_k : [-1,1]^2 \to P_k$ a homeomorphism and Z_k a subset of $\{1, \cdots, s\}$. We say $\{(P_k, \varphi_k, Z_k)\}_{k=1}^s$ is *a Markov cover* of Λ if the following conditions are satisfied:

1. $\Lambda \subset \bigcup_{k=1}^s P_k$,
2. $\mathrm{int}P_i \cap \mathrm{int}P_j = \emptyset$ for any $i, j \in Z_k$ with $i \neq j$,
3. $F(\Lambda \cap P_k) \subset \bigcup_{l\in Z_k} P_l$,
4. $F \circ \varphi_k([-1,1] \times \{-1,1\}) \cap \varphi_l([-1,1] \times (-1,1)) = \emptyset$ for any $l \in Z_k$,