AN ESTIMATE OF GENUS OF LINKS

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Abstract

By using the result of Meng-Taubes [1], we get an estimate of genus of links.

1 Introduction

Let $L = l_1 \cup \ldots \cup l_n$ be an oriented n-components link in S^3. We denote an oriented meridian of l_i by m_i, i.e. we put the orientation of m_i so that $lk(l_i, m_i) = 1$.

Let S be a connected oriented 2-surface smoothly embedded in S^3. The orientation of ∂S is induced by the orientation of S. When an oriented ∂S agrees with an oriented link L, we call S a *Seifert surface* of the L. Of course, there are many Seifert surfaces of the L, and if we reverse the orientation of some components of L, then Seifert surfaces are quite different. We call the minimum number of genus of Seifert surfaces of L *genus of* L , and we denote it by $g(L)$. In this article, we show an estimate of genus of links.

Denote by $\mathbf{Z}[H]$ a group ring of H, where $H = H_1(S^3 \setminus L; \mathbf{Z}) \cong \bigoplus_{i=1}^{n} \mathbf{Z}[m_i]$. Here we view H as a mutiplicative group, and there is a map, $^-; \mathbf{Z}[H] \to \mathbf{Z}[H]$, $\overline{h} = h^{-1}$, where $h \in H$. The next Theorem 1 follows Theorem 2 immediately. We will describe Theorem 2 in section 6.

Theorem 1. *Let L be an oriented n-components ($n \geq 2$) link in S^3, satisfing $lk(l_i, l_j) = 0$, $1 \leq i, j \leq n$. Let $\Delta_L(m_1, \ldots, m_n)$ be the Alexander polynomial of L, then there exists an element A of $\prod_{i=1}^{n}(m_i - 1) \cdot \Delta_L(m_1, \ldots, m_n)$ such that $\overline{A} = A$. A is unique up to multiplication by ± 1. Let S be a genus g Seifert surface of L. Then K satisfies*

$$2(g + n) - 2 \geq 2|K \cap S|,$$

where $K \in \{\ k \in H \mid A = \sum_{k \in H} a_k \cdot k,\ a_k = a_{k^{-1}} \neq 0\ \}$.

Since Kauffman [2] proved the equation $(t - t^{-1}) \cdot \Delta_L(t^2, \ldots, t^2) = det(tV - t^{-1\,T}V)$ up to multiplication by ± 1 and powers of t, where V is a Seifert matrix of an oriented n-components ($n \geq 2$) L and ${}^TV$ is the transpose of the V, it is known that genus of links are estimated from below. In section 6, we give an example which shows that estimate by Theorem 1 is better than this estimate.

Since an Alexander polynomial of a knot K may be defined by the equation $\Delta_K(t) = det(tV - {}^TV)$, where V and ${}^TV$ are as mentioned, the next theorem is well-known. We also give a new proof of it.

Theorem 3. *Let K be a knot in S^3, and let $\Delta_K(t)$ be the Alexander polynomial of K, satisfing $\Delta_K(t) = \Delta_K(t^{-1})$. Then we obtain $g(K) \geq$ degree of $\Delta_K(t)$.*

2 Alexander polynomial

Let X be a compact connected oriented 3-manifold, and let $p; \widetilde{X} \to X$ be a universal free abelian cover. Then $H := H_1(X; \mathbf{Z})/Torsion$ acts on $\widetilde{X}$. We can lift a cell-decomposition of X to $\widetilde{X}$, and we may regard $H_1(\widetilde{X}; \mathbf{Z})$ as a $\mathbf{Z}[H]$-module. Since $\mathbf{Z}[H]$ is Noetherian, $H_1(\widetilde{X}; \mathbf{Z})$ is a finitely generated $\mathbf{Z}[H]$-module. And $\mathbf{Z}[H]$ is factorial, we define the order of a $\mathbf{Z}[H]$-module $H_1(\widetilde{X}; \mathbf{Z})$ as follows.

Let A be a finitely generated module over a ring K, and let A is isomorphic to a $Coker f$, where $f : K^m \to K^n, m \geq n$, is a K-linear map. Then the ideal of K, generated by $n \times n$-minors of the matrix reprisenting f, depends only on A. When K is factorial, we can consider the greatest common divisor of elements of this ideal up to multiplication by a unit. We call the set of these greatest common divisors *the order* of a K-module A.

We call the order of a $\mathbf{Z}[H]$-module $H_1(\widetilde{X}; \mathbf{Z})$ the *Alexander polynomial* of X and denot it by Δ_X. Let L be an oriented n-components link in S^3. We call $\Delta_{S^3 \setminus nd(L)}$,where $nd(L)$ is a tubular neighborhood, the *Alexander polynomial of L*, denoted by Δ_L.

Example.

Figure 1.

Let s, s_1 be oriented meridians of h, h_1. Then $\Delta_{h \cup h_1} = (s-1)(s_1-1)$ up to multiplication by a unit.

The next formula due to Turaev [3] is a very important property of Alexander polynomials in 3-dimension.

Multiplicativity of Alexander polynomials in 3-dimension.

Let X_1, X_2, and Y be connected compact oriented 3-manifolds, and let $Y = X_1 \cap X_2$, $X = X_1 \cup_Y X_2$, and each Euler number ≤ 0. Let

$$A_{X_i} := \begin{cases} \Delta_{X_i}, & \text{when } rank H_1(X_i;\mathbf{Z}) \\ (t_i-1)^{-1}\cdot\Delta_{X_i}, & \text{where } t_i \text{ is a generator of } H_1(X_i;\mathbf{Z}) \\ & \text{,when } rank H_1(X_i;\mathbf{Z})=1, \text{ and } \partial X_i \neq \phi. \end{cases}$$

A_X, A_Y are the same definitions as above.
If $j_*A_Y \neq 0$, then we obtain $A_X = i_{1*}A_{X_1}\cdot i_{2*}A_{X_2}\cdot(j_*A_Y)^{-1}$, where $i_{l*}\colon \mathbf{Z}[H_1(X_l;\mathbf{Z})/Tor] \to \mathbf{Z}[H_1(X;\mathbf{Z})/Tor]$, $(l=1,2)$ and $j_*\colon \mathbf{Z}[H_1(Y;\mathbf{Z})/Tor] \to \mathbf{Z}[H_1(X;\mathbf{Z})/Tor]$ are the inclusion homomorphisms.

3 The calculations

We will calculate Alexander polynomials of some links and 3-manifolds so as to use the result of Meng-Taubes [1].

Lemma 1. *Let $L = l_1 \cup \ldots \cup l_n$ be an oriented n-components ($n \geq 2$) link in S^3, and let an oriented 2n-components link $L \cup \widetilde{L} = l_1 \cup \ldots \cup l_n \cup \widetilde{l_1} \cup \ldots \cup \widetilde{l_n}$ in S^3 be as in the Figure 2.*

Figure 2.

Let m_i, $\widetilde{m_i}$ be oriented meridians of l_i, $\widetilde{l_i}$. Then we obtain

$$\prod_{i=1}^{n}(m_i-1)\cdot\Delta_L(m_1,\ldots m_n) = \Delta_{L\cup\widetilde{L}}(m_1,\ldots,m_n,\overbrace{1,\ldots,1}^{n \text{ times}}).$$

Proof. Let $T_i \cong S^1\times D^2$ be a solid torus, and let ϕ_i be a homeomorphism, $\phi_i : \partial T_i \to \partial nd(\widetilde{l_i})$, sending $S^1 \times point$ to a natural longitude of $\partial nd(\widetilde{l_i})$, and $point \times \partial D^2$to a natural meridian of $\partial nd(\widetilde{l_i})$. Then $S^3 \setminus nd(L) \cong \{S^3 \setminus nd(L\cup\widetilde{L})\} \cup_{\phi_1} T_1 \cup_{\phi_2} \ldots \cup_{\phi_n} T_n$. We use the multiplicativity of Alexander polynomials and obtain

$$\Delta_L(m_1,\ldots,m_n) = \prod_{l=1}^{n}(i_{l*}[S^1\times point]-1)^{-1}\cdot i_*\Delta_{L\cup\widetilde{L}}(m_1,\ldots,m_n,\widetilde{m}_1,\ldots,\widetilde{m}_n).$$

where $i_*\colon \mathbf{Z}[H_1(S^3\backslash L\cup\widetilde{L};\mathbf{Z})] \to \mathbf{Z}[H_1(S^1\backslash L;\mathbf{Z})]$ and $i_{l*}\colon \mathbf{Z}[H_1(T_l;\mathbf{Z})] \to \mathbf{Z}[H_1(S^3\backslash L;\mathbf{Z})]$ $(l=1,\ldots,n)$ are the inclusion homomorphisms. Since $i_*[\widetilde{m}_l]=1$, $i_*[m_l]=[m_l]$ and $i_{l*}[S^1\times point]=[m_l]$, we obtain the equation in the lemma. ∎

Lemma 2. *Let $L = l_1 \cup \ldots \cup l_n$ be an oriented n-components ($n \geq 2$) link in S^3, and let an oriented 2n-components link $L \cup h_1 \cup \ldots \cup h_n$ in S^3 be as in the Figure 3.*

Figure 3.

Let m_i, s_i be oriented meridians of l_i, h_i. Then we obtain

$$\Delta_{L\cup h_1\cup\ldots\cup h_n}(m_1,\ldots,m_n,s_1,\ldots,s_n)=\prod_{i=1}^{n}(m_i-1)\cdot\prod_{i=1}^{n}(s_i-1)\cdot\Delta_{L\cup\widetilde{L}}(m_1,\ldots,m_n,\overbrace{1,\ldots,1}^{n\ times}).$$

where $\widetilde{L}$ is as in Lemma 1.

Proof. Let $T_i \setminus nd(h_i)$ be as in the Figure 1, and let f_i be a homomorphism, $f_i : \partial T_i \to \partial nd(\widetilde{l_i})$, such that $f_i(s) = \widetilde{l_i}$ and $f_i(h) = \widetilde{m_i}$. Then $S^3 \setminus nd(L \cup h_1 \cup \ldots \cup h_n) = \{S^3 \setminus nd(L \cup \widetilde{L})\} \cup_{f_1} \{T_1 \setminus nd(h_1)\} \cup_{f_2} \ldots \cup_{f_n} \{T_n \setminus nd(h_n)\}$. We use the multiplicativity of Alexander polynomials and we obtain

$$\Delta_{L\cup h_1\cup\ldots\cup h_n}(m_1,\ldots,m_n,s_1,\ldots,s_n)=\prod_{l=1}^{n}i_{l*}\Delta_{T_l\setminus nd(h_l)}(s,s_l)\cdot i_*\Delta_{L\cup\widetilde{L}}(m_1,\ldots,m_n,\widetilde{m_1},\ldots,\widetilde{m_n})$$

where $i_* : \mathbf{Z}[H_1(S^3 \setminus L \cup \widetilde{L}; \mathbf{Z})] \to \mathbf{Z}[H_1(S^3 \setminus L \cup h_1 \cup \ldots \cup h_n; \mathbf{Z})]$ and $i_{l*} : \mathbf{Z}[H_1(T_l \setminus h_l; \mathbf{Z})] \to \mathbf{Z}[H_1(S^3 \setminus L \cup h_1 \cup \ldots \cup h_n; \mathbf{Z})]$ $(l = 1, \ldots, n)$ are the inclusion homomorphisms.

Since $i_*[m_i] = [m_i]$, $i_{l*}[s_l] = [s_l]$, $i_*[\widetilde{m_i}] = [s_i]^0 = 1$, $i_{l*}[s] = [m_l]$, and $\Delta_{T_l \setminus nd(h_l)}(s, s_l) = (s-1)(s_l - 1)$, we obtain the equation in the lemma. ∎

Lemma 3. *Let X_L be a 3-manifold obtained by $-\sum_{j\neq i}^{n} lk(l_i, l_j)$-surgery on l_i and 0-surgery on h_i. Then $b_1(X_L) \geq n$ (≥ 2).*

Proof. The result follows from the Mayer-Vietoris homology exact sequence. And

$$H_1(X_L;\mathbf{Z}) \cong \bigoplus_{i=1}^{n}\mathbf{Z}[s_i] \oplus \Big\{\bigoplus_{i=1}^{n}\mathbf{Z}[m_i]/I\Big\},$$

where I is the submodule generated by $\{\sum_{k\neq i}^{n} lk(l_k, l_i)\cdot[m_k] - \sum_{j\neq i}^{n} lk(l_j, l_i)\cdot[m_i],\ 1 \leq i \leq n\}$. Then we obtain $\mathrm{rank} H_1(X_L; \mathbf{Z}) \geq n\,(\geq 2)$. ∎

Proposition 1. $\Delta_{X_L}(m_1,\ldots,m_n,s_1,\ldots,s_n)=\pi_*\Big\{\prod_{i=1}^{n}(m_i-1)\cdot\Delta_L(m_1,\ldots,m_n)\Big\}$, *where the homomorphism* $\pi_*:\mathbf{Z}[H_1(S^3\setminus L;\mathbf{Z})]\to\mathbf{Z}[H_1(X_L;\mathbf{Z})/Torsion]$ *is induced by the inclusion* $\pi: S^3\setminus L\hookrightarrow X_L$.

Proof. From Lemma 1 and Lemma 2 we obtain

$$\Delta_{L\cup h_1\cup\ldots\sqcup h_n}(m_1,\ldots,m_n,s_1,\ldots,s_n)=\prod_{i=1}^{n}(m_i-1)^2\cdot\prod_{i=1}^{n}(s_i-1)\cdot\Delta_L(m_1,\ldots,m_n).$$

We use the multiplicativity of Alexander polynomials and obtain the equation in the proposition by the construction of X_L. ∎

4 The result of Meng-Taubes

Meng and Taubes proved that, for a connected compact oriented and homology oriented 3-manifold with nonzero first Betti number and vanishing Euler number ,the Seiberg-Witten invariant is the same as the Milnor torsion. See Meng-Taubes [1].

Let X be a connected closed oriented 3-manifold with $b_1(X)>1$. For a spin^c structure of X, there exist a spin bundle W (a comlpex vector bundle with rank 2) and a determinant line bundle L. Seiberg-Witten equation is the following equations of a variable configuration (A,ϕ), A is a unitary connection of L and ϕ is a section of W.

$$\begin{cases} F_A &= q(\phi,\phi)+i\mu \\ D_A\phi &= 0 \end{cases}$$

where μ is a closed 2-form. Let (A,ϕ) be a solution of the above equation, then $(2g^{-1}dg+A, g^{-1}\phi)$ is also a solution, where g is a smooth map from X to S^1.

For a spin^c structure c, we denote the set of gauge equivalent classes of solutions of the Seiberg-Witten equation by $\mathcal{M}_c$. If we take μ generic, then $\mathcal{M}_c$ is a zero-dimensional, oriented and compact manifold.

We put

$\underline{S}:=\{(c,x)\mid c$; *a spin^c structure of* X *and* $x\in H^2(X;\mathbf{Z})/Torsion$ *satisfying* $c_1(L)=x\ mod\ Torsion\}$.

Let SW be the map from $\underline{S}$ to $\mathbf{Z}$, sending (c,x) to the sum of signs of the oriented points of $\mathcal{M}_c$. Since $b_1(X)>1$, for generic metrics, SW is well-defined.

We denote $H^2(X;\mathbf{Z})/Torsion\cong H_1(X;\mathbf{Z})/Torsion$ by H and it's group ring by $\mathbf{Z}[H]$. Fix $y=(c_0,x_0)\in\underline{S}$, then for $(c,x)\in\underline{S}$ we can define $z\in H$ with $x-x_0=2z$. We put $SW_y(z)$:=the sum of $SW(c,x)$, where $x-x_0=2z$. Since we consider the case that $b_1(X)>1$, SW_y is an element of $\mathbf{Z}[H]$, i.e. $SW_y:=\sum_{z\in H}^{finite}SW_y(z)\cdot z\in\mathbf{Z}[H]$. If we change y to y', then $SW_{y'}$ differs from SW_y by multiplication of an element of H. We denote the set $\{h\cdot SW_y\mid h\in H\}\subset\mathbf{Z}[H]$ by $\underline{SW}(X)$.

Meng-Taubes. *Let X be a connected closed oriented 3-manifold with $b_1 > 1$. Then $\Delta_X = \underline{SW}(X) \cup -\underline{SW}(X)$.*

In Meng-Taubes [1], They show exactly $\underline{SW}(X) = \tau(X)$ in general situation. By the nature of Seiberg-Witten invariants, there exists a unique element $SW_0 \in \underline{SW}(X)$ such that $\overline{SW_0} = SW_0$.

5 Generalized adjunction formula.

The next is a very important formula. See Kronheimer-Mrowka [4] and Fintushel-Stern [5]

Generalized adjunction formula. *Let M be a smooth connected closed and oriented 4-manifold with $b_2^+(M) > 1$, and let $K_i, (i = 1, \ldots, n)$, be the Seiberg-Witten basic classes of the M. Let $\Sigma \hookrightarrow M$ be a smoothly embedded 2-dimentional genus g surface with non-negative self-intersection number. If genus $g \geq 1$, then every basic class K_i of M satisfies $2g - 2 \geq |\langle K_i, [\Sigma]\rangle| + [\Sigma]\cdot[\Sigma]$.*

We can construct the 3-dimentional version of the above formula.

3-dimentional version. *Let X be a connected closed and oriented 3-manifold with $b_1(X) > 1$, and let $K_i, (i = 1, \ldots, n)$, be the Seiberg-Witten basic classes of X. Let $\Sigma \hookrightarrow X$ be a smoothly embedded 2-dimensional genus g surface. If genus $g \geq 1$, then every basic class K_i of X satisfies $2g - 2 \geq |K_i \cap \Sigma|$.*

Proof is analogus to the one in Kronheimer-Mrowka [4].

6 The estimate of genus of links

Theorem 2. *Let $L = l_1 \cup \ldots \cup l_n$ be an oriented n-components ($n \geq 2$) link in S^3, and let $\Delta_L(m_1, \ldots, m_n)$ be the Alexander polynomial of the L. Then there exists an element $A \in \pi_*\Big\{\prod_{i=1}^{n}(m_i - 1)\cdot\Delta_L(m_1, \ldots, m_n)\Big\}$ such that $\overline{A} = A$, where π_* is as mentioned. A is unique up to multiplication by ± 1. Let S be a genus g Seifert surface of the L, then K satisfies $2(g + n) - 2 \geq 2|K \cap S|$, where $K \in H$ satisfies $\pi_* K \in \{k \mid A = \sum_k a_k \cdot k,\ a_k = a_{k^{-1}} \neq 0\}$.*

When we reverse orientation of some components of L to $\widetilde{L}$, $g(L)$ and $g(\widetilde{L})$ are quite different. For example, suppose $L = l_1 \cup l_2$ has the Alexander polynomial $\Delta_L(m_1, m_2) = (m_1^2 m_2^{-1} - 1 + m_2 m_1^{-2})(m_1 - 1)(m_2 - 1)$, with $lk(l_1, l_2) = 0$. Then $g(L) \geq 2$ by Theorem 2. Now reverse the orientation of l_2, and denote $\widetilde{L} = \widetilde{l_1} \cup \widetilde{l_2}$,where $\widetilde{l_1} = l_1$, $\widetilde{l_2} = l_2^{-1}$, and $\widetilde{m_1} = m_1$, $\widetilde{m_2} = m_2^{-1}$, then $\Delta_{\widetilde{L}}(\widetilde{m_1}, \widetilde{m_2}) = (\widetilde{m_1}^2\widetilde{m_2}^1 - 1 + \widetilde{m_2}^{-1}\widetilde{m_1}^{-2})(\widetilde{m_1} - 1)(\widetilde{m_2} - 1)$, and $g(\widetilde{L}) \geq 4$ by Theorem 2.

Let V be a Seifert matrix of an oriented n-components (n$\geq$2) link in S^3. Of course, there are many Seifert matrixes of the L. Kauffman [2] proved the equation, $(t-t^{-1})\cdot\Delta_L(t^2,\ldots,t^2)=det(tV-t^{-1T}V)$ up to multiplication by ± 1 and powers of t, where ${}^TV$ is the transpose of the V, See Turaev [3] also. From this equation, if we put $h_L \in (t^2-1)^n\cdot\Delta_L(t^2,\ldots,t^2)$ with $h_L=a_0+a_1t^2+\ldots+a_dt^{2d}, a_0\neq 0$, then we obtain $2(g(L)+n)-2\geq d$. This estimate is induced from Theorem 2, also . Owing to the next result by Levine [6], we can see that the estimate of Theorem 2 is better.

Levine. *For a symmetric Laurent polynomial $F(m_1,m_2)$ with $F(1,1)=1$, there are some oriented 2-components link $L=l_1\cup l_2$ in S^3 with $lk(l_1,l_2)=0$ and $\Delta_L=F(m_1,m_2)(m_1-1)(m_2-1)$ up to multiplication by ± 1 and powers of m_1, m_2.*

For example, there is an oriented 2-components link $L=l_1\cup l_2$ in S^3 with $lk(l_1,l_2)=0$ and $\Delta_L(m_1,m_2)=(m_1^p-m_2^p+1-m_2^{-p}+m_1^{-p})(m_1-1)(m_2-1)$, $p\geq 0$. Then we obtain $g(L)\geq 1$ from the degree of the h_L, and $g(L)\geq 1+p$ from Theorem 2.

7 Proof of Theorem 2.

Let $L=l_1\cup\ldots\cup l_2$ be an oriented n-components (n$\geq$2) link in S^3, and let X_L be a 3-manifold as mentioned in section 3.

Lemma 4. *Let S be an Seifert surface of L. We put $S\cap\partial nd(l_i)=[l_i]+a_i[m_i]\in H_1(S^3\setminus L;\mathbf{Z})$. Then $a_i=-\sum_{j\neq i}^{n} lk(l_i,l_j)$.*

Proof.
$$\begin{aligned} 0=\partial S &= \sum_{i=1}^{n}\{[l_i]+a_i[m_i]\} \\ &= \sum_{i=1}^{n}\Big\{\sum_{j\neq i}^{n} lk(l_i,l_j)+a_i\Big\}[m_i] \quad \text{in } H_1(S^3;\mathbf{Z}). \end{aligned}$$
Then we obtain the equation in the lemma. ∎

Let S be a genus g Seifert surface of L, then we put a connected orientable closed and genus $(g+n)$ surface $\widetilde{S}$ in X_L as in the Figure 4.

Glue a meridian disk of surgery at l_i.
Figure 4.

Proof of Theorem 2. Let L be an oriented n-components (n$\geq$2) link in S^3, and let X_L be a 3-manifold as in section 3. Let S be a Seifert surface of the L, and let $\widetilde{S}$ be as in the Figure 4. Since $b_1(X_L) \geq 2$, there is a unique, up to multiplication by ± 1, element $A \in \pi_*\Big\{\prod_{i=1}^{n}(m_i-1)\cdot\Delta_L(m_1,\ldots,m_n)\Big\}$ such that $\overline{A} = A$, by the result of Meng-Taubes and the nature of Seiberg-Witten invariant. And $2K$ is a Seiberg-Witten basic class of X_L for $K \in \{k \mid A = \sum_k a_k \cdot k, \quad a_k = a_{k^{-1}} \neq 0\}$.

Notice that the intersection number $S \cap K = \widetilde{S} \cap \pi_* K$, where $K \in H_1(S^3 \setminus L; \mathbf{Z})$, because the cycle reprisenting K is winded around L. Since genus of $\widetilde{S}$ $\geq$n$\geq$2, we use the generalised adjunction fomula in 3-dimensional version, and we obtain Theorem 2. ∎

Proof of Theorem 3. Let K be a knot in S^3, and let X_K be the same 3-manifold as X_L, but now 0-surgeries on K and h. Then $b_1(X_K)=2$, and $\Delta_{X_K}(m,s)=\pi_*\Delta_K(m)$. Notice that genus of $\widetilde{S}$ $\geq$1. And we obtain the theorem 3. ∎

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