RELATIVE BOGOMOLOV'S INEQUALITY IN THE ARITHMETIC CASE

SHU KAWAGUCHI AND ATSUSHI MORIWAKI

ABSTRACT. In this paper, we will consider an arithmetic analogue of relative Bogomolov's inequality in [14]. We also establish arithmetic Riemann-Roch formulae for stable curves over regular arithmetic varieties and generically finite morphisms of arithmetic varieties.

INTRODUCTION

Let X and Y be smooth algebraic varieties over an algebraically closed field of characteristic zero, and $f : X \to Y$ a semi-stable curve. Let E be a vector bundle on X of rank r, and y a point of Y. In [14], the second author proved that if f is smooth over y and $E|_{X_{\bar{y}}}$ is semistable, then $\mathrm{dis}_{X/Y}(E) = f_*\left(2rc_2(E) - (r-1)c_1^2(E)\right)$ is weakly positive at y.

In this paper, we would like to consider an arithmetic analogue of the above result. Let us fix regular arithmetic varieties X and Y, and a semistable curve $f : X \to Y$. Since we have a good dictionary for translation from a geometric case to an arithmetic case, it looks like routine works. There are, however, two technical difficulties to work over the standard dictionary.

The first one is that how can we define a push-forward of arithmetic cycles in our situation. If $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is smooth, then, according to Gillet-Soulé's arithmetic intersection theory [4], we can get the push-forward $f_* : \widehat{\mathrm{CH}}^{p+1}(X) \to \widehat{\mathrm{CH}}^{p}(Y)$. We would not like to restrict ourself to the case where $f_{\mathbb{Q}}$ is smooth because in the geometric case, weak positivity of $\mathrm{dis}_{X/Y}(E)$ gives wonderful applications to analyses of the boundary of the moduli space of stable curves. Thus the usual push-forward for arithmetic cycles is insufficient for our purpose. A difficulty to define the push-forward arises from a fact: if $f_{\mathbb{C}} : X(\mathbb{C}) \to Y(\mathbb{C})$ is not smooth, then $(f_{\mathbb{C}})_*(\eta)$ is not necessarily C^∞ even for a C^∞ form η. This suggests us that we need to extend the usual arithmetic Chow groups defined by Gillet-Soulé [4]. For this purpose, we will introduce an arithmetic L^1-cycle of codimension p, namely, a pair (Z, g) such that Z is a cycle of codimension p, g is a current of type $(p-1, p-1)$, and g and $dd^c(g) + \delta_{Z(\mathbb{C})}$ are represented by locally integrable forms. Thus, dividing by the usual arithmetically rational equivalence, an arithmetic Chow group, denoted by $\widehat{\mathrm{CH}}^p_{L^1}$, consisting of arithmetic L^1-cycles of codimension p will be defined (cf. §2.2). In this way, we have the natural push-forward

$$f_* : \widehat{\mathrm{CH}}^{p+1}_{L^1}(X) \to \widehat{\mathrm{CH}}^{p}_{L^1}(Y)$$

as desired (cf. Corollary 2.2.3).

The second difficulty is existence of a suitable Riemann-Roch formula in our situation. As before, if $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is smooth, we know the arithmetic Riemann-Roch theorem

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due to Gillet-Soulé [6]. If we ignore Noether's formula, then, under the assumption that $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is smooth, their Riemann-Roch formula can be written in the following form:

$$\widehat{c}_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right) - \mathrm{rk}(E)\widehat{c}_1\left(\det Rf_*(\mathcal{O}_X), h_Q^{\overline{\mathcal{O}}_X}\right) \\ = f_*\left(\frac{1}{2}\left(\widehat{c}_1(\overline{E})^2 - \widehat{c}_1(\overline{E})\cdot\widehat{c}_1(\overline{\omega}_{X/Y})\right) - \widehat{c}_2(\overline{E})\right)$$

where $\overline{E} = (E, h)$ is a Hermitian vector bundle on X and $\overline{\omega}_{X/Y}$ is the dualizing sheaf of $f : X \to Y$ with a Hermitian metric. If we consider a general case, the right hand side in the above equation is well defined and sits in $\widehat{\mathrm{CH}}^1_{L^1}(X)_{\mathbb{Q}}$. On the other hand, the left hand side is rather complicated. If we admit singular fibers of $f_{\mathbb{C}} : X(\mathbb{C}) \to Y(\mathbb{C})$, then the Quillen metric $h_Q^{\overline{E}}$ is no longer C^∞. According to [1], it extends to a generalized metric. Thus, we may define $\widehat{c}_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right)$ (cf. §3.2). In general, this cycle is not a L^1-cycle. However, using Bismut-Bost formula [1], we can see that

$$\widehat{c}_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right) - \mathrm{rk}(E)\widehat{c}_1\left(\det Rf_*(\mathcal{O}_X), h_Q^{\overline{\mathcal{O}}_X}\right)$$

is an element of $\widehat{\mathrm{CH}}^1_{L^1}(Y)$. Thus, we have a way to establish a Riemann-Roch formula in the arithmetic Chow group $\widehat{\mathrm{CH}}^1_{L^1}(Y)_{\mathbb{Q}}$. Actually, we will prove the above formula in our situation (cf. Theorem 5.2.1). The idea to compare two sides in $\widehat{\mathrm{CH}}^1_{L^1}(Y)_{\mathbb{Q}}$ is a tricky Lemma 2.4.1.

Let us go back to our problem. First of all, we need to define an arithmetic analogue of weak positivity. Let α be an element of $\widehat{\mathrm{CH}}^1_{L^1}(Y)_{\mathbb{Q}}$, S a subset of $Y(\mathbb{C})$, and $y \in Y(\overline{\mathbb{Q}})$. We say α is semi-ample at y with respect to S if there are an arithmetic L^1-cycle (E, f) and a positive integer n such that (1) $dd^c(f) + \delta_{E(\mathbb{C})}$ is C^∞ around each $z \in S$, (2) E is effective, (3) $y \notin \mathrm{Supp}(E)$, (4) $f(z) \geq 0$ for all $z \in S$, and (5) $n\alpha$ coincides with the class of (E, f) in $\widehat{\mathrm{CH}}^1_{L^1}(Y)_{\mathbb{Q}}$. Moreover, α is said to be weakly positive at y with respect to S if it is the limit of semi-ample cycles at y with respect to S (for details, see §3.5). For example, if $Y = \mathrm{Spec}(O_K)$, y is the canonical K-valued point, and $S = Y(\mathbb{C})$, then, α is weakly positive at y with respect to S if and only if $\widehat{\deg}(\alpha) \geq 0$, where K is a number field and O_K is the ring of integers in K (cf. Proposition 3.6.1).

Let (E, h) be a Hermitian vector bundle of rank r on X, and $\widehat{\mathrm{dis}}_{X/Y}(E, h)$ an arithmetic discriminant divisor of (E, h) with respect to $f : X \to Y$, that is, an element of $\widehat{\mathrm{CH}}^1_{L^1}(Y)$ given by $f_*(2r\widehat{c}_2(E, h) - (r-1)\widehat{c}_1(E, h)^2)$. We assume that f is smooth over y and $E|_{X_y}$ is poly-stable. In a case where $\dim X = 2$ and $Y = \mathrm{Spec}(O_K)$, Miyaoka [10], Moriwaki [11, 12, 13], and Soulé [16] proved that $\widehat{\deg}\left(\widehat{\mathrm{dis}}_{X/Y}(E, h)\right) \geq 0$, consequently, $\widehat{\mathrm{dis}}_{X/Y}(E, h)$ is weakly positive at y with respect to $Y(\mathbb{C})$. The main theorem of this paper is the following generalization.

Theorem (cf. Theorem 8.2.1)**.** *Under the above assumptions, $\widehat{\mathrm{dis}}_{X/Y}(E, h)$ is weakly positive at y with respect to any subsets S of $Y(\mathbb{C})$ with the following properties:* (1) *S is finite, and* (2) *$f_{\mathbb{C}}^{-1}(z)$ is smooth and $E_{\mathbb{C}}|_{f_{\mathbb{C}}^{-1}(z)}$ is poly-stable for all $z \in S$. In particular, if*

$x \in X(K)$ for some number field K, and $x : \mathrm{Spec}(K) \to X$ extends to $\tilde{x} : \mathrm{Spec}(O_K) \to X$, then $\widehat{\deg}\left(\tilde{x}^*\left(\widehat{\mathrm{dis}}_{X/Y}(E,h)\right)\right) \geq 0$.

Finally, we would like to give a rough sketch of the proof of the above theorem, which, we believe, provides readers with some hints to read this paper.

Step 1. Using the Donaldson's Lagrangian, we reduce to a case where the Hermitian metric h of E along $f_{\mathbb{C}}^{-1}(z)$ is Einstein-Hermitian for each $z \in S$.

Step 2. We set

$$\overline{F}_n = \mathrm{Sym}^n\left(\mathcal{E}nd(\overline{E}) \otimes f^*(\overline{H})\right) \otimes \overline{A} \otimes f^*(\overline{H}),$$

where $\overline{A}$ is a Hermitian line bundle on X and $\overline{H}$ is a Hermitian line bundle on Y. Later we will specify these $\overline{A}$ and $\overline{H}$. By virtue of arithmetic Riemann-Roch for stable curves (cf. Theorem 5.2.1) and formulae of arithmetic Chern classes for symmetric powers (cf. §7.1), we can see that

$$\frac{1}{(r^2+1)!}\widehat{\mathrm{dis}}_{X/Y}(\overline{E}) = -\lim_{n\to\infty}\frac{\widehat{c}_1(\det Rf_*(F_n), h_n)}{n^{r^2+1}},$$

where h_n is a generalized metric of $\det Rf_*(F_n)$ such that $\widehat{c}_1(\det Rf_*(F_n), h_n) \in \widehat{\mathrm{CH}}^1_{L^1}(Y)$ and h_n coincides with the Quillen metric $h_Q^{\overline{F}_n}$ at each $z \in S$.

Step 3. We assume that A is very ample and $A \otimes \omega_{X/Y}^{-1}$ is ample. We choose an arithmetic variety $B \subset X$ such that $B \in |A^{\otimes 2}|$, $B \to Y$ is étale over y, and $B(\mathbb{C}) \to Y(\mathbb{C})$ is étale over each $z \in S$. (Exactly speaking, B is not realized as an element of $|A^{\otimes 2}|$. For simplicity, we assume it.) We set $\overline{G}_n = \overline{F}_n\big|_B$ and $g = f|_B$. Here we suppose that $g_*(\mathcal{E}nd(\overline{E})\big|_B) \otimes \overline{H}$ and $g_*(\overline{A}\big|_B) \otimes \overline{H}$ are generated by small sections at y with respect to S.

Applying Riemann-Roch formula for generically finite morphisms (cf. Theorem 4.2.1), we can find a generalized metric g_n of $\det g_*(G_n)$ such that g_n is equal to the Quillen metric of $\overline{G}_n$ at each $z \in S$, $\widehat{c}_1(\det g_*(G_n), g_n) \in \widehat{\mathrm{CH}}^1_{L^1}(Y)$, and

$$\lim_{n\to\infty}\frac{\widehat{c}_1(\det g_*(G_n), g_n)}{n^{r^2+1}} = 0.$$

Let us consider an exact sequence:

$$0 \to f_*(F_n) \to g_*(G_n) \to R^1 f_*(F_n \otimes A^{\otimes -2})$$

induced by $0 \to F_n \otimes A^{\otimes -2} \to F_n \to G_n \to 0$. Let Q_n be the image of

$$g_*(G_n) \to R^1 f_*(F_n \otimes A^{\otimes -2}).$$

The natural L^2-metric of $g_*(G_n)$ around z induces the quotient metric $\tilde{q}_n$ of Q_n around z for each $z \in S$. Thus, we can find a C^∞ metric q_n of $\det Q_n$ such that q_n is equal to $\det \tilde{q}_n$ at each $z \in S$.

Since

$$\det Rf_*(F_n) = \det g_*(G_n) \otimes (\det Q_n)^{\otimes -1} \otimes \left(\det R^1 f_*(F_n)\right)^{\otimes -1},$$

we have a generalized metric t_n of $\det R^1 f_*(F_n)$ such that

$$(\det Rf_*(F_n), h_n) = (\det g_*(G_n), g_n) \otimes (\det Q_n, q_n)^{\otimes -1} \otimes (\det R^1 f_*(F_n), t_n)^{\otimes -1}.$$

Step 4. We set $a_n = \max_{z\in S}\{\log t_n(s_n, s_n)(z)\}$, where s_n is the canonical section of $\det R^1 f_*(F_n)$. In this step, we will show that $\widehat{c}_1(\det Q_n, q_n)$ is semi-ample at y with respect to S and $a_n \leq O(n^{r^2}\log(n))$. Semi-ampleness of $\widehat{c}_1(\det Q_n, q_n)$ at y is derived from Proposition 3.7.1 and the fact that $g_*(\mathcal{E}nd(\overline{E})\big|_B) \otimes \overline{H}$ and $g_*(\overline{A}\big|_B) \otimes \overline{H}$ are generated by small sections at y with respect to S. The estimation of a_n involves asymptotic behavior of analytic torsion (cf. Corollary 6.5) and a comparison of sup-norm with L^2-norm (cf. Lemma 8.1.1).

Step 5. Thus, using the last equation in Step 3, we can get a decomposition

$$-\frac{\widehat{c}_1(\det Rf_*(F_n), h_n)}{n^{r^2+1}} = \alpha_n + \beta_n$$

such that α_n is semi-ample at y with respect to S and $\lim_{n\to\infty}\beta_n = 0$.

1. Locally integrable forms and their push-forward

1.1. **Locally integrable forms.** Let M be an n-dimensional orientable differential manifold. We assume that M has a countable basis of open sets. Let ω be a C^∞ volume element of M, and $C^0_c(M)$ a set of all complex valued continuous functions on M with compact supports. Then, there is a unique Radon measure μ_ω defined on the topological σ-algebra of M such that

$$L\int_M f d\mu_\omega = \int_M f\omega$$

for all $f \in C^0_c(M)$, where $L\int_M f d\mu_\omega$ is the Lebesgue integral arising from the measure μ_ω.

Let f be a complex valued function on M. We say f is *locally integrable*, denoted by $f \in L^1_{\mathrm{loc}}(M)$, if f is measurable and, for any compact set K,

$$L\int_K |f| d\mu_\omega < \infty.$$

Let ω' be another C^∞ volume form on M. Then, there is a positive C^∞ function a on M with $\omega' = a\omega$. Thus,

$$L\int_K |f| d\mu_{\omega'} = L\int_K |f| a d\mu_\omega,$$

which shows us that local integrability does not depend on the choice of the volume form ω. Moreover, it is easy to see that, for a measurable complex valued function f on M, the following are equivalent.

(a) f is locally integrable.
(b) For each point $x \in M$, there is an open neighborhood U of x such that the closure of U is compact and $L\int_U |f| d\mu_\omega < \infty$.

Let Ω^p_M be a C^∞ vector bundle consisting of C^∞ complex valued p-forms. Let $\pi_p : \Omega^p_M \to M$ be the canonical map. We denote $C^\infty(M, \Omega^p_M)$ (resp. $C^\infty_c(M, \Omega^p_M)$) by $A^p(M)$ (resp. $A^p_c(M)$). Let α be a section of $\pi_p : \Omega^p_M \to M$. We say α is *locally integrable*, or simply *an L^1-form* if, at any point of M, all coefficients of α in terms of local coordinates are locally integrable functions. A set of all locally integrable p-forms is denoted by $L^1_{\mathrm{loc}}(M, \Omega^p_M)$. For

an maximal form α on M, there is a unique function g on M with $\alpha = g\omega$. We denote this function g by $c_\omega(\alpha)$.

Let us define the Lebesgue integral of locally integrable n-form with compact support. Let α be an element of $L^1_{\mathrm{loc}}(M, \Omega^n_M)$ such that the support of α is compact. Then $c_\omega(\alpha) \in L^1_{\mathrm{loc}}(M)$ and $\mathrm{supp}(c_\omega(\alpha))$ is compact. Thus, $L\int_M c_\omega(\alpha) d\mu_\omega$ exists. Let ω' be another C^∞ volume element of M. Then, there is a positive C^∞ function a on M with $\omega' = a\omega$. Here $ac_{\omega'}(\alpha) = c_\omega(\alpha)$. Thus,

$$L\int_M c_{\omega'}(\alpha) d\mu_{\omega'} = L\int_M c_{\omega'}(\alpha) a d\mu_\omega = L\int_M c_\omega(\alpha) a d\mu_\omega.$$

Hence, $L\int_M c_\omega(\alpha) d\mu_\omega$ does not depend on the choice of the volume form ω. Thus, the Lebesgue integral of α is defined by

$$L\int_M \alpha = L\int_M c_\omega(\alpha) d\mu_\omega.$$

Moreover, we denote by $D^p(M)$ the space of currents of type p on M. Then, there is a natural homomorphism

$$[\ ] : L^1_{\mathrm{loc}}(M, \Omega^p_M) \to D^p(M)$$

given by $[\alpha](\phi) = L\int_M \alpha \wedge \phi$ for $\phi \in A^{n-p}_c(M)$. It is well known that the kernel of $[\ ]$ is $\{\alpha \in L^1_{\mathrm{loc}}(M, \Omega^p_M) \mid \alpha = 0 \text{ (a. e.)}\}$. A topology on $D^p(M)$ is defined in the following way. For an sequence $\{T_n\}_{n=1}^\infty$ in $D^p(M)$, $T_n \to T$ as $n \to \infty$ if and only if $T_n(\phi) \to T(\phi)$ as $n \to \infty$ for each $\phi \in A^{n-p}_c(M)$. For an element $T \in D^n(M)$, by abuse of notation, we denote by $c_\omega(T)$ a unique distribution g on M given by $T = g\omega$.

Proposition 1.1.1. *Let T be a current of type p on M. Then, the following are equivalent.*

(1) *T is represented by a L^1-form.*

(2) *For any $\phi \in A^{n-p}(M)$, $c_\omega(T \wedge \phi)$ is represented by a locally integrable function.*

Proof. (1) $\Longrightarrow$ (2): Let $\phi \in A^{n-p}(M)$. Then, by our assumption, for any point $x \in M$, there are an open neighborhood U of x, C^∞ functions $a_1, \ldots, a_r$ on U, and locally integrable functions $b_1, \ldots, b_r$ on U such that

$$c_\omega(T \wedge \phi)|_U = \sum_{i=1}^r [a_i b_i].$$

Thus, if K is a compact set in U, then

$$L\int_K \left| \sum_{i=1}^r a_i b_i \right| d\mu_\omega \leq L\int_K \sum_{i=1}^r |a_i||b_i| d\mu_\omega \leq \max_i \sup_{x \in K} \{|a_i(x)|\} \sum_{i=1}^r L\int_K |b_i| d\mu_\omega < \infty.$$

Thus, we get (2).

(2) $\Longrightarrow$ (1): Before starting the proof, we would like to claim the following fact. Let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of M such that A is a countable set at most. Let λ_α be a locally integrable form U_α with $\lambda_\alpha = \lambda_\beta$ (a. e.) on $U_\alpha \cap U_\beta$ for all $\alpha, \beta \in A$. Then, there is a

locally integrable form λ on M such that $\lambda = \lambda_\alpha$ (a. e.) on U_α for all $\alpha \in A$. Indeed, let us fix a map $a : M \to A$ with $x \in U_{a(x)}$ and define a form λ by $\lambda(x) = \lambda_{a(x)}(x)$. Then, λ is our desired form because for each $\alpha \in A$,

$$\{x \in U_\alpha \mid \lambda(x) \neq \lambda_\alpha(x)\} \subseteq \bigcup_{\beta \in A \setminus \{\alpha\}} \{x \in U_\alpha \cap U_\beta \mid \lambda_\beta(x) \neq \lambda_\alpha(x)\}$$

and the right hand side has measure zero.

Let U be an open neighborhood of a point $x \in M$ and $(x_1, \ldots, x_n)$ a local coordinate of U such that $dx_1 \wedge \cdots \wedge dx_n$ coincides with the orientation by ω. Then, there is a positive C^∞ function a on U with $\omega = a dx_1 \wedge \cdots \wedge dx_n$ over U. We set

$$T = \sum_{i_1 < \cdots < i_p} T_{i_1 \cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

for some distributions $T_{i_1 \cdots i_p}$. We need to show that $T_{i_1 \cdots i_p}$ is represented by a locally integrable function. Since M has a countable basis of open sets, by the above claim, it is sufficient to check that $T_{i_1 \cdots i_p}$ is represented by an integral function on every compact set K in U. Let f be a non-negative C^∞ function on M such that $f = 1$ on K and $\mathrm{supp}(f) \subset U$. Choose $i_{p+1}, \ldots, i_n$ such that $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$. Here we set $\phi = f a dx_{i_{p+1}} \wedge \cdots \wedge dx_{i_n}$. Then, $\phi \in A^{n-p}(M)$ and

$$T \wedge \phi = \epsilon T_{i_1 \cdots i_p} f a dx_1 \wedge \cdots \wedge dx_n = \epsilon T_{i_1 \cdots i_p} f \omega,$$

where $\epsilon = 1$ or -1 depending on the orientation of $\{x_{i_1}, \ldots, x_{i_n}\}$. By our assumption, there is a locally integrable function h on M with $c_\omega(T \wedge \phi) = [h]$. Thus, $[\epsilon h] = T_{i_1 \cdots i_p} f$. Therefore, $T_{i_1 \cdots i_p}$ is represented by ϵh on K because $f = 1$ on K. Thus, we get (2). □

1.2. **Push-forward of L^1-forms as current.** First of all, we recall push-forward of currents. Let $f : M \to N$ be a proper and surjective morphism of orientable manifolds with $d = \dim M - \dim N$. Then,

$$f_* : D^p(M) \to D^{p-d}(M)$$

is defined by $(f_*(T))(\phi) = T(f^*(\phi))$ for $\phi \in A_c^{\dim Y - p + d}(N)$. It is easy to see that f_* is a continuous homomorphism. Let us begin with the following lemma.

Lemma 1.2.1. *Let F be an orientable compact differential manifold and Y an orientable differential manifold. Let ω_F (resp. ω_Y) be a C^∞ volume element of F (resp. Y). Let $p : F \times Y \to Y$ be the projection to the second factor. Then, we have the following.*

(1) *If g is a continuous function on $F \times Y$, then $\int_F g\omega_F$ is a continuous function on Y.*
(2) *If α is a continuous maximal form on $F \times Y$, then $p_*([\alpha])$ is represented by a continuous from. This continuous form is denoted by $\int_p \alpha$.*
(3) *For a continuous function g on $F \times Y$,*

$$\left| c_{\omega_Y} \left(\int_p g\omega_F \wedge \omega_Y \right) \right| \leq c_{\omega_Y} \left(\int_p |g| \omega_F \wedge \omega_Y \right).$$

Proof. (1) This is standard.

(2) Since $\omega_F \wedge \omega_Y$ is a volume form on $F \times Y$, there is a continuous function g on $F \times Y$ with $\alpha = g\omega_F \wedge \omega_Y$. Thus, it is sufficient to show that

$$p_*([\alpha]) = \left[\left(\int_F g\omega_F\right)\omega_Y\right].$$

Indeed, by Fubini's theorem, for $\phi \in A_c^0(Y)$,

$$p_*([\alpha])(\phi) = \int_{F\times Y} \phi\alpha = \int_Y \left(\int_F g\omega_F\right)\phi\omega_Y = \left[\left(\int_F g\omega_F\right)\omega_Y\right](\phi).$$

(3) This is obvious because

$$\left|\int_F g\omega_F\right| \leq \int_F |g|\omega_F.$$

□

Corollary 1.2.2. *Let $f : X \to Y$ be a proper, surjective and smooth morphism of connected complex manifolds. Let ω_X and ω_Y be volume elements of X and Y respectively. Then,*

(1) *For a continuous maximal form α on X, $f_*([\alpha])$ is represented by a continuous form. We denote this continuous form by $\int_f \alpha$.*
(2) *For any continuous functions g on X,*

$$\left|c_{\omega_Y}\left(\int_f g\omega_X\right)\right| \leq c_{\omega_Y}\left(\int_f |g|\omega_X\right).$$

Proof. (1) This is a local question on Y. Thus, we may assume that there are a compact complex manifold F and a differomorphism $h : X \to F \times Y$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow[h]{\sim} & F \times Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & = & Y, \end{array}$$

where $p : F \times Y \to Y$ is the natural projection. Hence, (1) is a consequence of (2) of Lemma 1.2.1.

(2) First, we claim that if the above inequality holds for some special volume elements ω_X and ω_Y, then the same inequality holds for any volume elements. Let ω'_X and ω'_Y be another volume elements of X and Y respectively. We set $\omega'_X = a\omega_X$ and $\omega'_Y = b\omega_Y$. Then, a and b are positive C^∞ functions. Let g be any continuous function on X. Then, by our assumption,

$$\left|c_{\omega_Y}\left(\int_f g\omega'_X\right)\right| = \left|c_{\omega_Y}\left(\int_f ga\omega_X\right)\right| \leq c_{\omega_Y}\left(\int_f |g|a\omega_X\right) = c_{\omega_Y}\left(\int_f |g|\omega'_X\right).$$

On the other hand, for any maximal forms α on Y,

$$c_{\omega_Y}(\alpha) = bc_{\omega'_Y}(\alpha).$$

Thus, we get our claim.

Hence, as in the proof of (1), using the differomorphism h and (3) of Lemma 1.2.1, we can see (2). □

Proposition 1.2.3. *Let $f : X \to Y$ be a proper and surjective morphism of connected complex manifolds. Let U be a non-empty Zariski open set of Y such that f is smooth over U. Let α be a compactly supported continuous maximal form on X. If we set*

$$\lambda = \begin{cases} \displaystyle\int_{f^{-1}(U)\to U} \alpha & \text{on } U, \\ 0 & \text{on } Y \setminus U, \end{cases}$$

then λ is integrable. Moreover, $f_*([\alpha]) = [\lambda]$.

Proof. Let ω_X and ω_Y be volume forms of X and Y respectively. Let h be a function on Y with $\lambda = h\omega_Y$. Then, h is continuous on U by Corollary 1.2.2. Moreover, let g be a continuous function on X with $\alpha = g\omega_X$. We need to show that h is an integrable function. First note that $\int_X |g|\omega_X < \infty$ because g is a compactly supported continuous function. Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of open sets such that $\overline{U_n} \subset U$, $\overline{U_n}$ is compact, $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n \subseteq \cdots$, and $\bigcup_{n=1}^{\infty} U_n = U$. Here we set

$$h_n(y) = \begin{cases} |h(y)| & \text{if } y \in U_n \\ 0 & \text{otherwise.} \end{cases}$$

Then, $0 \leq h_1 \leq h_2 \leq \cdots \leq h_n \leq \cdots$ and $\lim_{n\to\infty} h_n(y) = |h(y)|$. By Corollary 1.2.2,

$$|h|_U| \leq c_{\omega_Y}\left(\int_{f^{-1}(U)\to U} |g|\omega_X\right).$$

Thus,

$$\begin{aligned} \int_{U_n} |h|\omega_Y &\leq \int_{U_n} c_{\omega_Y}\left(\int_{f^{-1}(U_n)\to U_n} |g|\omega_X\right)\omega_Y = \int_{U_n}\int_{f^{-1}(U_n)\to U_n} |g|\omega_X \\ &= \int_{f^{-1}(U_n)} |g|\omega_X \leq \int_X |g|\omega_X. \end{aligned}$$

Therefore,

$$L\int_Y h_n d\mu_{\omega_Y} = \int_{U_n} |h|\omega_Y \leq \int_X |g|\omega_X < \infty.$$

Thus, by Fatou's theorem,

$$L\int_Y |h| d\mu_{\omega_Y} = \lim_{n\to\infty} L\int_Y h_n d\mu_{\omega_Y} \leq L\int_X |g|\omega_X < \infty.$$

Hence, h is integral.

Let ϕ be any element of $A_c^0(Y)$. Then, since $\lim_{n\to\infty} \mu_{\omega_Y}(Y \setminus U_n) = 0$ and $h\phi$ is integrable, by absolute continuity of Lebesgue integral,

$$\lim_{n\to\infty} L\int_{Y\setminus U_n} h\phi d\mu_{\omega_Y} = 0.$$

Thus,

$$\begin{aligned} L\int_Y \lambda\phi &= \lim_{n\to\infty} \left(L\int_{U_n} h\phi d\mu_{\omega_Y} + L\int_{Y\setminus U_n} h\phi d\mu_{\omega_Y} \right) \\ &= \lim_{n\to\infty} L\int_{U_n} h\phi d\mu_{\omega_Y} = \lim_{n\to\infty} \int_{U_n} h\phi\omega_Y = \lim_{n\to\infty} \int_{U_n} \lambda\phi. \end{aligned}$$

In the same way,

$$\int_X \alpha f^*(\phi) = \lim_{n\to\infty} \int_{f^{-1}(U_n)} \alpha f^*(\phi).$$

On the other hand, we have

$$\int_{U_n} \lambda\phi = \int_{f^{-1}(U_n)} \alpha f^*(\phi).$$

Hence

$$\begin{aligned} f_*([\alpha])(\phi) = [\alpha](f^*(\phi)) &= \int_X \alpha \wedge f^*(\phi) = \lim_{n\to\infty} \int_{f^{-1}(U_n)} \alpha f^*(\phi) \\ &= \lim_{n\to\infty} \int_{U_n} \lambda\phi = L\int_Y \lambda\phi = [\lambda](\phi) \end{aligned}$$

Therefore, $f_*([\alpha]) = [\lambda]$. □

Let X be an equi-dimensional complex manifold, i.e., every connected component has the same dimension. We denote by $A^{p,q}(X)$ the space of C^∞ complex valued (p,q)-forms on X. Let $A_c^{p,q}(X)$ be the subspace of compactly supported forms. Let $D^{p,q}(X)$ be the space of currents on X of type (p,q). As before, there is a natural homomorphism

$$[\ ] : L^1_{\mathrm{loc}}(\Omega_X^{p,q}) \to D^{p,q}(X).$$

Then, as corollary of Proposition 1.2.3, we have the following main result of this section.

Proposition 1.2.4. *Let $f : X \to Y$ be a proper morphism of equi-dimensional complex manifolds. We assume that every connected component of X maps surjectively to a connected component of Y. Let α be a L^1-form of type $(p+d, q+d)$ on X, where $d = \dim X - \dim Y$. Then there is a $\lambda \in L^1_{\mathrm{loc}}(\Omega_Y^{p,q})$ with $f_*([\alpha]) = [\lambda]$.*

Proof. Clearly we may assume that Y is connected. Since f is proper, there are finitely many connected components of X, say, $X_1, \ldots, X_e$. If we set $\alpha_i = \alpha|_{X_i}$ and $f_i = f|_{X_i}$ for each i, then $f_*([\alpha]) = (f_1)_*([\alpha_1]) + \cdots + (f_e)_*([\alpha_e])$. Thus, we may assume that X is connected. Further, since $f_*([\alpha \wedge f^*(\phi)]) = f_*([\alpha]) \wedge \phi$ for all $\phi \in A^{\dim Y - p, \dim Y - q}(Y)$, we may assume that α is a maximal form by Proposition 1.1.1.

Let g be a locally integrable function on X with $\alpha = g\omega_X$. Since the question is local with respect to Y, we may assume that g is integrable. Thus, since $C_c^0(Y)$ is dense on $L^1(Y)$

(cf. [15, Theorem 3.14]), there is a sequence $\{g_n\}_{n=1}^{\infty}$ of compactly supported continuous functions on X such that

$$\lim_{n\to\infty} L\int_X |g_n - g| d\mu_{\omega_X} = 0.$$

By Proposition 1.2.3, for each n, there is an integrable function h_n on Y such that $f_*([g_n\omega_X]) = [h_n\omega_Y]$. Moreover, by (2) of Corollary 1.2.2,

$$|h_n - h_m| \leq c_{\omega_Y}\left(\int_{f^{-1}(U)\to U} |g_n - g_m|\omega_X\right)$$

over U. Thus, we can see

$$L\int_Y |h_n - h_m| d\mu_{\omega_Y} \leq L\int_X |g_n - g_m| d\mu_{\omega_X}$$

for all n, m. Hence, $\{h_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^1(Y)$. Therefore, by completeness of $L^1(Y)$, there is an integrable function h on Y with $h = \lim_{n\to\infty} h_n$ in $L^1(Y)$. Then, for any $\phi \in A_c^{0,0}(Y)$,

$$\lim_{n\to\infty} L\int_Y h_n\phi\omega_Y = L\int_Y h\phi\omega_Y \quad \text{and} \quad \lim_{n\to\infty} L\int_X g_n f^*(\phi)\omega_X = L\int_X gf^*(\phi)\omega_X.$$

Thus,

$$\begin{aligned} f_*([\alpha])(\phi) &= L\int_X gf^*(\phi)\omega_X = \lim_{n\to\infty} L\int_X g_n f^*(\phi)\omega_X \\ &= \lim_{n\to\infty} L\int_Y h_n\phi\omega_Y = L\int_Y h\phi\omega_Y = [h\omega_Y](\phi). \end{aligned}$$

Therefore, $f_*([\alpha]) = [h\omega_Y]$. □

2. Variants of arithmetic Chow groups

2.1. **Notation for arithmetic varieties.** An arithmetic variety X is an integral scheme which is flat and quasi-projective over $\mathrm{Spec}(\mathbb{Z})$, and has the smooth generic fiber $X_{\mathbb{Q}}$.

Let us consider a $\mathbb{C}$-scheme $X \otimes_{\mathbb{Z}} \mathbb{C}$. We denote the underling analytic space of $X \otimes_{\mathbb{Z}} \mathbb{C}$ by $X(\mathbb{C})$. We may think $X(\mathbb{C})$ as a set of all $\mathbb{C}$-valued points of X. Let $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ be an anti-holomorphic involution given by the complex conjugation. On the arithmetic variety X, every (p,p)-form α is always assumed to be compatible with F_∞, i.e., $F_\infty^*(\alpha) = (-1)^p\alpha$.

Let E be a locally free sheaf on X of finite rank, and $\pi : \boldsymbol{E} \to X$ the vector bundle associated with E, i.e., $\boldsymbol{E} = \mathrm{Spec}\,(\bigoplus_{n=0}^{\infty} \mathrm{Sym}^n(E))$. As before, we have an analytic space $\boldsymbol{E}(\mathbb{C})$ and an anti-holomorphic involution $F_\infty : \boldsymbol{E}(\mathbb{C}) \to \boldsymbol{E}(\mathbb{C})$. Then, $\pi_{\mathbb{C}} : \boldsymbol{E}(\mathbb{C}) \to X(\mathbb{C})$ is a holomorphic vector bundle on $X(\mathbb{C})$, and the following diagram is commutative:

$$\begin{array}{ccc} \boldsymbol{E}(\mathbb{C}) & \xrightarrow{F_\infty} & \boldsymbol{E}(\mathbb{C}) \\ {\scriptstyle \pi_{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathbb{C}}} \\ X(\mathbb{C}) & \xrightarrow[F_\infty]{} & X(\mathbb{C}) \end{array}$$

Here note that $F_\infty : \boldsymbol{E}(\mathbb{C}) \to \boldsymbol{E}(\mathbb{C})$ is anti-complex linear at each fiber. Let h be a C^∞ Hermitian metric of $\boldsymbol{E}(\mathbb{C})$. We can think h as a C^∞ function on $\boldsymbol{E}(\mathbb{C}) \times_{X(\mathbb{C})} \boldsymbol{E}(\mathbb{C})$. For simplicity, we denote by $F_\infty^*(h)$ a C^∞ function $\left(F_\infty \times_{X(\mathbb{C})} F_\infty\right)^*(h)$ on $\boldsymbol{E}(\mathbb{C}) \times_{X(\mathbb{C})} \boldsymbol{E}(\mathbb{C})$. Then, $\overline{F_\infty^*(h)}$ is a C^∞ Hermitian metric of $\boldsymbol{E}(\mathbb{C})$. We say h *is invariant under* F_∞ if $F_\infty^*(h) = \overline{h}$. Moreover, a pair (E, h) is called *a Hermitian vector bundle on* X if h is invariant under F_∞. Note that even if h is not invariant under F_∞, $h + \overline{F_\infty^*(h)}$ is an invariant metric.

2.2. **Variants of arithmetic cycles.** Let X be an arithmetic variety. We would like to define three types of arithmetic cycles, namely, arithmetic A-cycle, arithmetic L^1-cycle, and arithmetic D-cycle. In the following definition, g is compatible with F_∞ as mentioned in §2.1.

(a) (arithmetic A-cycle on X of codimension p) : a pair (Z, g) such that Z is a cycle on X of codimension p and g is represented by a Green form ϕ of $Z(\mathbb{C})$, namely, ϕ is a C^∞ form on $X(\mathbb{C}) \setminus \mathrm{Supp}(Z(\mathbb{C}))$ of logarithmic type along $\mathrm{Supp}(Z(\mathbb{C}))$ with $dd^c([\phi]) + \delta_{Z(\mathbb{C})} \in A^{p,p}(X(\mathbb{C}))$.

(b) (arithmetic L^1-cycle on X of codimension p) : a pair (Z, g) such that Z is a cycle on X of codimension p and, there are $\phi \in L^1_{\mathrm{loc}}(\Omega^{p-1,p-1}_{X(\mathbb{C})})$ and $\omega \in L^1_{\mathrm{loc}}(\Omega^{p,p}_{X(\mathbb{C})})$ with $g = [\phi]$ and $dd^c(g) + \delta_{Z(\mathbb{C})} = [\omega]$.

(c) (arithmetic D-cycle on X of codimension p) : a pair (Z, g) such that Z is a cycle on X of codimension p and $g \in D^{p-1,p-1}(X(\mathbb{C}))$.

A set of all arithmetic A-cycles (resp. L^1-cycles, D-cycles) of codimension p is denoted by $\widehat{Z}^p_A(X)$ (resp. $\widehat{Z}^p_{L^1}(X)$, $\widehat{Z}^p_D(X)$).

Let $\widehat{R}^p(X)$ be a subgroup of $\widehat{Z}^p(X)$ generated by the following elements:

(i) $((f), -[\log|f|^2])$, where f is a rational function on some subvariety Y of codimension $p-1$ and $[\log|f|^2]$ is the current defined by

$$[\log|f|^2](\gamma) = L\int_{Y(\mathbb{C})} (\log|f|^2)\gamma.$$

(ii) $(0, \partial(\alpha) + \bar{\partial}(\beta))$, where $\alpha \in D^{p-2,p-1}(X(\mathbb{C}))$, $\beta \in D^{p-1,p-2}(X(\mathbb{C}))$.

Here we define

$$\begin{cases} \widehat{\mathrm{CH}}^p_A(X) = \widehat{Z}^p_A(X)/\widehat{R}^p(X) \cap \widehat{Z}^p_A(X), \\ \widehat{\mathrm{CH}}^p_{L^1}(X) = \widehat{Z}^p_{L^1}(X)/\widehat{R}^p(X) \cap \widehat{Z}^p_{L^1}(X), \\ \widehat{\mathrm{CH}}^p_D(X) = \widehat{Z}^p_D(X)/\widehat{R}^p(X). \end{cases}$$

Proposition 2.2.1. *A natural homomorphism* $\widehat{\mathrm{CH}}^p_A(X) \to \widehat{\mathrm{CH}}^p(X)$ *is an isomorphism.*

Proof. Let $(Z, g) \in \widehat{Z}^p(X)$. By [4, Theorem 1.3.5], there is a Green form g_Z of $Z(\mathbb{C})$. Then, $dd^c(g - [g_Z]) \in A^{p,p}(X(\mathbb{C}))$. Hence, by [4, Theorem 1.2.2], there are $a \in A^{d,d}(X(\mathbb{C}))$ and $v \in \mathrm{Image}(\partial) + \mathrm{Image}(\bar{\partial})$ with $g - [g_Z] = [a] + v$. Since $g - [g_Z]$ is compatible with F_∞, replacing a and v by $(1/2)(a + (-1)^p F_\infty^*(a))$ and $(1/2)(v + (-1)^p F_\infty^*(v))$ respectively, we may assume that a and v are compatible with F_∞. Here, $g_Z + a$ is a Green form of Z. Thus, $(Z, [g_Z + a]) \in \widehat{Z}^p_A(X)$. Moreover, since $(Z, g) - (Z, [g_Z + a]) \in \widehat{R}^p(X)$, our proposition follows. □

Proposition 2.2.2. *Let $f : X \to Y$ be a proper and surjective morphism of arithmetic varieties of* $\dim f = d$. *Then, a homomorphism $f_* : \widehat{Z}_D^{p+d}(X) \to \widehat{Z}_D^p(Y)$ given by $f_*(Z, g) = (f_*(Z), f_*(g))$ gives rise to*

$$f_* : \widehat{Z}_{L^1}^{p+d}(X) \to \widehat{Z}_{L^1}^p(Y).$$

Proof. Clearly we may assume that $p \geq 1$. Let $(Z, g) \in \widehat{Z}_{L^1}^{p+d}(X)$. Then, by the definition of L^1-arithmetic cycles, g and $dd^c(g) + \delta_{Z(\mathbb{C})}$ are represented by L^1-forms. Thus, by Proposition 1.2.4, there is a $\omega \in L^1_{\mathrm{loc}}(\Omega^{p,p}_{Y(\mathbb{C})})$ with

$$f_* \left(dd^c(g) + \delta_{Z(\mathbb{C})} \right) = [\omega].$$

On the other hand,

$$f_* \left(dd^c(g) + \delta_{Z(\mathbb{C})} \right) = dd^c(f_*(g)) + \delta_{f_*(Z(\mathbb{C}))}.$$

Moreover, by Proposition 1.2.4, $f_*(g)$ is represented by a L^1-form on $Y(\mathbb{C})$. Thus, $(f_*(Z), f_*(g))$ is an element of $\widehat{Z}_{L^1}^p(Y)$. □

Corollary 2.2.3. *Let $f : X \to Y$ be a proper and surjective morphism of arithmetic varieties with* $\dim f = d$. *Then, there are homomorphisms*

$$f_* : \widehat{\mathrm{CH}}_D^{p+d}(X) \to \widehat{\mathrm{CH}}_D^p(Y) \quad \textit{and} \quad f_* : \widehat{\mathrm{CH}}_{L^1}^{p+d}(X) \to \widehat{\mathrm{CH}}_{L^1}^p(Y)$$

induced by $f_* : \widehat{Z}_D^{p+d}(X) \to \widehat{Z}_D^p(Y)$ and $f_* : \widehat{Z}_{L^1}^{p+d}(X) \to \widehat{Z}_{L^1}^p(Y)$ respectively. In particular, we have a homomorphism $f_* : \widehat{\mathrm{CH}}^{p+d}(X) \to \widehat{\mathrm{CH}}_{L^1}^p(Y)$.

Proof. In the same way as in the proof of [4, Theorem 3.6.1], we can see $f_*(\widehat{R}^{d+p}(X)) \subseteq \widehat{R}^p(Y)$. Thus, this is a consequence of Proposition 2.2.2. □

2.3. **Scalar product for arithmetic L^1-cycles and arithmetic D-cycles.** Let X be a regular arithmetic variety. The purpose of this subsection is to give a scalar product on $\widehat{\mathrm{CH}}_D^*(X)_{\mathbb{Q}} = \bigoplus_{p \geq 0} \widehat{\mathrm{CH}}_D^p(X)_{\mathbb{Q}}$ by the arithmetic Chow ring $\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}} = \bigoplus_{p \geq 0} \widehat{\mathrm{CH}}^p(X)_{\mathbb{Q}}$. Roughly speaking, the scalar product is defined by

$$(Y, f) \cdot (Z, g) = (Y \cap Z, f \wedge \delta_Z + \omega((Y, f)) \wedge g)$$

for $(Y, f) \in \widehat{Z}^p(X)$ and $(Z, g) \in \widehat{Z}_D^q(X)$. This definition, however, works only under the assumption that Y and Z intersect properly. Usually, by using Chow's moving lemma, we can avoid the above assumption. This is rather complicated, so that in this paper we try to use the standard arithmetic intersection theory to define the scalar product.

Let $x \in \widehat{\mathrm{CH}}^p(X)$, $(Z, g) \in \widehat{Z}_D^q(X)$, and g_Z a Green current of Z. First we check that

$$x \cdot [(Z, g_Z)] + [(0, \omega(x) \wedge (g - g_Z))]$$

in $\widehat{\mathrm{CH}}_D^{p+q}(X)_{\mathbb{Q}}$ does not depend on the choice of g_Z. For, let g'_Z be another Green current of Z. Then, there are $\eta \in A^{p-1,p-1}(X(\mathbb{C}))$, and $v \in \mathrm{Image}(\partial) + \mathrm{Image}(\bar{\partial})$ with with $g'_Z = g_Z + [\eta] + v$.

Then, since $[(0,[\eta]+v)] \in \widehat{\mathrm{CH}}^p(X)$,

$$\begin{aligned} x\cdot[(Z,g'_Z)]+[(0,\omega(x)\wedge(g-g'_Z))] &= x\cdot[(Z,g_Z)]+x\cdot[(0,[\eta]+v)] \\ &\quad +[(0,\omega(x)\wedge(g-g_Z-[\eta]-v))] \\ &= x\cdot[(Z,g_Z)]+[(0,\omega(x)\wedge([\eta]+v))] \\ &\quad +[(0,\omega(x)\wedge(g-g_Z-[\eta]-v))] \\ &= x\cdot[(Z,g_Z)]+[(0,\omega(x)\wedge(g-g_Z))]. \end{aligned}$$

Thus, we have a bilinear homomorphism

$$\widehat{\mathrm{CH}}^p(X)\times\widehat{Z}^q_D(X)\to\widehat{\mathrm{CH}}^{p+q}_D(X)_{\mathbb{Q}}$$

given by $x\cdot(Z,g)=x\cdot[(Z,g_Z)]+[(0,\omega(x)\wedge(g-g_Z))]$. Moreover, if $(Z,g)\in\widehat{R}^q(X)$, then, by [4, Theorem 4.2.3], $x\cdot(Z,g)=0$ in $\widehat{\mathrm{CH}}^{p+q}(X)_{\mathbb{Q}}$. Thus, the above induces

$$\widehat{\mathrm{CH}}^p(X)\otimes\widehat{\mathrm{CH}}^q_D(X)\to\widehat{\mathrm{CH}}^{p+q}_D(X)_{\mathbb{Q}}, \tag{2.3.1}$$

which may give rises to a natural scalar product of $\widehat{\mathrm{CH}}^*_D(X)_{\mathbb{Q}}$ over the arithmetic Chow ring $\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}}$. To see that this is actually a scalar product, we need to check that

$$(x\cdot y)\cdot z = x\cdot(y\cdot z)$$

for all $x\in\widehat{\mathrm{CH}}^p(X)$, $y\in\widehat{\mathrm{CH}}^q(X)$ and $z\in\widehat{\mathrm{CH}}^r_D(X)$. If $z\in\widehat{\mathrm{CH}}^r(X)$, then this is nothing more than associativity of the product of the arithmetic Chow ring (cf. [4, Theorem 4.2.3]). Thus, we may assume that $z=[(0,g)]$ for some $g\in D^{r-1,r-1}(X(\mathbb{C}))$. Then, since

$$(x\cdot y)\cdot z=[(0,\omega(x\cdot y)\wedge g)]=[(0,\omega(x)\wedge\omega(y)\wedge g)]$$

and

$$x\cdot(y\cdot z)=x\cdot[(0,\omega(y)\wedge g)]=[(0,\omega(x)\wedge\omega(y)\wedge g)],$$

we can see $(x\cdot y)\cdot z=x\cdot(y\cdot z)$. Therefore, we get the natural scalar product.

Moreover, (2.3.1) induces

$$\widehat{\mathrm{CH}}^p(X)\otimes\widehat{\mathrm{CH}}^q_{L^1}(X)\to\widehat{\mathrm{CH}}^{p+q}_{L^1}(X)_{\mathbb{Q}}. \tag{2.3.2}$$

Indeed, if $(Z,g)\in\widehat{Z}^q_{L^1}(X)$ and g_Z is a Green form of Z, then,

$$x\cdot[(Z,g)]=x\cdot[(Z,g_Z)]+[(0,\omega(x)\wedge(g-g_Z))].$$

Thus, in order to see that $x\cdot[(Z,g)]\in\widehat{\mathrm{CH}}^{p+q}_{L^1}(X)_{\mathbb{Q}}$, it is sufficient to check that

$$\begin{cases}\omega(x)\wedge(g-g_Z)\in L^1_{\mathrm{loc}}(\Omega^{p+q-1,p+q-1}_{X(\mathbb{C})}),\\ dd^c(\omega(x)\wedge(g-g_Z))\in L^1_{\mathrm{loc}}(\Omega^{p+q,p+q}_{X(\mathbb{C})}).\end{cases}$$

The first assertion is obvious because g and g_Z are L^1-forms. Further, we can easily see the second assertion because

$$dd^c(\omega(x)\wedge(g-g_Z))=\pm\omega(x)\wedge dd^c(g-g_Z)=\pm\omega(x)\wedge(\omega(g)-\omega(g_Z)).$$

Gathering all observations, we can conclude the following proposition, which is a generalization of [4, Theorem 4.2.3].

Proposition 2.3.3. *$\widehat{\mathrm{CH}}^*_{L^1}(X)_{\mathbb{Q}}$ and $\widehat{\mathrm{CH}}^*_D(X)_{\mathbb{Q}}$ has a natural module structure over the arithmetic Chow ring $\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}}$.*

Finally, let us consider the following projection formula.

Proposition 2.3.4. *Let $f : X \to Y$ be a proper and surjective morphism of arithmetic varieties. Let (L, h) be a Hermitian line bundle on Y, and $z \in \widehat{\mathrm{CH}}^p_D(X)$. Then*

$$f_*(\widehat{c}_1(f^*L, f^*h) \cdot z) = \widehat{c}_1(L, h) \cdot f_*(z).$$

Proof. Let (Z, g) be a representative of z. Clearly, we may assume that Z is reduced and irreducible. We set $T = f(Z)$ and $\pi = f|_Z : Z \to T$. If $\dim T < \dim Z$, then our assertion is trivial. Thus, we may assume that $\dim T = \dim Z$. Let s be a rational section of $L|_T$. Then, $\pi^*(s)$ gives rise to a rational section of $f^*(L)|_Z = \pi^*(L|_T)$. Thus, $\widehat{c}_1(f^*L, f^*h) \cdot z$ can be represented by

$$(\mathrm{div}(\pi^*(s)), [-\log \pi^*(h|_T)(\pi^*(s), \pi^*(s))] + c_1(f^*L, f^*h) \wedge g),$$

where $[-\log \pi^*(h|_T)(\pi^*(s), \pi^*(s))]$ is a current given by

$$[-\log \pi^*(h|_T)(\pi^*(s), \pi^*(s))](\phi) = \int_{Z(\mathbb{C})} (-\log \pi^*(h|_T)(\pi^*(s), \pi^*(s)))\, \phi.$$

Since

$$\begin{aligned}\int_{Z(\mathbb{C})} (-\log \pi^*(h|_T)(\pi^*(s), \pi^*(s)))\, f^*(\psi) &= \int_{Z(\mathbb{C})} \pi^*((-\log(h|_T)(s, s))\, \psi) \\ &= \deg(\pi) \int_{T(\mathbb{C})} (-\log(h|_T)(s, s))\, \psi\end{aligned}$$

for a C^∞-form ψ on $Y(\mathbb{C})$, we have

$$f_*[-\log \pi^*(h|_T)(\pi^*(s), \pi^*(s))] = \deg(\pi)[-\log(h|_T)(s, s)].$$

Therefore,

$$\begin{aligned}f_*(\widehat{c}_1(f^*L, f^*h) \cdot z) &= (\deg(\pi)\, \mathrm{div}(s), \deg(\pi)[-\log(h|_T)(s, s)] + c_1(L, h) \wedge f_*(g)) \\ &= \widehat{c}_1(L, h) \cdot (\deg(\pi)T, f_*(g)) = \widehat{c}_1(L, h) \cdot f_*(z).\end{aligned}$$

Hence, we get our proposition. □

2.4. **Injectivity of i^*.** Let X be an arithmetic variety, U a non-empty Zariski open set of X, and $i : U \to X$ the inclusion map. Then, there is a natural homomorphism

$$i^* : \widehat{Z}^1_{L^1}(X) \to \widehat{Z}^1_{L^1}(U)$$

given by $i^*(D, g) = (D|_U, g|_{U(\mathbb{C})})$. Since $i^*\left(\widehat{(f)}\right) = \widehat{(f|_U)}$ for any non-zero rational functions f on X, the above induces a homomorphism

$$i^* : \widehat{\mathrm{CH}}^1_{L^1}(X) \to \widehat{\mathrm{CH}}^1_{L^1}(U).$$

Then, we have the following useful lemma.

Lemma 2.4.1. *If $X \setminus U$ does not contain any irreducible components of fibers of $X \to \mathrm{Spec}(\mathbb{Z})$, then*

$$i^* : \widehat{\mathrm{CH}}^1_{L^1}(X) \to \widehat{\mathrm{CH}}^1_{L^1}(U).$$

is injective. In particular, $i^* : \widehat{\mathrm{CH}}^1_{L^1}(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^1_{L^1}(U)_{\mathbb{Q}}$ is injective.

Proof. Suppose that $i^*(\alpha) = 0$ for some $\alpha \in \widehat{\mathrm{CH}}^1_{L^1}(X)$. Let $(D, g) \in \widehat{Z}^1_{L^1}(X)$ be a representative of α. Since $i^*(\alpha) = 0$, there is a non-zero rational function f on X with

$$(D|_U\,,\, g|_{U(\mathbb{C})}) = ((f)|_U\,,\, -[\log|f|^2]\big|_{U(\mathbb{C})}).$$

Pick up $\phi \in L^1_{\mathrm{loc}}(X(\mathbb{C}))$ with $g = [\phi]$. Then, the above implies that $[\phi]|_{U(\mathbb{C})} = -[\log|f|^2]|_{U(\mathbb{C})}$. Thus, $\phi = -\log|f|^2$ (a. e.). Therefore, we have

$$g = [\phi] = -[\log|f|^2]. \tag{2.4.1.1}$$

Here, $dd^c(g) + \delta_{D(\mathbb{C})} = [h]$ for some $h \in L^1_{\mathrm{loc}}(\Omega^{1,1}_{X(\mathbb{C})})$ and $dd^c(-[\log|f|^2]) + \delta_{(f)(\mathbb{C})} = 0$. Thus, by (2.4.1.1), $\delta_{D(\mathbb{C})} - \delta_{(f)(\mathbb{C})} = [h]$. This shows us that $h = 0$ (a. e.) over $X(\mathbb{C}) \setminus (\mathrm{Supp}(D(\mathbb{C})) \cup \mathrm{Supp}((f)(\mathbb{C})))$. Hence $h = 0$ (a. e.) on $X(\mathbb{C})$. Therefore, we have $D(\mathbb{C}) = (f)(\mathbb{C})$, which implies $D = (f)$ on $X_{\mathbb{Q}}$. Thus, $D - (f)$ is a linear combination of irreducible divisors lying on finite fibers. On the other hand, $D = (f)$ on U and $X \setminus U$ does not contains any irreducible components of fibers. Therefore, $D = (f)$. Hence $\alpha = 0$ because $(D, g) = \widehat{(f)}$. □

3. Weakly positive arithmetic divisors

3.1. **Generalized metrics.** Let X be a smooth algebraic scheme over $\mathbb{C}$ and L a line bundle on X. We say h is *a generalized metric on L* if there is a C^∞ Hermitian metric h_0 of L over X and $\varphi \in L^1_{\mathrm{loc}}(X)$ with $h = e^{\varphi} h_0$.

To know when a metric of a line bundle defined over a dense Zariski open set extends to a generalized metric, the following lemma is useful.

Lemma 3.1.1. *Let X be a smooth algebraic variety over $\mathbb{C}$ and L a line bundle on X. Let U be a non-empty Zariski open set of X and h a C^∞ Hermitian metric of L over U. We fix a non-zero rational section s of L. Then, h extends to a generalized metric of L on X if and only if $\log h(s, s) \in L^1_{\mathrm{loc}}(X)$.*

Proof. If h extends to a generalized metric of L on X, then $\log h(s, s) \in L^1_{\mathrm{loc}}(X)$ by the definition of generalized metrics. Conversely, we assume that $\log h(s, s) \in L^1_{\mathrm{loc}}(X)$. Let h_0 be a C^∞ Hermitian metric of L over X. Here we consider a function ϕ given by

$$\phi = \frac{h(s, s)}{h_0(s, s)}.$$

Let $y \in U$ and ω be a local frame of L around y. If we set $s = f\omega$ for some meromorphic function f around y, then

$$\phi = \frac{h(s, s)}{h_0(s, s)} = \frac{|f|^2 h(\omega, \omega)}{|f|^2 h_0(\omega, \omega)} = \frac{h(\omega, \omega)}{h_0(\omega, \omega)}.$$

This shows us that ϕ is a positive C^∞ function on U and $h = \phi h_0$ over U. On the other hand,

$$\log \phi = \log h(s,s) - \log h_0(s,s).$$

Here since $\log h(s,s), \log h_0(s,s) \in L^1_{\mathrm{loc}}(X)$, we have $\log \phi \in L^1_{\mathrm{loc}}(X)$. Thus, if we set $\varphi = \log \phi$, then $\varphi \in L^1_{\mathrm{loc}}(X)$ and $h = e^{\varphi} h_0$. □

3.2. **Arithmetic D-divisors and generalized metrics.** Let X be an arithmetic variety, L a line bundle on X, and h a generalized metric of L on $X(\mathbb{C})$ with $F_\infty^*(h) = \overline{h}$ (a. e.). We would like to define $\widehat{c}_1(L,h)$ as an element of $\widehat{\mathrm{CH}}^1_D(X)$. Let s, s' be two non-zero rational sections of L, and f a non-zero rational function on X with $s' = fs$. Then, it is easy to see that

$$(\mathrm{div}(s'), [-\log h(s',s')]) = (\mathrm{div}(s), [-\log h(s,s)]) + \widehat{(f)}$$

in $\widehat{Z}^1_D(X)$. Thus, we may define $\widehat{c}_1(L,h)$ as the class of $(\mathrm{div}(s), [-\log h(s,s)])$ in $\widehat{\mathrm{CH}}^1_D(X)$.

Let us consider a homomorphism

$$\omega : \widehat{Z}^p_D(X) \to D^{p,p}(X(\mathbb{C}))$$

given by $\omega(Z,g) = dd^c(g) + \delta_{Z(\mathbb{C})}$. Since $\omega\left(\widehat{R}^p(X)\right) = 0$, the above homomorphism induces a homomorphism $\widehat{\mathrm{CH}}^p_D(X) \to D^{p,p}(X(\mathbb{C}))$. Hence, we get a homomorphism $\widehat{\mathrm{CH}}^p_D(X)_{\mathbb{Q}} \to D^{p,p}(X(\mathbb{C}))$ because $D^{p,p}(X(\mathbb{C}))$ has no torsion. By abuse of notation, we denote this homomorphism by ω.

Proposition 3.2.1. *Let X be a regular arithmetic variety and $(Z,[\phi]) \in \widehat{Z}^1_D(X)$ with $\phi \in L^1_{\mathrm{loc}}(X(\mathbb{C}))$, and 1 a rational section of $\mathcal{O}_X(Z)$ with $\mathrm{div}(1) = Z$. Then, there is a unique generalized metric h of $\mathcal{O}_X(Z)$ such that $F_\infty^*(h) = \overline{h}$ (a. e.) and $[-\log h(1,1)] = [\phi]$. (Here uniqueness of h means that if h' is another generated metric with the same property, then $h = h'$ (a. e.).) Moreover, $\omega(Z,[\phi])$ is C^∞ around $x \in X(\mathbb{C})$ if and only if h is C^∞ around x. We denote this line bundle $(\mathcal{O}_X(Z), h)$ with the generalized metric h by $\mathcal{O}_Z((Z,[\phi]))$. With this notation, for $(Z_1,[\phi_1]), (Z_2,[\phi_2]) \in \widehat{Z}^1_D(X)$ with $\phi_1, \phi_2 \in L^1_{\mathrm{loc}}(X(\mathbb{C}))$, if $(Z_1,[\phi_1]) \sim (Z_2,[\phi_2])$, then $\mathcal{O}_X((Z_1,[\phi_1]))$ is isometric to $\mathcal{O}_X((Z_2,[\phi_2]))$ at every point around which $\omega(Z_1,[\phi_1]) = \omega(Z_2,[\phi_2])$ is a C^∞ form.*

Proof. First, let us see uniqueness. Let h and h' be generalized metrics of $\mathcal{O}_X(Z)$ with $[-\log h(1,1)] = [-\log h'(1,1)] = [\phi]$. Take $a \in L^1_{\mathrm{loc}}(X(\mathbb{C}))$ with $h' = e^a h$. Then, by our assumption, $a = 0$ (a. e.). Thus, $h = h'$ (a. e.).

Pick up arbitrary point $x \in X(\mathbb{C})$. Let s be a local basis of $\mathcal{O}_X(Z)$ around x. Then, there is a non-zero rational rational function f on $X(\mathbb{C})$ with $1 = fs$. Let us consider

$$\exp(-\phi - \log|f|^2)$$

around x. Let s' be a another local basis of $\mathcal{O}_X(Z)$ around x. We set $s' = us$ and $1 = f's'$. Then,

$$\exp(-\phi - \log|f'|^2) = \exp(-\phi - \log|f/u|^2) = |u|^2 \exp(-\phi - \log|f|^2),$$

which means that if we define a generalized metric h by

$$h(s,s) = \exp(-\phi - \log|f|^2),$$

then h is patched globally, and h is a generalized metric by Lemma 3.1.1. Moreover,

$$-\log h(1,1) = -\log h(fs,fs) = -\log\left(|f|^2 h(s,s)\right) = \phi.$$

Here, since $F_\infty^*(\phi) = \phi$ (a. e.), we can see $F_\infty^*(h) = \overline{h}$ (a. e.). Thus, we can construct our desired metric.

We set $\omega(Z,[\phi]) = b \in D^{1,1}(X(\mathbb{C}))$. Then, since $1 = fs$ around x, we have $Z = (f)$ around x. Thus, since $dd^c([\phi]) + \delta_{Z(\mathbb{C})} = b$ and $dd^c(-[\log|f|^2]) + \delta_{(f)} = 0$,

$$dd^c(-[\phi + \log|f|^2]) = \delta_{Z(\mathbb{C})} - b - \delta_{(f)} = -b$$

around x. Therefore,

$$\begin{aligned} h \text{ is } C^\infty \text{ around } x &\iff -\phi - \log|f|^2 \text{ is } C^\infty \text{ around } x \\ &\iff dd^c(-[\phi + \log|f|^2]) \text{ is } C^\infty \text{ around } x \qquad (\because [4, \text{Theorem } 1.2.2]) \\ &\iff b \text{ is } C^\infty \text{ around } x \end{aligned}$$

Finally, let us consider the last assertion. By our assumption, there is a rational function z on X such that

$$(Z_1,[\phi_1]) = (Z_2,[\phi_2]) + \widehat{(z)}.$$

Then, $Z_1 = Z_2 + (z)$ and $\phi_1 = \phi_2 - \log|z|^2$. Let us consider a homomorphism $\alpha : \mathcal{O}_X(Z_1) \to \mathcal{O}_X(Z_2)$ defined by $\alpha(s) = zs$. Then, α is an isomorphism. Let 1 be the unit in the rational function field of X. Then, 1 gives rise to canonical rational sections of $\mathcal{O}_X(Z_1)$ and $\mathcal{O}_X(Z_2)$. Let x be a point of $X(\mathbb{C})$ such that $\omega(Z_1,[\phi_1])$ is C^∞ around x, and s a local basis of $\mathcal{O}_X(Z_1)$ around x. Then, $\alpha(s) = zs$ is a local basis of $\mathcal{O}_X(Z_2)$ around x. Choose a rational function f with $1 = fs$. Then, $1 = z^{-1}f\alpha(s)$. Thus, if h_1 and h_2 are metrics of $\mathcal{O}_X((Z_1,[\phi_1]))$ and $\mathcal{O}_X((Z_2,[\phi_2]))$ respectively, then

$$h_1(s,s) = \exp(-\phi_1 - \log|f|^2) = \exp(-\phi_2 - \log|z^{-1}f|^2) = h_2(\alpha(s),\alpha(s))$$

Hence, α is an isometry. □

3.3. **Semi-ampleness and small sections.** Let X be an arithmetic variety and S a subset of $X(\mathbb{C})$. We set

$$\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}} = \{\alpha \in \widehat{\mathrm{CH}}^1_{L^1}(X)_{\mathbb{Q}} \mid \omega(\alpha) \text{ is } C^\infty \text{ around } z \text{ for all } z \in S\}.$$

In the same way, we can define $\widehat{\mathrm{CH}}^1_{L^1}(X;S)$, $\widehat{Z}^1_{L^1}(X;S)$ and $\widehat{Z}^1_{L^1}(X;S)_{\mathbb{Q}}$. Let x be a $\overline{\mathbb{Q}}$-valued point of X, i.e., $x \in X(\overline{\mathbb{Q}})$. An element α of $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$ is said to be *semi-ample at x with respect to S* if there are a positive integer n and $(E,g) \in \widehat{Z}^1_{L^1}(X;S)$ with the following properties:

(a) E is effective and $x \notin \mathrm{Supp}(E)$.
(b) $g(z) \geq 0$ for each $z \in S$. (Note that $g(z)$ might be ∞.)
(c) $n\alpha$ coincides with (E,g) in $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$.

We remark that $\alpha \in \widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$ by the condition (c). Moreover, it is easy to see that if α_1 and α_2 are semi-ample at x with respect to S, so is $\lambda\alpha_1 + \mu\alpha_2$ for all non-negative rational numbers λ and μ.

In terms of a natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $X(\overline{\mathbb{Q}})$, we can consider the orbit $O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(x)$ of x. If we fix an embedding $\overline{\mathbb{Q}} \to \mathbb{C}$, we have an injection $X(\overline{\mathbb{Q}}) \to X(\mathbb{C})$. It is easy to see that the image of $O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(x)$ does not depend on the choice of the embedding $\overline{\mathbb{Q}} \to \mathbb{C}$. By abuse of notation, we denote this image by $O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(x)$. Then, $O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(x)$ is one of examples of S.

Let U be a Zariski open set of X, and F a coherent $\mathcal{O}_X$-module such that F is locally free over U. Let h_F be a C^∞ Hermitian metric of F over $U(\mathbb{C})$. We assume that $S \subseteq U(\mathbb{C})$. For a point $x \in U(\overline{\mathbb{Q}})$, we say (F, h_F) is *generated by small sections at x with respect to S* if there are sections $s_1, \ldots, s_n \in H^0(X, F)$ such that F_x is generated by $s_1, \ldots, s_n$ and $h_F(s_i, s_i)(z) \leq 1$ for all $1 \leq i \leq n$ and $z \in S$.

Proposition 3.3.1. *We assume that X is regular and $S \subseteq U(\mathbb{C})$. For an element (Z, g) of $\widehat{Z}^1_{L^1}(X;S)$, (Z, g) is semi-ample at x with respect to S if and only if there is a positive integer n such that $\mathcal{O}_X(n(Z, g))$ is generated by small sections at x with respect to S.*

Proof. First, we assume that (Z, g) is semi-ample at x with respect to S. Then, there is a $(E, f) \in \widehat{Z}^1_{L^1}(X;S)$ and positive integer n such that $n(Z, g) \sim (E, f)$, E is effective, $x \notin \mathrm{Supp}(E)$, and $f(z) \geq 0$ for each $z \in S$. Then, by Proposition 3.2.1, $\mathcal{O}_X(n(Z, g)) \simeq \mathcal{O}_X((E, f))$. Moreover, if h is the metric of $\mathcal{O}_X((E, f))$ and 1 is the canonical section of $\mathcal{O}_X(E)$ with $\mathrm{div}(1) = E$, then $-\log(h(1, 1)) = f$. Here $f(z) \geq 0$ for each $z \in S$. Thus, $h(1, 1)(z) \leq 1$ for each $z \in S$. Therefore, $\mathcal{O}_X((E, f))$ is generated by small sections at x with respect to S.

Next we assume that $\mathcal{O}_X(n(Z, g))$ is generated by small sections at x with respect to S for some positive integer n. Then, there is a section s of $\mathcal{O}_X(nZ)$ such that $h(s, s)(z) \leq 1$ for each $z \in S$. Thus, if we set $E = \mathrm{div}(s)$ and $f = -\log h(s, s)$, then we can see (Z, g) is semi-ample at x with respect to S. □

Proposition 3.3.2. *Let U be a Zariski open set of X, and L a line bundle on X. Let h be a C^∞ Hermitian metric of L over $U(\mathbb{C})$. Fix a $\overline{\mathbb{Q}}$-valued point x of U. If X is projective over $\mathbb{Z}$, then the followings are equivalent.*

(1) *(L, h) is generated by small sections at x with respect to $U(\mathbb{C})$.*
(2) *(L, h) is generated by small sections at x with respect to any finite subsets S of $U(\mathbb{C})$.*

Proof. Clearly, (1) implies (2). So we assume (2). First of all, we can easily find $z_1, \ldots, z_n \in U(\mathbb{C})$ such that, for any $s \in H^0(X(\mathbb{C}), L_{\mathbb{C}})$, if $s(z_1) = \cdots = s(z_n) = 0$, then $s = 0$. Thus, if we set

$$\|s\| = \sqrt{h(s, s)(z_1)} + \cdots + \sqrt{h(s, s)(z_n)}$$

for each $s \in H^0(X(\mathbb{C}), L_{\mathbb{C}})$, then $\| \ \|$ gives rise to a norm on $H^0(X(\mathbb{C}), L_{\mathbb{C}})$. Here we set

$$B_z = \{s \in H^0(X, L) \mid h(s, s)(z) \leq 1\}$$

for each $z \in U(\mathbb{C})$. Then, since $H^0(X, L)$ is a discrete subgroup of $H^0(X(\mathbb{C}), L_{\mathbb{C}})$, $\bigcap_{i=1}^n B_{z_i}$ is a finite set. Thus, adding finite points $z_{n+1}, \ldots, z_N \in U(\mathbb{C})$ to $z_1, \ldots, z_n$ if necessary, we have

$$\bigcap_{z \in U(\mathbb{C})} B_z = \bigcap_{i=1}^{N} B_{z_i}.$$

By our assumption, there is a section $s \in H^0(X, L)$ such that $s(x) \neq 0$ and $h(s, s)(z_i) \leq 1$ for all $i = 1, \ldots, N$. Then, $s \in \bigcap_{i=1}^N B_{z_i} = \bigcap_{z \in U(\mathbb{C})} B_z$. Thus, we get (2). □

3.4. **Restriction to arithmetic curves.** Let X be a regular arithmetic variety, S a subset of $X(\mathbb{C})$, K a number field, O_K the ring of integers in K, and $x \in X(K)$. We assume that the orbit of x by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is contained in S, namely, $O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(x) \subseteq S$, and that $x : \mathrm{Spec}(K) \to X$ extends to $\tilde{x} : \mathrm{Spec}(O_K) \to X$.

Proposition 3.4.1. *There is a natural homomorphism*

$$\tilde{x}^* : \widehat{\mathrm{CH}}^1_{L^1}(X; S)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^1(\mathrm{Spec}(O_K))_{\mathbb{Q}}$$

such that the restriction of $\tilde{x}^*$ to $\widehat{\mathrm{CH}}^1(X)_{\mathbb{Q}}$ coincides with the usual pull-back homomorphism.

Proof. Let $\alpha \in \widehat{\mathrm{CH}}^1_{L^1}(X; S)_{\mathbb{Q}}$. Choose $(Z, g) \in \widehat{Z}^1_{L^1}(X; S)$ and a positive integer e such that the class of $(1/e)(Z, g)$ in $\widehat{\mathrm{CH}}^1_{L^1}(X; S)_{\mathbb{Q}}$ coincides with α. We would like to define $\tilde{x}^*(\alpha)$ by

$$(1/e)\widehat{c}_1\left(\tilde{x}^*(\mathcal{O}_X((Z, g)))\right).$$

For this purpose, we need to check that the above does not depend on the choice (Z, g) and e. Let (Z', g') and e' be another L^1-cycle of codimension 1 and positive integer such that the class of $(1/e')(Z', g')$ is α. Then, there is a positive integer d such that $de'(Z, g) \sim de(Z', g')$. Thus, by Proposition 3.2.1, $\mathcal{O}_Z(de'(Z, g))$ is isometric to $\mathcal{O}_Z(de(Z', g'))$. Hence,

$$\begin{aligned} de'\widehat{c}_1\left(\tilde{x}^*(\mathcal{O}_X((Z, g)))\right) &= \widehat{c}_1\left(\tilde{x}^*(\mathcal{O}_X(de'(Z, g)))\right) \\ &= \widehat{c}_1\left(\tilde{x}^*(\mathcal{O}_X(de(Z', g')))\right) \\ &= de\widehat{c}_1\left(\tilde{x}^*(\mathcal{O}_X((Z', g')))\right). \end{aligned}$$

Therefore,

$$(1/e)\widehat{c}_1\left(\tilde{x}^*(\mathcal{O}_X((Z, g)))\right) = (1/e')\widehat{c}_1\left(\tilde{x}^*(\mathcal{O}_X((Z', g')))\right).$$

Thus, we can define $\tilde{x}^*$. □

3.5. **Weak positivity of arithmetic L^1-divisors.** Let X be an arithmetic variety, S a subset of $X(\mathbb{C})$, and $x \in X(\overline{\mathbb{Q}})$. Let $\alpha \in \widehat{\mathrm{CH}}^1_{L^1}(X; S)_{\mathbb{Q}}$ and $\{\alpha_n\}_{n=1}^{\infty}$ a sequence of elements of $\widehat{\mathrm{CH}}^1_{L^1}(X; S)_{\mathbb{Q}}$. We say α is the limit of $\{\alpha_n\}_{n=1}^{\infty}$ as $n \to \infty$, denoted by $\alpha = \lim_{n \to \infty} \alpha_n$, if there are (1) $Z_1, \ldots, Z_{l_1} \in \widehat{\mathrm{CH}}^1_{L^1}(X; S)_{\mathbb{Q}}$, (2) $g_1, \ldots, g_{l_2} \in L^1_{\mathrm{loc}}(X(\mathbb{C}))$ with $a(g_j) \in \widehat{\mathrm{CH}}^1_{L^1}(X; S)_{\mathbb{Q}}$ for all j, (3) sequences $\{a^1_n\}_{n=1}^{\infty}, \ldots, \{a^{l_1}_n\}_{n=1}^{\infty}$ of rational numbers, and (4) sequences $\{b^1_n\}_{n=1}^{\infty}, \ldots, \{b^{l_2}_n\}_{n=1}^{\infty}$ of real numbers with the following properties:

(a) l_1 and l_2 does not depend on n.
(b) $\lim_{n\to\infty} a_n^i = \lim_{n\to\infty} b_n^j = 0$ for all $1 \leq i \leq l_1$ and $1 \leq j \leq l_2$.
(c) $\alpha = \alpha_n + \sum_{i=1}^{l_1} a_n^i Z_i + \sum_{j=1}^{l_2} a(b_n^j g_j)$ in $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$ for all n.

It is easy to see that if $\alpha = \lim_{n\to\infty} \alpha_n$ and $\beta = \lim_{n\to\infty} \beta_n$ in $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$, then $\alpha + \beta = \lim_{n\to\infty} (\alpha_n + \beta_n)$.

An element α of $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$ is said to be *weakly positive at x with respect to S* if it is the limit of semi-ample $\mathbb{Q}$-cycles at x with respect to S, i.e., there is a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of elements of $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$ such that α_n's are semi-ample at x with respect to S and $\alpha = \lim_{n\to\infty} \alpha_n$.

Let K be a number field and O_K the ring of integers in K. We assume that $x \in X(K)$, $O_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(x) \subseteq S$, and $x : \mathrm{Spec}(K) \to X$ extends to $\tilde{x} : \mathrm{Spec}(O_K) \to X$. Then, we have the following proposition.

Proposition 3.5.1. *If X is regular and an element α of $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$ is weakly positive at x with respect to S, then $\widehat{\deg}(\tilde{x}^*(\alpha)) \geq 0$.*

Proof. Take $Z_1, \ldots, Z_{l_1}$, $g_1, \ldots, g_{l_2}$, $\{a_n^1\}_{n=1}^{\infty}, \ldots, \{a_n^{l_1}\}_{n=1}^{\infty}$, $\{b_n^1\}_{n=1}^{\infty}, \ldots, \{b_n^{l_2}\}_{n=1}^{\infty}$, and $\{\alpha_n\}_{n=1}^{\infty}$ as in the previous definition of weak positive arithmetic divisors. Then,

$$\widehat{\deg}(\tilde{x}^*(\alpha)) = \widehat{\deg}(\tilde{x}^*(\alpha_n)) + \sum_{i=1}^{l_1} a_n^i \widehat{\deg}(\tilde{x}^*(Z_i)) + \sum_{j=1}^{l_2} b_n^j \widehat{\deg}(\tilde{x}^* a(g_j)).$$

Thus, since $\lim_{n\to\infty} a_n^i = \lim_{n\to\infty} b_n^j = 0$ for all $1 \leq i \leq l_1$ and $1 \leq j \leq l_2$ and $\widehat{\deg}(\tilde{x}^*(\alpha_n)) \geq 0$ for all n, we have $\widehat{\deg}(\tilde{x}^*(\alpha)) \geq 0$. □

3.6. **Characterization of weak positivity.** Let X be an arithmetic variety, S a subset of $X(\mathbb{C})$, and x a $\overline{\mathbb{Q}}$-valued point of X. For an element $\alpha \in \widehat{\mathrm{CH}}^1_{L^1}(X)_{\mathbb{Q}}$, we say α is *ample at x with respect to S* if there are $(A, f) \in \widehat{Z}^1_{L^1}(X;S)$ and a positive integer n such that A is an effective and ample Cartier divisor on X, $x \notin \mathrm{Supp}(A)$, $f(z) > 0$ for all $z \in S$, and $n\alpha$ is equal to (A, f) in $\widehat{\mathrm{CH}}^1_{L^1}(X)_{\mathbb{Q}}$.

First, let us consider a case where $X = \mathrm{Spec}(O_K)$.

Proposition 3.6.1. *We assume that $X = \mathrm{Spec}(O_K)$, x the canonical K-valued point of X, and $S = X(\mathbb{C})$. For an element $\alpha \in \widehat{\mathrm{CH}}^1(X;S)_{\mathbb{Q}}$, we have the following.*

(1) *α is ample at x with respect to S if and only if $\widehat{\deg}(\alpha) > 0$.*
(2) *α is weakly positive at x with respect to S if and only if $\widehat{\deg}(\alpha) \geq 0$.*

Proof. (1) Clearly, if α is ample at x with respect to S, then $\widehat{\deg}(\alpha) > 0$. We assume that $\widehat{\deg}(\alpha) > 0$. We take a positive integer e and a Hermitian line bundle (L, h) on X such that $\widehat{c}_1(L, h) = e\alpha$. Then, $\widehat{\deg}(L, h) > 0$. Thus, by virtue of Riemann-Roch formula and Minkowski's theorem, there are a positive integer n and a non-zero element s of $L^{\otimes n}$ with $(h^{\otimes n})(s, s)(z) < 1$ for all $z \in S$. Thus, α is ample at x with respect to S.

(2) First, we assume that α is weakly positive at x with respect to S. Then, by Proposition 3.5.1, $\widehat{\deg}(\alpha) \geq 0$. Next, we assume that $\widehat{\deg}(\alpha) \geq 0$. Let β be an element of $\widehat{\mathrm{CH}}^1(X;S)_{\mathbb{Q}}$ such that β is ample at x with respect to S. Then, for any positive integer n, $\widehat{\deg}(\alpha + (1/n)\beta) > 0$. Thus, $\alpha + (1/n)\beta$ is ample at x with respect to S by (1). Hence, α is weakly positive at x with respect to S. □

Before starting a general case, let us consider the following lemma.

Lemma 3.6.2. *We assume that S is compact. Let α be an element of $\widehat{\mathrm{CH}}^1_{L^1}(X)_{\mathbb{Q}}$ such that α is ample at x with respect to S. Then, we have the following.*

(1) *Let β be an element of $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$. Then, there is a positive number ϵ_0 such that $\alpha + \epsilon\beta$ is semi-ample at x with respect to S for all rational numbers ϵ with $|\epsilon| \leq \epsilon_0$.*

(2) *Let g be a locally integrable function on $X(\mathbb{C})$ with $a(g) \in \widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$. Then, there is a positive number ϵ_0 such that $\alpha + a(\epsilon g)$ is semi-ample at x with respect to S for all real numbers ϵ with $|\epsilon| \leq \epsilon_0$.*

Proof. (1) First, we claim that there is a positive number t_0 such that $t\alpha+\beta$ is semi-ample at x with respect to S for all rational numbers $t \geq t_0$.

Let us choose $(A, f) \in \widehat{Z}^1_{L^1}(X;S)$ and a positive integer n_0 such that A is an effective and ample Cartier divisor on X, $x \notin \mathrm{Supp}(A)$, $f(z) > 0$ for all $z \in S$, and $n_0\alpha$ is equal to (A, f) in $\widehat{\mathrm{CH}}^1_{L^1}(X)_{\mathbb{Q}}$. Moreover, we choose $(D, g) \in \widehat{Z}^1_{L^1}(X;S)$ and a positive integer e such that $e\beta$ is equal to (D, g) in $\widehat{\mathrm{CH}}^1_{L^1}(X)_{\mathbb{Q}}$. Since A is ample, there is a positive integer n_1 such that $\mathcal{O}_X(n_1A) \otimes \mathcal{O}_X(D)$ is generated by global sections at x. Thus, there are $(Z, h) \in \widehat{Z}^1_{L^1}(X;S)_{\mathbb{Q}}$ such that Z is effective, $x \notin \mathrm{Supp}(Z)$ and $(Z, h) \sim n_1(A, f) + (D, g)$.

We would like to find a positive integer n_2 with $n_2 f(z) + h(z) \geq 0$ for all $z \in S$. Let U be an open set of $X(\mathbb{C})$ such that $S \subseteq U$, and $\omega(A, f)$ and $\omega(Z, h)$ are C^∞ over U. We set $\phi = \exp(-f)$ and $\psi = \exp(-h)$. Then, ϕ and ψ are continuous on U, and $0 \leq \phi < 1$ on S. Since $n_2 f + h = -\log(\phi^{n_2}\psi)$, it is sufficient to find a positive integer n_2 with $\phi^{n_2}\psi \leq 1$ on S. If we set $a = \sup_{z \in S} \phi(z)$ and $b = \sup_{z \in S} \psi(z)$, then $0 \leq a < 1$ and $0 \leq b$ because S is compact. Thus, there is a positive integer n_2 with $a^{n_2} b \leq 1$. Therefore, $\phi^{n_1}\psi \leq 1$ on S.

Here we set $t_0 = (n_1 + n_2)n_0 e^{-1}$. In order to see that $t\alpha + \beta$ is semi-ample at x with respect to S for $t \geq t_0$, it is sufficient to show that $(n_1+n_2)n_0\alpha + e\beta$ is semi-ample at x with respect to S because $et \geq (n_1 + n_2)n_0$. Here

$$\begin{aligned}(n_1 + n_2)n_0\alpha + e\beta &\sim n_2(A, f) + (n_1(A, f) + (D, g)) \\ &\sim n_2(A, f) + (Z, h) \\ &= (n_2A + Z, n_2 f + h),\end{aligned}$$

$x \notin \mathrm{Supp}(n_2A+Z)$, and $(n_2f+h)(z) \geq 0$ for all $z \in S$. Thus, $(n_1+n_2)n_0\alpha+e\beta$ is semi-ample at x with respect to S. Hence, we get our claim.

In the same way, we can find a positive number t_1 such that $t\alpha - \beta$ is semi-ample with respect to S for all $t \geq t_1$. Thus, if we set $\epsilon_0 = \min\{1/t_0, 1/t_1\}$, then we have (1).

(2) In the same way as in the proof of (1), we can find a positive number ϵ_0 such that $(f+\epsilon n_0 g)(z) \geq 0$ for all $z \in S$ and all real number ϵ with $|\epsilon| \leq \epsilon_0$. Thus we have (2) because $n_0(\alpha + a(\epsilon g)) \sim (A, f + \epsilon n_0 g)$. □

Proposition 3.6.3. *We assume that S is compact. Let β be an element of $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$. Then the following are equivalent.*

(1) *β is weakly positive at x with respect to S.*

(2) *$\beta + \alpha$ is semi-ample at x with respect to S for any ample $\alpha \in \widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$ at x with respect to S.*

Proof. (1) $\Longrightarrow$ (2): Since β is weakly positive at x with respect to S, there is a sequence of $\{\beta_n\}$ such that $\beta_n \in \widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$, β_n's are semi-ample at x with respect to S, and $\lim_{n\to\infty} \beta_n = \beta$. Take $Z_1, \ldots, Z_{l_1}, g_1, \ldots, g_{l_2}, \{a^1_n\}_{n=1}^{\infty}, \ldots, \{a^{l_1}_n\}_{n=1}^{\infty}$, and $\{b^1_n\}_{n=1}^{\infty}, \ldots, \{b^{l_2}_n\}_{n=1}^{\infty}$ as in the definition of the limit in $\widehat{\mathrm{CH}}^1_{L^1}(X;S)_{\mathbb{Q}}$. Then, by Lemma 3.6.2, there is a positive number ϵ_0 such that $\alpha + \epsilon Z_i$'s are semi-ample at x with respect to S for all rational numbers ϵ with $|\epsilon| \leq \epsilon_0$, and $\alpha + a(\epsilon g_j)$'s are semi-ample at x with respect to S for all real numbers ϵ with $|\epsilon| \leq \epsilon_0$. We choose n such that $(l_1+l_2)|a^i_n| \leq \epsilon_0$ and $(l_1+l_2)|b^j_n| \leq \epsilon_0$ for all i and j. Then,

$$\beta + \alpha = \beta_n + \sum_{i=1}^{l_1} \frac{\alpha + (l_1+l_2)a^i_n Z_i}{l_1+l_2} + \sum_{j=1}^{l_2} \frac{\alpha + a((l_1+l_2)b^j_n g_j)}{l_1+l_2}.$$

Here, $\alpha + (l_1+l_2)a^i_n Z_i$ and $\alpha + a((l_1+l_2)b^j_n g_j)$ are semi-ample x with respect to S. Thus, we get the direction (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (1): Let α be an element of $\widehat{\mathrm{CH}}^1_{L^1}(X)_{\mathbb{Q}}$ such that α is ample at x with respect to S. We set $\beta_n = \beta + (1/n)\alpha$. Then, by our assumption, β_n is semi-ample at x with respect to S. Further, $\beta = \lim_{n\to\infty} \beta_n$. □

3.7. **Small sections via generically finite morphisms.** Let $g : V \to U$ be a proper and étale morphism of complex manifolds. Let (E,h) be a Hermitian vector bundle on V. Then, a Hermitian metric $g_*(h)$ of $g_*(E)$ is defined by

$$g_*(h)(s,t)(y) = \sum_{x \in g^{-1}(y)} h(s,t)(x)$$

for any $y \in U$ and $s, t \in g_*(E)_y$.

Proposition 3.7.1. *Let X be a scheme such that every connected component of X is a arithmetic variety. Let Y be a regular arithmetic variety, and $g : X \to Y$ a proper and generically finite morphism such that every connected component of X maps surjectively to Y. Let U be a Zariski open set of Y such that g is étale over U. Let S be a subset of $U(\mathbb{C})$ and $y \in U(\overline{\mathbb{Q}})$. Then, we have the following.*

(1) *Let $\phi : E \to Q$ be a homomorphism of coherent $\mathcal{O}_X$-modules such that ϕ is surjective over $g^{-1}(U)$, and E and Q are locally free over $g^{-1}(U)$. Let h_E be a C^∞ Hermitian metric of E over $g^{-1}(U)(\mathbb{C})$, and h_Q the quotient metric of Q induced by h_E. If*

$(g_*(E), g_*(h_E))$ is generated by small sections at y with respect to S, then $(g_*(Q), g_*(h_Q))$ is generated by small sections at y with respect to S.

(2) *Let E_1 and E_2 be coherent $\mathcal{O}_X$-modules such that E_1 and E_2 are locally free over $g^{-1}(U)$. Let h_1 and h_2 be C^∞ Hermitian metrics of E_1 and E_2 over $g^{-1}(U)(\mathbb{C})$. If $(g_*(E_1), g_*(h_1))$ and $(g_*(E_2), g_*(h_2))$ are generated by small sections at y with respect to S, then so is $(g_*(E_1 \otimes E_2), g_*(h_1 \otimes h_2))$.*

(3) *Let E be a coherent $\mathcal{O}_X$-module such that E is locally free over $g^{-1}(U)$. Let h_E be a C^∞ Hermitian metric of E over $g^{-1}(U)(\mathbb{C})$. If $(g_*(E), g_*(h_E))$ is generated by small sections at y with respect to S, then $(g_*(\mathrm{Sym}^n(E)), g_*(\mathrm{Sym}^n(h_E)))$ is generated by small sections at y with respect to S. (For the definition of $\mathrm{Sym}^n(h_E)$, see §7.1.)*

(4) *Let F be a coherent $\mathcal{O}_Y$-module such that F is locally free over U. Let h_F be a C^∞-Hermitian metric of F over $U(\mathbb{C})$. Since $\det(F)|_U$ is canonically isomorphic to $\det(F|_U)$, $\det(h_F)$ gives rise to a C^∞ Hermitian metric of $\det(F)$ over $U(\mathbb{C})$. If (F, h_F) is generated by small sections at y with respect to S, then so is $(\det(F), \det(h_F))$.*

Proof. (1) By our assumption, $g_*(\phi) : g_*(E) \to g_*(Q)$ is surjective over U. Let $s_1, \ldots, s_l \in H^0(Y, g_*(E)) = H^0(X, E)$ such that $g_*(E)_y$ is generated by $s_1, \ldots, s_l$ and $g_*(h_E)(s_i, s_i)(z) \leq 1$ for all i and $z \in S$. Then, $g_*(Q)_y$ is generated by $g_*(\phi)(s_1), \ldots, g_*(\phi)(s_l)$. Moreover, by the definition of he quotient metric h_Q,

$$g_*(h_Q)(g_*(\phi)(s_i), g_*(\phi)(s_i))(z) = \sum_{x \in g^{-1}(z)} h_Q(\phi(s_i), \phi(s_i))(x) \leq \sum_{x \in g^{-1}(z)} h_E(s_i, s_i)(x) \leq 1$$

for all $z \in S$. Hence, $g_*(Q)$ is generated by small sections at y with respect to S.

(2) Since g is étale over U, $\alpha : g_*(E_1) \otimes g_*(E_2) \to g_*(E_1 \otimes E_2)$ is surjective over U. By our assumption, there are $s_1, \ldots, s_l \in H^0(Y, g_*(E_1))$ and $t_1, \ldots, t_m \in H^0(Y, g_*(E_2))$ such that $g_*(E_1)_y$ (resp. $g_*(E_2)_y$) is generated by $s_1, \ldots, s_l$ (resp. $t_1, \ldots, t_m$), and $g_*(h_1)(s_i, s_i)(z) \leq 1$ and $g_*(h_2)(t_j, t_j)(z) \leq 1$ for all i, j and $z \in S$. Then, $g_*(E_1 \otimes E_2)_y$ is generated by $\{\alpha(s_i \otimes t_j)\}_{i,j}$. Moreover,

$$\begin{aligned} g_*(h_1 \otimes h_2)(\alpha(s_i \otimes t_j), \alpha(s_i \otimes t_j))(z) &= \sum_{x \in g^{-1}(z)} (h_1 \otimes h_2)(s_i \otimes t_j, s_i \otimes t_j)(x) \\ &= \sum_{x \in g^{-1}(z)} h_1(s_i, s_i)(x) h_2(t_j, t_j)(x) \\ &\leq \left(\sum_{x \in g^{-1}(z)} h_1(s_i, s_i)(x) \right) \left(\sum_{x \in g^{-1}(z)} h_2(t_j, t_j)(x) \right) \\ &\leq 1 \end{aligned}$$

for all $z \in S$. Thus, we get (2).

(3) This is a consequence of (1) and (2).

(4) Let r be the rank of F. Since F is generated by small sections at y, there are $s_1, \ldots, s_r \in H^0(Y, F)$ such that $F \otimes \kappa(y)$ is generated by $s_1, \ldots, s_r$ and $h(s_i, s_i)(z) \leq 1$ for all i and $z \in S$. Let us consider an exact sequence:

$$0 \to F_{tor} \to F \to F/F_{tor} \to 0.$$

Then, $\det(F) = \det(F/F_{tor}) \otimes \det(F_{tor})$. Noting that $F_{tor} = 0$ on U, let g be a Hermitian metric of $\det(F/F_{tor})$ over $U(\mathbb{C})$ given by $\det(h_F)$. Then, there is a Hermitian metric k of $\det(F_{tor})$ over $U(\mathbb{C})$ such that $(\det(F), \det(h_F)) = (\det(F/F_{tor}), g) \otimes (\det(F_{tor}), k)$ over $U(\mathbb{C})$. If we identify $\det(F_{tor})$ with $\mathcal{O}_Y$ over U, k is nothing more than the canonical metric of $\mathcal{O}_Y$ over $U(\mathbb{C})$.

Let us fix a locally free sheaf P on Y and a surjective homomorphism $P \to F_{tor}$. Let P' be the kernel of $P \to F_{tor}$. Here $\left(\bigwedge^{\mathrm{rk}\, P'} P'\right)^*$ is an invertible sheaf on Y because Y is regular. Thus we may identify $\det(F_{tor})$ with

$$\bigwedge^{\mathrm{rk}\, P} P \otimes \left(\bigwedge^{\mathrm{rk}\, P'} P'\right)^*.$$

Further, a homomorphism $\bigwedge^{\mathrm{rk}\, P'} P' \to \bigwedge^{\mathrm{rk}\, P} P$ induced by $P' \hookrightarrow P$ gives rise to a non-zero section t of $\det(F_{tor})$ because

$$\mathcal{H}om\left(\bigwedge^{\mathrm{rk}\, P'} P', \bigwedge^{\mathrm{rk}\, P} P\right) = \mathcal{H}om\left(\bigwedge^{\mathrm{rk}\, P'} P', \mathcal{O}_Y\right) \otimes \bigwedge^{\mathrm{rk}\, P} P.$$

Here $F_{tor} = 0$ on U. Thus, $\det(F_{tor})$ is canonically isomorphic to $\mathcal{O}_Y$ over U. Since $P' = P$ over U, under the above isomorphism, t goes to the determinant of $P' \xrightarrow{\mathrm{id}} P$, namely $1 \in \mathcal{O}_Y$ over U. Thus, $k(t,t)(z) = 1$ for each $z \in S$.

Let $\overline{s}_i$ be the image of s_i in F/F_{tor}. Then, $\overline{s}_1 \wedge \cdots \wedge \overline{s}_r$ gives rise to a section s of $\det(F/F_{tor})$. Thus, $s \otimes t$ is a section of $\det(F)$. By our construction, $(s \otimes t)(y) \neq 0$. Moreover, using Hadamard's inequality,

$$\begin{aligned} \det(h_F)(s \otimes t, s \otimes t)(z) &= g(s,s)(z) \cdot k(t,t)(z) = \det\left(h(s_i, s_j)(z)\right) \\ &\leq h(s_1, s_1)(z) \cdots h(s_r, s_r)(z) \leq 1 \end{aligned}$$

for each $z \in S$. Thus, we get (4). □

4. Arithmetic Riemann-Roch for generically finite morphisms

4.1. **Quillen metric for generically finite morphisms.** Before starting Proposition 4.1.1, we recall the tensor product of two matrices, which we shall use in the proof. For an $r \times r$ matrix $A = (a_{ij})$ and an $n \times n$ matrix $B = (b_{kl})$, consider the following $rn \times rn$ matrix

$$\begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1r}B \\ a_{21}B & a_{22}B & \cdots & a_{2r}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1}B & a_{r2}B & \cdots & a_{rr}B \end{pmatrix}.$$

This matrix, denoted by $A \otimes B$, is called *the tensor product of A and B*. Then for $r \times r$ matrices A, A' and $n \times n$ matrices B, B', we see immediately

$$\begin{aligned} (A \otimes B)(A' \otimes B') &= AA' \otimes BB', \\ \det(A \otimes B) &= (\det A)^n (\det B)^r. \end{aligned}$$

Let X be a smooth algebraic scheme over $\mathbb{C}$, Y a smooth algebraic variety over $\mathbb{C}$, and $f : X \to Y$ a proper and generically finite morphism. We assume that every connected component of X maps surjectively to Y. Let W be the maximal open set of Y such that f is étale over there. Let (E, h) a Hermitian vector bundle on X such that on every connected component of X, E has the same rank r.

Proposition 4.1.1. *With notation and assumptions being as above, the Quillen metric $h_Q^{\overline{E}}$ on $\det Rf_*(E)$ over W extends to a generalized metric on $\det Rf_*(E)$ over Y.*

Proof. Let n be the degree of f. Since f is étale over W, $f_*(E)$ is a locally free sheaf of rank rn and $R^i f_*(E) = 0$ for $i \geq 1$ over there. Thus

$$\det Rf_*(E)|_W = \bigwedge^{rn} f_*(E)|_W.$$

If $y \in W$ is a complex point and $X_y = \{x_1, x_2, \cdots, x_n\}$ the fiber of f over y, then we have

$$\det Rf_*(E)_y = \det H^0(X_y, E).$$

The Quillen metric on $\det Rf_*(E)$ over W is defined as follows. On $H^0(X_y, E)$ the L^2-metric is defined by the formula:

$$h_{L^2}(s, t) = \sum_{\alpha=1}^{n} h(s, t)(x_\alpha),$$

where $s, t \in H^0(X_y, E)$. This metric naturally induces the L^2-metric on $\det H^0(X_y, E)$. Since X_y is zero-dimensional, there is no need for zeta function regularization to obtain the Quillen metric. Thus the Quillen metric $h_Q^{\overline{E}}$ on $\det Rf_*(E)|_W$ is defined by the family of hermitian line bundles $\{\det H^0(X_y, E)\}_{y \in W}$ with the induced L^2-metrics pointwisely.

To see that the Quillen metric over W extends to a generalized metric over Y, let $s_1, s_2, \cdots, s_r$ be rational sections of E such that at the generic point of every connected component of X, they form a basis of E. Also let $\omega_1, \omega_2, \cdots, \omega_n$ be rational sections of $f_*(\mathcal{O}_X)$ such that at the generic point they form a basis of $f_*(\mathcal{O}_X)$. Since

$$\det Rf_*(E) = \left(\bigwedge^{rn}(f_*(E))\right)^{**}. \tag{4.1.1.1}$$

over Y, we can regard $\bigwedge_{ik} s_i\omega_k = s_1\omega_1 \wedge s_1\omega_2 \wedge \cdots \wedge s_1\omega_n \wedge \cdots \wedge s_r\omega_n$ as a non-zero rational section of $\det Rf_*(E)$. Shrinking W, we can find a non-empty Zariski open set W_0 of W such that s_i's and ω_j's has no poles or zeros over $f^{-1}(W_0)$.

To proceed with our argument, we need the following lemma.

Lemma 4.1.2. *Let L be the total quotient field of X, and K the function field of Y. Then,*

$$\log h_Q^{\overline{E}}\left(\bigwedge_{ik} s_i\omega_k, \bigwedge_{ik} s_i\omega_k\right) = r \log\left|\det(\mathrm{Tr}_{L/K}(\omega_i \cdot \omega_j))\right| + f_* \log\det(h(s_i, s_j))$$

over W_0.

Proof. Let $y \in W_0$ be a complex point, and $\{x_1, x_2, \ldots, x_n\}$ the fiber of $f^{-1}(y)$ over y. Then,

$$\begin{aligned}
\log h_Q^{\overline{E}}\left(\bigwedge_{ik} s_i\omega_k, \bigwedge_{ik} s_i\omega_k\right)(y) &= \log\det\left(\sum_{\alpha=1}^{n} h(s_i\omega_k, s_j\omega_l)(x_\alpha)\right)_{ij,kl} \\
&= \log\det\left(\sum_{\alpha=1}^{n} \omega_k(x_\alpha)h(s_i, s_j)(x_\alpha)\overline{\omega_l(x_\alpha)}\right)_{ij,kl} \\
&= \log\det\left((I_r \otimes \Omega)\begin{pmatrix} H(x_1) & & 0 \\ & \ddots & \\ 0 & & H(x_n)\end{pmatrix}\overline{{}^t(I_r \otimes \Omega)}\right) \\
&= \log\det\left\{|\det(\Omega)|^{2r}\prod_{\alpha=1}^{n}\det\left(h(s_i, s_j)(x_\alpha)\right)_{ij}\right\} \\
&= r\log\det|\det(\Omega)|^2 + \sum_{\alpha=1}^{n}\log\det\left(h(s_i, s_j)\right)(x_\alpha),
\end{aligned}$$

where $\Omega = (\omega_k(x_\alpha))_{k\alpha}$ and $H(x_\alpha) = (h(s_i, s_j)(x_\alpha))_{ij}$. On the other hand, we have

$$\sum_{\alpha=1}^{n}\log\det\left(h(s_i, s_j)\right)(x_\alpha) = \left(f_*\log\det\left(h(s_i, s_j)\right)\right)(y).$$

Moreover, using the following Lemma 4.1.3, we have

$$|\det(\Omega)|^2 = |\det(\Omega{}^t\Omega)| = \left|\det\left(\sum_{\alpha=1}^{n}\omega_k(x_\alpha)\omega_l(x_\alpha)\right)_{kl}\right| = |\det\left(\mathrm{Tr}_{L/K}(\omega_k\cdot\omega_l)\right)_{kl}|.$$

Thus we get the lemma. □

Lemma 4.1.3. *Let $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be a finite étale morphism of regular affine schemes. Let $\mathfrak{m}$ be the maximal ideal of A and $\mathfrak{n}_1, \mathfrak{n}_2, \cdots, \mathfrak{n}_n$ the prime ideals lying over $\mathfrak{m}$. Assume that $\kappa(\mathfrak{m})$ is algebraically closed and hence $\kappa(\mathfrak{n}_i)$ is (naturally) isomorphic to $\kappa(\mathfrak{m})$ for each $1 \leq i \leq n$. Let b be an element of B and $b(\mathfrak{n}_i)$ the value of b in $\kappa(\mathfrak{n}_i) \cong \kappa(\mathfrak{m})$. Then*

$$\mathrm{Tr}_{B/A}(b)(\mathfrak{m}) = \sum_{i=1}^{n} b(\mathfrak{n}_i)$$

in $\kappa(\mathfrak{m})$, where $\mathrm{Tr}_{B/A}(b)(\mathfrak{m})$ is the value of $\mathrm{Tr}_{B/A}(b)$ in $\kappa(\mathfrak{m})$.

Proof. It is easy to see that every $\mathfrak{n}_i$ is the maximal ideal and that $\mathfrak{m}B = \mathfrak{n}_1\mathfrak{n}_2\cdots\mathfrak{n}_n$. Let $\widehat{A}$ be the completion of A with respect to $\mathfrak{m}$, $\widehat{B}$ the completion of B with respect to $\mathfrak{m}B$, and $\widehat{B_i}$ the completion of B with respect to $\mathfrak{n}_i$ for each $1 \leq i \leq n$. Then by Chinese remainder theorem, $\widehat{B} = \prod_{i=1}^{n}\widehat{B_i}$ as an $\widehat{A}$-algebra. Note that $\widehat{A}/\mathfrak{m}\widehat{A} = \kappa(\mathfrak{m})$ and $\widehat{B_i}/\mathfrak{n}_i\widehat{B_i} = \kappa(\mathfrak{n}_i)$. Since $\widehat{A} \to \widehat{B_i}$ is étale and $\kappa(\mathfrak{m}) \cong \kappa(\mathfrak{n}_i)$, we have $\widehat{A} \cong \widehat{B_i}$. Let $e_1 = (1, 0, \cdots, 0), e_2 = (0, 1, \cdots, 0), \cdots, e_n = (0, 0, \cdots, 1) \in \prod_{i=1}^{n}\widehat{B_i} = \widehat{B}$ be a free basis of

$\widehat{B}$ over $\widehat{A}$. We put $be_i = b_i e_i$ with $b_i \in \widehat{B_i} \cong \widehat{A}$ for each $1 \leq i \leq n$. Then $b_i \equiv b(\mathfrak{n}_i) \pmod{\mathfrak{n}_i}$. Now the lemma follows from

$$\mathrm{Tr}_{B/A}(b) = \mathrm{Tr}_{\widehat{B}/\widehat{A}}(b) = \sum_{i=1}^{n} b_i$$

in $\widehat{A}$. □

Let us go back to the proof of Proposition 4.1.1. Since $\det(\mathrm{Tr}_{L/K}(\omega_i \cdot \omega_j))\big|_{W_0}$ extends to a rational function $\det(\mathrm{Tr}_{L/K}(\omega_i \cdot \omega_j))$ on Y,

$$\log \left|\det(\mathrm{Tr}_{L/K}(\omega_i \cdot \omega_j))\right| \in L^1_{\mathrm{loc}}(Y).$$

Moreover, by Proposition 1.2.4, $f_* \log \det(h(s_i, s_j)) \in L^1_{\mathrm{loc}}(Y)$. Thus, by Lemma 4.1.2,

$$\log h_Q^{\overline{E}} \left(\bigwedge_{ik} s_i \omega_k, \bigwedge_{ik} s_i \omega_k \right) \Bigg|_{W_0}$$

extends to a locally integrable function on Y. Hence by Lemma 3.1.1 the Quillen metric over W extends to a generalized metric over Y. □

Remark 4.1.4. In the above situation, Let W' be a open set of Y such that f is flat and finite over there. Then the Quillen metric extends to a continuous function over W' by the same formula as in (4.1.2)

4.2. **Riemann-Roch for generically finite morphisms.** In this subsection, we shall formulate the arithmetic Riemann-Roch theorem for generically finite morphisms.

Theorem 4.2.1. *Let X be a scheme such that every connected component of X is an arithmetic variety. Let Y be a regular arithmetic variety, and $f : X \to Y$ a proper and generically finite morphism such that every connected component of X maps surjectively to Y. Let (E, h) a Hermitian vector bundle on X such that on each connected component of X, E has the same rank r. Then,*

$$\widehat{c}_1 \left(\det Rf_*(E), h_Q^{\overline{E}} \right) - r\widehat{c}_1 \left(\det Rf_*(\mathcal{O}_X), h_Q^{\overline{\mathcal{O}}_X} \right) \in \widehat{\mathrm{CH}}^1_{L^1}(Y)$$

and

$$\widehat{c}_1 \left(\det Rf_*(E), h_Q^{\overline{E}} \right) - r\widehat{c}_1 \left(\det Rf_*(\mathcal{O}_X), h_Q^{\overline{\mathcal{O}}_X} \right) = f_* \left(\widehat{c}_1(E, h) \right)$$

in $\widehat{\mathrm{CH}}^1_{L^1}(Y)_{\mathbb{Q}}$, where $h_Q^{\overline{E}}$ and $h_Q^{\overline{\mathcal{O}}_X}$ are the Quillen metric of $\det Rf_*(E)$ and $\det Rf_*(\mathcal{O}_X)$ respectively.

Proof. Let $X = \coprod_{\alpha \in A} X_\alpha$ be the decomposition into connected components of X. Since f is proper, A is a finite set. We set $f_\alpha = f|_{X_\alpha}$ and $(E_\alpha, h_\alpha) = (E, h)|_{X_\alpha}$. Then

$$Rf_*(E) = \bigoplus_{\alpha \in A} R(f_\alpha)_*(E_\alpha), \quad Rf_*(\mathcal{O}_X) = \bigoplus_{\alpha \in A} R(f_\alpha)_*(\mathcal{O}_{X_\alpha}), \quad \widehat{c}_1(E, h) = \sum_{\alpha \in A} \widehat{c}_1(E_\alpha, h_\alpha).$$

Hence we have the following:

$$\begin{cases} \widehat{c}_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right) = \sum_{\alpha\in A} \widehat{c}_1\left(\det R(f_\alpha)_*(E_\alpha), h_Q^{\overline{E}_\alpha}\right), \\ \widehat{c}_1\left(\det Rf_*(\mathcal{O}_X), h_Q^{\overline{\mathcal{O}}_X}\right) = \sum_{\alpha\in A} \widehat{c}_1\left(\det R(f_\alpha)_*(\mathcal{O}_{X_\alpha}), h_Q^{\overline{\mathcal{O}}_{X_\alpha}}\right), \\ f_*\left(\widehat{c}_1(E,h)\right) = \sum_{\alpha\in A} f_*\left(\widehat{c}_1(E_\alpha, h_\alpha)\right). \end{cases}$$

Thus, we may assume that X is connected, i.e., X is an arithmetic variety.

Let $K = K(Y)$ and $L = K(X)$ be the function fields of Y and X respectively. Let n be the degree of f and $\omega_1, \omega_2, \cdots, \omega_n$ rational functions on X such that at the generic point they form a basis of K-vector space L. Further, let $s_1, s_2, \ldots, s_r$ be rational sections of E such that at the generic point they form a basis of L-vector space E_L. Then $s_1\omega_1 \wedge s_1\omega_2 \wedge \cdots \wedge s_1\omega_n \wedge \cdots \wedge s_r\omega_n$, $s_1 \wedge \cdots \wedge s_r$ and $\omega_1 \wedge \cdots \wedge \omega_n$ are non-zero rational sections of $\det f_*(E)$, $\det(E)$ and $\det f_*(\mathcal{O}_X)$ respectively. Here we shall prove the following equality in $\widehat{Z}^1_D(Y)$:

$$\begin{aligned} (4.2.1.1)\quad & \left(\operatorname{div}\left(\bigwedge_{ik} s_i\omega_k\right), \left[-\log h_Q^{\overline{E}}\left(\bigwedge_{ik} s_i\omega_k, \bigwedge_{ik} s_i\omega_k\right)\right]\right) \\ & - r\left(\operatorname{div}\left(\bigwedge_{k} \omega_k\right), \left[-\log h_Q^{\overline{\mathcal{O}}_X}\left(\bigwedge_{k} \omega_k, \bigwedge_{k} \omega_k\right)\right]\right) \\ & = f_*\left(\operatorname{div}\left(\bigwedge_{i} s_i\right), \left[-\log \det h\left(\bigwedge_{i} s_i, \bigwedge_{i} s_i\right)\right]\right), \end{aligned}$$

where $\bigwedge_{ik} s_i\omega_k = s_1\omega_1 \wedge s_1\omega_2 \wedge \cdots \wedge s_1\omega_n \wedge \cdots \wedge s_r\omega_n$, $\bigwedge_k \omega_k = \omega_1 \wedge \cdots \wedge \omega_n$ and $\bigwedge_i s_i = s_1 \wedge \cdots \wedge s_r$.

First we shall show the equality of divisors. Let Y_0 be the maximal Zariski open set of X such that f is flat over Y_0. Then, $\operatorname{codim}_Y(Y \setminus Y_0) \geq 2$ by [7, III,Proposition 9.7]. Since f is generically finite, f is in fact finite over Y_0. Then $Z^1(Y) = Z^1(Y_0)$ and thus it suffices to prove the equality of divisors over Y_0. Since it suffices to prove it locally, let $U = \operatorname{Spec}(A)$ be an affine open set of Y_0 and $f^{-1}(U) = \operatorname{Spec}(B)$ the open set of $X_0 = f^{-1}(Y_0)$. Shrinking U if necessary, we may assume that B is a free A-module of rank n and that E is a free B-module of rank r. Let $d_1, d_2, \cdots, d_n$ be a basis of B over A, and $e_1, e_2, \cdots, e_r$ be a basis of E over B. Note that K and L are the quotient fields of A and B respectively. In the following we freely identify a rational function (or section) by the corresponding element at the generic point. In this sense, we set

$$\omega_k = \sum_{l=1}^{n} a^{kl} d_l \quad (k = 1, 2, \cdots, n)$$

$$s_i = \sum_{j=1}^{r} \sigma_{ij} e_j \quad (i = 1, 2, \cdots, r),$$

where $a^{kl} \in K\,(1 \leq k, l \leq n)$ and $\sigma_{ij} \in L\,(1 \leq i, j \leq r)$.

For each $\sigma_{ij}(1 \leq i, j \leq r)$, let $T_{\sigma_{ij}} : L \to L$ be multiplication by σ_{ij}. With respect to a basis $\omega_1, \omega_2, \cdots, \omega_n$ of L over K, $T_{\sigma_{ij}}$ gives rise to a matrix $(c_{ij}^{kl})_{1\leq k,l\leq n} \in M_n(K)$ defined by

$$\sigma_{ij}\omega_k = \sum_{l=1}^{n} c_{ij}^{kl}\omega_l \quad (k = 1, 2, \cdots, n).$$

We also denote this matrix by $T_{\sigma_{ij}}$. Then,

$$\begin{aligned}
\bigwedge_{ik} s_i\omega_k &= \bigwedge_{ik}\left(\sum_{j=1}^{r}\sigma_{ij}e_j\right)\omega_k = \bigwedge_{ik}\left(\sum_{j=1}^{r}\sum_{l=1}^{n} c_{ij}^{kl}\right)e_j\omega_l \\
&= \det(c_{ij}^{kl})_{ik,jl}\bigwedge_{ik} e_i\omega_k = \det(c_{ij}^{kl})_{ik,jl}\bigwedge_{ik} e_i\left(\sum_{l=1}^{n} a^{kl}d_l\right) \\
&= \det(c_{ij}^{kl})_{ik,jl}\bigwedge_{ik}\left(\sum_{j=1}^{r}\delta_{ij}a^{kl}\right)e_j\omega_l = \det(c_{ij}^{kl})_{ik,jl}\det(\delta_{ij}a^{kl})_{ik,jl}\bigwedge_{ik} e_id_l.
\end{aligned}$$

On the other hand, since the matrices $T_{\sigma_{ij}}$ and $T_{\sigma_{i'j'}}$ commute with each other, we have

$$\begin{aligned}
\det(c_{ij}^{kl})_{ik,jl} &= \det\begin{pmatrix} T_{\sigma_{11}} & T_{\sigma_{12}} & \cdots & T_{\sigma_{1r}} \\ T_{\sigma_{21}} & T_{\sigma_{22}} & \cdots & T_{\sigma_{2r}} \\ \vdots & \vdots & \ddots & \vdots \\ T_{\sigma_{r1}} & T_{\sigma_{r2}} & \cdots & T_{\sigma_{rr}} \end{pmatrix} \\
&= \det\left(\sum_{\tau\in\mathfrak{S}_r}\operatorname{sign}(\tau)T_{\sigma_{1\tau(1)}}\cdot\cdots\cdot T_{\sigma_{r\tau(r)}}\right) \\
&= \det(T_{\det(\sigma_{ij})_{ij}}) \\
&= \operatorname{Norm}_{L/K}(\det(\sigma_{ij})_{ij}).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\det(\delta_{ij}a^{kl})_{ik,jl} &= \det(I_r \otimes (a^{kl})_{kl}) \\
&= (\det(a^{kl})_{kl})^r.
\end{aligned}$$

From the above three equalities, $\operatorname{div}\left(\bigwedge_{ik} s_i\omega_k\right)$ is given by a rational function

$$\operatorname{Norm}_{L/K}(\det(\sigma_{ij})_{ij})(\det(a^{kl})_{kl})^r.$$

Further

$$\bigwedge_i s_i = (\det(\sigma_{ij})_{ij})\bigwedge_i e_k \quad \text{and} \quad \bigwedge_k \omega_k = (\det(a^{kl})_{kl})\bigwedge_k d_k.$$

Hence we have

$$\operatorname{div}\left(\bigwedge_{ik} s_i\omega_k\right) - r\left(\operatorname{div}\left(\bigwedge_k \omega_k\right)\right) = f_*\left(\operatorname{div}\left(\bigwedge_i s_i\right)\right).$$

Next we shall show the equality of currents. Since all the currents in the equality come from locally integrable functions by Proposition 1.2.4 and Proposition 4.1.1, it suffices to show the equality over a non-empty Zariski open set of every connected component of $Y(\mathbb{C})$.

So let W_0 be a non-empty Zariski open set of a connected component of $Y(\mathbb{C})$ such that $f_{\mathbb{C}}$ is étale and that $s_i\,(1 \leq i \leq r)$ or $\omega_k\,(1 \leq k \leq n)$ have no poles or zeroes over there. Then over W_0 all these currents are defined by C^∞ functions. Let $y \in Y(\mathbb{C})$ be a complex point and $x_1, x_2, \cdots, x_n$ be the fiber $f_{\mathbb{C}}^{-1}(y)$ over y. From the proof of Lemma 4.1.2, as C^∞ functions around y,

$$-\log h_Q^{\overline{E}}\left(\bigwedge_{ik} s_i\omega_k, \bigwedge_{ik} s_i\omega_k\right)(y) = -\log\det\left\{|\det(\Omega)|^{2r}\prod_{\alpha=1}^{n}\det\left(h(s_i,s_j)(x_\alpha)\right)_{ij}\right\},$$

where $\Omega = (\omega_k(x_\alpha))_{k\alpha}$ and $H(x_\alpha) = (h(s_i,s_j)(x_\alpha))_{ij}$. Also,

$$-\log h_Q^{\overline{\mathcal{O}}_X}\left(\bigwedge_{k}\omega_k, \bigwedge_{k}\omega_k\right)(y) = -\log\det|\det(\Omega)|^2.$$

On the other hand, by the definition of the push-forward f_*,

$$\begin{aligned} f_*\left(-\log\det h\left(\bigwedge_i s_i, \bigwedge_i s_i\right)\right)(y) &= \sum_{\alpha=1}^{n} -\log\det h\left(\bigwedge_i s_i, \bigwedge_i s_i\right)(x_\alpha) \\ &= \sum_{\alpha=1}^{n} -\log\det\left(h(s_i,s_j)(x_\alpha)\right)_{ij}. \end{aligned}$$

Hence we have the desired equality of currents by the above three equalities.

Thus we have showed the equality (4.2.1.1). Since the right hand side belongs in fact to $\widehat{Z}^1_{L^1}(Y)$, the left hand side must also belong to $\widehat{Z}^1_{L^1}(Y)$, and thus we have the equality in $\widehat{Z}^1_{L^1}(Y)$. □

5. Arithmetic Riemann-Roch for stable curves

5.1. **Bismut-Bost formula.** Let X be a smooth algebraic variety over $\mathbb{C}$, L a line bundle on X, and h a generalized metric of L over X. Let s be a rational section of L. Then, by the definition of the generalized metric h, $-\log h(s,s)$ gives rise to a current $-[\log h(s,s)]$. Moreover, it is easy to see that a current

$$dd^c(-[\log h(s,s)]) + \delta_{\mathrm{div}(s)}$$

does not depend on the choice of s. Thus, we define $c_1(L,h)$ by

$$c_1(L,h) = dd^c(-[\log h(s,s)]) + \delta_{\mathrm{div}(s)}.$$

Let $f : X \to Y$ be a proper morphism of smooth algebraic varieties $\mathbb{C}$ such that every fiber of f is a reduced and connected curve with only ordinary double singularities. We set $\Sigma = \{x \in X \mid f \text{ is not smooth at } x.\}$ and $\Delta = f_*(\Sigma)$. Let $|\Delta|$ be the support of Δ. We fix a Hermitian metric of $\omega_{X/Y}$. Then, in [1], Bismut and Bost proved the following.

Theorem 5.1.1. *Let $\overline{E} = (E,h)$ be a Hermitian vector bundle on X. Then, the Quillen metric $h_Q^{\overline{E}}$ of $\det Rf_*(E)$ on $Y \setminus |\Delta|$ gives rise to a generalized metric of $\det Rf_*(E)$ on Y. Moreover,*

$$c_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right) = -f_*\left[\mathrm{td}(\overline{\omega}_{X/Y}{}^{-1})\,\mathrm{ch}(\overline{E})\right]^{(2,2)} - \frac{\mathrm{rk}\,E}{12}\delta_\Delta.$$

5.2. **Riemann-Roch for stable curves.** In this subsection, we shall prove the arithmetic Riemann-Roch theorem for stable curves.

Theorem 5.2.1. *Let $f : X \to Y$ be a projective morphism of regular arithmetic varieties such that every fiber of $f_{\mathbb{C}} : X(\mathbb{C}) \to Y(\mathbb{C})$ is a reduced and connected curve with only ordinary double singularities. We fix a Hermitian metric of the dualizing sheaf $\omega_{X/Y}$. Let $\overline{E} = (E, h)$ be a Hermitian vector bundle on X. Then,*

$$\widehat{c}_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right) - \mathrm{rk}(E)\widehat{c}_1\left(\det Rf_*(\mathcal{O}_X), h_Q^{\overline{\mathcal{O}}_X}\right) \in \widehat{\mathrm{CH}}^1_{L^1}(Y)$$

and

$$\widehat{c}_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right) - \mathrm{rk}(E)\widehat{c}_1\left(\det Rf_*(\mathcal{O}_X), h_Q^{\overline{\mathcal{O}}_X}\right) = f_*\left(\frac{1}{2}\left(\widehat{c}_1(\overline{E})^2 - \widehat{c}_1(\overline{E})\cdot\widehat{c}_1(\overline{\omega}_{X/Y})\right) - \widehat{c}_2(\overline{E})\right)$$

in $\widehat{\mathrm{CH}}^1_{L^1}(Y)_{\mathbb{Q}}$, where $h_Q^{\overline{E}}$ and $h_Q^{\overline{\mathcal{O}}_X}$ are the Quillen metric of $\det Rf_*(E)$ and $\det Rf_*(\mathcal{O}_X)$ respectively.

Proof. We prove the theorem in two steps.

Step 1. First, we assume that $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is smooth. In this case, by [6],

$$\widehat{c}_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right) = f_*\left(\widehat{\mathrm{ch}}(E,h)\widehat{\mathrm{td}}(Tf, h_f) - a(\mathrm{ch}(E_{\mathbb{C}})\,\mathrm{td}(Tf_{\mathbb{C}})R(Tf_{\mathbb{C}}))\right)^{(1)}.$$

in $\widehat{\mathrm{CH}}^1(Y)_{\mathbb{Q}}$. Since

$$\widehat{\mathrm{ch}}(\overline{E}) = \mathrm{rk}(E) + \widehat{c}_1(\overline{E}) + \left(\frac{1}{2}\widehat{c}_1(\overline{E})^2 - \widehat{c}_2(\overline{E})\right) + (\text{higher terms})$$

and

$$\widehat{\mathrm{td}}(Tf, h_f) = 1 - \frac{1}{2}\widehat{c}_1(\overline{\omega}_{X/Y}) + \widehat{\mathrm{td}}_2(Tf, h_f) + (\text{higher terms}),$$

we have

$$\left(\widehat{\mathrm{ch}}(E,h)\widehat{\mathrm{td}}(Tf,h_f)\right)^{(2)} = \frac{1}{2}\left(\widehat{c}_1(\overline{E})^2 - \widehat{c}_1(\overline{E})\cdot\widehat{c}_1(\overline{\omega}_{X/Y})\right) - \widehat{c}_2(\overline{E}) + \mathrm{rk}(E)\widehat{\mathrm{td}}_2(Tf,h_f).$$

On the other hand, since the power series $R(x)$ has no constant term, $(1,1)$ part of

$$\mathrm{ch}(E_{\mathbb{C}})\,\mathrm{td}(Tf_{\mathbb{C}})R(Tf_{\mathbb{C}})$$

is $\mathrm{rk}(E)R_1(Tf_{\mathbb{C}})$, where $R_1(Tf_{\mathbb{C}})$ is the $(1,1)$ part of $R(Tf_{\mathbb{C}})$. Therefore, we obtain

$$\widehat{c}_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right) = f_*\left(\frac{1}{2}\left(\widehat{c}_1(\overline{E})^2 - \widehat{c}_1(\overline{E})\cdot\widehat{c}_1(\overline{\omega}_{X/Y})\right) - \widehat{c}_2(\overline{E})\right) + \mathrm{rk}(E)f_*\left(\widehat{\mathrm{td}}_2(Tf,h_f) - a(R_1(Tf_{\mathbb{C}}))\right). \tag{5.2.1.1}$$

Applying (5.2.1.1) to a case $(E,h) = (\mathcal{O}_X, h_{can})$, we have

$$\widehat{c}_1\left(\det Rf_*(\mathcal{O}_X), h_Q^{\overline{\mathcal{O}}_X}\right) = f_*\left(\widehat{\mathrm{td}}_2(Tf,h_f) - a(R_1(Tf_{\mathbb{C}}))\right). \tag{5.2.1.2}$$

Thus, combining (5.2.1.1) and (5.2.1.2), we have our formula in the case where $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is smooth.

Step 2. Next, we consider the general case. The first assertion is a consequence of Theorem 5.1.1 because using Theorem 5.1.1,

$$c_1\left(\det Rf_*(E), h_Q^{\overline{E}}\right) - \mathrm{rk}(E)c_1\left(\det Rf_*(\mathcal{O}_X), h_Q^{\overline{\mathcal{O}}_X}\right) = -f_*\left[\mathrm{td}(\overline{\omega}_{X/Y}{}^{-1})\,\mathrm{ch}(\overline{E})\right]^{(2,2)} + \mathrm{rk}(E) f_*\left[\mathrm{td}(\overline{\omega}_{X/Y}{}^{-1})\,\mathrm{ch}(\overline{\mathcal{O}}_X)\right]^{(2,2)}$$

belongs to $L^1_{\mathrm{loc}}(\Omega^{1,1}_{Y(\mathbb{C})})$ by Proposition 1.2.4. The second assertion is a consequence of the useful Lemma 2.4.1. In fact, both sides of the assertion are arithmetic L^1-cycles on Y by the first assertion and the Proposition 2.2.2: If we take $\Delta = \{y \in Y_{\mathbb{Q}} \mid f_{\mathbb{Q}} \text{ is not smooth over } y\}$ and set $\overline{\Delta}$ as the closure of Δ in Y, then the compliment $U = Y \setminus \overline{\Delta}$ contains no irreducible components of fibers of $Y \to \mathrm{Spec}(\mathbb{Z})$ and $f_{\mathbb{C}}$ is smooth over $U(\mathbb{C})$: The arithmetical linear equivalence of both sides restricted to U is a consequence of Step 1. Thus by Lemma 2.4.1, we also have our formula in the general case. □

6. Asymptotic behavior of analytic torsion

Let M be a compact Kähler manifold of dimension d, $\overline{E} = (E, h_E)$ a flat vector bundle on M of rank r with a flat metric h_E, and $\overline{A} = (A, h_A)$ a Hermitian vector bundle on M. For $0 \leq q \leq d$, let $\Delta_{q,n}$ be the Laplacian on $A^{0,q}\left(\mathrm{Sym}^n(\overline{E}) \otimes \overline{A}\right)$ and $\Delta'_{q,n}$ the restriction of $\Delta_{q,n}$ to $\mathrm{Image}\,\partial \oplus \mathrm{Image}\,\overline{\partial}$. Let $\sigma(\Delta'_{q,n}) = \{0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ be the sequence of eigenvalues of $\Delta'_{q,n}$. Here we count each eigenvalue up to its multiplicity. Then, the associated zeta function $\zeta_{q,n}(s)$ is given by

$$\zeta_{q,n}(s) = \mathrm{Tr}\left[(\Delta'_{q,n})^{-s}\right] = \sum_{i=1}^{\infty} \lambda_i^{-s}.$$

$\zeta_{q,n}(s)$ converges absolutely for $\Re(s) > d$ and has a meromorphic continuation to the whole complex plane and has no pole at $s = 0$. The analytic torsion $T\left(\mathrm{Sym}^n(\overline{E}) \otimes \overline{A}\right)$ is defined by

$$T\left(\mathrm{Sym}^n(\overline{E}) \otimes \overline{A}\right) = \sum_{q=0}^{d} (-1)^q q \zeta'_{q,n}(0).$$

In the following we closely follow [17, §2].

The Theta function associated with $\sigma(\Delta'_{q,n})$ is defined by

$$\Theta_{q,n}(t) = \mathrm{Tr}\left[\exp(-t\Delta'_{q,n})\right] = \sum_{i=1}^{\infty} e^{-\lambda_i t}.$$

By taking Mellin transformation, we have, for $\Re(s) > d$,

$$\zeta_{q,n}(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \Theta_{q,n}(t) t^s \frac{dt}{t}.$$

We put

$$\tilde{\zeta}_{q,n}(s) = \frac{1}{\mathrm{rk}(\mathrm{Sym}^n(E))} n^{-d} \frac{1}{\Gamma(s)} \int_0^\infty \Theta_{q,n}\left(\frac{t}{n}\right) t^s \frac{dt}{t}.$$

Then we have

$$\frac{1}{\mathrm{rk}(\mathrm{Sym}^n(E))} n^{-d} \zeta_{q,n}(s) = n^{-s} \tilde{\zeta}_{q,n}(s)$$

and thus

$$\frac{1}{\mathrm{rk}(\mathrm{Sym}^n(E))} n^{-d} \zeta'_{q,n}(0) = -(\log n)\tilde{\zeta}_{q,n}(0) + \zeta'_{q,n}(0) \tag{6.1}$$

Bismut and Vasserot [3, (14),(19)] showed that $\Theta_{q,n}(t)$ has the following properties (note that these parts of [3] do not depend on the assumption of positivity of a line bundle, as indicated in Vojta [17, Proposition 2.7.3]):

(a) For every $k \in \mathbb{N}$, $0 \le q \le d$ and $n \in \mathbb{N}$, there are real numbers $a^j_{q,n}$ $(-d \le j \le k)$ such that

$$\frac{1}{\mathrm{rk}(\mathrm{Sym}^n(E))} n^{-d} \Theta_{q,n}\left(\frac{t}{n}\right) = \sum_{j=-d}^{k} a^j_{q,n} t^j + o(t^k)$$

as $t \downarrow 0$, with $o(t^k)$ uniform with respect to $n \in \mathbb{N}$.

(b) For every $0 \le q \le d$ and $j \ge -d$, there are real numbers a^j_q such that

$$a^j_{q,n} = a^j_q + O\left(\frac{1}{\sqrt{n}}\right)$$

as $n \to \infty$.

Also by (b), we can replace the $o(t^k)$ in (a) by $O(t^{k+1})$ and still have the uniformity statement. Thus we can write, for every $k \in \mathbb{N}$,

$$\frac{1}{\mathrm{rk}(\mathrm{Sym}^n(E))} n^{-d} \Theta_{q,n}\left(\frac{t}{n}\right) = \sum_{j=-d}^{k} a^k_{q,n} t^j + \rho^k_{q,n}(t)$$

with $\rho^k_{q,n}(t) = o(t^{k+1})$. Then

$$\begin{aligned}
\tilde{\zeta}_{q,n}(s) &= \frac{1}{\mathrm{rk}(\mathrm{Sym}^n(E))} n^{-d} \frac{1}{\Gamma(s)} \int_1^\infty \Theta_{q,n}\left(\frac{t}{n}\right) t^s \frac{dt}{t} \\
&\quad + \frac{a^j_{q,n}}{\Gamma(s)} \int_0^1 t^{j+s-1} dt + \sum_{j=-d}^{k} \frac{1}{\Gamma(s)} \int_0^1 \rho^k_{q,n}(t) dt \\
&= \frac{1}{\mathrm{rk}(\mathrm{Sym}^n(E))} n^{-d} \frac{1}{\Gamma(s)} \int_1^\infty \Theta_{q,n}\left(\frac{t}{n}\right) t^s \frac{dt}{t} \\
&\quad + \sum_{j=-d}^{k} \frac{a^j_{q,n}}{\Gamma(s)(j+s)} + \frac{1}{\Gamma(s)} \int_0^1 \rho^k_{q,n}(t) t^s \frac{dt}{t}.
\end{aligned}$$

In the last expression, the first integral is holomorphic for all $s \in \mathbb{C}$, while the second integral is holomorphic for $\Re(s) > -k-1$; the middle term is a meromorphic function in the whole complex plane.

Putting $k=0$ and $s=0$ in the above equation, we have

$$\tilde{\zeta}_{q,n}(0) = a^0_{q,n}. \tag{6.2}$$

Moreover, by differentiating the above equation when $k=0$, we have

$$\begin{aligned}\tilde{\zeta}'_{q,n}(0) = \frac{1}{\mathrm{rk}(\mathrm{Sym}^n(E))} n^{-d} \int_1^\infty \Theta_{q,n}\left(\frac{t}{n}\right)\frac{dt}{t} \\ + \sum_{j=-d}^{-1} \frac{a^j_{q,n}}{j} - a^0_{q,n}\Gamma'(1) + \frac{1}{\Gamma(s)}\int_0^1 \rho^0_{q,n}(t)\frac{dt}{t}.\end{aligned} \tag{6.3}$$

We have now the following Proposition.

Proposition 6.4. *There exists a constant c such that for all $n \in \mathbb{N}$,*

$$\zeta'_{q,n}(0) \geq -cn^{d+r-1}\log n$$

Proof. By (6.1), (6.2) and (6.3), we have

$$\begin{aligned}\zeta'_{q,n}(0) = &-\mathrm{rk}(\mathrm{Sym}^n(E))n^d(\log n)a^0_{q,n} \\ &+ \mathrm{rk}(\mathrm{Sym}^n(E))n^d\left(\frac{1}{\mathrm{rk}(\mathrm{Sym}^n(E))}n^{-d}\int_1^\infty \Theta_{q,n}\left(\frac{t}{n}\right)\frac{dt}{t}\right. \\ &\left. + \sum_{j=-d}^{-1}\frac{a^j_{q,n}}{j} - a^0_{q,n}\Gamma'(1) + \frac{1}{\Gamma(s)}\int_0^1 \rho^0_{q,n}(t)\frac{dt}{t}\right)\end{aligned}$$

In the first term of the right hand side, $a^0_{q,n}$ is bounded with respect to n by (b). In the second term of the right hand side, the first integral is non-negative; the sum of $a^j_{q,n}$'s is bounded with respect to n by (b); the term $-a^0_{q,n}\Gamma'(1)$ is also bounded with respect to n by (b); the second integral is also bounded with respect to n, for $\rho^0_{q,n}(t) = O(t)$ uniformly with respect to n. Moreover,

$$\mathrm{rk}(\mathrm{Sym}^n(E)) = \binom{n+r-1}{r-1} = O(n^{r-1})$$

as $n \to \infty$. Thus, there is a constant c such that for all $n \in \mathbb{N}$,

$$\zeta'_{q,n}(0) \geq -cn^{d+r-1}\log n.$$

□

In the following sections, we only need the case of $d=1$, namely when M is a compact Riemann surface. In this case, the above Proposition 6.4 gives an asymptotic upper bound of analytic torsion.

Corollary 6.5. *Let C be a compact Riemann surface, $\overline{E} = (E, h_E)$ a flat vector bundle on C of rank r with a flat metric h, and $\overline{A} = (A, h_A)$ a Hermitian vector bundle on C. Then, there is a constant c such that for all $n \in \mathbb{N}$,*

$$T\left(\mathrm{Sym}^n(\overline{E}) \otimes \overline{A}\right) \leq cn^r \log n.$$

Proof. Since $\dim C = 1$

$$T\left(\mathrm{Sym}^n(\overline{E}) \otimes \overline{A}\right) = -\zeta'_{1,n}(0).$$

Now the corollary follows from Proposition 6.4. □

7. Formulae for arithmetic Chern classes

7.1. **Arithmetic Chern classes of symmetric powers.** Let M be a complex manifold and (E, h) a Hermitian vector bundle on M. Since $E^{\otimes n}$ has the natural Hermitian metric $h^{\otimes n}$, we can define a Hermitian metric $\mathrm{Sym}^n(h)$ of $\mathrm{Sym}^n(E)$ as the quotient metric of $E^{\otimes n}$ in terms of the natural surjective homomorphism $E^{\otimes n} \to \mathrm{Sym}^n(E)$. We denote $(\mathrm{Sym}^n(E), \mathrm{Sym}^n(h))$ by $\mathrm{Sym}^n(E, h)$. If $x \in M$ and $\{e_1, \ldots, e_r\}$ is an orthonormal basis of E_x with respect to h_x, then it is easy to see that

$$(\mathrm{Sym}^n(h))_x\left(e_1^{\alpha_1} \cdots e_r^{\alpha_r}, e_1^{\beta_1} \cdots e_r^{\beta_r}\right) = \begin{cases} \dfrac{\alpha_1! \cdots \alpha_r!}{n!} & \text{if } (\alpha_1, \ldots, \alpha_r) = (\beta_1, \ldots, \beta_r), \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the following proposition.

Proposition 7.1.1. *Let X be an arithmetic variety and $\overline{E} = (E, h)$ a Hermitian vector bundle on X of rank r. Then, we have the following.*

(1) $$\widehat{c}_1\left(\mathrm{Sym}^n(\overline{E})\right) = \frac{n}{r}\binom{n+r-1}{r-1}\widehat{c}_1(\overline{E}) + a\left(\sum_{\substack{\alpha_1+\cdots+\alpha_r=n,\\ \alpha_1\geq 0,\ldots,\alpha_r\geq 0}} \log\left(\frac{n!}{\alpha_1!\cdots\alpha_r!}\right)\right).$$

(2) *If X is regular, then*

$$\widehat{\mathrm{ch}}_2\left(\mathrm{Sym}^n(\overline{E})\right) = \binom{n+r}{r+1}\widehat{\mathrm{ch}}_2(\overline{E}) + \frac{1}{2}\binom{n+r-1}{r+1}\widehat{c}_1(\overline{E})^2 + \left(\frac{n}{r}\sum_{\substack{\alpha_1+\cdots+\alpha_r=n,\\ \alpha_1\geq 0,\ldots,\alpha_r\geq 0}} \log\left(\frac{n!}{\alpha_1!\cdots\alpha_r!}\right)\right) a(c_1(\overline{E})).$$

Proof. In [16], C. Soulé gives similar formulae in implicit forms. We follow his idea to calculate them.

(1) First of all, we fix notation. We set

$$S_{r,n} = \{(\alpha_1, \ldots, \alpha_r) \in (\mathbb{Z}_+)^r \mid \alpha_1 + \cdots + \alpha_r = n\},$$

where $\mathbb{Z}_+ = \{x \in \mathbb{Z} | x \geq 0\}$. For $I = (\alpha_1, \ldots, \alpha_r) \in S_{r,n}$ and rational sections $s_1, \ldots, s_r$ of E, we denote $s_1^{\alpha_1} \cdots s_r^{\alpha_r}$ by s^I and $\alpha_1! \cdots \alpha_r!$ by $I!$.

Let $s_1, \ldots, s_r$ be independent rational sections of E. Then, $\{s^I\}_{I \in S_{r,n}}$ forms independent rational sections of $\mathrm{Sym}^n(E)$. First, let us see that

$$\mathrm{div}\left(\bigwedge_{I \in S_{r,n}} s^I\right) = \frac{n}{r}\binom{n+r-1}{r-1}\mathrm{div}(s_1 \wedge \cdots \wedge s_r). \tag{7.1.1.1}$$

This is a local question. So let $x \in X$ and $\{\omega_1, \ldots, \omega_r\}$ be a local basis of E around x. We set $s_i = \sum_{j=1}^r a_{ij}\omega_j$. Then, $s_1 \wedge \cdots \wedge s_r = \det(a_{ij})\omega_1 \wedge \cdots \wedge \omega_r$. Let K be a rational function field of X. Since the characteristic of K is zero, any 1-dimensional representation of $\mathrm{GL}_r(K)$ is a power of the determinant. Thus, there is an integer N with

$$\bigwedge_{I \in S_{r,n}} s^I = \det(a_{ij})^N \bigwedge_{I \in S_{r,n}} \omega^I.$$

Here, by an easy calculation, we can see that

$$N = \frac{n}{r}\binom{n+r-1}{r-1}.$$

Thus, we get (7.1.1.1).

Next, let us see that

$$-\log\det\left(\mathrm{Sym}^n(h)(s^I, s^J)\right)_{I,J \in S_{r,n}} = -\frac{n}{r}\binom{n+r-1}{r-1}\log\det(h(s_i, s_j))_{i,j} + \sum_{I \in S_{r,n}} \log\left(\frac{n!}{I!}\right). \tag{7.1.1.2}$$

Let $x \in X(\mathbb{C})$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of $E \otimes \kappa(x)$. We set $s_i = \sum_{i=1}^r b_{ij}e_j$. Moreover, we set $s^I = \sum_{J \in S_{r,n}} b_{IJ}e^J$. Then, in the same way as before, $\det(b_{IJ}) = \det(b_{ij})^N$. Further, since

$$\mathrm{Sym}^n(h)(s^I, s^J) = \sum_{I', J' \in S_{r,n}} b_{II'}\,\mathrm{Sym}^n(h)(e^{I'}, e^{J'})\overline{b_{J'J}},$$

we have

$$\begin{aligned}\det\left(\mathrm{Sym}^n(h)(s^I, s^J)\right)_{I,J \in S_{r,n}} &= |\det(b_{IJ})|^2 \det\left(\mathrm{Sym}^n(h)(e^I, e^J)\right)_{I,J \in S_{r,n}} \\ &= |\det(b_{ij})|^{2N} \prod_{I \in S_{r,n}} \frac{I!}{n!}.\end{aligned}$$

Thus, we get (7.1.1.2). Therefore, combining (7.1.1.1) and (7.1.1.2), we obtain (1).

(2) First, we recall an elementary fact. Let $\Phi \in \mathbb{R}[X_1, \ldots, X_r]$ be a symmetric homogeneous polynomial, and $M_r(\mathbb{C})$ the algebra of complex $r \times r$ matrices. Then, there is a unique polynomial map $\underline{\Phi} : M_r(\mathbb{C}) \to \mathbb{C}$ such that $\underline{\Phi}$ is invariant under conjugation by $\mathrm{GL}_r(\mathbb{C})$ and its value on a diagonal matrix $\mathrm{diag}(\lambda_1, \ldots, \lambda_r)$ is equal to $\Phi(\lambda_1, \ldots, \lambda_r)$.

Let us consider a natural homomorphism

$$\rho_{r,n} : \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^r) \to \mathrm{Aut}_{\mathbb{C}}(\mathrm{Sym}^n(\mathbb{C}^r))$$

as complex Lie groups, which induces a homomorphism

$$\gamma_{r,n} = d(\rho_{r,n})_{\mathrm{id}} : \mathrm{End}_{\mathbb{C}}(\mathbb{C}^r) \to \mathrm{End}_{\mathbb{C}}(\mathrm{Sym}^n(\mathbb{C}^r))$$

as complex Lie algebras. Let $\{e_1, \dots, e_r\}$ be the standard basis of $\mathbb{C}^r$. Then, $\{e_I\}_{I \in S_{r,n}}$ forms a basis of $\mathrm{Sym}^n(\mathbb{C}^r)$, where $e_I = e_1^{\alpha_1} \cdots e_r^{\alpha_r}$ for $I = (\alpha_1, \dots, \alpha_r)$. Let us consider a symmetric polynomial

$$\mathrm{ch}_2^{r,n} = \frac{1}{2} \sum_{I \in S_{r,n}} X_I^2$$

in $\mathbb{R}[X_I]_{I \in S_{r,n}}$. Then, by the previous remark, using the basis $\{e_I\}_{I \in S_{r,n}}$, we have a polynomial map

$$\underline{\mathrm{ch}_2^{r,n}} : \mathrm{End}_{\mathbb{C}}(\mathrm{Sym}^n(\mathbb{C}^r)) \to \mathbb{C}$$

such that $\underline{\mathrm{ch}_2^{r,n}}$ is invariant under conjugation by $\mathrm{Aut}_{\mathbb{C}}(\mathrm{Sym}^n(\mathbb{C}^r))$ and

$$\underline{\mathrm{ch}_2^{r,n}}\left(\mathrm{diag}(\lambda_I)_{I \in S_{r,n}}\right) = \mathrm{ch}_2^{r,n}(\dots, \lambda_I, \dots).$$

Here we consider a polynomial map given by

$$\theta_{r,n} = \underline{\mathrm{ch}_2^{r,n}} \cdot \gamma_{r,n} : \mathrm{End}_{\mathbb{C}}(\mathbb{C}^r) \to \mathbb{C}.$$

Since $\gamma_{r,n}(PAP^{-1}) = \rho_{r,n}(P)\gamma_{r,n}(A)\rho_{r,n}(P)^{-1}$ for all $A \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^r)$ and $P \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^r)$, $\theta_{r,n}$ is invariant under conjugation by $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^r)$. Let us calculate

$$\theta_{r,n}(\mathrm{diag}(\lambda_1, \dots, \lambda_r)).$$

First of all,

$$\gamma_{r,n}(\mathrm{diag}(\lambda_1, \dots, \lambda_r)) = \mathrm{diag}\left(\dots, (\alpha_1\lambda_1 + \cdots + \alpha_r\lambda_r), \dots\right)_{(\alpha_1, \dots, \alpha_r) \in S_{r,n}}.$$

Thus,

$$\theta_{r,n}(\mathrm{diag}(\lambda_1, \dots, \lambda_r)) = \frac{1}{2} \sum_{(\alpha_1, \dots, \alpha_r) \in S_{r,n}} (\alpha_1\lambda_1 + \cdots + \alpha_r\lambda_r)^2.$$

On the other hand, by an easy calculation, we can see that

$$\sum_{(\alpha_1, \dots, \alpha_r) \in S_{r,n}} (\alpha_1\lambda_1 + \cdots + \alpha_r\lambda_r)^2 = \binom{n+r}{r+1}\left(\lambda_1^2 + \cdots + \lambda_r^2\right) + \binom{n+r-1}{r+1}(\lambda_1 + \cdots + \lambda_r)^2.$$

Therefore, we get

$$\theta_{r,n}(\mathrm{diag}(\lambda_1, \dots, \lambda_r)) = \frac{1}{2}\binom{n+r}{r+1}\left(\lambda_1^2 + \cdots + \lambda_r^2\right) + \frac{1}{2}\binom{n+r-1}{r+1}(\lambda_1 + \cdots + \lambda_r)^2.$$

Hence,

$$\theta_{r,n} = \binom{n+r}{r+1}\underline{\mathrm{ch}_2} + \frac{1}{2}\binom{n+r-1}{r+1}\underline{(c_1)^2}, \tag{7.1.1.3}$$

where $\mathrm{ch}_2(X_1, \dots, X_r) = \frac{1}{2}(X_1^2 + \cdots + X_r^2)$ and $c_1(X_1, \dots, X_r) = X_1 + \cdots + X_r$.

Let M be a complex manifold and $\overline{F} = (F, h_F)$ a Hermitian vector bundle of rank r on M. Let $K_{\overline{F}}$ be the curvature form of $\overline{F}$, and $K_{\mathrm{Sym}^n(\overline{F})}$ the curvature form of $\mathrm{Sym}^n(\overline{F})$. Then,

$$K_{\mathrm{Sym}^n(\overline{F})} = \left(\gamma_{r,n} \otimes \mathrm{id}_{A^{1,1}(M)}\right)(K_{\overline{F}}).$$

Thus, by (7.1.1.3),

$$\mathrm{ch}_2\left(\mathrm{Sym}^n(F, h_F)\right) = \binom{n+r}{r+1}\mathrm{ch}_2(F, h_F) + \frac{1}{2}\binom{n+r-1}{r+1}c_1(F, h_F)^2. \tag{7.1.1.4}$$

Now let $\overline{E} = (E, h)$ be a Hermitian vector bundle on a regular arithmetic variety X. Let h' be another Hermitian metric of E. Then, using the definition of Bott-Chern secondary characteristic classes and (7.1.1.4),

$$\widehat{\mathrm{ch}}_2\left(\mathrm{Sym}^n(E, h)\right) - \widehat{\mathrm{ch}}_2\left(\mathrm{Sym}^n(E, h')\right) = a\left(\binom{n+r}{r+1}\widetilde{\mathrm{ch}}_2(E, h, h') + \frac{1}{2}\binom{n+r-1}{r+1}\widetilde{c}_1^2(E, h, h')\right).$$

Thus,

$$\widehat{\mathrm{ch}}_2\left(\mathrm{Sym}^n(E, h)\right) - \binom{n+r}{r+1}\widehat{\mathrm{ch}}_2(E, h) - \frac{1}{2}\binom{n+r-1}{r+1}\widehat{c}_1(E, h)^2$$

does not depend on the choice of the metric h. Therefore, by using splitting principle [5, 3.3.2], we may assume that

$$(E, h) = \overline{L}_1 \oplus \cdots \oplus \overline{L}_r,$$

where $\overline{L}_i = (L_i, h_i)$'s are Hermitian line bundles. Then,

$$\mathrm{Sym}^n(\overline{E}) = \bigoplus_{\substack{\alpha_1+\cdots+\alpha_r=n,\\ \alpha_1\geq 0,\ldots,\alpha_r\geq 0}} \overline{L}_1^{\otimes\alpha_1} \otimes \cdots \otimes \overline{L}_r^{\otimes\alpha_r} \otimes \left(\mathcal{O}_X, \frac{\alpha_1!\cdots\alpha_r!}{n!}h_{can}\right).$$

Therefore, $\widehat{\mathrm{ch}}_2\left(\mathrm{Sym}^n(\overline{E})\right)$ is equal to

$$\sum_{\substack{\alpha_1+\cdots+\alpha_r=n,\\ \alpha_1\geq 0,\ldots,\alpha_r\geq 0}} \left\{\widehat{\mathrm{ch}}_2\left(\overline{L}_1^{\otimes\alpha_1} \otimes \cdots \otimes \overline{L}_r^{\otimes\alpha_r}\right) - \log\left(\frac{\alpha_1!\cdots\alpha_r!}{n!}\right) a\left(c_1\left(\overline{L}_1^{\otimes\alpha_1} \otimes \cdots \otimes \overline{L}_r^{\otimes\alpha_r}\right)\right)\right\}.$$

On the other hand, since

$$\sum_{(\alpha_1,\cdots,\alpha_r)\in S_{r,n}} \log\left(\frac{n!}{\alpha_1!\cdots\alpha_r!}\right)(\alpha_1 X_1 + \cdots + \alpha_r X_r) = \left(\frac{n}{r}\sum_{(\alpha_1,\cdots,\alpha_r)\in S_{r,n}} \log\left(\frac{n!}{\alpha_1!\cdots\alpha_r!}\right)\right)(X_1 + \cdots + X_r),$$

we have

$$\widehat{\mathrm{ch}}_2\left(\mathrm{Sym}^n(\overline{E})\right) = \binom{n+r}{r+1}\widehat{\mathrm{ch}}_2(\overline{E}) + \frac{1}{2}\binom{n+r-1}{r+1}\widehat{c}_1(\overline{E})^2$$

$$
\begin{aligned}
&+ \sum_{(\alpha_1,\cdots,\alpha_r)\in S_{r,n}} \log\left(\frac{n!}{\alpha_1!\cdots\alpha_r!}\right) a\left(\alpha_1 c_1(\overline{L_1}) + \cdots + \alpha_r c_1(\overline{L_r})\right) \\
=&\binom{n+r}{r+1}\widehat{\mathrm{ch}}_2(\overline{E}) + \frac{1}{2}\binom{n+r-1}{r+1}\widehat{c}_1(\overline{E})^2 \\
&+ \left(\frac{n}{r}\sum_{(\alpha_1,\cdots,\alpha_r)\in S_{r,n}} \log\left(\frac{n!}{\alpha_1!\cdots\alpha_r!}\right)\right) a(c_1(\overline{E})).
\end{aligned}
$$

Thus, we get (2). □

7.2. **Arithmetic Chern classes of $\overline{E}\otimes\overline{E}^{\vee}$.** Here, let us consider arithmetic Chern classes of $\overline{E}\otimes\overline{E}^{\vee}$.

Proposition 7.2.1. *Let X be a regular arithmetic variety and (E,h) a Hermitian vector bundle on X of rank r. Then,*

$$\widehat{\mathrm{ch}}_2(E\otimes E^{\vee}, h\otimes h^{\vee}) = 2r\widehat{\mathrm{ch}}_2(E,h) - \widehat{c}_1(E,h)^2 = (r-1)\widehat{c}_1(E,h)^2 - 2r\widehat{c}_2(E,h).$$

Proof. Since $\widehat{\mathrm{ch}}_i(E^{\vee},h^{\vee}) = (-1)^i\widehat{\mathrm{ch}}_i(E,h)$ and $\widehat{\mathrm{ch}}(E\otimes E^{\vee}, h\otimes h^{\vee}) = \widehat{\mathrm{ch}}(E,h)\cdot\widehat{\mathrm{ch}}(E^{\vee},h^{\vee})$, we have

$$
\begin{aligned}
\widehat{\mathrm{ch}}_2(E\otimes E^{\vee}, h\otimes h^{\vee}) &= r\widehat{\mathrm{ch}}_2(E,h) + \widehat{c}_1(E,h)\cdot\widehat{c}_1(E^{\vee},h^{\vee}) + r\widehat{\mathrm{ch}}_2(E^{\vee},h^{\vee}) \\
&= 2r\widehat{\mathrm{ch}}_2(E,h) - \widehat{c}_1(E,h)^2.
\end{aligned}
$$

The last assertion is derived from the fact

$$\widehat{\mathrm{ch}}_2(E,h) = \frac{1}{2}\widehat{c}_1(E,h)^2 - \widehat{c}_2(E,h).$$

□

8. The proof of Relative Bogomolov's inequality in the arithmetic case

In this section, we will give the proof of the main theorem of this paper.

8.1. **Preliminaries.** First of all, we will prepare three lemmas for the proof of the main theorem.

Lemma 8.1.1. *Let M be a d-dimensional compact Kähler manifold, $\overline{E} = (E,h)$ a flat Hermitian vector bundle on M of rank r, and $\overline{V} = (V,k)$ a Hermitian line bundle. Then, there is a constant c such that, for any $n > 0$ and any $s \in H^0(M, \mathrm{Sym}^n(E)\otimes V)$,*

$$\|s\|_{\sup} \leq cn^{d+r-1}\|s\|_{L^2}.$$

Proof. Let $f : P = \mathbb{P}(E) \to M$ be the projective bundle of E, and $L = \mathcal{O}_P(1)$ the tautological line bundle of E on P. Let h_L be the quotient metric of L induced by the surjective homomorphism $f^*(E) \to L$ and the Hermitian metric $f^*(h)$ of $f^*(E)$. Let Ω_M be a Kähler form of M. Since $\overline{E}$ is flat, $c_1(L,h_L)$ is positive semi-definite of rank $r-1$. Thus,

$f^*(\Omega_M) + c_1(L, h_L)$ gives rise to a fundamental 2-form Ω_P on P. Moreover, by virtue of flatness of $\overline{E}$, we have $c_1(L, h_L)^r = 0$. Thus,

$$\Omega_P^{d+r-1} = \binom{d+r-1}{d} f^*(\Omega_M^d) \wedge c_1(L, h_L)^{r-1}.$$

By [6, Lemma 30], there is a constant c such that

$$\|s'\|_{\sup} \leq cn^{d+r-1}\|s'\|_{L^2}$$

for any $n > 0$ and any $s' \in H^0(P, L^{\otimes n} \otimes f^*(V))$, where $\|s'\|_{L^2} = \int_P |s'|^2 \Omega_P^{d+r-1}$. We denote a homomorphism

$$f^*(\mathrm{Sym}^n(E)) \otimes f^*(V) \to L^{\otimes n} \otimes f^*(V)$$

by α_n. As in the proof of [6, (44)], we can see that, for any $s \in H^0(M, \mathrm{Sym}^n(E) \otimes V)$,

$$|s|^2 = \binom{n+r-1}{r-1} \int_{P \to M} |\alpha_n(s)|^2 c_1(L, h_L)^{r-1}.$$

Thus,

$$|s|^2 \leq \binom{n+r-1}{r-1} \int_{P \to M} \|\alpha(s)\|_{\sup}^2 c_1(L, h_L)^{r-1} = \binom{n+r-1}{r-1} \|\alpha(s)\|_{\sup}^2.$$

Therefore, we get

$$\|s\|_{\sup}^2 \leq \binom{n+r-1}{r-1} \|\alpha_n(s)\|_{\sup}^2$$

for all $s \in H^0(M, \mathrm{Sym}^n(E) \otimes V)$.

On the other hand,

$$\begin{aligned}
\|\alpha_n(s)\|_{L^2}^2 &= \int_P |\alpha_n(s)|^2 \Omega_P^r \\
&= \binom{d+r-1}{d} \int_M \int_{P \to M} |\alpha_n(s)|^2 f^*(\Omega_M^d) \wedge c_1(L, h_L)^{r-1} \\
&= \binom{d+r-1}{d} \int_M \Omega_M^d \int_{P \to M} |\alpha_n(s)|^2 c_1(L, h_L)^{r-1} \\
&= \binom{d+r-1}{d} \binom{n+r-1}{r-1}^{-1} \int_M |s|^2 \Omega_M^d \\
&= \binom{d+r-1}{d} \binom{n+r-1}{r-1}^{-1} \|s\|_{L^2}^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}\|s\|_{\sup}^2 &\leq \binom{n+r-1}{r-1}\|\alpha_n(s)\|_{\sup}^2 \\ &\leq \binom{n+r-1}{r-1}c^2 n^{2(d+r-1)}\|\alpha_n(s)\|_{L^2}^2 \\ &= \binom{d+r-1}{d}c^2 n^{2(d+r-1)}\|s\|_{L^2}^2.\end{aligned}$$

Thus, we get our lemma. □

Here we recall Einstein-Hermitian metrics of vector bundles. Let M be a d-dimensional compact Kähler manifold with a Kähler form Ω_M, and E a vector bundle on M. We say E is *stable* (resp. *semistable*) *with respect to* Ω_M if, for any subsheaves F of E with $0 \subsetneq F \subsetneq E$,

$$\frac{1}{\operatorname{rk} F}\int_M c_1(F)\wedge\Omega_M^{d-1} < \frac{1}{\operatorname{rk} E}\int_M c_1(E)\wedge\Omega_M^{d-1}.$$

$$\left(\text{resp.}\quad \frac{1}{\operatorname{rk} F}\int_M c_1(F)\wedge\Omega_M^{d-1} \leq \frac{1}{\operatorname{rk} E}\int_M c_1(E)\wedge\Omega_M^{d-1}.\right)$$

Moreover, E is said to be *poly-stable with respect to* Ω_M if E is semistable with respect to Ω_M and E has a decomposition $E = E_1 \oplus \cdots \oplus E_s$ of vector bundles such that each E_i is stable with respect to Ω_M. Let h be a Hermitian metric of E. We say h is *Einstein-Hermitian with respect to* Ω_M if there is a constant ρ such that $K(E,h)\wedge\Omega_M^{d-1} = \rho\Omega_M^d \otimes \mathrm{id}_E$, where $K(E,h)$ is the curvature form given by (E,h) and id_E is the identity map in $\mathcal{H}om(E,E)$. The Kobayashi-Hitchin correspondence says us that E has an Einstein-Hermitian metric with respect to Ω_M if and only if E is poly-stable with respect to Ω_M.

Lemma 8.1.2. *Let M be a compact Kähler manifold with a Kähler form Ω_M, and E a poly-stable vector bundle with respect to Ω_M on M. If h and h' are Einstein-Hermitian metrics of E with respect to Ω_M, then so is $h+h'$.*

Proof. Let $E = E_1 \oplus \cdots \oplus E_s$ be a decomposition into stable vector bundles. If we set $h_i = h|_{E_i}$ and $h_i' = h'|_{E_i}$ for each i, then h_i and h_i' are Einstein-Hermitian metrics of E_i and we have the following orthogonal decompositions:

$$(E,h) = \bigoplus_{i=1}^{s}(E_i,h_i) \quad\text{and}\quad (E,h') = \bigoplus_{i=1}^{s}(E_i,h_i')$$

(cf. [9, Chater IV, § 3]). Thus, we may assume that E is stable. In this case, by virtue of uniqueness of Einstein-Hermitian metric, there is a positive constant c with $h' = ch$. Thus, $h+h' = (1+c)h$. Hence $h+h'$ is Einstein-Hermitian. □

Lemma 8.1.3. *Let C be a compact Riemann surface. Considering C as a projective variety over $\mathbb{C}$, let $\overline{C} = C \otimes_{\mathbb{C}} \mathbb{C}$ be the tensor product via the complex conjugation. Let E be a vector bundle on C, and $\overline{E} = E \otimes_{\mathbb{C}} \mathbb{C}$ on $\overline{C}$. Then, E is poly-stable on C if and only if $\overline{E}$ is poly-stable on $\overline{C}$.*

Proof. This is an easy consequence of the fact that if F is a vector bundle on C, then $\deg(F) = \deg(\overline{F})$. □

8.2. **Proof of the main theorem.** Let us start the proof of the main theorem.

Theorem 8.2.1. *Let $f : X \to Y$ be a projective morphism of regular arithmetic varieties such that every fiber of $f_{\mathbb{C}} : X(\mathbb{C}) \to Y(\mathbb{C})$ is a reduced and connected curve with only ordinary double singularities. Let (E, h) be a Hermitian vector bundle on X of rank r, and $y \in Y(\overline{\mathbb{Q}})$. If f is smooth over y and $E|_{X_y}$ is semi-stable, then*

$$\widehat{\mathrm{dis}}_{X/Y}(E, h) = f_* \left(2r\widehat{c}_2(E, h) - (r-1)\widehat{c}_1(E, h)^2\right)$$

is weakly positive at y with respect to any subsets S of $Y(\mathbb{C})$ with the following properties: (1) S is finite, and (2) $f_{\mathbb{C}}^{-1}(z)$ is smooth and $E_{\mathbb{C}}|_{f_{\mathbb{C}}^{-1}(z)}$ is poly-stable for all $z \in S$.

Proof. Considering $S \cup F_\infty(S)$ instead of S, we may assume that $F_\infty(S) = S$ by virtue of Lemma 8.1.3. For each $z \in S$, let Ω_z be a Kähler form induced by the metric of $\overline{\omega}_{X/Y}$ along $f_{\mathbb{C}}^{-1}(z)$. Since $E_{\mathbb{C}}|_{f_{\mathbb{C}}^{-1}(z)}$ is poly-stable for all $z \in S$, there is a C^∞ Hermitian metric h' of $E_{\mathbb{C}}$ such that $h'|_{f_{\mathbb{C}}^{-1}(z)}$ is Einstein-Hermitian with respect to Ω_z for all $z \in S$. It is easy to see that $\overline{F_\infty^*(h')}\Big|_{f_{\mathbb{C}}^{-1}(z)}$ is Einstein-Hermitian with respect to Ω_z for all $z \in S$. Thus, if h' is not invariant under F_∞, then, considering $h' + \overline{F_\infty^*(h')}$, we may assume that h' is invariant under F_∞. For, by Lemma 8.1.2, $h' + \overline{F_\infty^*(h')}$ is Einstein-Hermitian with respect to Ω_z on $f_{\mathbb{C}}^{-1}(z)$ for each $z \in S$.

Here we claim:

Claim 8.2.1.1. *There is a $\gamma \in L^1_{\mathrm{loc}}(Y(\mathbb{C}))$ such that $a(\gamma) \in \widehat{\mathrm{CH}}^1_{L^1}(Y; S)$ and $\gamma(z) \geq 0$ for each $z \in S$, and*

$$\widehat{\mathrm{dis}}_{X/Y}(E, h) = \widehat{\mathrm{dis}}_{X/Y}(E, h') + a(\gamma).$$

Proof. We set $\phi = \sqrt[r]{\det(h')/\det(h)}$. Then, it is easy to see that $\widehat{\mathrm{dis}}_{X/Y}(E, \phi h) = \widehat{\mathrm{dis}}_{X/Y}(E, h)$. Thus, we may assume that $\det(h) = \det(h')$. Then, we have

$$\widehat{\mathrm{dis}}_{X/Y}(E, h) - \widehat{\mathrm{dis}}_{X/Y}(E, h') = a\left(f_*(2r\widetilde{\mathrm{ch}}_2(E, h, h'))\right).$$

Hence if we set $\gamma = f_*(2r\widetilde{\mathrm{ch}}_2(E, h, h'))$, then $a(\gamma) \in \widehat{\mathrm{CH}}^1_{L^1}(Y; S)$. On the other hand, by [2, (ii) of Corollary 1.30], $f_*(\widetilde{\mathrm{ch}}_2(E, h, h'))(z)$ is nothing more than Donaldson's Lagrangian (for details, see [11, §6]). Thus, we get $\gamma(z) \geq 0$ for each $z \in S$. □

By the above claim, we may assume that $h|_{f_{\mathbb{C}}^{-1}(z)}$ is Einstein-Hermitian for each $z \in S$. Let $\overline{A} = (A, h_A)$ be a Hermitian line bundle on X such that A is very ample, and $A \otimes \omega_{X/Y}^{\otimes -1}$ is ample. If we take a general member M' of $|A_{\mathbb{Q}}^{\otimes 2}|$, then, by Bertini's theorem (cf. [8, Theorem 6.10]), M' is smooth over $\mathbb{Q}$, and $M' \to Y_{\mathbb{Q}}$ is étale over y. Note that if Z is an algebraic set of $\mathbb{P}^N_{\mathbb{C}}$, U is a non-empty Zariski open set of $\mathbb{P}^N_{\mathbb{Q}}$, and $U(\mathbb{Q}) \subseteq Z(\mathbb{C})$, then $Z = \mathbb{P}^N_{\mathbb{C}}$. Hence, we may assume that $M'(\mathbb{C}) \to Y(\mathbb{C})$ is étale over z for all $z \in S$. Let $M' = M'_1 + \cdots + M'_{l_1} + M'_{l_1+1} + \cdots + M'_{l_2}$ be the decomposition of M' into irreducible components

(actually, a decomposition into connected components because M' is smooth over $\mathbb{Q}$) such that $f_{\mathbb{Q}}(M'_i) = Y_{\mathbb{Q}}$ for $1 \leq i \leq l_1$ and $f_{\mathbb{Q}}(M'_j) \subsetneq Y_{\mathbb{Q}}$ for $l_1+1 \leq j \leq l_2$. Let M_i $(i = 1, \ldots, l_1)$ be the closure of M'_i in X. We set $M = M_1 + \cdots + M_{l_1}$ and $B = M_1 \coprod \cdots \coprod M_{l_1}$ (disjoint union). Then, there is a line bundle L on X with $M \in |A^{\otimes 2} \otimes L|$. Note that $L|_{X_y} \simeq \mathcal{O}_{X_y}$ and $L_{\mathbb{C}}|_{f_{\mathbb{C}}^{-1}(z)} \simeq \mathcal{O}_{f_{\mathbb{C}}^{-1}(z)}$ for all $z \in S$ because $y \notin \bigcup_{j=l_1+1}^{l_2} f_{\mathbb{Q}}(M'_j)$ and $z \notin \bigcup_{j=l_1+1}^{l_2} f_{\mathbb{C}}(M'_j(\mathbb{C}))$. We denote a morphism $B \to M \to X$ by ι, and a morphism $B \xrightarrow{\iota} X \xrightarrow{f} Y$ by g. We remark that the morphism $B \to M$ is an isomorphism over $\mathbb{Q}$. Further, we set

$$\overline{F} = \mathcal{E}nd(E, h) = (E \otimes E^{\vee}, h \otimes h^{\vee}).$$

Then, $h \otimes h^{\vee}$ is a flat metric along $f_{\mathbb{C}}^{-1}(z)$ for each $z \in S$ because $h \otimes h^{\vee}$ is Einstein-Hermitian and $\deg(E \otimes E^{\vee}) = 0$ along $f_{\mathbb{C}}^{-1}(z)$. We choose a Hermitian line bundle $\overline{H} = (H, h_H)$ on Y such that $g_*(\iota^*(A)) \otimes H$ and $g_*(\iota^*(F)) \otimes H$ are generated by small sections at y with respect to S. Moreover, we set

$$\overline{F}_n = \mathrm{Sym}^n\left(\overline{F} \otimes f^*(\overline{H})\right) \otimes \overline{A} \otimes f^*(\overline{H}) = \left(\mathrm{Sym}^n\left(F \otimes f^*(H)\right) \otimes A \otimes f^*(H), k_n\right).$$

Claim 8.2.1.2. *There are $Z_0, \ldots, Z_{r^2} \in \widehat{\mathrm{CH}}^1_{L^1}(Y; S)_{\mathbb{Q}}$ and $\beta \in L^1_{\mathrm{loc}}(Y(\mathbb{C}))$ such that $a(\beta) \in \widehat{\mathrm{CH}}^1_{L^1}(Y; S)$, and*

$$f_*\left(\widehat{\mathrm{ch}}_2(\overline{F}_n) - \frac{1}{2}\widehat{c}_1(\overline{F}_n) \cdot \widehat{c}_1(\overline{\omega}_{X/Y})\right) = \frac{n^{r^2+1}}{(r^2+1)!} f_*(\widehat{\mathrm{ch}}_2(\overline{F})) + \sum_{i=0}^{r^2} Z_i n^i + a(b_n \beta),$$

where $b_n = \displaystyle\sum_{\substack{\alpha_1 + \cdots + \alpha_{r^2} = n, \\ \alpha_1 \geq 0, \ldots, \alpha_{r^2} \geq 0}} \log\left(\frac{n!}{\alpha_1! \cdots \alpha_{r^2}!}\right)$.

Proof. Since $\mathrm{Sym}^n(\overline{F} \otimes f^*(\overline{H})) \otimes \overline{A} \otimes f^*(\overline{H})$ is isometric to $\mathrm{Sym}^n(\overline{F}) \otimes f^*(\overline{H})^{\otimes(n+1)} \otimes \overline{A}$,

$$\widehat{\mathrm{ch}}_2(\overline{F}_n) = \widehat{\mathrm{ch}}_2(\mathrm{Sym}^n(\overline{F})) + \widehat{c}_1(\mathrm{Sym}^n(\overline{F})) \cdot \widehat{c}_1(f^*(\overline{H})^{\otimes(n+1)} \otimes \overline{A}) + \binom{n+r^2-1}{r^2-1} \widehat{\mathrm{ch}}_2(f^*(\overline{H})^{\otimes(n+1)} \otimes \overline{A}).$$

Here since $\det(\overline{F}) = \overline{\mathcal{O}}_X$, by Proposition 7.1.1,

$$\widehat{c}_1(\mathrm{Sym}^n(\overline{F})) = a(b_n) \quad \text{and} \quad \widehat{\mathrm{ch}}_2(\mathrm{Sym}^n(\overline{F})) = \binom{n+r^2}{r^2+1} \widehat{\mathrm{ch}}_2(\overline{F}).$$

Thus, by Proposition 2.3.4,

$$\begin{aligned} f_*\left(\widehat{c}_1(\mathrm{Sym}^n(\overline{F})) \cdot \widehat{c}_1(f^*(\overline{H})^{\otimes(n+1)} \otimes \overline{A})\right) &= f_*\left(b_n a\left((n+1)f^*(c_1(\overline{H})) + c_1(\overline{A})\right)\right) \\ &= a\left(b_n f_*(c_1(\overline{A}))\right). \end{aligned}$$

On the other hand, using the projection formula (cf. Proposition 2.3.4),

$$\begin{aligned} f_*\left(\widehat{\mathrm{ch}}_2(f^*(\overline{H})^{\otimes(n+1)} \otimes \overline{A})\right) &= \frac{1}{2} f_*\left[\left((n+1)\widehat{c}_1(f^*(\overline{H})) + \widehat{c}_1(\overline{A})\right)^2\right] \\ &= \frac{1}{2} f_*\left[(n+1)^2 \widehat{c}_1(f^*(\overline{H}))^2 + 2(n+1)\widehat{c}_1(f^*(\overline{H})) \cdot \widehat{c}_1(\overline{A}) + \widehat{c}_1(\overline{A})^2\right] \\ &= (n+1)\deg_f(A)\widehat{c}_1(\overline{H}) + \frac{1}{2} f_*\left(\widehat{c}_1(\overline{A})^2\right), \end{aligned}$$

where $\deg_f(A)$ is the degree of A on the generic fiber of f. Therefore, we have

$$\begin{aligned} f_*\widehat{\mathrm{ch}}_2(\overline{F}_n) = &\binom{n+r^2}{r^2+1} f_*\widehat{\mathrm{ch}}_2(\overline{F}) + \\ &\binom{n+r^2-1}{r^2-1}\left((n+1)\deg_f(A)\widehat{c}_1(\overline{H}) + \frac{1}{2} f_*\left(\widehat{c}_1(\overline{A})^2\right)\right) + a\left(b_n f_*(c_1(\overline{A}))\right). \end{aligned}$$

Thus, there are $Z'_0, \ldots, Z'_{r^2} \in \widehat{\mathrm{CH}}^1(Y;S)_{\mathbb{Q}}$ such that

$$f_*\widehat{\mathrm{ch}}_2(\overline{F}_n) = \frac{n^{r^2+1}}{(r^2+1)!} f_*\widehat{\mathrm{ch}}_2(\overline{F}) + \sum_{i=0}^{r^2} Z'_i n^i + a\left(b_n f_*(c_1(\overline{A}))\right). \tag{8.2.1.3}$$

Further, since $\widehat{c}_1(\overline{F}_n) \cdot \widehat{c}_1(\overline{\omega}_{X/Y})$ is equal to

$$\left(\widehat{c}_1(\mathrm{Sym}^n(\overline{F})) + \binom{n+r^2-1}{r^2-1}\left((n+1)\widehat{c}_1(f^*(\overline{H})) + \widehat{c}_1(\overline{A})\right)\right) \cdot \widehat{c}_1(\overline{\omega}_{X/Y}),$$

we have

$$\begin{aligned} f_*\left(\widehat{c}_1(\overline{F}_n) \cdot \widehat{c}_1(\overline{\omega}_{X/Y})\right) = &\, a\left(b_n f_*(c_1(\overline{\omega}_{X/Y}))\right) + \\ &\binom{n+r^2-1}{r^2-1}\left((n+1)(2g-2)\widehat{c}_1(\overline{H}) + f_*\left(\widehat{c}_1(\overline{A}) \cdot \widehat{c}_1(\overline{\omega}_{X/Y})\right)\right). \end{aligned}$$

Thus, there are $Z''_0, \ldots, Z''_{r^2} \in \widehat{\mathrm{CH}}^1(Y;S)_{\mathbb{Q}}$ such that

$$f_*\left(\widehat{c}_1(\overline{F}_n) \cdot \widehat{c}_1(\overline{\omega}_{X/Y})\right) = \sum_{i=0}^{r^2} Z''_i n^i + a\left(b_n f_*(c_1(\overline{\omega}_{X/Y}))\right). \tag{8.2.1.4}$$

Thus, combining (8.2.1.3) and (8.2.1.4), we get our claim. □

Let $h_{X/Y}$ be a C^∞ Hermitian metric of $\det Rf_*\mathcal{O}_X$ over $Y(\mathbb{C})$ such that $h_{X/Y}$ is invariant under F_∞. Then, since the Quillen metric $h_Q^{\overline{\mathcal{O}_X}}$ of $\det Rf_*\mathcal{O}_X$ is a generalized metric, there is a real valued $\phi \in L^1_{\mathrm{loc}}(Y(\mathbb{C}))$ such that $h_Q^{\overline{\mathcal{O}_X}} = e^{\phi} h_{X/Y}$ and $F_\infty^*(\phi) = \phi$ (a. e.). Adding a suitable real valued C^∞ function ϕ' with $F_\infty^*(\phi') = \phi'$ to ϕ (replace $h_{X/Y}$ by $e^{-\phi'}h_{X/Y}$ accordingly), we may assume that $\phi(z) = 0$ for all $z \in S$. Here, we set $h_n = \exp\left(-\binom{n+r^2-1}{r^2-1}\phi\right) h_Q^{\overline{F}_n}$.

Then, h_n is a generalized metric of $\det Rf_*F_n$ with $F_\infty^*(h_n) = \overline{h}_n$ (a. e.). Moreover,

$$\widehat{c}_1\left(\det Rf_*F_n, h_n\right) - \binom{n+r^2-1}{r^2-1}\widehat{c}_1\left(\det Rf_*\mathcal{O}_X, h_{X/Y}\right) = \widehat{c}_1\left(\det Rf_*F_n, h_Q^{\overline{F}_n}\right) - \binom{n+r^2-1}{r^2-1}\widehat{c}_1\left(\det Rf_*\mathcal{O}_X, h_Q^{\overline{\mathcal{O}_X}}\right).$$

Here, since

$$\widehat{c}_1\left(\det Rf_*F_n, h_Q^{\overline{F}_n}\right) - \binom{n+r^2-1}{r^2-1}\widehat{c}_1\left(\det Rf_*\mathcal{O}_X, h_Q^{\overline{\mathcal{O}_X}}\right) \in \widehat{\mathrm{CH}}^1_{L^1}(Y;S)_{\mathbb{Q}}$$

by Theorem 5.2.1 and $\widehat{c}_1(\det Rf_*\mathcal{O}_X, h_{X/Y}) \in \widehat{\mathrm{CH}}^1(Y;S)$, we have

$$\widehat{c}_1\left(\det Rf_*F_n, h_n\right) \in \widehat{\mathrm{CH}}^1_{L^1}(Y;S)_{\mathbb{Q}}.$$

Further, by the arithmetic Riemann-Roch theorem for stable curves (cf. Theorem 5.2.1),

$$\widehat{c}_1\left(\det Rf_*(F_n), h_n\right) - \binom{n+r^2-1}{r^2-1}\widehat{c}_1\left(\det Rf_*(\mathcal{O}_X), h_{X/Y}\right) = f_*\left(\widehat{\mathrm{ch}}_2(\overline{F}_n) - \frac{1}{2}\widehat{c}_1(\overline{F}_n)\cdot\widehat{c}_1(\overline{\omega}_{X/Y})\right).$$

Therefore, by Claim 8.2.1.2, there are $W_0, \ldots, W_{r^2} \in \widehat{\mathrm{CH}}^1_{L^1}(Y;S)_{\mathbb{Q}}$ and $\beta \in L^1_{\mathrm{loc}}(Y(\mathbb{C}))$ such that $a(\beta) \in \widehat{\mathrm{CH}}^1_{L^1}(Y;S)$, and

$$\widehat{c}_1\left(\det Rf_*(F_n), h_n\right) = \frac{n^{r^2+1}}{(r^2+1)!}f_*(\widehat{\mathrm{ch}}_2(\overline{F})) + \sum_{i=0}^{r^2} W_i n^i + a(b_n\beta). \tag{8.2.1.5}$$

Claim 8.2.1.6. $\dfrac{1}{(r^2+1)!}\widehat{\mathrm{dis}}_{X/Y}(\overline{E}) = -\lim_{n\to\infty}\dfrac{\widehat{c}_1\left(\det Rf_*(F_n), h_n\right)}{n^{r^2+1}}$ *in* $\widehat{\mathrm{CH}}^1_{L^1}(Y;S)_{\mathbb{Q}}$.

Proof. By virtue of Proposition 7.2.1, $f_*(\widehat{\mathrm{ch}}_2(\overline{F})) = -\widehat{\mathrm{dis}}_{X/Y}(\overline{E})$. Thus, by (8.2.1.5), it is sufficient to show that $0 \leq b_n \leq O(n^{r^2})$.

It is well known that

$$\frac{\log(\theta_1)+\cdots+\log(\theta_N)}{N} \leq \log\left(\frac{\theta_1+\cdots+\theta_N}{N}\right)$$

for positive numbers $\theta_1, \ldots, \theta_N$. Thus, noting $\displaystyle\sum_{\substack{\alpha_1+\cdots+\alpha_{r^2}=n,\\ \alpha_1\geq 0,\ldots,\alpha_{r^2}\geq 0}} \frac{n!}{\alpha_1!\cdots\alpha_{r^2}!} = (r^2)^n$, we have

$$0 \leq \sum_{\substack{\alpha_1+\cdots+\alpha_{r^2}=n,\\ \alpha_1\geq 0,\ldots,\alpha_{r^2}\geq 0}} \log\left(\frac{n!}{\alpha_1!\cdots\alpha_{r^2}!}\right) \leq \binom{n+r^2-1}{r^2-1}\log\left(\frac{(r^2)^n}{\binom{n+r^2-1}{r^2-1}}\right) \leq O(n^{r^2}).$$

□

We set $\overline{G}_n = \iota^*(\overline{F}_n)$. Then, by Theorem 4.2.1,

$$\widehat{c}_1\left(\det Rg_*(G_n), h_Q^{\overline{G}_n}\right) - \binom{n+r^2-1}{r^2-1}\widehat{c}_1\left(\det Rg_*(\mathcal{O}_B), h_Q^{\overline{\mathcal{O}}_B}\right) \in \widehat{\mathrm{CH}}^1_{L^1}(Y;S)$$

and

$$\widehat{c}_1\left(\det Rg_*(G_n), h_Q^{\overline{G}_n}\right) - \binom{n+r^2-1}{r^2-1}\widehat{c}_1\left(\det Rg_*(\mathcal{O}_B), h_Q^{\overline{\mathcal{O}}_B}\right) = g_*\left(\widehat{c}_1(\overline{G}_n)\right).$$

As before, we can take a C^∞ Hermitian metric $h_{B/Y}$ of $\det Rg_*(\mathcal{O}_B)$ over $Y(\mathbb{C})$ and a real valued $\varphi \in L^1_{\mathrm{loc}}(Y(\mathbb{C}))$ such that $h_Q^{\overline{\mathcal{O}}_B} = e^{\varphi} h_{B/Y}$, $F_\infty^*(h_{B/Y}) = \overline{h}_{B/Y}$, $F_\infty^*(\varphi) = \varphi$ (a. e.), and $\varphi(z) = 0$ for all $z \in S$. We set

$$g_n = \exp\left(-\binom{n+r^2-1}{r^2-1}\varphi\right) h_Q^{\overline{G}_n}.$$

Then,

$$\widehat{c}_1\left(\det Rg_*(G_n), g_n\right) - \binom{n+r^2-1}{r^2-1}\widehat{c}_1\left(\det Rg_*(\mathcal{O}_B), h_{B/Y}\right) = g_*\left(\widehat{c}_1(\overline{G}_n)\right)$$

and $\widehat{c}_1\left(\det Rg_*(G_n), g_n\right) \in \widehat{\mathrm{CH}}^1_{L^1}(Y;S)$. Moreover, in the same as in Claim 8.2.1.2, we can see that

$$g_*\left(\widehat{c}_1(\overline{G}_n)\right) = a(\deg(g)b_n) + \binom{n+r^2-1}{r^2-1}\left((n+1)g_*\widehat{c}_1(g^*(\overline{H})) + g_*\widehat{c}_1(\iota^*(\overline{A}))\right).$$

Thus, there are $W'_0, \ldots, W'_{r^2} \in \widehat{\mathrm{CH}}^1_{L^1}(Y;S)_{\mathbb{Q}}$ such that

$$\widehat{c}_1\left(\det Rg_*(G_n), g_n\right) = \sum_{i=0}^{r^2} W'_i n^i + a(b_n \deg(g)).$$

Therefore, we have

$$\lim_{n\to\infty} \frac{\widehat{c}_1\left(\det Rg_*(G_n), g_n\right)}{n^{r^2+1}} = 0 \tag{8.2.1.7}$$

in $\widehat{\mathrm{CH}}^1_{L^1}(Y;S)_{\mathbb{Q}}$.

Let us consider an exact sequence:

$$0 \to F_n \otimes A^{\otimes -2} \otimes L^{\otimes -1} \to F_n \to F_n|_M \to 0.$$

Since F is semi-stable and of degree 0 along X_y and $L|_{X_y} = \mathcal{O}_{X_y}$, we have

$$f_*(F_n \otimes A^{\otimes -2} \otimes L^{\otimes -1}) = 0$$

on Y. Thus, the above exact sequence gives rise to

$$0 \to f_*(F_n) \to (f|_M)_*(F_n|_M) \to R^1 f_*(F_n \otimes A^{\otimes -2} \otimes L^{\otimes -1}) \to R^1 f_*(F_n).$$

Let Q_n be the cokernel of

$$f_*(F_n) \to (f|_M)_*(F_n|_M) \to g_*(G_n).$$

Let U be the maximal Zariski open set of Y such that f is smooth over U and g is étale over U. Moreover, let U_n be the maximal Zariski open set of Y such that

$$\begin{cases} \text{(a) } U_n \subset U, \\ \text{(b) } (f|_M)_*(F_n|_M) \text{ coincides with } g_*(G_n) \text{ over } U_n, \\ \text{(c) } R^1 f_*(F_n) = 0 \text{ over } U_n, \text{ and} \\ \text{(d) } f_*(F_n),\ g_*(G_n) \text{ and } Q_n \text{ are locally free over } U_n. \end{cases}$$

Then, $y \in U_n(\overline{\mathbb{Q}})$ and $S \subseteq U_n(\mathbb{C})$. For, since $A \otimes \omega_{X/Y}^{-1}$ is ample on X_y and E is semi-stable on X_y, we can see that $R^1 f_*(F_n) = 0$ around y, which implies that $f_*(F_n)$ is locally free around y. Further, since $f_*(F_n)$ and $(f|_M)_*(F_n|_M)$ are free at y, $R^1 f_*(F_n) = 0$ around y, and $(f|_M)_*(F_n|_M)$ coincides with $g_*(G_n)$ around y, we can easily check that Q_n is free at y. Thus, $y \in U_n(\overline{\mathbb{Q}})$. In the same way, we can see that $S \subseteq U_n(\mathbb{C})$.

Next let us consider a metric of $\det Q_n$. $g_*(G_n)$ has the Hermitian metric $(f|_M)_*(k_n|_M)$ over $U_n(\mathbb{C})$, where k_n is the Hermitian metric of $\overline{F}_n$. Let $\tilde{q}_n$ be the quotient metric of Q_n over $U_n(\mathbb{C})$ induced by $(f|_M)_*(k_n|_M)$. Let q_n be a C^∞ Hermitian metrics of $\det Q_n$ over $Y(\mathbb{C})$ such that $F_\infty^*(q_n) = q_n$ and $q_n(z) = \det \tilde{q}_n(z)$ for all $z \in S$. (If q_n is not invariant under F_∞, then consider $(1/2)\left(q_n + \overline{F_\infty^*(q_n)}\right)$.)

Here since $\det Rf_*(F_n) \simeq \det f_*(F_n) \otimes (\det R^1 f_*(F_n))^{-1}$ and $\det f_*(F_n) \simeq \det g_*(G_n) \otimes (\det Q_n)^{-1}$, we have

$$\det Rf_*(F_n) \simeq \det g_*(G_n) \otimes (\det Q_n)^{-1} \otimes \left(\det R^1 f_*(F_n)\right)^{-1}.$$

Further, we have metrics h_n, g_n and q_n of $\det Rf_*(F_n)$, $\det g_*(G_n)$ and $\det Q_n$. Thus, there is a generalized metric t_n of $\det R^1 f_*(F_n)$ such that the above is an isometry.

As in the proof of Proposition 3.7.1, let us construct a section of $\det R^1 f_*(F_n)$. First, we fix a locally free sheaf P_n on Y and a surjective homomorphism $P_n \to R^1 f_*(F_n)$. Let P_n' be the kernel of $P_n \to R^1 f_*(F_n)$. Then, P_n' is a torsion free sheaf and has the same rank as P_n because $R^1 f_*(F_n)$ is a torsion sheaf. Noting that $\left(\bigwedge^{\mathrm{rk}\, P_n'} P_n'\right)^*$ is an invertible sheaf on Y, we can identify $\det R^1 f_*(F_n)$ with

$$\bigwedge^{\mathrm{rk}\, P_n} P_n \otimes \left(\bigwedge^{\mathrm{rk}\, P_n'} P_n'\right)^*.$$

Moreover, a homomorphism $\bigwedge^{\mathrm{rk}\, P_n'} P_n' \to \bigwedge^{\mathrm{rk}\, P_n} P_n$ induced by $P_n' \hookrightarrow P_n$ gives rise to a non-zero section s_n of $\det R^1 f_*(F_n)$. Note that $s_n(y) \neq 0$ and $s_n(z) \neq 0$ for all $z \in S$ because $R^1 f_*(F_n) = 0$ at y and z.

Here we set

$$a_n = \max_{z \in S}\{\log t_n(s_n, s_n)(z)\}.$$

By our construction, we have

$$\widehat{c}_1(\det R^1 f_*(F_n), e^{-a_n} t_n) \in \widehat{\mathrm{CH}}^1_{L^1}(Y; S).$$

and an isometry

$$(8.2.1.8)\quad (\det Rf_*(F_n), h_n) \simeq (\det g_*(G_n), g_n) \otimes (\det Q_n, q_n)^{-1} \otimes (\det R^1 f_*(F_n), e^{-a_n} t_n)^{-1} \otimes (\mathcal{O}_Y, e^{-a_n} h_{can}).$$

Here we claim:

Claim 8.2.1.9. $(\det Q_n, q_n)$ *is generated by small sections at* y *with respect to* S.

Proof. First of all,

$$g_*(\iota^*(F) \otimes g^*(H)) = g_*(\iota^*(F)) \otimes H \quad \text{and} \quad g_*(\iota^*(A) \otimes g^*(H)) = g_*(\iota^*(A)) \otimes H$$

are generated by small section at y with respect to S. Thus, by (2) and (3) of Proposition 3.7.1,

$$g_*(G_n) = g_*(\mathrm{Sym}^n(\iota^*(F) \otimes g^*(H)) \otimes \iota^*(A) \otimes g^*(H))$$

is generated by small sections at y with respect to S. Thus, by (1) of Proposition 3.7.1, $(Q_n, \tilde{q}_n)$ is generated by small sections at y with respect to S. Hence, by (4) of Proposition 3.7.1, $(\det Q_n, q_n)$ is generated by small sections at y with respect to S because $q_n(z) = \det \tilde{q}_n(z)$ for all $z \in S$. □

Next we claim:

Claim 8.2.1.10. $a_n \leq O(n^{r^2} \log(n))$.

Proof. It is sufficient to show that $\log t_n(s_n, s_n)(z) \leq O(n^{r^2} \log(n))$ for each $z \in S$. Let $\{e_1, \ldots, e_{l_n}\}$ be an orthonormal basis of $g_*(G_n) \otimes \kappa(z)$ with respect to $g_*(k_n|_B)(z)$ such that $\{e_1, \ldots, e_{m_n}\}$ forms a basis of $f_*(F_n) \otimes \kappa(z)$. Then, $e_1 \wedge \cdots \wedge e_{m_n}$, $e_1 \wedge \cdots \wedge e_{l_n}$ and $\bar{e}_{m_n+1} \wedge \cdots \wedge \bar{e}_{l_n}$ form bases of $\det(f_*(F_n)) \otimes \kappa(z)$, $\det(g_*(G_n)) \otimes \kappa(z)$, and $\det(Q_n) \otimes \kappa(z)$ respectively, and $(e_1 \wedge \cdots \wedge e_{m_n}) \otimes (\bar{e}_{m_n+1} \wedge \cdots \wedge \bar{e}_{l_n}) = e_1 \wedge \cdots \wedge e_{l_n}$, where $\bar{e}_{m_n+1}, \ldots, \bar{e}_{l_n}$ are images of $e_{m_n+1}, \ldots, e_{l_n}$ in $Q_n \otimes \kappa(z)$. Then,

$$\left|(e_1 \wedge \cdots \wedge e_{m_n}) \otimes s_n^{\otimes -1}\right|^2_{h_n}(z) = \frac{|e_1 \wedge \cdots \wedge e_{l_n}|^2_{g_n}(z)}{|\bar{e}_{m_n+1} \wedge \cdots \wedge \bar{e}_{l_n}|^2_{q_n}(z) |s_n|^2_{t_n}(z)} = |s_n|^{-2}_{t_n}(z),$$

where $|a|_\lambda = \sqrt{\lambda(a,a)}$ for $\lambda = h_n, g_n, q_n, t_n$. Moreover, let Ω_z be the Kähler form induced by the metric of $\overline{\omega}_{X/Y}$ along $f_{\mathbb{C}}^{-1}(z)$. Then, there is a Hermitian metric v_n of $H^0(f_{\mathbb{C}}^{-1}(z), F_n)$ defined by

$$v_n(s, s') = \int_{f_{\mathbb{C}}^{-1}(z)} k_n(s, s') \Omega_z.$$

Here $R^1 f_*(F_n) = 0$ at z. Thus, $(\det R^1 f_*(F_n))_z$ is canonically isomorphic to $\mathcal{O}_{Y(\mathbb{C}),z}$. Since $(P'_n)_z = (P_n)_z$, under the above isomorphism, s_n goes to the determinant of $(P_n)_z \xrightarrow{\mathrm{id}} (P_n)_z$, namely $1 \in \mathcal{O}_{Y(\mathbb{C}),z}$. Hence, by the definition of Quillen metric,

$$\left|(e_1 \wedge \cdots \wedge e_{m_n}) \otimes s_n^{\otimes -1}\right|^2_{h_n}(z) = \det(v_n(e_i, e_j)) \exp\left(-T\left(\overline{F}_n\big|_{f_{\mathbb{C}}^{-1}(z)}\right)\right).$$

Therefore,

$$\log |s_n|^2_{t_n}(z) = T\left(\overline{F}_n\big|_{f_{\mathbb{C}}^{-1}(z)}\right) - \log\det(v_n(e_i, e_j)).$$

By Corollary 6.5,

$$T\left(\overline{F}_n\big|_{f_{\mathbb{C}}^{-1}(z)}\right) \leq O(n^{r^2}\log(n)).$$

Thus, in order to get our claim, it is sufficient to show that

$$-\log\det(v_n(e_i,e_j)) \leq O(n^{r^2-1}\log(n)).$$

Let s be an arbitrary section of $H^0(f_{\mathbb{C}}^{-1}(z),F_n)$. Then, by Lemma 8.1.1,

$$g_*\left(k_n|_B\right)(s,s) = \sum_{x\in g_{\mathbb{C}}^{-1}(z)} |s|_{k_n}^2(x) \leq \deg(g)\sup_{x\in f_{\mathbb{C}}^{-1}(z)}\{|s|_{k_n}^2(x)\} \leq \deg(g)c^2n^{2r^2}\|s\|_{L^2}^2$$

for some constant c independent of n. Thus, by [11, Lemma 3.4] and our choice of e_i's,

$$1 = \det\left(g_*\left(k_n|_B\right)(e_i,e_j)\right) \leq \left(\deg(g)c^2n^{2r^2}\right)^{\dim_{\mathbb{C}} H^0(f_{\mathbb{C}}^{-1}(z),F_n)}\det\left(v_n(e_i,e_j)\right).$$

Using Riemann-Roch theorem, we can easily see that

$$\dim_{\mathbb{C}} H^0(f_{\mathbb{C}}^{-1}(z),F_n) \leq O(n^{r^2-1}).$$

Thus, we have

$$-\log\det(v_n(e_i,e_j)) \leq O(n^{r^2-1}\log(n)).$$

Hence, we obtain our claim. □

Let us go back to the proof of our theorem. By the isometry (8.2.1.8), we get

$$\begin{aligned}-\widehat{c}_1(\det Rf_*(F_n),h_n) &= -\widehat{c}_1(\det g_*(G_n),g_n) + \widehat{c}_1(\det Q_n,q_n) + \widehat{c}_1(\det R^1f_*(F_n),e^{-a_n}t_n) - a(a_n)\\ &= \left[\widehat{c}_1(\det Q_n,q_n) + \widehat{c}_1(\det R^1f_*(F_n),e^{-a_n}t_n) + a\left(\max\{-a_n,0\}\right)\right]\\ &\qquad + \left[-\widehat{c}_1(\det g_*(G_n),g_n) + a(\min\{-a_n,0\})\right].\end{aligned}$$

Here we set

$$\begin{cases}\alpha_n = \dfrac{(r^2+1)!}{n^{r^2+1}}\left[\widehat{c}_1(\det Q_n,q_n) + \widehat{c}_1(\det R^1f_*(F_n),e^{-a_n}t_n) + a\left(\max\{-a_n,0\}\right)\right],\\[2ex] \beta_n = \dfrac{(r^2+1)!}{n^{r^2+1}}\left[-\widehat{c}_1(\det g_*(G_n),g_n) + a(\min\{-a_n,0\})\right].\end{cases}$$

Then,

$$\frac{-(r^2+1)!\widehat{c}_1(\det Rf_*(F_n),h_n)}{n^{r^2+1}} = \alpha_n+\beta_n.$$

By (8.2.1.7) and Claim 8.2.1.10, $\lim_{n\to\infty}\beta_n = 0$ in $\widehat{\mathrm{CH}}^1_{L^1}(Y;S)_{\mathbb{Q}}$. Therefore, by Claim 8.2.1.6,

$$\widehat{\mathrm{dis}}_{X/Y}(\overline{E}) = \lim_{n\to\infty}\frac{-(r^2+1)!\widehat{c}_1\left(\det Rf_*(F_n),h_n\right)}{n^{r^2+1}} = \lim_{n\to\infty}(\alpha_n+\beta_n) = \lim_{n\to\infty}\alpha_n$$

in $\widehat{\mathrm{CH}}^1_{L^1}(Y;S)_{\mathbb{Q}}$. On the other hand, it is obvious that

$$\widehat{c}_1(\det R^1f_*(F_n),e^{-a_n}t_n) \quad\text{and}\quad a\left(\max\{-a_n,0\}\right)$$

is semi-ample at y with respect to S. By Claim 8.2.1.9, $\widehat{c}_1(\det Q_n, q_n)$ is semi-ample at y with respect to S. Thus, α_n is semi-ample at y with respect to S. Hence we get our theorem. $\square$

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Department of Mathematics, Faculty of Science, Kyoto University, Kyoto, 606-01, Japan

E-mail address, Shu Kawaguchi: kawaguch@kusm.kyoto-u.ac.jp

E-mail address, Atsushi Moriwaki: moriwaki@kusm.kyoto-u.ac.jp