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Theorem 4.0.3 *Let M be a QHS and K be a null homologous knot in M. We construct +1 surgery M', 0 surgery M" as in ZHS case. The Euler characteristic $\chi(f)$ for the Chern-Simons gradient vector field f on the admissible bundle $P_{M"}$ is to be formulated as above. Then the following holds:*

$$2|\bar{\lambda}'(K)| \ = \ |\chi(f)|$$

Here $\bar{\lambda}'(K)$ denotes the unnormalized Casson-Walker knot invariant for K. ([W] Corollary (3.7))

Remark 1

In the definition of Walker's invariant for QHS and knot, the following normalization is to be made (in order to get the compatibility with connected sum formula for knot invariant): for a QHS M, we put

$$\lambda(M) \ = \ \bar{\lambda}(M) \ /|H_1(M \ ; \ Z)|$$

here the numerator denotes the unnormalized Casson-Walker invariant which is described above. And the denominator denotes the cardinality of $H_1(M \ ; \ Z)$. Similarly, for a null homologous knot K in M, we put

$$\lambda'(K) \ = \ \bar{\lambda}'(K)/|H_1(M \ ; \ Z)|$$

$$where \ \bar{\lambda}'(K) \ = \ \bar{\lambda}(M') - \bar{\lambda}(M)$$

Remark 2

We can also formulate the difference cycle δ for the knot with self linking number 0 ([W], 3.34). So theorem 4.0.3 is also valid for this case.

From the above theorem, we can see that, for the extention of Floer's triangle to QHS case, it is necessary to consider how large the twisting effect of h on ν is.

$$h_t(\rho)(\alpha) \;=\; \rho(\alpha) \;\in\; SU(2)$$
$$and\ for\ \alpha \in \pi_1(bottom\ F)\ (the\ other\ side)$$
$$h_t(\rho)(\alpha) \;=\; \rho(K)^{g(\rho(K))t}\rho(\alpha)\rho(K)^{-g(\rho(K))t} \in SU(2)$$

Clearly h_t $(0 \leq t \leq 1)$ descends to an isotopy of the space of conjugacy classes, $\hat{R}(\Sigma)$. Note that h_1 coincides with h outside a small neighborhood of $R_-(K)$. As we saw in chapter 1, h can not be extended continuously to $R_-(K)$, but h is well defined as a map all over $\hat{R}(\Sigma)$. By the definitions of the isotopies h and h_t, we can formulate the difference between $h^*\hat{R}(W_1)$ and $h_1^*\hat{R}(W_1)$ as difference cycle δ, as in section 1.3. In particular, δ resides outside of S.

For the proof of framing independence of λ'(K), it is crucial that h_1 is a special isotopy. (This is a result of Goldman [G]) From this, the following equality of the unnormalized Walker invariant $(*)$ holds:

$$< h_1^*\hat{R}(W_1),\ \hat{R}(W_2) > = < \hat{R}(W_1),\ \hat{R}(W_2) > \qquad (**)$$

Here we used special isotopy invariance of Casson-Walker invariant. Both sides are equal to $\bar{\lambda}(M)$. On the other hand, in a completely similar manner as in the proof of proposition 1.3.1, the following holds:

$$< h^*\hat{R}(W_1),\ \hat{R}(W_2) > \;-\; < h_1^*\hat{R}(W_1),\ \hat{R}(W_2) > = < \delta,\ \hat{R}(W_2) > \qquad (***)$$

here δ denotes the difference cycle. In the left hand side, the first term is equal to $\bar{\lambda}(M')$, and the second term is equal to $\bar{\lambda}(M)$. Hence

$$\bar{\lambda}'(K) \;=\; \bar{\lambda}(M') \;-\; \bar{\lambda}(M)$$
$$= < \delta,\ \hat{R}(W_2) >$$

We must remark that, though the equality $(***)$ concerns only irreducible representations, the equality $(**)$ contains information about reducible representations. In the case of ZHS, the equality $(**)$ is based on the fact that the signed cardinality of the boundary of a compact 1 dimensional oriented manifold is 0. (All inter section points of isotoped Lagrangian submanifolds reside outside of S. So, as t moves, the trace of intersection points of $h_t^*R(W_1)$ and $R(W_2)$ makes up a compact 1 dimensional oriented manifold.) However, in the case of QHS, this compactness is broken because of the existence of the intersection points $T_1 \cap T_2$ in the singular strata S. From this, h may well give rise to generation or annihilation of irreducible intersection points. (this corresponds to generation or annihilation of nondegenerate critical points of Chern-Simons functional as mentioned before) The counter terms I(p) compensates for this effect of h, and the equality $(**)$ holds.

Similarly as in the proof of our main theorem 3.5.2, the space of flat connections on the relatively nontrivial bundle $P_{M''}|_{W_1'}$ coinsides with the difference cycle δ which is formulated above. Thus we obtain

This is nothing but the difficulty pointed out above. So counting (with sign) only irreducible representations do not give a topological invariant. Walker gave a topological invariant (in particular, independent of the choice of special isotopy) with the following form:

$$(*) \qquad \bar{\lambda}(M) \;=\; \sum_{p \in R(W_1) \cap R(W_2) - S} sign(p) \;+\; \sum_{p \in R(W_1) \cap R(W_2) \cap S} I(p)$$

We call it unnormalized Walker invariant for M. Here I(p) is a counter term which carries information about twisting in ν around p. ([W], chapter 2)

Now we return to the description of the knot invariant λ'(K). We use the same notations as in section 1.3. Performing +1 surgery along K' in W_1, the representation space $R(W_1)$ is transformed, as the subspace of $R(\Sigma)$, in the following manner (see section 1.3):

$$h \;:\; R(W_1) \longrightarrow R(W_1')$$
$$(\rho_1,\ \rho_1) \longmapsto (\rho_1,\ Ad\rho_1(K)\rho_1)$$
$$here\ \rho_1 \in R(F)$$

We denote h for this map (h is also well defined as a self map on $R(\Sigma)$). We define one more map:

$$h_t \;:\; R(\Sigma) \longrightarrow R(\Sigma)$$

For definition of it, we first fix an Ad-invariant, smooth cut-off function

$$g \;:\; SU(2) \longrightarrow [0\ ,\ 1]$$

such that g=1 outside a small neighborhood of -1, and g=0 at -1. Then h_t $(0 \leq t \leq 1)$ is defined as follows:

$$for\ \rho \in R(\Sigma)\ and\ \alpha \in \pi_1(upper\ F)\ (base\ point\ side)$$

knot invariant. In QHS case, it is not necessarily possible to take such a surface for general knot. So, we must assume that the knot K is null homologous. (See the next figure)

We denote the singular strata of the representation space R(Σ) by S, and it's Zariski normal bundle in R(Σ) by ν. We also denote the Lagrangian subspace of S composed of reducible representations

$$\pi_1(W_1)\ resp.\ \pi_1(W_2) \longrightarrow\ S_0^1\ \in\ SU(2)$$

by T_1 resp. T_2, and it's Zariski normal bundle in R(W_1) resp. R(W_2) by $\eta_1,\ \eta_2$. Here S_0^1 denotes an S^1 subgroup of SU(2) which is characterized by the holonomy of each reducible representation. As is easily seen, the fibers at p, η_{1p} and η_{2p}, belong to special class $\mathcal{L}_p^C$ of Lagrangian subspaces of ν_p : complex Lagrangian subspaces ([W], section 1C).

The complex Lagrangian subspaces η_{1p} and η_{2p} determine unique elements of $\det^1(\nu_p)$ (unit determinant) respectively. We denote these elements by $\det^1(\eta_{1p})$, $\det^1(\eta_{2p})$. Performing special isotopy, the section $\det^1(\eta_1)$ rotates around p$\in$S and the irreducible intersection points $\det(\eta_{1p})\cap\det(\eta_{2p})\subset\det(\nu_p)$ are generated or annihilated. This rotation corresponds to the fact that $\pi_1(\mathcal{L}_p^C)$=Z.(See the next figure)

Since K is a null homologous knot in M, the holonomy along K of any reducible flat connection is always 1. From this observation, every flat connection we consider is necessarily irreducible. (Moreover, in the space of flat connections on the trivial SU(2) bundle over M, the space of reducible flat connections is apart from the space of flat connections on $P_{M''}$. The reason of this is nothing but the choice of our admissible bundle.) It is at this point that the assumption that K is null homologous is necessary from Gauge theoretic point of view. Therefore, we can formulate the Euler characteristic $\chi(f)$ as follows: Fix a flat connection on $P_{M''}$, A_0. For each flat connection A on $P_{M''}$, we can take a path of connections on $P_{M''}$ which connects A_0 and A. Since both A_0 and A are nondegenerate critical points of f, the mod 2 spectral data of the Chern-Simons Hessian ∇(f) along this path is well defined. It is independent of the choice of path connecting A_0 and A. (see Proposition 3.2.1) So we denote this data by $\Delta(A_0, A)$. Let Z be the finite set of zero points of f. We put

$$|\chi(f)| \ = \ | \sum_{[A]\in Z} \Delta(A_0, A)|$$

Next we review the formulation of the Casson-Walker knot invariant λ'(K). Originally the knot invariant $\lambda'(K)$ was defined by

$$\lambda'(K) \ = \ \lambda(M') \ - \lambda(M)$$

Here λ denotes Casson-Walker invariant for QHS. In the extention of λ invariant from ZHS case to QHS case, the essential point is the treatment of reducible representations. In the case of ZHS, λ invariant is defined by counting (with sign) irreducible representations of the fundamental group. In particular, transversality of Lagrangian submanifolds (representation spaces of the fundamental groups of handle bodies) was guaranteed by performing compactly supported isotopy of the representation space of Heegard surface. This transversality condition is equivalent to the nondegeneracy condition of Chern-Simons functional. On the other hand, in the case of QHS, to guarantee this transversality (nondegeneracy) condition, it is needed to perform so called special isotopy ([W], section 1E).

The main difficulty lies in that irreducible intersection points of Lagrangian submanifolds (nondegenerate critical points) are generated or annihilated if the choice of special isotopy is altered. (So, in the case of QHS, Floer homology is not well defined if we do not take into account of the choice of special isotopy.)

First, we fix a Heegard decomposition corresponding to the knot K

$$M \ = \ W_1 \ \cup_\Sigma \ W_2$$

In chapter 2, we used a Seifert surface F of K to construct the above Heegard splitting corresponding to K, and this splitting is essential for the formulation of Casson-Walker

Chapter 4

Rational homology sphere case

The aim of this chapter is to extend our main theorem 3.5.2 to the case of null homologous knot in a rational homology 3 sphere. (We denote those 3 manifolds by QHS.) So, let M be a QHS, and let K be a null homologous knot in M. As in chapter 2, we denote +1 Dehn surgery of M along K by M', and 0 Dehn surgery by M".

First of all, we formulate (the absolute value of) the "Euler characteristic for the Floer homology group" of the 0-surgered rational homology $S^1 \times S^2$, M". We define admissible SO(3) bundle $P_{M"}$ over M" and Chern-Simons functional on it as in section 3.1. Our Euler characteristic is formulated as the Euler characteristic $\chi(f)$ of the Chern-Simons gradient vector field f for $P_{M"}$. In chapter 2, we described the space of gauge equivalence classes of flat connections on the nontrivial SO(3) bundle $P_{M"}$ over M". (In that case, M was a ZHS) Those connections can be identified with the SU(2) flat connections on the (trivial) SU(2) bundle over the complement in M" of the core circle K, with holonomy -1 around a small linking circle. Briefly speaking, we count those flat connections with sign. If there exist reducible flat connections on $P_{M"}$ (these correspond to degenerate critical points of Chern-Simons functional), we ought to formulate the spectral flow data concerning these degenerate critical points (i.e. the treatment of the 0-eigenvalue of the Chern-Simons Hessian is difficult). We claim that this difficulty does not occur.

As we saw in chapter 2, under the 0-surgery description,

$$M" = (M - N(K)) \cup N(K)$$

the small linking circle in N(K) is attached to the "longitude curve" K in M-N(K) (see chapter 2). This is also the boundary circle of the Seifert surface F in section 1.3. Heegard surface Σ corresponding to the knot K, is the double of F.

We consider the space of flat connections on the trivial SU(2) bundle over M whose holonomy along this null homologous knot K is -1. The space of the above mentioned flat connections on $P_{M"}$ is contained in this space.

Here we regard V as subbundle of $\lambda_0^* \mathcal{T}_0$. ($\lambda_0$, V, Π) is nothing but the triple which gives the relative intersection numbers of δ and $\mathrm{m'}_2^0$ in $\mathrm{m'}_0^0$.

The connection family $\lambda \in \mathrm{B}_0$ (in fact, if they are flat on Σ, then it is not necessary for them to be flat on W_2) and the bundle $\ker \nabla f_1'|_{\lambda_1}$, $\ker \nabla f_2'|_{\lambda_2}$ pick up the information concerning the information about spectre of Chern-Simons Hessian on the infinite dimensional connection space. By restriction to Σ, the flat connection family $\lambda_0 \in m_0'^0$ pick up the information concerning intersection numbers in the finite dimensional representation space. □

By inspection, the above mod 2 cardinality coincides with the mod 2 spectral data of ∇f'. This completes the proof of Main Theorem.

Remark

To tell the truth, we must consider transversality in the requirement (2) and triviality of coker($\nabla \mathrm{f'}_\alpha$). About these considerations, see [T].

Require that this triple has the following properties:

(1) *At* $t = 0$ *and* $t = 1$, $h : V \cong \lambda^*(TX_1 \oplus TX_2)$ *is an orientation preserving isomorphism.*

(2) *The section* $det(h)$ *of the real line bundle* $\wedge^{d_0}(\lambda^* TX_0) \otimes \wedge^{d_0} V$ *is transverse to the zero section.*

Then $(\det h)^{-1}(0)$ is a finite point set, and the relative orientation of $h(V)=\lambda^*(TX_1 \oplus TX_2)$ and $\lambda^*(TX_0)$ at t=0,1, differs by the mod2 cardinality of $(\det h)^{-1}(0)$ (i.e. at each zero point of det(h), h loses isomorphy, and the relative orientation of h(V) and $\lambda^*(TX_0)$ is reversed across the point).

We introduce this picture into our setting associated to the Heegard decomposition. Suppose [A], [A']$\in$m'^{0} are given. Needless to say, (X_0, X_1, X_2) is to be (δ, m'^0_1, m'^0_2). Choose a smooth path λ_0 in m'^0_0 from [A] to [A'] (family of "flat" connections on Σ). It can be lifted to a path λ_2 in m'^0_2 (because m'^0_2 is path connected). On the other hand, λ_2 can not be lifted to a path in m'^{0} (because points of m'^{0} are isolated). So we lift λ_2 to a path in B^0. We denote λ_1 for it's restriction $i^*\lambda \subset B^0_1$ to W_1'.

Then we can make the following identification:

$$ker \nabla f'_0|_{\lambda_0} \cong \lambda_0^* T m'^0_0$$

$$ker \nabla f'_2|_{\lambda_2} \cong \lambda_2^* T m'^0_2$$

$$at\ the\ end\ points,\ ker \nabla f'_1 \cong T\delta \quad (as\ vector\ spaces)$$

As the bundle V$\longrightarrow$[0,1], we take

$$\lambda_1^*(ker \nabla f'_1|_{\lambda_1}) \oplus \lambda_2^*(ker \nabla f'_2|_{\lambda_2})$$

At last, as the bundle homomorphism h:V$\longrightarrow$ λ^*TX$_0$, we take the L^2-orthogonal projection:

$$\Pi : \lambda_0^* \mathcal{T}_0 \longrightarrow \lambda_0^* T m'^0_0$$

$$Q_A\omega \ = (Q_{1A}(a_1, a_2),\ Q_{2A}(a_1, a_2),\ Y(a_1, a_2) - Q^*_{0A}\ v_0)$$

Concerning these operators, the following proposition holds. This proposition translates spectral data in the order H→Q→ ∇f'.

Proposition 3.5.2 *Let f' satisfy nondegeneracy condition. Then for each [A]∈B^0, there exist the above bounded operators such that Q_A is an index 0, Fredholm operator. Furthermore*

1. *Q_A-H_A is compact (so index is not influenced)*
2. *If f'(A)=0, then Q_A=H_A*
3. *The assignment of Q_A to A defines a smooth section Q of $Fred_0(\mathcal{E}^0, \mathcal{E}^1)$ over B^0. Q qnd H are homotopic sections relative to $f'^{-1}(0)$*
4. *The following commutative diagram (of Fredholm bundle maps) holds:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{T}_1 \oplus \mathcal{T}_2 & \longrightarrow & \mathcal{T}_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{L}_1 \oplus \mathcal{L}_2 & \longrightarrow & \mathcal{L}_0 & \longrightarrow & 0 \end{array}$$

The last diagram is concerned with linearization of restriction of connections. Remark that we saw similar commutative diagram in Lemma 3.3.3. For definition of linearization Φ, Ψ, we must have taken into account of not only the Coulomb gauge condition but also the Neumann condition. From this, we must take Q_1, Q_2, Q_0 instead of $\nabla f'_1$, $\nabla f'_2$, $\nabla f'_0$. For detailes, see [T] Proposition 6.1. □

The background of the KEY OBSERVATION (Proposition 3.5.1) is as follows:

Suppose X_1, X_2 are finite dimensional oriented submanifolds of X_0 with complementary dimension. Let p_0, $p_1 \in X_1 \cap X_2$ are transverse intersections. Relative sign difference between the intersection numbers at p_0 and p_1 is calculated by choosing a smooth triple (λ, V, h): here

$$\lambda \ : \ [0,\ 1] \longrightarrow X_0$$

$$\lambda(0) = p_0, \quad \lambda(1) = p_1$$

$$V \longrightarrow [0,\ 1] \ : \ \textit{smooth oriented vector bundle}$$

$$h \ : \ V \longrightarrow \lambda^* TX_0 \ : \ \textit{bundle homomorphism}$$

Here the left hand side denotes the fiber bundle whose fiber over [A]$\in B_0$ is a Banach manifold of bounded, index 0, Fredholm operators from $\mathcal{E}^0_A$ to $\mathcal{E}^1_A$. As one sees easily, this bundle is trivial. So, fixing a trivialization, H defines a smooth map h from B^0 to the Banach space $\mathcal{F}_0$ of real, bounded, index 0, Fredholm operators on a separable Hilbert space.

As is easily seen, $\mathcal{F}_0$ is the classifying space for real K-theory. By the map h, the first Stiefel-Whitney class $\omega_1 \in H^1(\mathcal{F}_0\ ;\ Z/2)$ is pulled back to the class $h^*\omega_1 \in H^1(B^0\ ;\ Z/2)$. In our case, this class vanishes. The Poincaré dual of ω_1 is the subspace $\mathcal{F}^1_0 \in \mathcal{F}_0$ of operators with non empty kernel. (codimension 1)

When we choose a closed loop

$$\varphi\ :\ [0,\ 1] \longrightarrow B^0$$

then the closed loop

$$h\varphi\ :\ [0,\ 1] \longrightarrow \mathcal{F}_0$$

intersects with $\mathcal{F}^1_0$ (after generic perturbation) transversely in a finite point set. The mod 2 cardinality of this set is nothing but the evaluatioon of $h^*\omega_1$ on the loop φ. In our case, this class vanishes. Hence, the mod 2 cardinality of intersections between $\mathcal{F}^1_0$ and the image by h of a path in B^0 from [A] to [A'] depends only on the end points [A], [A'] $\in B^0$ (here we take [A], [A'] such that kernel$(H_{(*)})$=0).

The KEY OBSERVATION is the following:

Proposition 3.5.1 *Let [A], [A']*$\in m^0$.

Then the above mod 2 cardinality gives the sign difference between the intersection numbers of δ with $j_2^* m'^{\,0}_{\,2}$ in $m'^{\,0}_{\,0}$ at the points [A_0], [A_0'].

The proof is straightforward and similar to that of [T] Proposition 5.6. So, instead of giving a detailed proof, we see it's outline later.

This proposition translates a relative intersection number on $m'^{\,0}_{\,0}$ into spectral data for a family of Fredholm operators $H_{(*)}$. To obtain Theorem 3.5.1, we must compare the latter with the spectral data of a family of Fredholm operators $\nabla f'_{(*)}$. For this comparison, we introduce the following operators associated with Heegard decomposition : for [A]$\in B^0$

$$Q_A\ :\ \mathcal{E}^0_A \longrightarrow \mathcal{E}^1_A$$

$$Q_{1,2A}\ :\ \mathcal{T}_{1A} \oplus \mathcal{T}_{2A} \longrightarrow \mathcal{L}_{1,2A}$$

$$Q_{0A}\ :\ \mathcal{T}_{0A} \longrightarrow \mathcal{L}_{0A}$$

here $Q_{1,2A}$ and Q_{0A} are determined by the following proposition. With these determined, we define the operator Q_A as follows:

$$for\ \omega =\ (a_1,\ a_2,\ v_0) \in \mathcal{E}^0_A$$

Let [A], [A']$\in B^0$ be two points in $f'^{-1}(0)$, and $[A_0]=j_1^* i_1^*[A]$, $[A_0']=j_1^* i_1^*[A']$ be their restrictions (contained in $\delta \cap m_2'^0 \subset m_0'^0$). Then the mod 2 spectral data for ∇f' between [A] and [A'] vanishes if and only if the local intersection number at $[A_0]$ of $\delta \cap m_2'^0 \subset m_0'^0$ agrees with that at $[A_0']$.

Combining this with Proposition 2.2.1, we obtain immediately

Theorem 3.5.2 (Main Theorem) *The following equality holds:*

$$2|\lambda'(K)| \ = |\chi(f)|$$

The reason of taking perturbed section f' is that we want to guarantee nondegeneracy of f.We take an admissibly and compatively perturbed gradient f'. Here admissibility is understood to mean that of section 1 and compatibility is to mean that of section 3.

On the other hand, in the definition of λ(M), we must have taken a compactly supported isotopy in m_0^0 to satisfy transversality condition.

Following [T], we introduce a Fredholm operator which comes with the Heegard splitting. Using this operator, we can easily show that these nondegeneracy and transversality conditions coincide (see [T] Lemma 5.5). Moreover, we can use this operator to reinterpret the finite dimensional intersection theory on m_0^0 as a spectral invariant of a family of elliptic operators (on an infinite dimensional Hilbert space).

The Hilbert spaces concerned are

$$\mathcal{E}_A^0 = \ \mathcal{T}_{1A} \ \oplus \ \mathcal{T}_{2A} \ \oplus \ \mathcal{L}_{0A}$$

$$\mathcal{E}_A^1 = \ \mathcal{L}_{1A} \ \oplus \ \mathcal{L}_{2A} \ \oplus \ \mathcal{T}_{0A}$$

here we do not distinguish a connection and it's restriction

and the operator we consider is

$$H_A\omega \ : \ \mathcal{E}_A^0 \longrightarrow \mathcal{E}_A^1$$

$$H_A\omega = (\nabla f_1'|_A \, a_1, \ \nabla f_2'|_A \, a_2, \ Y(a_1,a_2) - (\nabla f_0'|_A)^* \, u_0)$$

$$here \ \omega = (a_1, \ a_2, \ u_0)$$

Y is a modification of Ψ (in section 3) which is compact, and the adjoint

$$(\nabla f_0'|_A)^* \ : \ \mathcal{L}_{0A} \longrightarrow \mathcal{T}_{0A}$$

is defined using integration by parts.

As in [T] Proposition 5.4, we can show that H_A is a Fredholm operator with index 0 and defines smooth section H of the bundle

$$Fred_0(\mathcal{E}^0, \ \mathcal{E}^1) \longrightarrow B^0$$

is twisted in accordance with the generator of $\pi_1(\mathrm{SO}(3))\cong \mathrm{Z}_2$. (Please refer to the argument in section 2.1.) This can not be a pull back form by the deformation retract.) From this, we can obtain

$$H^2_A(W_1') = 0$$

Poincaré duality implies the assertion. □

This proposition shows that transversality holds (without perturbation) for the cycle of flat connections on $\mathrm{P}_{M"}|_{W_1'}$ calculated in chapter 2.

Next we can see the following:

Proposition 3.4.2

$$ker\ \nabla f_{1A} \ = \ H^1_A(W_1')$$

So, combined with the above argument, we conclude

$$dim\ ker\ \nabla f_{1A} \ = \ 3g-3$$

This is an easy consequence of the variational problem.

3.5 Comparison of $|\lambda'(K)|$ and $|\chi(f)|$

To show the coincidence of $|\lambda'(K)|$ and $|\chi(f)|$, we first use the fact that ther is a 1to1 correspondence between $\mathrm{f}^{-1}(0)$ (the set of flat connections) and the representation space Hom(π_1(M"),SO(3)) with prescribed nontriviality as in chapter 2. We next use the fact that this set coincides with the intersection $\delta \cap m_2^0$ (Here δ denotes the difference cycle). The final step is to show that two different ways to determine the relative sign coincide. The proof of this fact is similar to [T], and proceed as follows. Fredholm properties of the maps in the equality in Lemma 3.3.3:

$$m^0(=\delta \cap m_2^0)$$

$$= f^{-1}(0) = I^{-1}((f_1^{-1}(0) \times f_2^{-1}(0)) \cap J^{-1}(\Delta))$$

has been established in section 3.3. From this, we can compare spectral data on B^0 and intersection number in m_0^0.

The theorem we aim at is the following. Beforehand we take the perturbation f' of f stated later.

Theorem 3.5.1 *Let M be ZHS and K be a knot in M.*

As in the preceding sections, we take the Heegard splitting (W_1, W_2) associated to K, make 0-surgery in W_1 to get (W_1', W_2), take relatively nontrivial SO(3) bundle on W_1', and gauge theoretic setting on them.

Proposition 3.4.1

$$H^1_A(W_1',\ \partial W_1') \ = \ 0$$

Proof

In the situation of [T] (1/0 Dehn surgery case), the corresponding equality

$$H^1_A(W_1,\ \partial W_1) \ = \ 0$$

follows from the fact that $H^2_A(W_1)$=0 and Poincaré duarity ($H^2_A(W_1)$=0 holds since W_1 is a handlebody, so it is homotopy equivalent to 1-complex).

In our W_1' case, we also take irreducible connection A, and use Meyer-Vietoris exact sequence for the pair (W_1'-N(K), N(K)). We remark that the equality

$$H^1_A(W_1',\ \partial W_1') \ = \ 0$$

does not hold for the relatively trivial SO(3) bundle on W_1'. We can see this immediately using Meyer-Vietoris exact sequence. We pick up the following part:

$$H^1_A(W_1' - N(K)) \oplus H^1_A(N(K)) \longrightarrow H^1_A(\partial N(K)) \longrightarrow H^2_A(W_1')$$

Here the first term is

$$H^1_A(W_1' - N(K)) \cong R^{3g-3} \oplus R(resp.R^3), \quad H^1_A(N(K)) \cong R(resp.R^3)$$

and the second term is

$$H^1_A(\partial N(K)) \cong R \oplus R\ (resp.R^3 \oplus R^3)$$

Of course, the choice of R or R^3 depends on the holonomy along the core circle of N(K).

From the definition of 0-surgery (section 1.2), the component R⊕R (resp. R^3⊕R^3) in the first terms goes to only one component of R⊕R (resp. R^3⊕R^3) in the second term. So in this case we can see

$$H^2_A(W_1') \cong R\ (resp.R^3)$$

On the other hand, if we use our nontrivial bundle P_{M^n}, then the two R (resp.R^3) component in the first term do not go to the same R (resp.R^3) component in the second term. This is caused by the construction of P_{M^n}. (i.e. H^1_A(N(K)) ≅R (resp.R^3) is represented by the "constant" 1-form with values in the centralizer of the holonomy along the core circle of N(K). This is a differential form pulled back by deformation retract to the core circle. Although the generator of R(resp. R^3) component in H^1_A(W_1'-N(K)) is also represented by the "constant" 1-form, the value of it's image in H^1_A(∂W_1')

3.4 Transversality and index of ∇f_1'

For each irreducible connection A on $P_{M^*}|_{W_1'}$, a Fredholm operator $\nabla f_{1A}'$ (admissibly perturbed Chern Simons Hessian) was defined in the last section. We compute the index of this operator.

At first, from Stokes' theorem we can conclude the following immediately. We omit it's proof:

Lemma 3.4.1 *We take an irreducible flat connection A. Then*

$$Coker\ \nabla f_{1A} \ = \ H_A^1(W_1',\ \partial W_1')$$

here H_A^1 denotes twisted cohomology defined by the flat connection A.

The index of ∇f_{1A} (or the twisted signature operator associated to it) is independent of choice of irreducible flat connection A. So in particular, we can assume A is irreducible on $W_0 \cap W_1$'. Then we can get the following long exact sequence of d_A-twisted cohomology for the pair (W_1', ∂W_1') :

$$\begin{aligned}
&\to H_A^2(W_1',\ \partial W_1') \longrightarrow H_A^2(W_1') \longrightarrow H_A^2(\partial W_1') \\
&\to H_A^1(W_1',\ \partial W_1') \longrightarrow H_A^1(W_1') \longrightarrow H_A^1(\partial W_1') \\
0 \to &H_A^0(W_1',\ \partial W_1') \longrightarrow H_A^0(W_1') \longrightarrow H_A^0(\partial W_1')
\end{aligned}$$

Here H^0 are all 0 (because A is irreducible). Applying Poincaré duality to ∂W_1' and combining the fact that the Euler number of ∂W_1' is 2-2g, we can conclude:

$$H_A^2(\partial W_1') \ = \ 0, \qquad dim\ H_A^1(\partial W_1') \ = \ 6g-6$$

Applying Poincaré duality to the pair (W_1', ∂W_1') once more, we also see

$$dim H_A^1(W_1') \ - dim H_A^1(W_1', \partial W_1') \ = \ 3g-3$$

Later we show that

$$ker\ \nabla f_{1A} \ = H_A^1(W_1')$$

Hence, from above, we can conclude:

$$\begin{aligned}
index\ \nabla f_{1A} \ &= \ dim\ ker\ \nabla f_{1A} \ - \ dim\ coker\ \nabla f_{1A} \\
&= \ 3g-3 \qquad (Proposition\ 3.3.1\ (1))
\end{aligned}$$

From the above argument, we can easily see that

$$index\ \nabla f_{1A} \ = \ -3/2\ \chi_{\partial W_1'}$$

We remark that the above equality holds without assuming anything on the topology of W_1'.

Proposition 3.3.2 was equivalent to the vanishing of Coker ∇f_{1A}. We next show this:

We suppose that the admissible perturbation of f (to ensure nondegeneracy) is compatible in the following sense:

$$v(a) \in \Omega^1(adP_{M^n})$$

which appeared in section 1 has support contained in W_0. (Then the perturbation is well defined even if the connection is defined only on W_0) In the above situation, the following holds:

Lemma 3.3.3 *f_α' is a smooth section of*

$$\mathcal{L}_\alpha \longrightarrow B^0_\alpha$$

and f' can be understood completely by it's restriction to subsets of M". i.e.

$$\underline{\Phi} f' = (f'_1, f'_2)I$$
$$(\underline{\Psi}(f'_1, 0),\ \underline{\Psi}(0, f'_2)) = (f'_0,\ f'_0)J$$
$$f'^{-1}(0) = I^{-1}((f'^{-1}_1(0) \times f'^{-1}_2(0)) \cap J^{-1}(\Delta))$$

The proof is similar to [T] Lemma 4.7.

Covariant derivative of f_α' (perturbation of Chern-Simons Hessian)

$$\nabla f'_\alpha :\ \mathcal{T}_\alpha \longrightarrow \mathcal{T}_\alpha$$

can be described by adding correction term, which is corresponding to Coulomb gauge condition and Neumann condition similarly as in [T], section 4. About this covariant derivative the following holds:

Proposition 3.3.1 *For an irreducible connection A_α on $P_{M^n}|_{W_\alpha}$*

1. *$\nabla\ f_{\alpha A}$': $\mathcal{T}_{\alpha A} \longrightarrow \mathcal{T}_{\alpha A}$ is a bounded Fredholm operator (with metric L^2_1, L^2 respectively) whose index is 0 ($\alpha = \phi$) , 3g-3 (α=1,2) , 6g-6 (α=0)*
2. *$\nabla\ f_{\alpha A}$'- $\nabla\ f_{\alpha A}$ is compact*
3. *the assignment $[A] \longmapsto \nabla\ f_{\alpha A}$' is a smooth map from B^0_α to the space of Fredholm operators.*
4. *If $f_{\alpha A}$'=0 then*

$$\nabla f'_{\alpha A} a = \frac{d}{ds}|_{s=0}\ (f'_\alpha[A + sa])$$

The point to take care of is the index for the case α=1 (relatively non trivial bundle case). Finally we consider the transversality of the zero set.

Proposition 3.3.2 *Introducing the differential structure into m^0_α by the identification with representation space, then it is an embedded submanifold of B^0_α*

This proposition is equivalent to the claim that cokernel of $\nabla f_{\alpha A}$ vanishes.

We treat these propositions in the next section.

$$J_* : \mathcal{T}_{1A_1} \times \mathcal{T}_{2A_2} \longrightarrow \mathcal{T}_{0A_0} \times \mathcal{T}_{0A_0}$$

here we denote A_α for restrictions.

Comparing with Coulomb gauge condition, we can get the explicit expression of these maps similarly as in [T], section 4. As in [T] Lemma 4.3,

Lemma 3.3.2 *Linearization of (1)*

$$0 \longrightarrow \mathcal{T}_A \longrightarrow \mathcal{T}_{1A_1} \times \mathcal{T}_{2A_2} \longrightarrow \mathcal{T}_{0A_0} \longrightarrow 0$$

$$here\ \Phi = I_* \qquad \Psi = (j_1^*)_* - (j_2^*)_*$$

is an exact Fredholm complex.

For closed manifold like M", Chern-Simons gradient vector field $f_A = {}^*F_A$ can be regarded as a section of tangent bundle and it's zero set can be identified to the representation space of π_1(M"). However in open manifold case as in this section, f does not necessarily satisfy the Neumann boundary condition.

$$i^*(*a) = 0$$

Hence f is not necessarily a section of the tangent bundle of B^0_α.

For [A] $\in B^0_\alpha$ we put

$$\mathcal{L}_{\alpha A} = \{a \in \Omega^1(adP_{M"}|_{W_\alpha}) : \ d_A^* a = 0\}$$

Then these are gauge equivariant subspaces of Ω^1, so they make up subbundle

$$\mathcal{L}_\alpha \longrightarrow B^0_\alpha$$

of

$$(A_\alpha \times \Omega^1)/G_\alpha$$

We must regard f_α as a section of this bundle. Then restriction maps make the following exact Fredholm complex:

$$0 \longrightarrow \mathcal{L}_A \longrightarrow \mathcal{L}_{1A_1} \times \mathcal{L}_{2A_2} \longrightarrow \mathcal{L}_{0A_0} \longrightarrow 0$$

We denote zero set of f_α's by

$$m^\alpha = f_\alpha^{-1}(0)$$

and the irreducible part of it by

$$m^0_\alpha = f_\alpha^{-1}(0) \cap B^0_\alpha$$

For each suffix α (=0,1,2) which distinguish W_0, W_1' and W_2, we denote the space of connections on the SO(3) bundle $P_{M"}|_{W_\alpha}$ by A_α. Inclusions

$$W_0 \longrightarrow W_1',\ W_2 \longrightarrow M"$$

induce pull back sequence which is exact:

$$A \longrightarrow A_1 \times A_2 \longrightarrow A_0 \times A_0 \qquad (1)$$

$$here \quad I = (i_{1*},\ i_{2*}) \qquad J = (j_{1*},\ j_{2*})$$

The exactness means that I is embedding, J is submersion, and that

$$Image(I) = J^{-1}(\Delta)$$

$$here\ \Delta\ denotes\ diagonal\ in\ A_0 \times A_0$$

On the bundles $P_{M"}|_{W_1'}$, $P_{M"}|_{W_2}$, $P_{M"}|_{W_0}$ the gauge groups $G_s(P_{M"}|_{W_1'})$, $G(P_{M"}|_{W_2})$, $G(P_{M"}|_{W_0})$ acts respectively. (the first of three groups is the restricted gauge group and the rest are identified with $\Gamma(W_\alpha,SU(2))$) We denote these groups by G_α, and the quotients by B_α. In these connection and quotient spaces, the subspace consisting of irreducible connections and orbits are dense, and their codimensions are infinite. To distinguish from the space of all connections, we denote them by A^0_α, B^0_α respectively. The α=ϕ case is to mean the M" case. Then similarly as in [T] Proposition 4.1,

Lemma 3.3.1 *B^0, B^0_α are Hilbert manifolds modelled on L^2_1 Hilbert spaces. The projections*

$$A^0 \longrightarrow B^0 \qquad A^0_\alpha \longrightarrow B^0_\alpha$$

become principal $G/\pm 1$, $G^{\,0}_\alpha/\pm 1$ bunbles respectively. The tangent spaces at [A] is

$$\mathcal{T}_{\alpha A} = \{a \in \Omega^1(adP_{M"}|_{W_\alpha}) :\ d^*_A a = 0,\ i^*(*a) = 0\}$$

here i_* denotes restriction. For each [A]$\in B^{\,0}_\alpha$, there exists $\varepsilon > 0$ such that the correspondence

$$[A + a] \longrightarrow a \in \ \{b \in \mathcal{T}_{\alpha A},\ \int_{M_\alpha} |\nabla b|^2 + |b|^2 < \varepsilon\}$$

gives local coordinate of B^0_α.

Taking gauge quotients of (1), we get the following exact sequence

$$B^0 \longrightarrow B^0_1 \times B^0_2 \longrightarrow B^0_0 \times B^0_0 \qquad (2)$$

Taking linearlization, the following sequences are induced:

$$I_* : \mathcal{T}_A \longrightarrow \mathcal{T}_{1A_1} \times \mathcal{T}_{2A_2}$$

Definition 3.2.1 *Suppose we take an admissibly perturbed section f' of f, let Z be the finite zero points of f' in B, and for each element [A], [A'] of Z, let*

$$\Delta(A,\ A') = \pm 1$$

be the parity of spectral data determined by A and A'. Then we put

$$|\chi(f)| = |\sum_{[A']\in Z} \Delta(A,\ A')|$$

This is the quantity depending only on the differential structure of M".

Δ(A,A') is to be regarded as the relative sign of det∇f' between A and A'. And $|\chi(f)|$ is to be regarded as the absolute value of Euler characteristic of HF_*(M"). In [T] (trivial bundle case), we can take the trivial connection as A, for considering spectral flows. Hence [T] could determine χ(f) itself. We only consider $|\chi(f)|$ in this article.

3.3 Gauge theory associated to Heegard decomposition

For a null homologous knot K in QHS M, Casson-Walker's knot invariant λ'(K) was described using Heegard decomposition associated with K as in Casson's case. Hence, to consider the invariant from Gauge theoretic viewpoint, we must take into account of it. Let

$$M = W_1 \cup_\Sigma W_2$$

be Heegard decomposition associated to K.(as in section 1.3) Then

$$M" = W_1' \cup_\Sigma W_2$$

For later use we take a small product neighborhood of Σ in M"

$$W_0 = \Sigma \times I$$

and we consider Σ coincides with $\Sigma \times \{1/2\}$. We take product metric.

Next we take trivial SO(3) bundle on Σ and fix it's trivialization. Since $H^2(W_2\ ;\ Z)$ is 0, this trivialization extends to interior of W_2. On the other hand, $H^2(W_1'\ ;\ Z)$ is not 0. So, such extention is not necessarily possible. We take the SO(3) bundle which the above extention is impossible. If we attach these bundles using the above trivialization on Σ, then we obtain the nontrivial SO(3) bundle $P_{M"}$ over M". From the above choice, on the W_2 side, there is no need of consideration.

3.2 Spectral flow and Relative sign

As mentioned in the last section, spectrum of ∇f_A is unbounded. So we can not formulate it's determinant itself. However, given two zero points of f' (critical points of perturbed Chern-simons functional), we can formulate the relative sign of det ∇f' between the two critical points by using spectral flow of ∇f'.

The concept of spectral flow was originally used in [APS]. Given a 1-parameter family of self-adjoint Fredholm operators, the spectrum near 0 is similar to that of self-adjoint matrix on a finite dimensional vector space. Assume that the operators on the end points have no 0-eigenvalue. If the path of the family is smooth in the operator space, then, as running along the path, the spectrum near zero moves smoothly. For example the movement of spectrum is as in the following picture:

The movement like this is called "spectral flow". The number

$$(the\ number\ of\ curves\ which\ cross\ 0\ level\ from\ -\ to\ +)$$

$$-(the\ number\ of\ curves\ which\ cross\ 0\ level\ from\ +\ to\ -)$$

is well defined and finite for generic path. We shall call this spectral data along the path. If we take two homotopic paths, their spectral datas coincide.

From the general L^2 theory

Proposition 3.2.1 *The assignment to each equivalence class of irreducible connection on P_{M^n} of (admissibly perturbed) Chern-Simons Hessian associated to the class*

$$[A] \longrightarrow \nabla f'_A$$

gives a smooth map from B to the space of self adjoint operators on a separable Hilbert space. If A_1, A_2 are nondegenerate zero points, spectral data gives a locally constant function on the space of paths in B from $[A_1]$ to $[A_2]$. It is constant mod 8.

annihilates the tangent vectors to the G_s orbits through A. Although if we differentiate this, it does not necessarily do so.

Tangent space of B at [A] can be identified with the orthogonal complement $\mathcal{T}_A$ of $T_A(G_sA)$ in T_AA.

$$\mathcal{T}_A = \{a \in T_AA | d_A^* a = 0\}$$

Composing orthogonal projection to $\mathcal{T}_A$, ∂f_A defines an element ∇f_A of $\mathrm{End}(\mathcal{T}_A)$. Specifically, taking Coulomb gauge

$$for\ a \in\ \mathcal{T}_A$$

$$\nabla f_A a = *d_A a - d_A u(a) \quad (Chern - Simons\ Hessian)$$

$$here\ u(a) \in \Omega^0(adP_{M"})\ is\ uniquely\ determined\ (A : irreducible)\ by$$

$$d_A^* d_A u(a) = *(a \wedge F_A - F_A \wedge a)$$

As in the trivial bundle case,

$$\nabla f_A :\ \mathcal{T}_A \longrightarrow\ \mathcal{T}_A$$

is a closed, essentially self-adioint, Fredholm operator. Here L^2 completion is understood and the domain is L_1^2. In particular, spectrum is a unbounded discrete subset of R which has no accumulation points, and multiplicity is finite.

Zero set of f in B (flat connections) is said to be nondegenerate if ∇f has no 0-eigenvalue. Later we show that index(∇f_A) is 0 on M", so non-degeneracy holds after making generic perturbation. Especially, zero set is discrete.

In case when M is a rational homology sphere (QHS) and K is a null homologous knot, reducible zero points are discrete and separated from irreducible zero points.

Necessary property for the perturbation of f is similar to the trivial bundle case, so we only say "making admissible compatible perturbation".(For precise description of this, see [T], Definition 1.4 and Definition 4.6) We denote f' for perturbed f. The perturbation is to be taken in W_1' side. (i.e. it's value at a connection is determined by restriction of it to W_1' side). The difference between ∇f' and ∇f is described as follows: corresponding to the perturbation, a section is determined as

$$v[A](x) \in \Omega^1(adP_{M"}) \qquad (A \in A(P_{M"}),\ x \in M")$$

which has a compact support in W_1' , and Hessian is changed by

$$\nabla f'_A a = \nabla f_A a + \partial v[A]a + Coulomb\ gauge\ correction$$

here

$$a \in T_AA$$

$$\partial v[A]a = \frac{d}{ds}|_{s=0}\ v([A + sa]) \in \Omega^1(adP_{M"})$$

$$B(P_{M"}) := A(P_{M"})/G_s(P_{M"})$$
$$For A \in A(P_{M"})$$
$$c(A) := \int_{M"} tr(A - A_0) \wedge F_A - 2/3(A - A_0)^3$$

Here both A-A_0 and F_A are differential forms with values in the adjoint bundle $adP_{M"}$. As in the trivial bundle case, exterior differentiation f of c becomes curvature F_A. As a 1-form on $A(P_{M"})$

$$for\, a \in T_A A(P_{M"}) \cong \Omega^1(adP_{M"})$$
$$f_A(a) = \int_{M"} tr(a \wedge F_A)$$

This is a pull back of a 1-form on B. From this, the critical point set of c is

$$f^{-1}(0) = \{[A] \in B : F_A = 0\} \quad (space\, of\, flat\, connections)$$

Fixing metric of M" and invariant inner product of so(3), we can define L^2 metric on $T_A A(P_{M"})$ automatically. Then we can regard f as a vector field on A. From this viewpoint

$$f_A = *F_A \in T_A A(P_{M"})$$

This corresponds to the vector field v mentioned at the beginning of this section.

Next we must consider covariant differentiation of the vector field f. The result is the same as in trivial bundle case : using the affine structure of $A(P_{M"})$

$$for\ a, b \in T_A A(P_{M"})$$
$$\partial_b f|_A(a) = \frac{d}{dt}(f_{A+tb}(a))|_{t=0}$$
$$= \int_{M"} tr(a \wedge d_A b)$$

In the case of trivial bundle, a and b were su(2) valued 1-forms. Here in nontrivial bundle case we can calculate completely in the same manner over each open set of covering of M". The correspondence

$$(a, b) \longmapsto \partial_b f|_A(a)$$

defines a symmetric bilinear form on $T_A A(P_{M"})$. Using L^2 metric, we can regard ∂_b is a symmetric linear endmorphism of $T_A A(P_{M"})$:

$$\partial f_A(a) = *d_A a \quad a \in T_A A(P_{M"})$$

From the definition of the gauge group action on connections, this endmorphism is equivariant with respect to G_s action. Since f is a pull back of a 1-form on B, f

Chapter 3

Gauge theoretic arguments

3.1 Setting of the argument

In this chapter we want to see the method of Taubes[T] can be used to our case of non trivial SO(3) bundle over homology $S^1 \times S^2$, M". What was treated in [T] is (trivial) SU(2) bundle over ZHS.

In the case of compact (finite dimensional) manifolds, taking a nondegenerate vector field v we can calculate the Euler characteristic χ of the base manifold as follows :

$$\chi = \sum_{p:v(p)=0} sign\, det \nabla v|_p$$

$$here \nabla v\, means\, covariant\, differentiation\, of\, the\, vector\, field\, v.$$

At the zeros of v, it is independent of choice of connection. We would like to apply this formalism to 3 dimensional gauge theory. The space we treat is infinite dimensional, so it is difficult to formulate the above determinant itself. Though we can formulate the relative sign for two critical points (flat connections). By the way, [APS 1,2] gave a formulation of η function for some self-adjoint operators over infinite dimensional Hilbert spaces. This is a modification of ς function. In particular, Dedekind sums come out in the case of signature operators.

Now there is no trivial flat connection on the bundle which we are considering, so the situation is slightly different. But what were used in [T] are gradient and Hessian of Chern-Simons functional. So we do not need such a special "origin". That is, we take an arbitrary irreducible flat connection A_0 and use this as the origin.

Let us denote

$$A(P_{M^n}) : space\, of\, irreducible\, connections$$

$$G_s(P_{M^n}) : restricted\, gauge\, group$$

is not a Heegard decomposition. If we insiste on describing using Heegard decomposition, we can not derive the above description of R(W_1'). Because, in 0-surgery case, the attaching map of Σ giving M" can not be described as

$$(attaching\ map\ giving\ M)\ (Dehn\ twist\ along\ K)$$

In +1-surgery case, this description was possible. So our change from Dehn twist description to Dehn surgery description is rather essential.

the beginning of this chapter: they were +1-surgery and 1/0-surgery. We are going to calculate the representation space for the 0-surgery W$_1$'.)

Recall the next relation in section 1.3:

$$\rho_2(g") = \rho(m)\ \rho_1(g)\ \rho(m^{-1})$$

$$= Ad\rho(m)\ \rho_1(g)$$

(*The notation is preserved from section* 1.3)

Here m denotes the small linking circle around the tubular neighborhood of K', N(K). Instead of 1/0-surgery case, we consider 0-surgery case here. From the definition of 0-surgery (section 1.2), this circle m$\subset W_1' - N(K)$ is identified with the core circle in N(K).

As remarked in section 2.1, this circle represents the generator of $H_1(W_1'; Z) \cong Z$. It is a torsion free element, so we can make a free choice for the value on m of an element of Hom(π_1(W$_1$', SU(2)). So we denote it as u$\in$SU(2).

Then the above relation means the following:

Lemma 2.2.1 *For* $\rho = (\rho_1,\ \rho_2) \in R(W_1') \subset R(\Sigma)$, *the next equation holds:*

$$\rho_2(g") = Adu\ \rho_1(g)$$

Here we can take arbitrary u$\in$SU(2).

Compared with the description of the difference cycle δ in section 1.3, this lemma implies Proposition 2.2.1 immediately. (Proof of Proposition 2.2.1 is also completed) □

Remark that (as in +1-surgery case), the decomposition

$$M" = W_1' \cup W_2$$

These cobordisms C_1, C_2 induce homomorphisms on the Floer homologies of three manifolds M', M, M" :

$$HF_*(M') \longrightarrow HF_*(M) \longrightarrow HF_*(M")$$

The Floer's triangle claims that

this sequence is exact

To prove this, we must treat the gluing argument of (projectively) ASD connections over these cobordisms. For more details, we refer to [BD].

2.2 Flat connections on W_1'

The aim of this section is to show the following proposition:

Proposition 2.2.1 *The following two spaces coincide:*

(1) The difference cycle δ in chapter 1
(2) The space of gauge equivalence classes of SU(2) flat connections on the (trivial) SU(2) bundle over the complement in W_1' of the core circle in N(K) whose holonomy around a small linking circle is -1. (open version of the second setting in section 2.1)

Remark

Combining this with the Taubes' interpretation of relative intersection number in representation space as some spectral data (Theorem 3.5.1), we can prove the main theorem (Theorem 3.5.2).

Proof

We have obtained a description of δ in chapter 1. Here we compute (2). We use our method to describe $R(W_1')$, $R(W_1)$ in section 1.3. (Recall the notational remark at

On the other hand, K is bounded by D^2 in N(K)$\cong D^2 \times S^1$. If we attach these surfaces along K using boundary identification of 0-surgery, we obtain a closed surface. As easily seen, the fundamental class of this closed surface gives the generator of $H_2(W_1'; Z) \cong Z$. (Usual argument for Poincaré duality. See for example the text of Bott-Tu) Take a trivialization over $\partial N(K)$ and extend it to W_1-N(K) and N(K) (always possible). To obtain the relatively nontrivial SO(3) bundle over W_1', we use the above trivialization over ∂N(K) and clutching function (over ∂N(K)):

$$\chi \ : \ \partial N(K) \longrightarrow SO(3)$$

whose restriction to ∂D^2 in N(K)$\cong D^2 \times S^1$ gives the nontrivial element of $\pi_1(SO(3)) \cong Z_2$.

Evaluation of relative second Stiefel-Whitney class on the above closed surface is nothing but this element of $\pi_1(SO(3))$.

We return to the Floer's triangle. Using the above setting of gauge theory over M", we obtain Floer homology of M",HF_*(M"). (It does depend on the choice of $\hat{P}$) There are standard, oriented, surgery cobordisms C_1 from M' to M and C_2 from M to M". It is drawn simply as follows.

Remark on the third setting

From the definition of 0-surgery, in the construction of M",

$$M" = M - N(K) \cup N(K)$$

the above core circle in N(K) is identified to the "longitude" curve. This is nothing but the boundary circle K of the Seifert surface F in section 1.3.

In our case of 0-surgery M", construction of the above nontrivial SO(3) bundle (in the first setting) can be described explicitly. For later use (Proposition 3.4.1), we treat this with the case of W_1' (open manifold).

Since $H^2(M"; Z_2)=Z_2$, there are two isomorphism classes of SO(3) bundle on M". One is the trivial bundle and the other is nontrivial. We treat the latter, of course. Although we can also see $H^2(W_1'; Z_2)=Z_2$, W_1' is open. So any SO(3) bundle over W_1' is trivial. But if we fix a trivialization over $\partial W_1'$, it does not necessarily extend to a trivialization over W_1'. If it is impossible, we call this bundle (with boundary trivialization) relatively nontrivial. Using this boundary trivialization, we can clutch this relatively nontrivial bundle over W_1' and (relatively) trivial bundle over W_2. Then we obtain the nontrivial SO(3) bundle over M" stated above.

Next we consider the relatively nontrivial bundle over W_1'. We fix a trivialization over ∂W_1'. From the definition of W_1' as 0-surgery:

$$W_1' = W_1 - N(K) \cup N(K)$$

the generator of $H_1(W_1'; Z) \cong Z$ is represented by the core circle of N(K) (as in M" case). On the one hand, K (in fact, a "inner knot" K') is bounded by the Seifert surface F in W_1-N(K). (The shaded part of the figure)

(3) Consider the action of (2) on (1)

But the choice (2) raises some trouble. That is, holonomy group of a flat connection may be contained in a Kleinian subgroup

$$Z_2 \times Z_2$$

of an O(2) subgroup. Thus, the space of gauge equivalence classes of flat connections on $P_{M"}$ becomes an orbifold. So, instead of (2), we choose a little smaller subgroup which we call restricted gauge group $G_s(P_{M"})$.

Definition 2.1.1 *The restricted gauge group $G_s(P_{M"})$ is composed of gauge transformations which are homotopic to the identity over the 1-skeleton of M".*

In other words, these are gauge transformations of $P_{M"}$ which can be lifted to SU(2) gauge transformations.

If we choose this gauge group, then stabilizer of each flat connection becomes trivial. Hence the space of gauge equivalence classes of flat connections on $P_{M"}$ becomes a manifold.

The above setting is equivalent to the following:

(1) We take a lift of $w_2(P_{M"})$ to an integral cohomology class $c_1(\hat{P}) \in H^2(M")/\text{Torsion}$ (odd element). Let $\hat{P}$ be an U(2) bundle which is classified by $c_1(\hat{P})$.

We consider projectively flat connections on $\hat{P}$. That is, under the decomposition $u(2) \cong u(1) \oplus so(3)$, we consider those connections whose curvature has 0 as the last component. Here u(1) represents the central part, and we fix a determinant U(1) connection whose curvature belongs to the class $c_1(\hat{P})$.

(2) Gauge transformations on $\hat{P}$.

(3) Consider the action of (2) on (1).

One more equivalent setting is as follows:

(1) We take a (trivial) SU(2) bundle over the complement in M" of the circle whose homology class is the generator of $H_1(M"; Z)$. This circle is nothing but the core circle of the tubular neighborhood N(K) of the knot K. (See the setting in section 1.2. From Meyer-Vietoris argument, we can see easily that this circle gives the generator class of $H_1(M"; Z)$.)

We consider flat connections on the above bundle whose holonomy around a small linking circle of the above core circle is -1.

(2) SU(2) gauge transformations on the above bundle.

(3) Consider the action of (2) on (1).

Chapter 2

Difference cycle from Gauge theoretic viewpoint

For simplicity, in chapters 2 and 3, we denote W_1' for the 0-surgery along a "little inner knot" K' in W_1. In chapter 1, it was used for +1 surgery.

2.1 Floer's triangle

As mentioned in section 1.3, for the computation of topological invariant of 3 manifolds, it is important to describe the behavior of it under the Dehn surgery. In this section, we review an attempt by Floer to obtain this description for his instanton homology for 3 manifolds. This section owes heavily to [BD].

Let M, M', M" be as in section 1.2. For ZHS' M, M', Floer homology group HF_* is already defined. On the other hand M" is a homology $S^1 \times S^2$, so we must formulate the Floer homology of this 3 manifold. We give the formulation below. For the definition of HF_* of ZHS, we use trivial SU(2) bundle on it. For our M" case, we use so called "admissible bundle" which is nontrivial.

Let $P_{M"} \longrightarrow$M" be a principal SO(3) bundle. We call $P_{M"}$ admissible iff $w_2(P_{M"})$ is not 0 as a map

$$H_2(M"; Z) \longrightarrow Z_2$$

Then, $P_{M"}$ carries no flat connection with holonomy group contained in an S^1 subgroup of SO(3). The author guesses this is the motivation Floer introduced the above kind of bundles. (i.e. if we use trivial bundle, it is hard to treat reducible flat connections)

At first sight, the setting seems to become as follows:

(1) Flat connections on the admissible SO(3) bundle $P_{M"}$
(2) Gauge transformations on $P_{M"}$

$$(C, \rho', u) \longmapsto (AdC\rho', AdCu)$$
$$SO(3) \times R_-(K) \longrightarrow R_-(K)$$
$$(C, (\rho_1, \rho_2)) \longmapsto (AdC\rho_1, AdC\rho_2)$$

With respect to this action, f is equivariant. We denote $\hat{f}$ for the induced map on the quotient spaces. Then we obtain the next description of the difference cycle:

Proposition 1.3.1 (A-M) *Let us put*

$$\delta_- = \hat{f}(R'_- \widehat{\times} SU(2)) \in H_{3g}(\hat{R}_-(K); Z)$$

and denote the homomorphism induced by inclusion as

$$i_* : H_{3g-3}(\hat{R}_-(K); Z) \longrightarrow H_{3g-3}(\hat{R}(\Sigma); Z)$$

Then

$$\delta = i_*(\delta_-) \in H_{3g-3}(\hat{R}(\Sigma); Z)$$

Remark

As easily seen, the map

$$\hat{f} : (R'_- \widehat{\times} SU(2)) \longrightarrow \hat{R}_-(K) = (R'_- \widehat{\times} R'_-)$$
$$[(\rho', u)] \longmapsto [(\rho', Adu\rho')]$$

(here we denote [ ] for conjugacy class of the Ad action) is a double covering onto it's image. So the intersection number of δ and $\hat{R}(\mathrm{W}_2)$ is always an even integer. Combined this with surgery formula, we can conclude that the signed summation

$$< \hat{R}(W_1), \hat{R}(W_2) >$$

is always an even integer. On the other hand, we show in the next chapter, that the space of flat connections on M" coincides with the above δ. Hence we can give an explanation of the fact that Euler characteristic of HF_*(M") is always an even integer.

isotopy which carries R(W_1') to R(W_1) as follows:

$$H : (R - R_-(K)) \times I \longrightarrow R - R_-(K)$$

$$\{(\rho', \rho"), t\} \mapsto (\rho', Ad(\rho'(K))^t \rho")$$

As we move t from 1 to 0 and trace the points (ρ, Ad(ρ(K))$^t\rho$) ($\rho\in$R(F)-R$_-$'), we can make an isotopy, outside of a small neighborhood of R$_-$(K), from R(W_1') to R(W_1). The shaded region in the following figure is the trace of this part. Although H does not extend continuously onto R$_-$(K), originally R(W_1') and R(W_1) coincide on R$_-$(K).

Because of the existence of H, the shaded region is null homologous. So it has no contribution to intersection number with R(W_2). Since H was defined everywhere in the complement of R$_-$(K), we can take the above mentioned neighborhood of R$_-$(K) arbitrarily small. Hence the homology class to which the difference cycle δ belongs is in the image of the homomorphism

$$i_* : H_*(R_-(K); Z) \longrightarrow H_*(R(\Sigma); Z)$$

induced by inclusion. This fact was crucial for the framing invariance of λ'(K). [A-M] gave an explicit description of δ as a cycle on R$_-$(K) as follows. We define the map f by

$$f : R'_- \times SU(2) \longrightarrow R_-(K)$$

$$(\rho', u) \longmapsto (\rho', Adu\rho')$$

SO(3) acts on both R'$_-\times$SU(2) and R$_-$(K) in the following manner:

$$SO(3) \times R'_- \times SU(2) \longrightarrow R'_- \times SU(2)$$

Here $<,>$ means the sum of topological intersection number. So in principle λ'(K) can be obtained by summing up intersection numbers between R(W_2) and the "difference cycle" R(W_1')-R(W_1). At first sight, the pair (W_1', W_2) seems not to be a Heegard splitting. However, W_1' is homeomorphic to W_2 and, under the identification of these handlebodies, the gluing map on Σ suffer a change corresponding to the above change of representation space.

Here let us denote R$_-$(K) for the space of elements of R(Σ) whose values on K are -identity. (In this article, we denote identity as 1.) Remark that R(Σ) is the subspace of R(F)$\times$ R(F) consisting of representations (ρ_1,ρ_2) such that ρ_1(K)=ρ_2(K). So if we denote

$$R'_- = \{\rho \in R(F) | \rho(K) = -1\}$$

then we can see

$$R_-(K) = R'_- \times R'_-$$

We can also see, in R(Σ)

$$R_-(K) \cap R(W_1) = \{(\rho', \rho') | \rho' \in R'_-\}$$

$$= R_-(K) \cap R(W'_1)$$

As we have seen, any element of R(W_1')-R$_-$(K) can be obtained by making Ad-action of an element of SU(2)-{-1} on the second component of diagonal elements in R(F)$\times$R(F). Remark that every element of SU(2)-{-1} can be diagonalized in the following form:

$$\forall C \in SU(2) - \{-1\} \quad \exists A \in SU(2)$$

$$ACA^{-1} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

Here we can take θ in $0 \leq \theta < \pi$. Hence for any A$\in$SU(2)-{-1}, we can clearly define it's exponential A^t (t$\in$R). From this, on the complement of R$_-$(K), we can define an

The difference between g' and g" is nothing but a return trip along the meridian curve m of K'(See the next figure): as an element of $\pi_1(\mathrm{W}_1')$, g" is the same as $\mathrm{mg'm}^{-1}$.

So with respect to $\rho=(\rho_1, \rho_2) \in \mathrm{R}(\mathrm{W}_1') \subset \mathrm{R}(\Sigma)$

$$\rho_2(g") = \rho(m)\rho_1(g)\rho(m)^{-1}$$

$$= Ad\rho(m)\rho_1(g)$$

From the definition of +1-surgery, the above curve m is identified with a curve which belongs to the homology class $\mu+\lambda$ on $\partial(\mathrm{K}\times\mathrm{D}^2)\cong\mathrm{T}^2$. From this identification we obtain

$$\rho(m) = \rho(\lambda)$$

$$= \rho_1(K)$$

After all, $\mathrm{R}(\mathrm{W}_1')$ consists of (conjugacy classes of) representations with the form

$$(\rho_1, Ad\rho_1(K)\rho_1) \qquad \rho_1 \in R(F)$$

In the sequel of this chapter, let us use the notation $\wedge$ to make it clear that we consider space of conjugacy classes. From the definition of Casson's invariant

$$\lambda(M') = 1/2 < \hat{R}(W_1'), \hat{R}(W_2) >$$

$$\lambda(M) = 1/2 < \hat{R}(W_1), \hat{R}(W_2) >$$

We take a copy K' of K in interior of W_1 near it's boundary. (We consider K' and K are "parallel") The "+1-surgery" of W_1 along K' is denoted by W_1'. As is remarked before, F is an open surface with boundary circle K.

Then we consider the characterization of $R(W_1')$ and $R(W_1)$ as subspaces in $R(\Sigma)$. $\Sigma=\partial W_1$ is composed of two open surfaces F attaching them together on the boundary K's. So $\pi_1(\Sigma)$ has generators "symmetric" with respect to K. Since any representation of $\pi_1(\Sigma)$ is uniquely determined by giving values on these generators, we describe them by values on symmetrically ordered generators. In other words, we describe them as elements of $R(F)\times R(F)$

First of all, to determine $R(W_1)$, we take a base point as in the following figure. Because W_1 is a direct product $F\times I$, we can see immediately that $R(W_1)$ is the subspace of $R(\Sigma)$ composed of the representations of the form (ρ, ρ). $(\rho\in R(F))$

Fix an element g of generator set of π_1 of "upper" boundary component F. As in the figure, let g' be the element corresponding to g running through inner side of K'.

Of course any representation has the same value on g and g'. But what we must consider is to describe $R(W_1')$ as a subspace of $R(\Sigma)$. Hence when we calculate the second component of an element of $R(W_1')$ whose first component is $\rho \in R(W_1)$, we must take value of ρ on the generator g" of $\pi_1(\Sigma)$ running along Σ corresponding to g. (See the next figure)

the description of the difference cycle δ we must choose appropriate Heegard splitting corresponding to K. We review this particular choice first ([AM]). Let us take a Seifert surface F of K and identify it's tubular neighborhood in M with F×I. Here We denote I for an interval [0,1]. F×I and it's complement in M are handlebodies. Boundary of F×I is a surface composed from 3 parts:

$$\partial(F \times I) = F \times 0 \cup K \times I \cup F \times 1$$

We identify the knot K itself with K× $\{1/2\}$. The following fact is known: if necessary, attaching many 1-handles to F × { 1 } and drill from F × { 0 } to F × { 1 } and through the cores of 1- handles which were attached to F × { 1 }, we get an Heegard decomposition of M. The newly obtained handlebody becomes also direct product. We also see that this handlebody is identified with F×I. (We abuse the notation F. This F is quite different from the former F. See the next figure.) We call this handlebody W_1 and call it's complement W_2.

We denote the boundary surface of this handlebody by Σ. As is easily seen, genus of Σ is an even integer. We denote this genus by g. Also in this case we can identify K with K× $\{1/2\}$. It is a bounding simple closed curve of Σ. In the rest of this chapter, we describe difference cycle δ explicitly. Our method, in principle, is similar to [AM] completely, however it is hard to apply their method to deduce this description in the case of 0/1-surgery. So we modify some part of the method in [AM]. Briefly speaking, their method persisted in Heegard splitting given by the above procedure. In our method we regard each representation of π_1(M) as a flat connection and use Dehn surgery instead of Dehn twist. In the result, $\lambda(M^{n+1})$-$\lambda(M^n)$ is independent of n, so we only consider n=0 case. We write M' for M^1. So

$$\lambda'(K) = \lambda(M') - \lambda(M).$$

(with Z coefficient) . Here the generator of H_1(M") corresponds to the core circle of N(K). Generally the diffeomorphism h which gives so called 1/n-surgery M^n is described by

$$h_*(\mu) = \mu + n\lambda$$

$$h_*(\lambda) = \lambda$$

We can see that for all integer n, M^n are ZHS'. In general, M^n is not diffeomorphic to each other.

1.3 Casson's knot invariant λ'(K) (Description of difference cycle)

It is well known that every 3 manifold can be obtained by Dehn surgery along a framed link in S^3. So for us to compute the invariant of 3 manifolds, it is very important how the invariant behaves under Dehn surgery. One of the reason why many topologists are paying much attention to Casson's invariant and it's extention(Walker's invariant for rational homology 3 spheres) is that their behavior under Dehn surgery is known.

For Casson's invariant it is known that this behavior can be described as follows: Let M, M', M^n and K be as in the preceding section. We denote Δ_K(t) for symmetrized Alexander polynomial of K. Then

$$\lambda(M^{n+1}) - \lambda(M^n) = (1/2)\Delta''_K(t)|_{t=1}$$

Here the right hand side is the value at t=1 of the second derivative of Δ_K(t). Above description implies also the fact that the quantity of the left hand side is independent of the framing n. This quantity is usually called Casson's knot invariant λ'(K) for the knot K in M. The well known fact that

$$\lambda(M) \cong \mu(M)/;(mod/;2)$$

$$here\ we\ denote\ \mu(M)\ for\ Rohlin\ invariant$$

was obtained from the above surgery formula and Gordon's theorem.

The most crucial step for the proof of framing independence of λ invariant was introduction of so called difference cycle δ. Briefly speaking this δ is the cycle which gives λ'(K) by counting intersection number with R(W_2). Walker showed that the same framing independence holds for null homologous knots in rational homology 3 spheres in a similar manner ([W]). So we also consider rational case.

From here, let M be a rational homology 3 sphere and K be a null homologous knot in M. We can choose any Heegard splitting of M to define λ(M). On the other hand, for

1.2 Dehn surgery along a knot

In this section, let M be a 3 manifold and K be a knot in M. We denote N(K) for tubular neighborhood of the knot K. This is diffeomorphic to $S^1 \times D^2$ and it's boundary ∂ N(K) is diffeomorphic to T^2. So

$$H_1(\partial N(K); Z) \cong Z \oplus Z$$

We can take canonical basis of this group called meridian(μ) and longitude(λ). They are characterized as follows. μ is the generator of the kernel of

$$i_{1_*} : H_1(\partial N(K); Z) \longrightarrow H_1(N(K); Z)$$

On the other hand, λ is the generator of the kernel of

$$i_{2_*} : H_1(\partial N(K); Z) \longrightarrow H_1(M - N(K); Z)$$

Taking N(K) sufficiently small, μ is ∂ D^2.

Dehn surgery of M along K is a 3 manifold constructed from N(K) and M-N(K) by using an orientation preserving diffeomorphism of their boundaries to attach them together.

In general, λ winds around K. But if K is a null homologous knot, K itself is λ.

As is well known, diffeomorphism of T^2 to itself is determined up to isotopy by specifying the images of λ and μ. In particular, the diffeomorphism h which gives +1-Dehn surgery M' is described by

$$h_*(\mu) = \mu + \lambda$$

$$h_*(\lambda) = \lambda$$

the diffeomorphism h which gives 0/1-Dehn surgery M" is described by

$$h_*(\mu) = \lambda$$

$$h_*(\lambda) = -\mu$$

In case where M is a ZHS (then K is automatically null homologous), we can see, by using Meyer-Vietoris exact sequence, that M' is a ZHS, and M" is a homology $S^1 \times S^2$

Outside the singular point set (reducible representations) $R(W_1)$ and $R(W_2)$ are Lagrangian submanifolds of $R(\Sigma)$. If M is a ZHS, trivial representation is the unique intersection of $R(W_1)$ and $R(W_2)$ in the singular point set. In the definition of λ(M), we do not take into account of it.

In general, intersections of $R(W_1)$ and $R(W_2)$ may not be transverse. But, from the fact that M is a ZHS, we can perturb these subspaces into transversal position by compactly supported ambient isotopy. We can also conclude that the intersection is compact, so (after performing isotopy) the intersection consists of finitely many points. From now on, we also denote $R(W_2)$ for isotoped subspace of itself and we always assume that transversality holds.

Fixing orientations of M and SU(2), orientations of $R(W_1)$, $R(W_2)$, and $R(\Sigma)$ are determined roughly in the following manner : Fixing orientation of M, a compatible order of generators of $\pi_1(\Sigma)$ can be chosen. Let g be the genus of Σ , and Σ^* be $\Sigma-$neighborhood of base point. Then

$$Hom(\pi_1(\Sigma^*), SU(2)) \cong SU(2)^{2g}$$

and under this identification Hom($\pi_1(\Sigma)$,SU(2)) is the subset of representations whose value for boundary circle is identity. From this, we can introduce an orientation for $R(\Sigma)$. After fixing orientation, at each intersection point of $R(W_1)$ and $R(W_2)$, topological intersection number(± 1) becomes well defined. Then we put

$$\lambda(M) := 1/2 \sum \quad [topological\ intersection\ number\ of\ \ R(W_1)\ and\ R(W_2) at\ p]$$

$$p \in R(W_1) \cap R(W_2), p \neq trivial\ representation$$

This is the definition of Casson's invariant of M.

Precise description of orientation is not necessary for consideration of the absolute value, so we do not review it. I would like to comment only the following fact: Let -M denote M with reversed orientation, then

$$\lambda(-M) = -\lambda(M)$$

That λ(M) is a topological invariant means that it is independent of choices in the definition of it (for example, choice of Heegard splitting).

Chapter 1

Casson's invariant for knots

1.1 Casson's invariant for homology 3 spheres

Casson's invariant is a topological invariant of oriented homology 3 spheres. For simplicity we call such manifold ZHS. This invariant assigns an integer λ(M) to each ZHS M. To define this invariant, we use Heegard splitting of M. As is well known, any 3 manifold can be described by Heegard model as follows.

$$W_1 \cong W \cong W_2 \qquad : \text{handle body and it's copies}$$

$$\text{h : orientation reversing diffeomorphism from } \partial W_1 \text{ to } \partial W_2$$

$$M = W_1 \cup_h W_2$$

We denote the common boundary by Σ. This decomposition is the Heegard splitting of M. For any ZHS M, λ(M) is defined roughly in the following way [AM]: We take an arbitrary Heegard splitting of M ,(W_1,W_2,h). By inclusions, the following sequences of fundamental groups are induced.

$$\pi_1(\Sigma) \longrightarrow \begin{matrix} \pi_1(W_1) \\ \\ \pi_1(W_2) \end{matrix} \longrightarrow \pi_1(M)$$

Taking the space of conjugacy classes of representations from these fundamental groups to SU(2), we obtain the sequence.

$$\mathrm{R}(\Sigma) \longrightarrow \begin{matrix} \mathrm{R}(W_1) \\ \\ \mathrm{R}(W_2) \end{matrix} \longrightarrow \mathrm{R(M)}$$

Regarding as the subsets of R(Σ), we can consider

$$R(M) = R(W_1) \cap R(W_2)$$

Remark

> The assumption that the knot K is null homologous is necessary for the formulation of both Casson-Walker's knot invariant λ'(K) and the Euler characteristic χ(f).

What must be commented here is that Floer homology groups for rational homology 3 spheres have not yet been completely formulated, because of the existence of reducible SU(2) flat connections (i.e. degenerate critical points of Chern-Simons functional). As seen in the definition of the Casson-Walker invariant, we can make all the critical points of Chern-Simons functional nondegenerate, if we perform a general positioning perturbation to the functional. However, the critical point set obtained in this way, depends on the choice of the above mentioned perturbation. On the other hand, when we consider Casson-Walker's knot invariant and it's gauge theoretic setting, we can neglect the contribution from the reducible flat connections, and so, they are independent of the choice of the above perturbation. This is the meaning of Theorem 0.0.2. From the method for the proof of this theorem, we can easily extend it to the case of the knot with self linking number 0 (see [W] Lemma 3.34), but we do not refer to it here.

The author hopes that theorem 0.0.2 serves topologists as the starting point for the study concerning the behavior of Floer homology groups under Dehn surgery in rational homology 3 sphere case.

Acknowledgements

The writer would like to express special thanks to Professor Fukaya for much helpful discussion and heartful support. He also gave the author the suggestion to study the relationship between Casson's invariant and Floer's triangle. Also, the author would like to express thanks to Professor Furuta for his acceptance of my troublesome consultation.

here the right hand side is to be regarded as the Euler characteristic of the Floer homology group for M, $HF_*(M)$. It has informations of infinite dimensional space (connection space), and the degree in HF_* is calculated by a transcendental method (index of Atiyah-Hitchin-Singer complex). On the other hand, the left hand side has informations of finite dimensional space (representation space of the fundamental group of homology 3 sphere), and the intersection number defining $\lambda(M)$ is calculated by an algebraic topology method.

We apply the method of [T] to the knot invariant case. As a result, we obtain the following main theorem (Theorem 3.5.2):

Theorem 0.0.1 *Let K be a knot in an integral homology 3 sphere M. We denote $\lambda'(K)$ for the Casson's knot invariant of K. Moreover, let M" be the 0/1-Dehn surgery along K in M, and $P_{M"}$ be the nontrivial SO(3) bundle over M". Then we can set up Chern-Simons gauge theory for $P_{M"}$, and we can formulate the Euler characteristic for Chern-Simons gradient vector field f on $P_{M"}$. If we denote this by $\chi(f)$, then we obtain the next equality:*

$$2|\lambda'(K)| \ = \ |\chi(f)|$$

For the proof of this, we need the following two steps:

> (1)We find an appropriate setting of Chern-Simons gauge theory to define the above mentioned Euler characteristic corresponding to the knot K. (Chapter 1, 2, Section 3.1)
> (2) Apply the method of Taubes with a suitable modification to show that the above Euler characteristic coincides with the Casson's knot invariant. (the rest of Chapter 3)

Concerning (1), we use the method of [BD]. Briefly speaking, our choice is admissible bundles, and we consider SU(2) flat connections with singularity along the knot K. We compare the space of the above mentioned flat connections and the "difference cycle"(which gives Casson's knot invariant) in the space of representations of fundamental group. Concerning (2), we treat nontrivial SO(3) bundle. So we must make topological modification in the method of [T]. This modification is most apparent in the argument of transversality (section 3.4). (Recall that [T] used the trivial SU(2) bundle.)

We remark that Theorem 0.0.1 follows from Floer's triangle for Dehn surgery. So our result gives an alternative proof of it. However the following generalization to rational homology 3 spheres is new. (Chapter 4, Theorem 4.0.1):

Theorem 0.0.2 *Let K be a null homologous knot in a rational homology 3 sphere M. Then we can formulate the same setting as the above theorem, and we also obtain*

$$2|\bar{\lambda}'(K)| \ = \ |\chi(f)|$$

here $\bar{\lambda}'(K)$ denotes unnormalized Casson-Walker's knot invariant of K ([W]).

INTRODUCTION

As is well known, every 3 dimensional manifold can be obtained by Dehn surgery along a framed link in S^3. So, for us to compute the invariant of 3 manifolds, it is very important how it behaves under Dehn surgery.

About Casson's invariant for integral homology 3 spheres (3 manifolds having the same homology with integer coefficient as S^3), it is known that this behavior can be described by using so called Casson's knot invariant ([AM], also see section 1.3). On the other hand, about Floer homology groups for integral homology 3 spheres ([F]), it is known that such behavior can be described by Floer's triangle ([BD], also see section 2.1). To clarify the meaning of our result, we briefly review these descriptions. So, let M be an integral homology 3 sphere, and let K be a knot in M. Dividing M to the tubular neighborhood of K and it's complement, and regluing them together by an homeomorphism on their boundary, we can obtain various 3 manifolds. Among them, let M' be the +1 Dehn surgery of M along K, and M" be the 0 surgery. M' is an integral homology 3 sphere, and M" is a integral homology $S^1 \times S^2$. Then the former description is as follows:

$$\lambda(M') \ - \ \lambda(M) \ = \ \lambda'(K)$$

Here λ denotes Casson's invariant for integral homology 3 sphere

and λ' denotes Casson's knot invariant

On the other hand, the latter description is that the following sequence of Floer homology groups induced by the canonical cobordisms:

$$HF_*(M') \ \longrightarrow \ HF_*(M) \ \longrightarrow \ HF_*(M'') \ \longrightarrow$$

becomes a long exact sequence. This is called the Floer triangle for Dehn surgery.

Both of these results cover only restricted class of 3 manifolds. However the former result has been extended to more general 3 manifolds (rational homology 3 spheres, i.e. 3 manifolds having the same homology with rational coefficient as S^3) by Kevin Walker ([W]). The behavior of his extended invariant under Dehn surgery is known to be described by so called Dehn surgery formula. The author is studying the corresponding extention of the latter result. He hopes that this extention will lead to the algorithm for computation of Floer homology groups.

The aim of this paper is to describe the relationship between Casson's knot invariant λ'(K) and Floer homology group of the 0 Dehn surgery of M along the knot K, M".

In [T], Taubes showed the following: for an integral homology 3 sphere M,

$$2\lambda(M) \ = \ \chi(f)$$

CASSON'S KNOT INVARIANT AND GAUGE THEORY

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April 23, 1997