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$TM|_\Lambda$ and a positive constant λ such that

$$\|DF_r|_{E^s}\| < \lambda < \|DF_r|_{E^u}\|$$

for any $r \in \Lambda$. We call $E^s \oplus E^u$ *the pseudo-hyperbolic splitting*. It is known that if Λ is a pseudo-hyperbolic set then there exist neighborhoods U of Λ and V of G in the space of C^1 diffeomorphisms such that if $\tilde{G} \in V$, then $\Lambda(\tilde{G}) = \bigcap_{n\in\mathbf{N}} \tilde{G}^n(U)$ is a pseudo-hyperbolic set for $\tilde{G}$. This can be shown analogously to the case of hyperbolic sets. See *e.g* [6]. If the non-wandering set of a diffeomorphism G is a pseudo-hyperbolic, then G has no homoclinic tangency since a homoclinic point is a non-wandering point and the stable or unstable manifold of a hyperbolic periodic point is tangent to E^s or E^u respectively at a point in the non-wandering set. Therefore, the pseudo-hyperbolicity of the non-wandering set of G implies that G cannot be approximated by a diffeomorphism with homoclinic tangencies.

If there exists a splitting $T\mathbf{R}^2|_R = E^s \oplus E^u$ and a positive number λ such that $E^s(F(r)) = DF(E^s(r))$ and $E^u(F(r)) = DF(E^u(r))$ and $\|DF_r|_{E^s}(r)\| < \lambda < \|DF_r|_{E^u}(r)\|$ for any $r \in R \cap F^{-1}(R)$, then the non-wandering set of F is pseudo-hyperbolic by Lemma 4, and hence, the proof of the main theorem will be completed. The following lemma shows that this is in fact the case.

Lemma 5 *Let E^s and E^u be the vector bundle parallel to the x-axis and y-axis, respectively. Then, a splitting $T\mathbf{R}^2|_{[0,1]^2} = E^s \oplus E^u$ is the semi-hyperbolic splitting.*

(Proof) By the definition of F, we have $[0,1]^2 \cap F([0,1]^2) = [0,1] \times ([0, \frac{1}{3}] \cup [\frac{2}{3}, 1])$. Then,

$$DF_{(x,y)} = \begin{pmatrix} \pm c & 0 \\ 0 & f'(y) \end{pmatrix}$$

for any $(x,y) \in [0,1]^2 \cap F([0,1]^2)$. This implies that E^s and E^u is F-invariant and $\|DF_{(x,y)}|_{E^s}\| = c < |f'(y)| = \|DF_{(x,y)}|_{E^u}\|$. □

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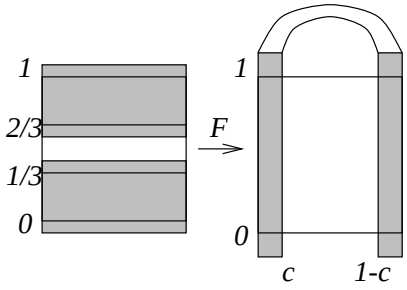

Figure 2: a diffeomorphism F

(Proof) By the definition of F, we have $F((x,y)) = (cx, f(y))$ for $(x,y) \in M_1$, and hence $F(M_1) = [-2c^2\epsilon, c(1+2c)\epsilon] \times (-\infty, f(1)]$. The inequality $c < \frac{1}{2}$ implies that $[-2c^2\epsilon, c(1+2c)\epsilon] \subset [-2c\epsilon, 1+2c\epsilon]$. Since $f'|_{(-\infty,0)} > 1$ and $f(0) = 0$, we have $f^n(1)$ strictly decrease and goes to $-\infty$ as $n \to \infty$. These facts implies that $F(M_1) \subset \mathrm{int}M_1$ and $\bigcap_{n\in\mathbf{Z}} F^n(M_1) = \emptyset$.

It is trivial that $F(M_3) \subset \mathrm{int}M_3$. From (F2),(F3) and the form of F, we have $F^{-1}(M_3 \backslash \mathrm{int}M_2) \subset (-\infty, -\epsilon) \times \mathbf{R}$. This implies $F^{-1}(M_3 \backslash \mathrm{int}M_2) \subset M_3 \backslash M_2$. Hence, $F(M_2) \subset \mathrm{int}M_2$.

We also have $F^{-1}((-\infty,-\epsilon) \times \mathbf{R}) \subset (-\infty,-\epsilon) \times (-\infty, \frac{7}{12})$. If $(x,y) \in (-\infty,-\epsilon) \times (-\infty, \frac{7}{12})$, then $F^{-1}((x,y)) = (c^{-1}x, f^{-1}(y))$, where $f^{-1}(y)$ is one of the inverse images of y less than $\frac{1}{2}$. This implies that $\bigcap_{n\in\mathbf{Z}} F^n(M_3 \backslash M_2) = \emptyset$.

Finally, we show $\bigcap_{n\in\mathbf{Z}} F^n(M_2 \backslash M_1)$ is contained in $[0,1]^2$. Let $(x,y) \in M_2$. If $y < 0$, then $F((x,y)) = (cx, f(y))$ and $f(y) < 0$. Since $c^n x$ goes to zero and $f^n(y)$ decreases and goes to $-\infty$ as $n \to \infty$, we have $F^n((x,y)) \in M_1$ for some large n. If $(x,y) \in M_2$ and $y > 1$, then $F((x,y)) = (1-cx, f(y))$. Since $F((x,y)) \in M_2$ and $f(y) < 0$, we also have $F^n((x,y)) \in M_1$ for some large n. If $x < 0$ and $0 \le y \le 1$, then $F^{-1}((x,y)) = (c^{-1}x, f^{-1}(y))$, $c^{-1}x < 0$ and $0 \le f^{-1}(y) \le 1$, where $f^{-n}(y)$ is one of the inverse images of y less than $\frac{1}{2}$. This implies $F^{-n}(x,y) = (c^{-n}(x), f^{-n}(y))$. Since $c < 1$, we have $F^{-n}((x,y)) \notin M_2$ for some large n. If $x > 1$ and $0 \le y \le 1$, then $F^{-1}((x,y)) = (c^{-1}(1-x), f^{-1}(y))$. We also have $F^{-n}((x,y)) \notin M_2$ since $c^{-1}(1-x) < 0$ and $0 \le f^{-1}(y) \le 1$.□

3.2 A pseudo-hyperbolic splitting

Recall the definition of pseudo-hyperbolic sets. Let G be a diffeomorphism on a 2-dimensional manifold M and Λ be a G-invariant subset of M. We say Λ is *a pseudo-hyperbolic set* when there exists a DG-invariant splitting $E^u \oplus E^s$ of

$1, 2, \cdots, n-2$. In addition, $|f_k'(f_k^{n-1}(p_n))| = |f_1'(f_0^{n-1}(p_n))| < \frac{1}{2}$ and $|f_k'(p_n)| = |f_0'(p_n)| < 2$. Therefore, $|(f_k^n)'| = \prod_{i=0}^{i=n-1} |f_k'(f_k^i(p_n))| < 1$.

Finally, we show f_k converges to a C^1 map f. Once it is done, $f = f_k$ on $[a_k + 1, 1]$ for any $k \in \mathbf{N}$, and hence, p_n is an attractive periodic point of f for each $n \in \mathbf{N}$. For two integers k and k' with $k < k'$, C^1 maps f_k and $f_{k'}$ are identical on $\mathbf{R}\backslash[a_{k'+1}, a_k]$. The difference of f_k and $f_{k'}$ on $[a_{k'+1}, a_k]$ is equal to the maximum of the difference of f_{j+1} and f_j for $j = k, k+1, \cdots, k'$. Therefore, we have

$$\begin{aligned} \|f_k - f_{k'}\|_{C^1} &\leq \sup_{k \leq j < k'} \|f_{j+1} - f_j\|_{C^1} \\ &\leq 12\left(\left(\frac{k+1}{k-1}\right)^2 - 1\right). \end{aligned}$$

and the last term goes to zero as $k \to \infty$. This implies that $\{f_k\}$ is a Cauchy sequence in C^1 topology. □

3 A diffeomorphism with infinitely many periodic attractors

Let $c = \frac{1}{2}\inf_{x \notin (\frac{5}{12}, \frac{7}{12})} |f'(x)|$ and ϵ be a positive number with $\epsilon < \frac{1}{4}$. Note that $0 < c < \frac{1}{2}$. We define a diffeomorphism F as a C^1 diffeomorphism on $\mathbf{R}^2$ with

$$F(x,y) = \begin{cases} (cx, f(y)) & \text{if} \quad x \leq 1+\epsilon \text{ and } y < \frac{5}{12} \\ (1-c(x), f(y)) & \text{if} \quad x \leq 1+\epsilon \text{ and } y > \frac{7}{12} \end{cases}$$

and the *filtration conditions*:

(F1) $F((\infty, -\epsilon) \times (\frac{5}{12}, \frac{7}{12})) \subset \mathbf{R} \times (f(\frac{5}{12}), \infty)$,

(F2 $F([-\epsilon, 1+\epsilon] \times (\frac{5}{12}, \frac{7}{12})) \subset [-c\epsilon, 1 + c\epsilon \times (1, \infty)$ and

(F3) $F((1+\epsilon, \infty) \times \mathbf{R}) \subset (c, 1-c) \times \mathbf{R}$.

We show F satisfies the assertion of Main theorem.

3.1 A filtration associated to F

A sequence $\emptyset = M_0 \subset M_1 \cdots \subset M_{N-1} \subset M_N = M$ of subsets of a manifold M is called a filtration associated to a homeomorphism H when $H(M_i) \subset \mathrm{int} M_i$. It is known that the non-wandering set of $\tilde{H}$ which is C^0-near H is contained in a small neighborhood of $\bigcup_{i=1}^N \bigcap_{n \in \mathbf{Z}} H^n(M_i \backslash M_{i-1})$ (See *e.g.* [6]).

Let $M_0 = \emptyset$, $M_1 = [-2c\epsilon, 1 + 2c\epsilon] \times (-\infty, 1]$, $M_2 = [-\epsilon, 1+\epsilon] \times \mathbf{R}$ and $M_3 = \mathbf{R}^2$.

Lemma 4 *A sequence of subsets $M_0 \subset M_1 \subset M_2 \subset M_3$ of $\mathbf{R}^2$ is a filtration and $\bigcup_{i=1}^3 \bigcap_{n \in \mathbf{Z}} F^n(M_i \backslash M_{i-1}) \subset [0,1]^2$.*

Lemma 3 *Let $h : \mathbf{R} \to \mathbf{R}$ with $1 \leq |h'| \leq 1+\delta$ for some $\delta < 1$. Assume $p \in \mathbf{R}$ is a fixed point of h. Then, for any positive number ϵ, there exists a C^1 map $\tilde{h} : \mathbf{R} \to \mathbf{R}$ such that $\tilde{h}(x) = g(x)$ on $|x-p| > \epsilon$, $\|\tilde{h} - h\|_{C^1} < 12\delta$ and p is a fixed point of $\tilde{h}$ with $0 < \tilde{h}'(p) < 1$.*

(Proof) Without loss of generality, we may assume a fixed point $p = 0$. We choose C^1 map $\phi : \mathbf{R} \to \mathbf{R}$ such that

$$\phi(x) = \begin{cases} 1 & |x| > \epsilon \\ \frac{1}{1+2\delta} & |x| < \frac{\epsilon}{4} \end{cases}$$

and

$$|\phi'| < \left(1 - \frac{1}{1+2\delta}\right)\frac{2}{\epsilon}.$$

Let $\tilde{h}(x) = \phi(x) \cdot h(x)$. Obviously, $\tilde{h}(x) = h(x)$ for $|x| \geq \epsilon$. For any x with $|x| < \epsilon$, we have

$$|h(x) - \tilde{h}(x)| = |1 - \phi(x)| \cdot |h(x)| \leq \frac{2\delta}{1+2\delta} \cdot 2\epsilon < 12\delta$$

and

$$\begin{aligned} |\tilde{h}'(x) - h'(x)| &= |h'(x) - \phi(x)h'(x) - \phi'(x)h(x)| \\ &\leq |1 - \phi(x)| \cdot |h'(x)| + |\phi'(x)| \cdot |h(x)| \\ &\leq \frac{2\delta}{1+2\delta}(1+\delta) + \frac{2\delta}{1+2\delta} \cdot \frac{2}{\epsilon} \cdot 2\epsilon \\ &< 12\delta. \end{aligned}$$

Finally, $p = 0$ is a fixed point of $\tilde{h}$ and $|\tilde{h}'(0)| = \frac{1+\delta}{1+2\delta} < 1$.□

(Proof of Proposition 1) We construct f_k of which $\{p_n\}_{n \leq k}$ is attractive periodic points and show f_k converges to a C^1 map f.

Firstly, we define f_k. Since the cardinality of $\{f_0^{n-1}(p_n)\} \cap [a_2, b_1]$ is finite, we may have a C^1 unimodal map f_1 such that $f_1(f_0^{n-1}(p_n)) = p_n$, ${f_1}'(f^{n-1}(p_n)) < \frac{1}{2}$ and $f_1 = f_0$ on $\mathbf{R} \backslash (a_2, b_1)$, by a small perturbation of f_0 on a neighborhood of $\{f_0^{n-1}(p_n)\} \cap [a_2, b_1]$. When f_{k-1} is defined, Finiteness of the cardinality of $\{f_{k-1}^{n-k}(p_n)\} \cap [a_{k+1}, b_k]$ and Lemma 3 allow us to choose a C^1 map f_k such that $f_k(f_{k-1}^{n-k}(p_n)) = f_{k-1}(f_{k-1}^{n-k}(p_n))$ and ${f_k}'(f_{k-1}^{n-k}(p_n)) < 1$ for any $n \geq k$, $\|f_k - f_{k-1}\|_{C^1} < 12\left(\left(\frac{k+1}{k-1}\right)^2 - 1\right)$ and $f_k = f_{k-1}$ on $\mathbf{R} \backslash (a_{k+1}, b_k)$. One can easily have $f_k^i(p_n) = f_0^i(p_n)$ for any k and n.

Secondly, we show p_n is an attractive periodic point of f_k if $n \leq k$. Since $f_k^i(p_n) \in (a_{k+1}, a_k)$, we have $|f_k'(f_k^i(p_n))| = |f_{n-k}'(f_{n-i}(p_n))| \leq 1$ for $i =$

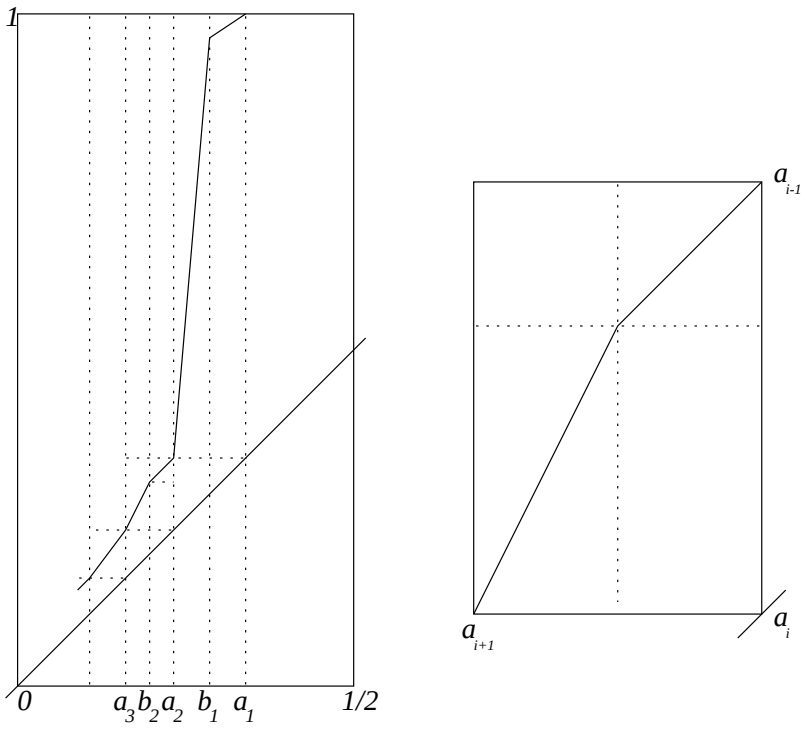

Figure 1: the graph of g

(Proof) One can easily see $f^i((1-a_n, 1-a_{n+1})) = (a_{n-i+1}, a_{n-i})$ for $i = 1, 2, \cdots, n-1$ and $f^n((1-a_n, 1-a_{n+1})) = (a_1, 1)$. Since $(a_1, 1) \supset [1-a_n, 1-a_{n+1}]$ and $f^n|_{(1-a_n, 1-a_{n+1})}$ is a diffeomorphism, there exists a unique n-periodic point p_n which satisfies the assertion of Lemma.□

Remark The sequence p_n converges to 1 as $n \to \infty$ since a_n goes to 0. This implies $f^{n-k}(p_n)$ converges to a_k for each $k \in \mathbf{N}$. Since $f^{n-k}(p_n) \in (a_{k+1}, a_k)$, we have

$$\#\left(\{f^{n-k}(p_n)\}_{n>k} \cap [a_{k+1}, b_k]\right) < \infty$$

for each $k \in \mathbf{N}$.

Notice that $|f_0{}'(f_0^i(p_n))| \leq 1$ if $f_0^i(p_n) \notin \bigcup_{k=2}^{\infty}(a_{k+1}, b_k)$. To make infinitely many periodic attractors, we perturb f_0 on $\bigcup_{k=2}^{\infty}(a_{k+1}, b_k)$. The next lemma is needed to estimate the size of perturbation in order to make a periodic point attracting.

Proposition 1 *There is a C^1 map $f : \mathbf{R} \to \mathbf{R}$ which satisfies following properties:*

1. $f(0) = f(1) = 0$.
2. $c = \frac{1}{2}$ *is a unique critical point of* f *with* $f(c) > 1$.
3. *There is a sequence of periodic attractors which are contained in* $[0, 1]$.
4. $f'(x) > 1$ *for all* $x < 0$.

To prove this proposition, we construct a C^1 map f_0 which has infinitely many "almost non-repelling" periodic points and perturb f_0 in a neighborhood of their orbits.

Let a positive constant $A = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$ and choose two sequences of numbers $a_n = \sum_{k=n}^{\infty} \frac{1}{3Ak^2}$ and $b_n = \frac{1}{2}(a_{n+1} + a_n)$. We define a continuous map $g : \mathbf{R} \to \mathbf{R}$ as follows:

1. g is increasing on $\left(\infty, \frac{1}{2}\right)$.
2. g maps $[b_1, a_1]$ to $\left[1 - \frac{1}{2}(a_1 - b_1), 1\right]$ affinely.
3. g maps $[b_i, a_i]$ to $[a_{i-1} - (a_i - b_i), a_{i-1}]$ and $[a_{i+1}, b_i]$ to $[a_i, a_{i-1} - (a_i - b_i)]$ affinely for any $i \geq 2$.
4. $g|_{(-\infty,0)}$ is C^1. $g'(x) > 1$ for any $x < 0$ and $\lim_{x \to 0-} g'(x) = 1$.
5. $g(1 - x) = x$ for any x.
6. $g|_{[a_1, 1-a_1]}$ is C^1.

The slopes of g on (a_{i+1}, b_i) and (b_i, a_i) are $\frac{a_i - (a_{i-1} - (a_i - b_i))}{b_i - a_{i+1}} = 2\left(\frac{i}{i-1}\right)^2 - 1$ and 1 respectively. Since $1 < 2\left(\frac{i}{i-1}\right)^2 - 1 < \left(\frac{i+1}{i-1}\right)^2$, the slope of g on (a_{i+1}, b_i) converges to 1 as $i \to \infty$. This means that a map g is C^1 on $\mathbf{R} \backslash \{a_i, b_i\}_{i \in \mathbf{N}}$ and $g'(0) = 1$. We can choose a C^1 unimodal map f as follows by a deformation of g on a small neighborhood of $\{a_i, b_i, 1 - a_i, 1 - b_i\}_{i \in \mathbf{N}}$.

1. $f_0(x) = f_0(1 - x)$ for any x.
2. $f_0(0) = 0$ and ${f_0}'(x) > 1$ for any $x < 0$.
3. $f_0([b_1, a_1]) = \left[1 - \frac{1}{2}(a_1 - b_1), 1\right]$ and ${f_0}'|_{[b_1, a_1]} = \frac{1}{2}$.
4. $f_0([a_{i+1}, a_i]) = [a_i, a_{i-1}]$ for any $i \geq 2$.
5. ${f_0}'|_{[b_i, a_i]} = 1$.
6. $1 \leq {f_0}'(x) \leq \left(\frac{i+1}{i-1}\right)^2$ for any $x \in [a_{i+1}, b_i]$.

Lemma 2 *For any $n \geq 2$, there exists a unique n-periodic point p_n such that $p_n \in (1 - a_n, 1 - a_{n+1})$ and $f_0^i(p_n) \in (a_{n-i+1}, a_{n-i})$ for $i = 1, 2, \cdots, n - 1$.*

Main theorem *There is a diffeomorphism $F \in \mathrm{Diff}^r(\mathbf{R}^2)$ which satisfies the following properties: (1) F has infinitely many periodic attractors with unbounded periods. (2) there is a neighborhood $V \subset \mathrm{Diff}^r(\mathbf{R}^2)$ of F such that any diffeomorphism $\tilde{F} \in V$ has no homoclinic tangency.*

To prove this theorem, we construct a C^1 map f on the real line such that f has a unique critical point outside the non-wandering set of f and has infinitely many periodic attractors with unbounded periods. Once it is done, it is not so difficult to construct a C^1 diffeomorphism on $\mathbf{R}^2$ which satisfies the condition in the main theorem by the method similar to constructing Smale's horseshoe.

Our construction can work only in C^1 topology since such a 1-dimensional map f cannot be C^2 by the following theorem.

Theorem (Mañé) *Assume that I is a compact interval and $f : I \to I$ is a C^1-map such that the variation of $\log |DF|$ is bounded. Denote the set of critical points of f by $c(f)$. If a closed f-invariant set Λ does not intersects $c(f)$, then the periods of the periodic attractors in Λ are bounded.*

Our example is pseudo-hyperbolic. Namely, let F be our example and Λ be the non-wandering set of F. Then, there exists a DF-invariant splitting $E^u \oplus E^s$ of $TM|_\Lambda$ and a positive constant λ such that

$$\|DF_r|_{E^s}\| < \lambda < \|DF_r|_{E^u}\|$$

for any $r \in \Lambda$. One can see that a pseudo-hyperbolic diffeomorphism cannot be approximated by a diffeomorphism with homoclinic tangencies. See section 3.2. It is natural to ask the finiteness of periodic attractors for C^2 pseudo-hyperbolic diffeomorphisms. The author [1] recently showed that the periods of periodic attractors are bounded for C^2 pseudo-hyperbolic diffeomorphism. This result implies that the conjecture due to Palis results in the following conjecture.

Conjecture *If C^2 diffeomorphism cannot be approximated by a diffeomorphism with homoclinic tangencies, then is the closure of periodic orbits pseudo-hyperbolic?*

Our example has a fixed point which has one as its eigenvalue. This means that our example is not generic. This yields the following question. "Is the periods of periodic attractors bounded for a *generic* C^1 pseudo-hyperbolic diffeomorphism?"

2 A 1-dimensional map with infinitely many periodic attractors

In this section, we prove following proposition.

A counterexample to C^1 Palis conjecture

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Abstract

In this paper, we construct a C^1 surface diffeomorphism which has infinitely many periodic attractors with unbounded periods and cannot be approximated by a diffeomorphism with homoclinic tangency. This diffeomorphism gives a counterexample to a conjecture due to Palis on coexistence of infinitely many periodic attractors.

1 Introduction and main result

We say that a diffeomorphism has a homoclinic tangency when the stable and unstable manifolds of a hyperbolic periodic point intersects tangentially. It is known that homoclinic tangencies imply complicated dynamics, such as Hénon-like strange attractors and coexistence of infinitely many periodic attractors (see *e.g.* [5]). Palis conjectured that complicated dynamics implies existence of a homoclinic tangency for 2-dimensional dynamics. The C^r Palis conjecture on the coexistence of infinitely many periodic attractors is formulated as follows:

Conjecture(Palis) *Let M be a 2-dimensional C^r manifold and $\mathrm{Diff}^r(M)$ be the space of C^r diffeomorphisms on M with C^r topology. Assume that $F \in \mathrm{Diff}^r(M)$ has infinitely many periodic attractors with unbounded periods. Then, for any neighborhood U of F in $\mathrm{Diff}^r(M)$, there is a diffeomorphism G which has a homoclinic tangency.*

In this paper, we show that there is a counterexample to the C^1 case of this conjecture, namely,