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Proof: The quotient morphism induces the natural transformation τ. (i) follows from the fact that the quotient morphism gives a categorical quotient. For (ii) we easily see that $\tau_{\operatorname{Spec}\Omega}$ is bijective, since the quotient morphism gives a universally geometric quotient and PGL_n principal fiber bundle. We have thus proved the statement. □

We name the universally geometric quotient of the a.i.r. part by PGL_n, or the coarse moduli scheme of equivalence classes of a.i.r.

Definition 6.14 (Definition of the character variety)**.** $\mathrm{Ch}_n(\Gamma)_{a.i.r.}$ is called the (a.i.r.) *character variety* of degree for Γ.

Remark 6.15. Suppose that Γ is a finitely generated group. In general theory we see that $\mathrm{Rep}_n(\Gamma) \to \mathrm{Ch}_n(\Gamma)$ is a uniform categorical quotient. For example, see [4] or [7]. We also call $\mathrm{Ch}_n(\Gamma)$ the character variety of degree n for Γ.

We end this paper by introducing the representation variety and the character variety in SL_n. The same statement holds for SL_n case.

Definition 6.16 (Definition for SL_n version)**.** J is defined as an ideal of $\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}$ generated by $\{\mathrm{d}(\gamma) - 1 \mid \gamma \in \Gamma\}$. Then we define

$$\begin{aligned} \overline{\mathrm{A}}_n^{\mathrm{Ch}}(\Gamma) &:= \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}/J \\ \overline{\mathrm{A}}_n(\Gamma) &:= \mathrm{A}_n(\Gamma)/J \cdot \mathrm{A}_n(\Gamma). \end{aligned}$$

We call

$$\begin{aligned} \overline{\mathrm{Ch}}_n(\Gamma) &:= \operatorname{Spec} \overline{\mathrm{A}}_n^{\mathrm{Ch}}(\Gamma) \\ \overline{\mathrm{Rep}}_n(\Gamma) &:= \operatorname{Spec} \overline{\mathrm{A}}_n(\Gamma) \end{aligned}$$

the character variety of degree n for Γ in SL_n and the representation variety of degree n for Γ in SL_n, respectively. The a.i.r. part $\overline{\mathrm{Ch}}_n(\Gamma)_{\mathrm{a.i.r.}}$ and $\overline{\mathrm{Rep}}_n(\Gamma)_{\mathrm{a.i.r.}}$ is defined in the same way as GL_n version.

Remark 6.17. $\overline{\mathrm{Rep}}_n(\Gamma)$ is a closed subscheme of $\mathrm{Rep}_n(\Gamma)$ consisting of representations in SL_n, while $\overline{\mathrm{Rep}}_n(\Gamma)_{\mathrm{a.i.r.}}$ consists of absolutely irreducible representations in SL_n. PGL_n acts on $\overline{\mathrm{Rep}}_n(\Gamma)$ and $\overline{\mathrm{Rep}}_n(\Gamma)_{\mathrm{a.i.r.}}$ by conjugation. There are natural morphisms $\overline{\mathrm{Rep}}_n(\Gamma) \to \overline{\mathrm{Ch}}_n(\Gamma)$ and $\overline{\mathrm{Rep}}_n(\Gamma)_{\mathrm{a.i.r.}} \to \overline{\mathrm{Ch}}_n(\Gamma)_{\mathrm{a.i.r.}}$.

Theorem 6.18. $\overline{\mathrm{Rep}}_n(\Gamma)_{\mathrm{a.i.r.}} \to \overline{\mathrm{Ch}}_n(\Gamma)_{\mathrm{a.i.r.}}$ *is a universal geometric quotient by the adjoint action of* PGL_n*. Moreover the quotient morphism is a* PGL_n *principal fiber bundle.*

Proof: The morphism is obtained by base extension of the quotient morphism in GL_n case. The statement follows from the Main Theorem. □

Remark 6.8. We can not expect to have a Zariski open covering of $\mathrm{Ch}_n(\Gamma)_{\mathrm{a.i.r.}}$ over which the PGL_n principal fiber bundle is trivial. However we can choose such an open covering in étale topology.

Remark 6.9. In this paper we deal with representations of a group. The Main Theorem also holds for the representation variety for a semigroup or a monoid. In monoid case, the prototype of degree n is the representation variety for the free monoid of rank n^2.

The Main Theorem includes the following statement in terms of representation theory.

Theorem 6.10. *Let R be a commutative ring and Γ a group. Suppose that $\rho, \rho' : \Gamma \to \mathrm{GL}_n(R)$ are absolutely irreducible representations. Then*

$$\rho \sim \rho' \Leftrightarrow c_i(\rho(\gamma)) = c_i(\rho'(\gamma)) \text{ for } 1 \leq i \leq n, \gamma \in \Gamma.$$

Remark 6.11. We say that Skolem-Noether Theorem holds in R if for any R-algebra isomorphism $\sigma : \mathrm{M}_n(R) \to \mathrm{M}_n(R)$ there exists $P \in \mathrm{GL}_n(R)$ such that $\sigma = \mathrm{Ad}(P)$. If Skolem-Noether Theorem holds in R, then for two equivalent a.i.r. ρ, ρ' there exists $P \in \mathrm{GL}_n(R)$ such that $\mathrm{Ad}(P)(\rho) = \rho'$. We remark that if $\mathrm{Pic}(\mathrm{Spec}\, R)$ is a torsion free group, then Skolem-Noether Theorem holds in R.

Now we consider the coarse moduli scheme of equivalence classes of a.i.r. We see that $\mathrm{Ch}_n(\Gamma)_{\mathrm{a.i.r.}}$ is the coarse moduli scheme.

Definition 6.12. We define the following moduli problem $\mathrm{EqAIR_n}(\Gamma)$ as follows:

$$\begin{array}{rccl} \mathrm{EqAIR_n}(\Gamma) & : & \textbf{(Sch)} & \to & \textbf{(Sets)} \\ & & X & \mapsto & \{\rho : \Gamma \to \mathrm{GL}_n(\Gamma(X, \mathcal{O}_X)) \text{ an a.i.r. } \}/\sim, \end{array}$$

where $\rho \sim \rho' \Longleftrightarrow \rho = \mathrm{Ad}(\sigma)(\rho')$ for some $\sigma \in h_{\mathrm{PGL}_n}(X)$.

Theorem 6.13. *The moduli problem $\mathrm{EqAIR_n}(\Gamma)$ has a coarse moduli scheme $\mathrm{Ch}_n(\Gamma)_{\mathrm{a.i.r.}}$. In other words, there exists a natural transformation $\tau : \mathrm{EqAIR_n}(\Gamma) \to \mathrm{Ch}_n(\Gamma)_{\mathrm{a.i.r.}}$ such that*

(i) *for any scheme Z there exists an induced isomorphism:*

$$\tau : \mathrm{Hom}\,(\mathrm{EqAIR_n}(\Gamma), h_Z) \cong \mathrm{Hom}\,(h_{\mathrm{Ch}_n(\Gamma)_{\mathrm{a.i.r.}}}, h_Z)$$

(ii) *τ induces a bijective map*

$$\tau_{\mathrm{Spec}\,\Omega} : \mathrm{EqAIR_n}(\Gamma)(\mathrm{Spec}\,\Omega) \to \mathrm{Ch}_n(\Gamma)_{\mathrm{a.i.r.}}(\mathrm{Spec}\,\Omega)$$

for any algebraic closed field Ω.

Proof: From Corollary 6.2 we have $\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta} \to \mathrm{A}_n(\Gamma)_{\Delta}$ is faithfully flat. Hence we get the following exact sequence by Corollary 6.3.

$$0 \longrightarrow \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta} \longrightarrow \mathrm{A}_n(\Gamma)_{\Delta} \longrightarrow \mathrm{A}_n(\Gamma)_{\Delta} \otimes_{\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta}} \mathrm{A}_n(\Gamma)_{\Delta}$$

From Theorem 5.7 there is an isomorphism $\mathrm{A}_n(\Gamma)_{\Delta} \otimes_{\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta}} \mathrm{A}_n(\Gamma)_{\Delta} \cong \mathrm{A}_n(\Gamma)_{\Delta} \otimes_{\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta}} \mathrm{A}(\mathrm{PGL}_n)$, so we get the next exact sequence.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta} & \longrightarrow & \mathrm{A}_n(\Gamma)_{\Delta} & \longrightarrow & \mathrm{A}_n(\Gamma)_{\Delta} \otimes_{\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta}} \mathrm{A}(\mathrm{PGL}_n) \\ & & & & b & \mapsto & b \otimes 1 - 1 \otimes \mathrm{Ad}(b) \end{array}$$

Tensoring $\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta}$-algebra R, we also get the exact sequence. The statement follows from the exact sequence. □

Now what we have discussed crystallizes to form the Main Theorem. We state the Main Theorem.

Theorem 6.5 (The Main Theorem)**.** $\mathrm{Rep}_n(\Gamma)_{\mathrm{a.i.r.}} \to \mathrm{Ch}^0_n(\Gamma)_{\mathrm{a.i.r.}}$ *is a universally geometric quotient by the adjoint action of* PGL_n*. Moreover the quotient morphism is a* PGL_n *principal fiber bundle.*

Proof: The proof has been almost completed. Faithfully flatness implies that the morphism is universally submersive. Recalling the previous theorem, we see that the morphism gives a universally geometric quotient. Corollary 5.8 implies that the quotient morphism is a PGL_n principal fiber bundle. We have thus proved the Main Theorem. □

Corollary 6.6. $\mathrm{Ch}_n(\Gamma)_{\mathrm{a.i.r.}} = \mathrm{Ch}^0_n(\Gamma)_{\mathrm{a.i.r.}}$.

Proof: This follows from that $\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta}$ is the universal invariant ring of $\mathrm{A}_n(\Gamma)_{\Delta}$ by PGL_n. □

From the above statement we see that the PGL_n invariant ring is generated by coefficients of characteristic polynomials on the a.i.r. part. This fact can be considered as an extension of Donkin's Theorem. From now on, we identify $\mathrm{Ch}^0_n(\Gamma)_{\mathrm{a.i.r.}}$ with $\mathrm{Ch}_n(\Gamma)_{\mathrm{a.i.r.}}$. Remark that if Γ is a finitely generated group, then $\mathrm{Ch}_n(\Gamma)_{\mathrm{a.i.r.}}$ is of finite type over $\mathbf{Z}$.

Corollary 6.7. *Let* F_m *be the free group of rank* m*. Then* $\mathrm{Ch}_n(\mathrm{F}_m)_{\mathrm{a.i.r.}}$ *is smooth over* $\mathbf{Z}$*.*

Proof: The statement follows from the fact that $\mathrm{Rep}_n(\mathrm{F}_m)_{\mathrm{a.i.r.}}$ is smooth over $\mathbf{Z}$. □

Proof: At first we prove that the standard morphism is surjective. It suffices to prove that for any algebraic closed field Ω the morphism $\mathrm{Rep}_n(\mathrm{F}_{n^2})_\Delta \otimes_{\mathbf{Z}} \Omega \to \mathrm{Ch}^0_n(\mathrm{F}_{n^2})_\Delta \otimes_{\mathbf{Z}} \Omega$ is surjective. From Donkin's Theorem (Theorem 1.10) we get an isomorphism $(\mathrm{A}_n(\mathrm{F}_{n^2}) \otimes_{\mathbf{Z}} \Omega)^{\mathrm{PGL}_n}_\Delta \cong \mathrm{A}_n(\mathrm{F}_{n^2})^{\mathrm{PGL}_n}{}_\Delta \otimes_{\mathbf{Z}} \Omega$. Hence $\phi_\Omega : \mathrm{Rep}_n(\mathrm{F}_{n^2})_\Delta \otimes_{\mathbf{Z}} \Omega \to \mathrm{Ch}^0_n(\mathrm{F}_{n^2})_\Delta \otimes_{\mathbf{Z}} \Omega$ is a categorical quotient. From Corollary 5.9 the action of PGL_n is free, so the quotient map gives a geometric quotient. In particular the quotient map is surjective. This completes the first step.

Secondly it is sufficient to prove that the morphism is flat. We recall that $\mathrm{Rep}_n(\mathrm{F}_{n^2})_\Delta$ is smooth over $\mathbf{Z}$ from Proposition 1.5. Since $\mathrm{Rep}_n(\mathrm{F}_{n^2})_\Delta \otimes_{\mathbf{Z}} \Omega \to \mathrm{Ch}^0_n(\mathrm{F}_{n^2})_\Delta \otimes_{\mathbf{Z}} \Omega$ is a geometric quotient and the action is free, $\mathrm{Ch}^0_n(\mathrm{F}_{n^2})_\Delta \otimes_{\mathbf{Z}} \Omega$ is nonsingular. Therefore $\mathrm{Ch}^0_n(\mathrm{F}_{n^2})_\Delta$ is smooth over $\mathbf{Z}$. In particular $\mathrm{Ch}^0_n(\mathrm{F}_{n^2})_\Delta$ is a regular scheme. Any fiber of the morphism is isomorphic to PGL_n, so dimensions of fibers is constant. This implies that the morphism is flat.

Now we completed the proof. □

Corollary 6.2. *The morphism* $\mathrm{Rep}_n(\Gamma)_{\mathrm{a.i.r.}} \to \mathrm{Ch}^0_n(\Gamma)_{\mathrm{a.i.r.}}$ *is faithfully flat.*

Proof: For any $\alpha_1, \alpha_2, \ldots, \alpha_{n^2} \in \Gamma$ we only have to prove that $\mathrm{Rep}_n(\Gamma)_{\Delta(\alpha_1,\alpha_2,\ldots,\alpha_{n^2})} \to \mathrm{Ch}^0_n(\Gamma)_{\Delta(\alpha_1,\alpha_2,\ldots,\alpha_{n^2})}$ is faithfully flat. This morphism is obtained by base change of the standard morphism. Therefore it is faithfully flat.

□

Now we remark the following fundamental lemma. The proof may be found, for example, in Milne [3].

Lemma 6.3. *Let $\phi : A \to B$ be a ring homomorphism of commutative rings. Suppose that ϕ is faithfully flat. Then*

$$\begin{array}{ccccccc} 0 & \to & A & \stackrel{\phi}{\to} & B & \to & B \otimes_A B \\ & & & & b & \mapsto & b \otimes 1 - 1 \otimes b \end{array}$$

is exact.

The previous lemma allows us to get the next theorem.

Theorem 6.4. $\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_\Delta$ *is the universal* PGL_n *invariant ring of* $\mathrm{A}_n(\Gamma)_\Delta$ *for any* $\alpha_1, \alpha_2, \ldots, \alpha_{n^2} \in \Gamma$ *and* $\Delta := \Delta(\alpha_1, \alpha_2, \ldots, \alpha_{n^2})$. *In other words, for any* $\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_\Delta$*-algebra* R

$$(\mathrm{A}_n(\Gamma)_\Delta \otimes_{\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_\Delta} R)^{\mathrm{PGL}_n} \cong R$$

holds.

The same equation also holds for ρ'. Hence $\phi(\rho(\gamma)) = \rho'(\gamma), \gamma \in \Gamma$. In particular

$$\begin{aligned} \phi(\rho(\alpha_i\alpha_j)) &= \rho'(\alpha_i\alpha_j) \\ &= \rho'(\alpha_i)\rho'(\alpha_j) \\ &= \phi(\rho(\alpha_i))\phi(\rho(\alpha_j)). \end{aligned}$$

Therefore ϕ is a R-isomorphism.

From Theorem 5.4 for the $\mathcal{O}_X$-algebra isomorphism $\phi : \mathrm{M}_n(R) \to \mathrm{M}_n(R)$ there exists $\sigma \in h_{\mathrm{Rep}_n(\Gamma)_\Delta}(X)$ such that $\mathrm{Ad}(\sigma) = \phi$. In particular $\mathrm{Ad}(\sigma)(\rho) = \rho'$. This implies surjectivity.

We have thus proved the theorem. □

The previous theorem can be extended to the next statement. The adjoint action of PGL_n is good on the absolutely irreducible representation part.

Corollary 5.8 (The action of PGL_n)**.** *In the same notation as above,* $(p_1, \mathrm{Ad}) : \mathrm{Rep}_n(\Gamma)_{\mathrm{a.i.r.}} \times_{\mathrm{Spec}\,\mathbf{Z}} \mathrm{PGL}_n \to \mathrm{Rep}_n(\Gamma)_{\mathrm{a.i.r.}} \times_{\mathrm{Ch}^0_n(\Gamma)_{\mathrm{a.i.r.}}} \mathrm{Rep}_n(\Gamma)_{\mathrm{a.i.r.}}$ *is a isomorphism.*

Proof: It suffices to prove that the morphism is an isomorphism on any open set such that $\Delta(\alpha_1, \alpha_2, \ldots, \alpha_{n^2}) \neq 0$ for arbitrary $\alpha_1, \alpha_2, \ldots, \alpha_{n^2} \in \Gamma$. This follows from the previous theorem. □

In particular we get the following corollary.

Corollary 5.9. PGL_n *acts freely on* $\mathrm{Rep}_n(\Gamma)_{\mathrm{a.i.r.}}$*. In other words,* $(p_1, \mathrm{Ad}) : \mathrm{Rep}_n(\Gamma)_{\mathrm{a.i.r.}} \times_{\mathrm{Spec}\,\mathbf{Z}} \mathrm{PGL}_n \to \mathrm{Rep}_n(\Gamma)_{\mathrm{a.i.r.}} \times_{\mathrm{Spec}\,\mathbf{Z}} \mathrm{Rep}_n(\Gamma)_{\mathrm{a.i.r.}}$ *is a closed immersion.*

6. The Main Theorem

We see that the adjoint action of PGL_n on the a.i.r. part is free in the previous section. In this section we see that there is a universal geometric quotient of the a.i.r. part by PGL_n. Moreover from Corollary 5.8 we see that the quotient morphism is a PGL_n principal fiber bundle. The quotient is also the coarse moduli scheme of equivalence classes of absolutely irreducible representations. This is the end of this paper.

For the proof of the main theorem we begin by showing that $\mathrm{Rep}_n(\Gamma)_\Delta \to \mathrm{Ch}^0_n(\Gamma)_\Delta$ is faithfully flat.

Theorem 6.1. *The standard morphism* $\mathrm{Rep}_n(\mathrm{F}_{n^2})_\Delta \to \mathrm{Ch}^0_n(\mathrm{F}_{n^2})_\Delta$ *is faithfully flat.*

Proof: It suffices to prove that for any scheme X the map $(p_1, \mathrm{Ad})_X : h_{\mathrm{Rep}_n(\Gamma)_\Delta}(X) \times h_{\mathrm{Rep}_n(\Gamma)_\Delta}(X) \to h_{\mathrm{Rep}_n(\Gamma)_\Delta}(X) \times_{h_{\mathrm{Ch}^0_n(\Gamma)_\Delta}(X)} h_{\mathrm{Rep}_n(\Gamma)_\Delta}(X)$ is bijective.

First we show that this map is injective. Suppose that $(p_1, \mathrm{Ad})(\rho_1, \sigma_1) = (p_1, \mathrm{Ad})(\rho_2, \sigma_2)$. Then it is clear that $\rho_1 = \rho_2$. Put $\rho := \rho_1 = \rho_2$. $\{ \rho(\gamma) \mid \gamma \in \Gamma\}$ generates $\Gamma(X, \mathcal{O}_X)$-algebra $\mathrm{M}_n(\Gamma(X, \mathcal{O}_X))$, since $\Delta(\rho(\alpha_1), \ldots, \rho(\alpha_{n^2})) \in \Gamma(X, \mathcal{O}_X)^\times$. Therefore $\mathrm{Ad}(\sigma_1)(\rho) = \mathrm{Ad}(\sigma_2)(\rho)$ implies that $\mathrm{Ad}(\sigma_1) = \mathrm{Ad}(\sigma_2)$ as $\Gamma(X, \mathcal{O}_X)$-algebra isomorphisms of $\mathrm{M}_n(\Gamma(X, \mathcal{O}_X))$. From Theorem 5.4 we have $\sigma_1 = \sigma_2$. So injectivity has been proved.

Next we show that the map is surjective. It suffices to show that for any $(\rho, \rho') \in h_{\mathrm{Rep}_n(\Gamma)_\Delta}(X) \times_{h_{\mathrm{Ch}^0_n(\Gamma)_\Delta}(X)} h_{\mathrm{Rep}_n(\Gamma)_\Delta}(X)$ there exists some $\sigma \in h_{\mathrm{Rep}_n(\Gamma)_\Delta}(X)$ such that $\mathrm{Ad}(\sigma)(\rho) = \rho'$. Put $R := \Gamma(X, \mathcal{O}_X)$. We recall that $\Delta(\rho(\alpha_1), \rho(\alpha_2), \ldots, \rho(\alpha_{n^2})), \Delta(\rho'(\alpha_1), \rho'(\alpha_2), \ldots, \rho'(\alpha_{n^2})) \in R^\times$. Thus $\{\rho(\alpha_1), \rho(\alpha_2), \ldots, \rho(\alpha_{n^2})\}$ and $\{\rho'(\alpha_1), \rho'(\alpha_2), \ldots, \rho'(\alpha_{n^2})\}$ are R-bases of $\mathrm{M}_n(R)$. Now we define a R-linear map ϕ as

$$\begin{array}{rccc} \phi & : & \mathrm{M}_n(R) & \to & \mathrm{M}_n(R) \\ & & \rho(\alpha_i) & \mapsto & \rho'(\alpha_i). \end{array}$$

We claim that ϕ is a R-algebra isomorphism. Suppose that $\psi, \psi' : \mathrm{A}_n(\Gamma)_\Delta \to R$ are the ring homomorphisms corresponding to ρ and ρ' respectively. Then we get a commutative diagram.

$$\begin{array}{ccc} \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_\Delta & \longrightarrow & \mathrm{A}_n(\Gamma)_\Delta \\ \downarrow & & \downarrow{\scriptstyle \psi} \\ \mathrm{A}_n(\Gamma)_\Delta & \xrightarrow[\psi']{} & R \end{array}$$

Denote the above ring homomorphism by $\chi : \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_\Delta \to R$. From Theorem 3.3 the following equation holds. For $\gamma \in \Gamma$

$$\rho(\gamma) = (\rho(\alpha_1), \ldots, \rho(\alpha_{n^2}))\, T \begin{pmatrix} \chi(\mathrm{s}(\alpha_1\gamma)) \\ \vdots \\ \chi(\mathrm{s}(\alpha_{n^2}\gamma)) \end{pmatrix},$$

where

$$T = \begin{pmatrix} \chi(\mathrm{s}(\alpha_1\alpha_1)) & \ldots & \chi(\mathrm{s}(\alpha_1\alpha_{n^2})) \\ \vdots & \chi(\mathrm{s}(\alpha_i\alpha_j)) & \vdots \\ \chi(\mathrm{s}(\alpha_{n^2}\alpha_1)) & \ldots & \chi(\mathrm{s}(\alpha_{n^2}\alpha_{n^2})) \end{pmatrix}^{-1} \in \mathrm{GL}_n(R).$$

In particular PGL_n *represents the functor of automorphisms of the full matrix ring.*

Proof: We take an element $\sigma \in \mathcal{PGL}_n(X)$ corresponding to a given element ϕ of h_{PGL_n}. (cf. Lemma 5.3.) There exists an open covering $\{U_i\}_{i\in I}$ of X such that σ is represented by elements σ_i of GL_n on each U_i. Then on each U_i we define an automorphism of $\mathrm{M}_n(\mathcal{O}_X\mid_{U_i})$ as $\mathcal{O}_X\mid_{U_i}$-algebras as corresponding to $\sigma_i^{-1}A\sigma_i$ for each section A of $\mathrm{M}_n(\mathcal{O}_X\mid_{U_i})$. Because of compatibility we obtain an automorphism of the full matrix rings on the total space X by gluing automorphisms on each U_i. This correspondence defines a natural transformation τ from h_{PGL_n} to F. In order to prove that τ is an isomorphism, we introduce the following Theorem without proof.

Theorem 5.5 (Skolem-Noether)**.** *Let R be a local ring. Then for any R-algebra isomorphism $\sigma : \mathrm{M}_n(R) \to \mathrm{M}_n(R)$ there exists an element P of $\mathrm{GL}_n(R)$ such that $\sigma = \mathrm{Ad}(P)$.*

It is easy to see that Skolem-Noether Theorem implies τ is an isomorphism. Therefore we have proved Theorem 5.4. □

Notation 5.6. For any scheme X we have the map $\tau_X : h_{\mathrm{PGL}_n}(X) \to \mathrm{F}(X)$ from the natural transformation τ. For $\sigma \in h_{\mathrm{PGL}_n}(X)$ we denote $\tau_X(\sigma)$ by $\mathrm{Ad}(\sigma)$.

From now on we show that PGL_n acts freely on the a.i.r. part. For $\alpha_1, \alpha_2, \ldots, \alpha_{n^2} \in \Gamma$ we put $\Delta := \Delta(\alpha_1, \alpha_2, \ldots, \alpha_{n^2})$ and $\mathrm{Rep}_n(\Gamma)_\Delta := \mathrm{Spec}\, \mathrm{A}_n(\Gamma)_\Delta$. $\mathrm{Rep}_n(\Gamma)_\Delta$ is the scheme representing the functor

$$\begin{array}{ccc} \textbf{(Sch)} & \to & \textbf{(Sets)} \\ X & \mapsto & \{\rho : \Gamma \to \mathrm{GL}_n(\Gamma(X,\mathcal{O}_X)) \mid \rho \text{ is a representation} \\ & & \quad \text{such that } \Delta(\rho(\alpha_1), \ldots, \rho(\alpha_{n^2})) \in \Gamma(X,\mathcal{O}_X)^\times\}. \end{array}$$

Then the action of PGL_n on $\mathrm{Rep}_n(\Gamma)_\Delta$ can be interpreted as

$$\begin{array}{rccc} \mathrm{Ad}_X \ : & h_{\mathrm{Rep}_n(\Gamma)_\Delta}(X) \times h_{PGL_n}(X) & \to & h_{\mathrm{Rep}_n(\Gamma)_\Delta}(X) \\ & (\rho\ ,\ \sigma) & \mapsto & \mathrm{Ad}(\sigma)(\rho) \end{array}$$

for a scheme X. We remark that $h_{\mathrm{PGL}_n}(X)$ is the set of automorphism of the full matrix ring $\mathrm{M}_n(\Gamma(X,\mathcal{O}_X))$. Then $\mathrm{Ad}(\sigma)(\rho)$ is a representation of Γ in $\mathrm{GL}_n(\Gamma(X,\mathcal{O}_X))$ defined as $(\mathrm{Ad}(\sigma)(\rho))(\gamma) := \mathrm{Ad}(\sigma)(\rho(\gamma))$ for $\gamma \in \Gamma$.

Theorem 5.7. *In the same notation as above,*

$$(p_1, \mathrm{Ad}) : \mathrm{Rep}_n(\Gamma)_\Delta \times_{\mathrm{Spec}\,\mathbf{Z}} \mathrm{PGL}_n \to \mathrm{Rep}_n(\Gamma)_\Delta \times_{\mathrm{Ch}^0_n(\Gamma)_\Delta} \mathrm{Rep}_n(\Gamma)_\Delta$$

is an isomorphism.

5. The action of PGL_n

In this section we study the adjoint action of PGL_n on the representation variety. We show that the action of PGL_n on the absolutely irreducible representation part is free. At first we characterize PGL_n as the functor representing automorphisms of the full matrix ring. This follows from Skolem-Noether Theorem. This fact may be found, for example, in [3]. Secondly we see that the Stabilizer of a point of the a.i.r. part is trivial. The reason why the action is free is that on the a.i.r. part the image of the representation of a group in GL_n generates the full matrix ring. Therefore we will see that there exists the quotient of the a.i.r. part by PGL_n in the next section.

Definition 5.1. We define contravariant functors from the category of schemes to the category of sets as follows:

$$\begin{array}{rccc} h_{\mathrm{PGL}_n} : & \textbf{(Sch)} & \to & \textbf{(Sets)} \\ & X & \mapsto & \mathrm{Hom}(X, \mathrm{PGL}_n) \end{array}$$

$$\begin{array}{rccc} \mathrm{F} : & \textbf{(Sch)} & \to & \textbf{(Sets)} \\ & X & \mapsto & \mathrm{Aut}_{\mathcal{O}_X-\mathrm{alg}}(\mathrm{M}_n(\mathcal{O}_X)). \end{array}$$

Remark: $\mathrm{M}_n(\mathcal{O}_X)$ is a sheaf of $\mathcal{O}_X$-algebras on X defined by corresponding to $\mathrm{M}_n(\mathcal{O}_X(U))$ for each open set U of X. $\mathrm{Aut}_{\mathcal{O}_X-\mathrm{alg}}(\mathrm{M}_n(\mathcal{O}_X))$ is a set of isomorphisms of $\mathrm{M}_n(\mathcal{O}_X)$ as a sheaf of $\mathcal{O}_X$-algebras.

Definition 5.2 (Definition of $\mathcal{PGL}_n$)**.** $\mathcal{PGL}_n$ is a sheaf of groups on a scheme X defined as the sheafification associated to the presheaf corresponding to $\mathrm{GL}_n(\mathcal{O}_X(U))/\mathcal{O}_X(U)^\times$ for each Zariski open set U of X.

We introduce an easy lemma.

Lemma 5.3. *There is a natural isomorphism for any scheme X:*

$$h_{\mathrm{PGL}_n}(X) \cong \mathcal{PGL}_n(X).$$

Proof: For an element $\sigma \in \mathcal{PGL}_n(X)$ there exists an open covering $\{U_i\}_{i\in I}$ of X such that σ is represented by elements of GL_n on each U_i. σ defines a morphism from U_i to GL_n for each i. By composition with the projection from GL_n to PGL_n we get a morphism from X to PGL_n because of compatibility. It is easy that this correspondence defines an isomorphism between $\mathcal{PGL}_n(X)$ and $h_{\mathrm{PGL}_n}(X)$. □

Now we characterize PGL_n as the functor representing automorphisms of full matrix ring.

Theorem 5.4. *There is a natural isomorphism :*

$$\tau : h_{\mathrm{PGL}_n} \to \mathrm{F}.$$

Proof of Step 5 : By definition we have

$$\sigma(e) = (A_1, \ldots, A_{n^2})\, T \begin{pmatrix} \psi(\mathrm{s}(\alpha_1)) \\ \vdots \\ \psi(\mathrm{s}(\alpha_{n^2})) \end{pmatrix}.$$

Using Theorem 3.3, we also obtain

$$I_n = (A_1, \ldots, A_{n^2})\, T \begin{pmatrix} \mathrm{tr}(A_1) \\ \vdots \\ \mathrm{tr}(A_{n^2}) \end{pmatrix}.$$

This implies that the statement is true.

Now we have showed that a map $\sigma : \Gamma \to \mathrm{GL}_n(R)$ is a representation.

(Step 6)

$$\mathrm{tr}(\sigma(\gamma)) = \psi(\mathrm{s}(\gamma)), \text{ for } \forall\, \gamma \in \Gamma.$$

Proof of Step 6 : By Step 2 we have

$$\mathrm{tr}(A_i \sigma(\gamma)) = \psi(\mathrm{s}(\alpha_i \gamma)).$$

Substituting $\alpha_i^{-1}\gamma$ for γ, we can prove Step 6. We can also prove Step 6 by the definition of $\sigma(\gamma)$ and using Corollary 3.5.

(Step 7)

$$\begin{array}{ccc} \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta} & \longrightarrow & \mathrm{A}_n(\Gamma)_{\Delta} \\ & \searrow^{\psi} & \downarrow \tilde{\sigma} \\ & & R \end{array}$$

commutes.

Proof of Step 7 : We show that two ring homomorphisms from $\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta}$ to R coincide. By Corollary 3.10 $\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta}$ is generated by $\mathrm{s}(\gamma)$, $\gamma \in \Gamma$ and $\mathrm{Im}(\phi)$ over $\mathbf{Z}$, where $\phi : \mathrm{A}_n(F_{n^2})^{\mathrm{Ch}}{}_{\Delta} \to \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta}$ is the standard morphism. Step 6 implies that two morphisms coincide for $\mathrm{s}(\gamma)$'s. On the other hand, the fact $\rho(\tilde{\alpha}_i) = \sigma(\alpha_i)$ implies that they coincide for elements of $\mathrm{Im}(\phi)$. Therefore the diagram is commutative.

We have thus proved the theorem. □

Proof of Step 4 : Put

$$C := \begin{pmatrix} \mathrm{tr}(A_1\sigma(\gamma)\sigma(\delta)) \\ \cdots \\ \mathrm{tr}(A_{n^2}\sigma(\gamma)\sigma(\delta)) \end{pmatrix}.$$

By Theorem 3.3

$$\begin{pmatrix} A_1\sigma(\gamma)\sigma(\delta) \\ \vdots \\ A_{n^2}\sigma(\gamma)\sigma(\delta) \end{pmatrix} = \begin{pmatrix} A_1\sigma(\gamma)A_1 & \cdots & A_1\sigma(\gamma)A_{n^2} \\ \vdots & A_i\sigma(\gamma)A_j & \vdots \\ A_{n^2}\sigma(\gamma)A_1 & \cdots & A_{n^2}\sigma(\gamma)A_{n^2} \end{pmatrix} \times T \begin{pmatrix} \psi(\mathrm{s}(\alpha_1\delta)) \\ \vdots \\ \psi(\mathrm{s}(\alpha_{n^2}\delta)) \end{pmatrix}.$$

Taking traces both side and using Step 3, we obtain

$$\begin{aligned} C &= \begin{pmatrix} \mathrm{tr}(A_1\sigma(\gamma)A_1) & \cdots & \mathrm{tr}(A_1\sigma(\gamma)A_{n^2}) \\ \vdots & \mathrm{tr}(A_i\sigma(\gamma)A_j) & \vdots \\ \mathrm{tr}(A_{n^2}\sigma(\gamma)A_1) & \cdots & \mathrm{tr}(A_{n^2}\sigma(\gamma)A_{n^2}) \end{pmatrix} T \begin{pmatrix} \psi(\mathrm{s}(\alpha_1\delta)) \\ \vdots \\ \psi(\mathrm{s}(\alpha_{n^2}\delta)) \end{pmatrix} \\ &= \begin{pmatrix} \psi(\mathrm{s}(\alpha_1\gamma\alpha_1)) & \cdots & \psi(\mathrm{s}(\alpha_1\gamma\alpha_{n^2})) \\ \vdots & \psi(\mathrm{s}(\alpha_i\gamma\alpha_j)) & \vdots \\ \psi(\mathrm{s}(\alpha_{n^2}\gamma\alpha_1)) & \cdots & \psi(\mathrm{s}(\alpha_{n^2}\gamma\alpha_{n^2})) \end{pmatrix} T \begin{pmatrix} \psi(\mathrm{s}(\alpha_1\delta)) \\ \vdots \\ \psi(\mathrm{s}(\alpha_{n^2}\delta)) \end{pmatrix}. \end{aligned}$$

Consider the universal representation of Γ (cf. Corollary 3.5) and we get

$$C = \begin{pmatrix} \psi(\mathrm{s}(\alpha_1\gamma\delta)) \\ \vdots \\ \psi(\mathrm{s}(\alpha_1\gamma\delta)) \end{pmatrix}.$$

By Step 2, we have

$$\begin{pmatrix} \psi(\mathrm{s}(\alpha_1\gamma\delta)) \\ \vdots \\ \psi(\mathrm{s}(\alpha_1\gamma\delta)) \end{pmatrix} = \begin{pmatrix} \mathrm{tr}(A_1\sigma(\gamma\delta)) \\ \vdots \\ \mathrm{tr}(A_{n^2}\sigma(\gamma\delta)) \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} \mathrm{tr}(A_1\sigma(\gamma)\sigma(\delta)) \\ \vdots \\ \mathrm{tr}(A_{n^2}\sigma(\gamma)\sigma(\delta)) \end{pmatrix} = \begin{pmatrix} \mathrm{tr}(A_1\sigma(\gamma\delta)) \\ \vdots \\ \mathrm{tr}(A_{n^2}\sigma(\gamma\delta)) \end{pmatrix}.$$

Since $\Delta(A_1, \ldots, A_{n^2}) \in R^\times$, we conclude that $\sigma(\gamma)\sigma(\delta) = \sigma(\gamma\delta)$.

(Step 5)

$$\sigma(e) = I_n.$$

Then

$$\begin{pmatrix} A_i\sigma(\gamma)A_1 \\ A_i\sigma(\gamma)A_2 \\ \vdots \\ A_i\sigma(\gamma)A_{n^2} \end{pmatrix} = \begin{pmatrix} A_iA_1A_1 & A_iA_2A_1 & \dots & A_iA_{n^2}A_1 \\ A_iA_1A_2 & A_iA_2A_2 & \dots & A_iA_{n^2}A_2 \\ \vdots & \vdots & \dots & \vdots \\ A_iA_1A_{n^2} & A_iA_2A_{n^2} & \dots & A_iA_{n^2}A_{n^2} \end{pmatrix} T \begin{pmatrix} \psi(\mathrm{s}(\alpha_1\gamma)) \\ \psi(\mathrm{s}(\alpha_2\gamma)) \\ \vdots \\ \psi(\mathrm{s}(\alpha_{n^2}\gamma)) \end{pmatrix}.$$

Taking traces of both sides, we obtain

$$\begin{aligned} \begin{pmatrix} \mathrm{tr}(A_i\sigma(\gamma)A_1) \\ \vdots \\ \mathrm{tr}(A_i\sigma(\gamma)A_{n^2}) \end{pmatrix} &= \begin{pmatrix} \mathrm{tr}(A_iA_1A_1) & \dots & \mathrm{tr}(A_iA_{n^2}A_1) \\ \vdots & \dots & \vdots \\ \mathrm{tr}(A_iA_1A_{n^2}) & \dots & \mathrm{tr}(A_iA_{n^2}A_{n^2}) \end{pmatrix} T \begin{pmatrix} \psi(\mathrm{s}(\alpha_1\gamma)) \\ \psi(\mathrm{s}(\alpha_2\gamma)) \\ \vdots \\ \psi(\mathrm{s}(\alpha_{n^2}\gamma)) \end{pmatrix} \\ &= \begin{pmatrix} \psi(\mathrm{s}(\alpha_i\alpha_1\alpha_1)) & \dots & \psi(\mathrm{s}(\alpha_i\alpha_{n^2}\alpha_1)) \\ \vdots & \dots & \vdots \\ \psi(\mathrm{s}(\alpha_i\alpha_1\alpha_{n^2})) & \dots & \psi(\mathrm{s}(\alpha_i\alpha_{n^2}\alpha_{n^2})) \end{pmatrix} \\ &\qquad \times T \begin{pmatrix} \psi(\mathrm{s}(\alpha_1\gamma)) \\ \psi(\mathrm{s}(\alpha_2\gamma)) \\ \vdots \\ \psi(\mathrm{s}(\alpha_{n^2}\gamma)) \end{pmatrix} \\ &= \begin{pmatrix} \psi(\mathrm{s}(\alpha_i\gamma\alpha_1)) \\ \psi(\mathrm{s}(\alpha_i\gamma\alpha_2)) \\ \vdots \\ \psi(\mathrm{s}(\alpha_i\gamma\alpha_{n^2})) \end{pmatrix}. \end{aligned}$$

Remark that the last equality is obtained by considering the universal representation σ_Γ of Γ:

$$\begin{aligned} \begin{pmatrix} \sigma_\Gamma(\alpha_i\gamma\alpha_1) \\ \vdots \\ \sigma_\Gamma(\alpha_i\gamma\alpha_{n^2}) \end{pmatrix} &= \begin{pmatrix} \sigma_\Gamma(\alpha_i\alpha_1\alpha_1) & \dots & \sigma_\Gamma(\alpha_i\alpha_{n^2}\alpha_1) \\ \vdots & \dots & \vdots \\ \sigma_\Gamma(\alpha_i\alpha_1\alpha_{n^2}) & \dots & \sigma_\Gamma(\alpha_i\alpha_{n^2}\alpha_{n^2}) \end{pmatrix} \\ &\qquad \times (\mathrm{s}(\alpha_i\alpha_j))^{-1}_{1\le i,j,\le n^2} \begin{pmatrix} \mathrm{s}(\alpha_1\gamma) \\ \mathrm{s}(\alpha_2\gamma) \\ \vdots \\ \mathrm{s}(\alpha_{n^2}\gamma) \end{pmatrix}. \end{aligned}$$

Taking traces of both sides and applying ψ, we obtain the last equality. This completes the proof of Step 3.

(Step 4)

$$\sigma(\gamma)\sigma(\delta) = \sigma(\gamma\delta), \quad \text{for } \forall\, \gamma, \delta \in \Gamma.$$

This implies the uniqueness.

Next we prove that there exists $\sigma : \Gamma \to \mathrm{GL}_n(R)$ such that the diagram is commutative. We define σ as

$$\sigma(\gamma) = (A_1, \ldots, A_{n^2})\, T \begin{pmatrix} \psi(\mathrm{s}(\alpha_1\gamma)) \\ \vdots \\ \psi(\mathrm{s}(\alpha_{n^2}\gamma)) \end{pmatrix},$$

for $\gamma \in \Gamma$, where A_i and T is defined as before. We will prove that σ as above is what we want.

(Step 1)

$$(\sigma(\alpha_1), \ldots, \sigma(\alpha_{n^2})) = (A_1, \ldots, A_{n^2}).$$

Proof of Step 1:

$$\begin{aligned} L.H.S &= (A_1, \ldots, A_{n^2})\, T \begin{pmatrix} \psi(\mathrm{s}(\alpha_1\alpha_1)) & \cdots & \psi(\mathrm{s}(\alpha_1\alpha_{n^2})) \\ \vdots & \psi(\mathrm{s}(\alpha_i\alpha_j)) & \vdots \\ \psi(\mathrm{s}(\alpha_{n^2}\alpha_1)) & \cdots & \psi(\mathrm{s}(\alpha_{n^2}\alpha_{n^2})) \end{pmatrix} \\ &= (A_1, \ldots, A_{n^2}). \end{aligned}$$

(Step 2)

$$\begin{pmatrix} \mathrm{tr}(A_1\sigma(\gamma)) \\ \vdots \\ \mathrm{tr}(A_{n^2}\sigma(\gamma)) \end{pmatrix} = \begin{pmatrix} \psi(\mathrm{s}(\alpha_1\gamma)) \\ \vdots \\ \psi(\mathrm{s}(\alpha_{n^2}\gamma)) \end{pmatrix} \text{ for } \forall\, \gamma \in \Gamma.$$

Proof of Step 2 :

$$\begin{aligned} L.H.S &= \begin{pmatrix} \mathrm{tr}(A_1A_1) & \cdots & \mathrm{tr}(A_1A_{n^2}) \\ \vdots & \mathrm{tr}(A_iA_j) & \vdots \\ \mathrm{tr}(A_{n^2}A_1) & \cdots & \mathrm{tr}(A_{n^2}A_{n^2}) \end{pmatrix} T \begin{pmatrix} \psi(\mathrm{s}(\alpha_1\gamma)) \\ \vdots \\ \psi(\mathrm{s}(\alpha_{n^2}\gamma)) \end{pmatrix} \\ &= \begin{pmatrix} \psi(\mathrm{s}(\alpha_1\gamma)) \\ \vdots \\ \psi(\mathrm{s}(\alpha_{n^2}\gamma)) \end{pmatrix} = R.H.S. \end{aligned}$$

(Step 3)

$$\mathrm{tr}(A_i\sigma(\gamma)A_j) = \psi(\mathrm{s}(\alpha_i\gamma\alpha_j)),\ \forall\, \gamma \in \Gamma,\ 1 \le i, j \le n^2.$$

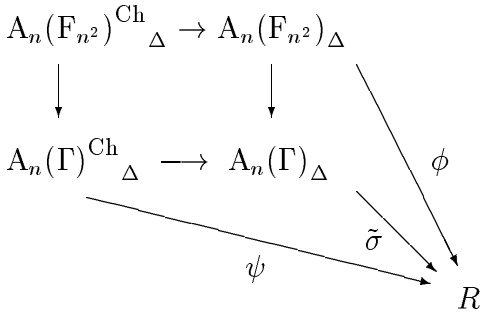
Proof of Step 3 : Multiply both sides of the next equation by A_i from left and by $A_1, \ldots, A_{n^2}$ from right.

$$\sigma(\gamma) = (A_1, \ldots, A_{n^2})\, T \begin{pmatrix} \psi(\mathrm{s}(\alpha_1\gamma)) \\ \vdots \\ \psi(\mathrm{s}(\alpha_{n^2}\gamma)) \end{pmatrix}.$$

More precisely, for any commutative diagram :

$$\begin{array}{ccc} \mathrm{A}_n(\mathrm{F}_{n^2})^{\mathrm{Ch}}{}_{\Delta} & \longrightarrow & \mathrm{A}_n(\mathrm{F}_{n^2})_{\Delta} \\ \downarrow & & \downarrow \phi \\ \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta} & \xrightarrow[\psi]{} & R \end{array}$$

there exists a unique ring homomorphism $\tilde{\sigma} : \mathrm{A}_n(\Gamma)_{\Delta} \to R$ *such that the following diagram is commutative.*



Suppose that $\rho : \mathrm{F}_{n^2} \to \mathrm{GL}_n(R)$ *is the representation corresponding to* $\phi : \mathrm{A}_n(\mathrm{F}_{n^2})_{\Delta} \to R$. *Then the representation* $\sigma : \Gamma \to \mathrm{GL}_n(R)$ *corresponding to* $\tilde{\sigma}$ *is given by*

$$\sigma(\gamma) = (A_1, \ldots, A_{n^2})\, T \begin{pmatrix} \psi(\mathrm{s}(\alpha_1\gamma)) \\ \vdots \\ \psi(\mathrm{s}(\alpha_{n^2}\gamma)) \end{pmatrix}$$

for $\gamma \in \Gamma$, *where* $A_i := \rho(\tilde{\alpha}_i)$, $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{n^2}$ *is a free generator of* F_{n^2}, *and* $T := (\psi(\mathrm{s}(\alpha_i\alpha_j)))^{-1}_{1\le i,j\le n^2}$.

Proof: It suffices to prove the latter part of the statement. At first we prove the uniqueness of σ or $\tilde{\sigma}$. By Theorem 3.3 we have

$$\sigma(\gamma) = (\sigma(\alpha_1), \ldots, \sigma(\alpha_{n^2}))\, \tilde{T} \begin{pmatrix} \mathrm{tr}(\sigma(\alpha_1\gamma)) \\ \vdots \\ \mathrm{tr}(\sigma(\alpha_{n^2}\gamma)) \end{pmatrix},$$

where $\tilde{T} := (\mathrm{tr}(\sigma(\alpha_i\alpha_j)))^{-1}_{1\le i,j\le n^2}$. By commutativity, we obtain $\sigma(\alpha_i) = \rho(\tilde{\alpha}_i) = A_i, \tilde{T} = (\psi(\mathrm{s}(\alpha_i\alpha_j)))^{-1}_{1\le i,j\le n^2} =: T$, and $\mathrm{tr}(\sigma(\alpha_i\gamma)) = \psi(\mathrm{s}(\alpha_i\gamma))$. Therefore

$$\sigma(\gamma) = (A_1, \ldots, A_{n^2})\, T \begin{pmatrix} \psi(\mathrm{s}(\alpha_1\gamma)) \\ \vdots \\ \psi(\mathrm{s}(\alpha_{n^2}\gamma)) \end{pmatrix}.$$

Proof: There exists a surjective group homomorphism $\phi : \mathrm{F}_m \to \Gamma$. Take $\tilde{\alpha}_i \in \mathrm{F}_m$ such that $\phi(\tilde{\alpha}_i) = \alpha_i$ for $1 \leq i \leq n^2$. Then ϕ induces a surjective homomorphism $\mathrm{A}_n(\mathrm{F}_m)^{\mathrm{Ch}}{}_{\Delta(\tilde{\alpha}_1,\dots,\tilde{\alpha}_{n^2})} \to \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta}$. The statement follows from Theorem 3.8. □

The following corollary will be used in the next section.

Corollary 3.10. *Let Γ be a group. For $\alpha_1, \alpha_2, \dots, \alpha_{n^2} \in \Gamma$ we put $\Delta := \Delta(\alpha_1, \alpha_2, \dots, \alpha_{n^2}) \in \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}$. Assume that $\phi : \mathrm{A}_n(\mathrm{F}_{n^2})^{\mathrm{Ch}}{}_{\Delta} \to \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta}$ be the standard morphism with respect to $\alpha_1, \alpha_2, \dots, \alpha_{n^2}$. Then $\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta}$ is generated by $\{\mathrm{s}(\gamma) \mid \gamma \in \Gamma\}$ and $\mathrm{Im}\phi$ as a $\mathbf{Z}$-algebra.*

Proof: It suffices to show that any $\mathrm{c}_i(\gamma)$ is generated by $\{\mathrm{s}(\gamma) \mid \gamma \in \Gamma\}$ and $\mathrm{Im}\phi$. Put $f_i(x_1, \dots, x_{n^2}) := \mathrm{c}_i(x_1\sigma_{\mathrm{F}_{n^2}}(\tilde{\alpha}_1) + \cdots + x_{n^2}\sigma_{\mathrm{F}_{n^2}}(\tilde{\alpha}_{n^2})) \in \mathrm{A}_n(\mathrm{F}_{n^2})^{\mathrm{Ch}}{}_{\Delta}[x_1, \dots, x_{n^2}]$, where $\tilde{\alpha}_1, \dots, \tilde{\alpha}_{n^2}$ is a free generator of F_{n^2}. Sending coefficients of f_i by ϕ, we have $f_i^{\phi}(x_1, \dots, x_{n^2}) = \mathrm{c}_i(x_1\sigma_\Gamma(\alpha_1) + \cdots + x_{n^2}\sigma_\Gamma(\alpha_{n^2})) \in \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta}[x_1, \dots, x_{n^2}]$. By Corollary 3.5 we have

$$\sigma_\Gamma(\gamma) = (\sigma_\Gamma(\alpha_1), \dots, \sigma_\Gamma(\alpha_{n^2}))\, T \begin{pmatrix} \mathrm{s}(\alpha_1\gamma) \\ \vdots \\ \mathrm{s}(\alpha_{n^2}\gamma) \end{pmatrix} \text{ for } \forall \gamma \in \Gamma,$$

Therefore any $\mathrm{c}_i(\gamma)$ is generated by $\{\mathrm{s}(\gamma) \mid \gamma \in \Gamma\}$, coefficients of f_i^{ϕ} and entries of T. However coefficients of f_i^{ϕ} and entries of T is contained in $\mathrm{Im}\phi$, so any $\mathrm{c}_i(\gamma)$ is generated by $\{\mathrm{s}(\gamma) \mid \gamma \in \Gamma\}$ and $\mathrm{Im}\phi$. □

4. Reconstruction of Representations

In this section we reconstruct representations from invariants and n^2 matrices. It follows from Reconstruction of Matrices. We prove that for any group the quotient morphism of the representation variety is obtained by base change of the standard morphism. This result allows us to reduce arbitrary cases to free groups cases.

Theorem 4.1 (Reconstruction of Representations)**.** *For $\alpha_1, \alpha_2, \dots, \alpha_{n^2} \in \Gamma$ we put $\Delta := \Delta(\alpha_1, \alpha_2, \dots, \alpha_{n^2}) \in \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}$. Assume that $\phi : \mathrm{A}_n(\mathrm{F}_{n^2})^{\mathrm{Ch}}{}_{\Delta} \to \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta}$ be the standard morphism with respect to $\alpha_1, \alpha_2, \dots, \alpha_{n^2}$. Then ϕ induces the following diagram which is a fiber product.*

$$\begin{array}{ccc} \mathrm{Spec}\,\mathrm{A}_n(\mathrm{F}_{n^2})_{\Delta} & \longrightarrow & \mathrm{Spec}\,\mathrm{A}_n(\mathrm{F}_{n^2})^{\mathrm{Ch}}{}_{\Delta} \\ \uparrow & & \uparrow \\ \mathrm{Spec}\,\mathrm{A}_n(\Gamma)_{\Delta} & \longrightarrow & \mathrm{Spec}\,\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta} \end{array}$$

Proof: The i-th invariant of the matrix $x_1\sigma_{\mathrm{F}_m}(\beta_1)+\cdots+x_l\sigma_{\mathrm{F}_m}(\beta_l) \in \mathrm{M}_n(\mathrm{A}_n(\mathrm{F}_m)[x_1,\ldots,x_l])$ is PGL_n invariant. Therefore the lemma follows from Donkin's Theorem (Theorem 1.10) and Lemma 3.6. $\square$

Theorem 3.8. *Let F_m be the free group of rank m. Suppose that $\alpha_1,\alpha_2,\ldots,\alpha_{n^2} \in \mathrm{F}_m$ and put $\Delta := \Delta(\alpha_1,\alpha_2,\ldots,\alpha_{n^2}) \in \mathrm{A}_n(\mathrm{F}_m)^{\mathrm{Ch}}$. Then $\mathrm{A}_n(\mathrm{F}_m)^{\mathrm{Ch}}{}_{\Delta}$ is a finitely generated algebra over $\mathbf{Z}$.*

Proof: Put $f_i(x_1,\ldots,x_{n^2}) := \mathrm{c}_i(x_1\sigma_{\mathrm{F}_m}(\alpha_1)+\cdots+x_{n^2}\sigma_{\mathrm{F}_m}(\alpha_{n^2}))$. Then $f_i \in \mathrm{A}_n(\mathrm{F}_m)^{\mathrm{PGL}_n}[x_1,\ldots,x_{n^2}]$ by the previous lemma. By Corollary 3.5 we have

$$\sigma_{\mathrm{F}_m}(\gamma) = (\sigma_{\mathrm{F}_m}(\alpha_1),\ldots,\sigma_{\mathrm{F}_m}(\alpha_{n^2}))\ T \begin{pmatrix} \mathrm{s}(\alpha_1\gamma) \\ \vdots \\ \mathrm{s}(\alpha_{n^2}\gamma) \end{pmatrix} \quad for\ \forall\gamma \in \Gamma,$$

Therefore there exist n^2 polynomials $y_1,\ldots,y_{n^2}$ in entries of T and $\{\mathrm{s}(\delta) \mid \delta \in \mathrm{F}_m\}$ such that

$$\mathrm{c}_i(\gamma) = \mathrm{c}_i(\sigma_{\mathrm{F}_m}(\gamma)) = f_i(y_1,\ldots,y_{n^2}).$$

We have thus showed that $\mathrm{A}_n(\mathrm{F}_m)^{\mathrm{Ch}}{}_{\Delta}$ is generated by entries of T, coefficients of f_i $(1 \le i \le n)$ and $\{\mathrm{s}(\delta) \mid \delta \in \mathrm{F}_m\}$ over $\mathbf{Z}$. It suffices to prove that $\{\mathrm{s}(\delta) \mid \delta \in \mathrm{F}_m\}$ is finitely generated over $\mathbf{Z}$.

We take a free generator $\beta_1,\ldots,\beta_m$ of F_m. For $\gamma \in \mathrm{F}_m$ we define the length of γ with respect to $\beta_1,\ldots,\beta_m$ as the shortest number of β_i or β_i^{-1} which express γ. Put $m := \max\{\ \mathrm{length}(\alpha_i) \mid 1 \le i \le n^2\} + 1$. We claim that $\gamma \in \mathrm{F}_m$ of length $\ge 2m$ is generated by entries of T and $\{\mathrm{s}(\delta) \mid \mathrm{length}(\delta) < 2m\}$. Theorem follows from this claim.

Assume that γ is of length $\ge 2m$. Then we can choose $\gamma_1,\gamma_2 \in \mathrm{F}_m$ such that $\gamma = \gamma_1\gamma_2$ and $\mathrm{length}(\gamma_1) \ge m, \mathrm{length}(\gamma_2) \ge m, \mathrm{length}(\gamma_1) + \mathrm{length}(\gamma_2) = \mathrm{length}(\gamma)$. By Corollary 3.5 we have

$$\mathrm{s}(\gamma) = \mathrm{s}(\gamma_1\gamma_2) = (\mathrm{s}(\alpha_1\gamma_1),\ldots,\mathrm{s}(\alpha_{n^2}\gamma_1))\ T \begin{pmatrix} \mathrm{s}(\alpha_1\gamma_2) \\ \vdots \\ \mathrm{s}(\alpha_{n^2}\gamma_2) \end{pmatrix}$$

Since $\mathrm{length}(\alpha_i\gamma_1) < \mathrm{length}(\gamma)$ and $\mathrm{length}(\alpha_i\gamma_2) < \mathrm{length}(\gamma)$ hold, we can show that $\mathrm{s}(\gamma)$ is generated by entries of T and $\{\mathrm{s}(\delta) \mid \mathrm{length}(\delta) < 2m\}$ by induction. This completes the proof. $\square$

Corollary 3.9. *Let Γ be a finitely generated group. Assume that $\alpha_1,\alpha_2,\ldots,\alpha_{n^2} \in \Gamma$ and that $\Delta := \Delta(\alpha_1,\alpha_2,\ldots,\alpha_{n^2}) \in \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}$. Then $\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta}$ is a finitely generated algebra over $\mathbf{Z}$.*

$\mathrm{GL}_n(\mathrm{A}_n(\Gamma)) \to \mathrm{GL}_n(\mathrm{A}_n(\Gamma)_\Delta)$ *is the universal representation. Then for any* $\gamma \in \Gamma$ *we have*

$$\sigma_\Gamma(\gamma) = (\sigma_\Gamma(\alpha_1), \dots, \sigma_\Gamma(\alpha_{n^2}))\ T \begin{pmatrix} \mathrm{s}(\alpha_1\gamma) \\ \vdots \\ \mathrm{s}(\alpha_{n^2}\gamma) \end{pmatrix} \in \mathrm{GL}_n(\mathrm{A}_n(\Gamma)_\Delta),$$

where $T = (\mathrm{s}(\alpha_i\alpha_j))^{-1}_{1\le i,j,\le n^2}$. *Moreover*

$$\mathrm{s}(\gamma) = (\mathrm{s}(\alpha_1), \dots, \mathrm{s}(\alpha_{n^2}))\ T \begin{pmatrix} \mathrm{s}(\alpha_1\gamma) \\ \vdots \\ \mathrm{s}(\alpha_{n^2}\gamma) \end{pmatrix} \ for\ \forall\gamma \in \Gamma.$$

$$\mathrm{s}(\gamma\delta) = (\mathrm{s}(\alpha_1\gamma), \dots, \mathrm{s}(\alpha_{n^2}\gamma))\ T \begin{pmatrix} \mathrm{s}(\alpha_1\delta) \\ \vdots \\ \mathrm{s}(\alpha_{n^2}\delta) \end{pmatrix} \ for\ \forall\gamma, \delta \in \Gamma.$$

The last two equations hold in $\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_\Delta$.

Proof: Using Theorem 3.3 and Corollary 3.4, we can prove the statement. □

We end this section by showing that $\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_\Delta$ is finitely generated over $\mathbf{Z}$ if Γ is a finitely generated group.

At first we define the natural action of PGL_n on the polynomial rings over the representation ring.

$$\begin{array}{rccc} \mathrm{Ad} : & \mathrm{A}_n(\Gamma)[x_1, \dots, x_m] & \to & \mathrm{A}_n(\Gamma)[x_1, \dots, x_m] \otimes_{\mathbf{Z}} \mathrm{A}(\mathrm{PGL}_n) \\ & \sigma_\Gamma(\gamma) & \mapsto & P^{-1}\sigma_\Gamma(\gamma)P \\ & x_j & \mapsto & x_j. \end{array}$$

We define the invariant ring in the same way as Definition 1.9.

$$(\mathrm{A}_n(\Gamma)[x_1, \dots, x_m])^{\mathrm{PGL}_n} := \{x \in \mathrm{A}_n(\Gamma)[x_1, \dots, x_m] \mid \mathrm{Ad}(x) = x \otimes 1\}.$$

The next lemma is obvious.

Lemma 3.6. *In the above situation, we have*

$$(\mathrm{A}_n(\Gamma)[x_1, \dots, x_m])^{\mathrm{PGL}_n} = \mathrm{A}_n(\Gamma)^{\mathrm{PGL}_n}[x_1, \dots, x_m].$$

Lemma 3.7. *Let* F_m *be the free group of rank* m. *We denote the universal representation for* F_m *by* σ_{F_m}. *Suppose that* $\beta_1, \dots, \beta_l \in \mathrm{F}_m$. *Then for* $1 \le i \le n$ *we have*

$$\mathrm{c}_i(x_1\sigma_{\mathrm{F}_m}(\beta_1) + \cdots + x_l\sigma_{\mathrm{F}_m}(\beta_l)) \in \mathrm{A}_n(\mathrm{F}_m)^{\mathrm{PGL}_n}[x_1, \dots, x_l].$$

In particular they can be expressed by a polynomial with coefficients in $\{c_k(\gamma) \mid 1 \le k \le n, \gamma \in \mathrm{F}_m.\}$

Theorem 3.3 (Reconstruction of Matrices)**.** *Suppose that* $A_1, A_2, \ldots, A_{n^2} \in \mathrm{M}_n(R)$ *and* $\Delta(A_1, A_2, \ldots, A_{n^2}) \in R^\times$*. Then for any* $X \in \mathrm{M}_n(R)$ *we have*

$$X = (A_1, A_2, \ldots, A_{n^2})\, T \begin{pmatrix} \mathrm{tr}(A_1X) \\ \mathrm{tr}(A_2X) \\ \vdots \\ \mathrm{tr}(A_{n^2}X) \end{pmatrix},$$

where

$$T = \begin{pmatrix} \mathrm{tr}(A_1A_1) & \mathrm{tr}(A_1A_2) & \ldots & \mathrm{tr}(A_1A_{n^2}) \\ \mathrm{tr}(A_2A_1) & \mathrm{tr}(A_2A_{n^2}) & \ldots & \mathrm{tr}(A_2A_{n^2}) \\ \vdots & \vdots & \cdots & \vdots \\ \mathrm{tr}(A_{n^2}A_1) & \mathrm{tr}(A_{n^2}A_2) & \ldots & \mathrm{tr}(A_{n^2}A_{n^2}) \end{pmatrix}^{-1} \in \mathrm{GL}_{n^2}(R).$$

Proof: Put $S := (\mathrm{tr}(A_iA_j))_{1 \le i,j \le n^2}$. There exists the inverse matrix $T := S^{-1}$, since $\det(S) = (-1)^{\frac{n(n-1)}{2}} \Delta(A_1, \ldots, A_{n^2})^2 \in R^\times$ by lemma 3.1. We use f, g in the same sense as Lemma 3.2. Let $h := g \circ f$. With respect to the basis $\{A_1, \ldots, A_{n^2}\}$ the matrix expression of h is S and that of h^{-1} is T. For any $X \in \mathrm{M}_n(R)$, we have

$$\begin{aligned} X &= h^{-1} \circ h(X) \\ &= h^{-1}(\mathrm{tr}(A_1X)A_1 + \cdots + \mathrm{tr}(A_{n^2}X)A_{n^2}) \\ &= R.H.S. \end{aligned}$$

This completes the proof. □

Corollary 3.4. *In the above situation, assume that* $X, Y \in \mathrm{M}_n(R)$*. Then*

$$\mathrm{tr}(XY) = (\mathrm{tr}(A_1X), \ldots, \mathrm{tr}(A_{n^2}X))\, T \begin{pmatrix} \mathrm{tr}(A_1Y) \\ \vdots \\ \mathrm{tr}(A_{n^2}Y) \end{pmatrix}.$$

Proof: Using Theorem 3.3, we have

$$Y = (A_1, \ldots, A_{n^2})\, T \begin{pmatrix} \mathrm{tr}(A_1Y) \\ \vdots \\ \mathrm{tr}(A_{n^2}Y) \end{pmatrix}.$$

Multiply X from left and take traces of both sides. We have thus proved the statement. □

Corollary 3.5. *Let* Γ *be a group and let* $\alpha_1, \alpha_2, \ldots, \alpha_{n^2}$ *be elements of* Γ*. We denote* $\Delta(\alpha_1, \alpha_2, \ldots, \alpha_{n^2})$ *by* Δ*. Suppose that* $\sigma_\Gamma : \Gamma \to$

Lemma 3.1. *Assume that* $A_1, A_2, \ldots, A_{n^2} \in \mathrm{M}_n(R)$. *Then*

$$\det \begin{pmatrix} \mathrm{tr}(A_1A_1) & \mathrm{tr}(A_1A_2) & \ldots & \mathrm{tr}(A_1A_{n^2}) \\ \mathrm{tr}(A_2A_1) & \mathrm{tr}(A_2A_2) & \ldots & \mathrm{tr}(A_2A_{n^2}) \\ \vdots & \vdots & \cdots & \vdots \\ \mathrm{tr}(A_{n^2}A_1) & \mathrm{tr}(A_{n^2}A_2) & \ldots & \mathrm{tr}(A_{n^2}A_{n^2}) \end{pmatrix}$$

$$= (-1)^{\frac{n(n-1)}{2}} \Delta(A_1, A_2, \cdots, A_{n^2})^2.$$

Proof: We put $A_k = (a_{ij}^{(k)})_{1 \leq i,j \leq n}$. Taking the determinants of both sides of the following equations, we can easily obtain the statement.

$$\begin{pmatrix} \mathrm{tr}(A_1A_1) & \mathrm{tr}(A_1A_2) & \ldots & \mathrm{tr}(A_1A_{n^2}) \\ \mathrm{tr}(A_2A_1) & \mathrm{tr}(A_2A_2) & \ldots & \mathrm{tr}(A_2A_{n^2}) \\ \vdots & \vdots & \cdots & \vdots \\ \mathrm{tr}(A_{n^2}A_1) & \mathrm{tr}(A_{n^2}A_2) & \ldots & \mathrm{tr}(A_{n^2}A_{n^2}) \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \ldots & a_{nn}^{(1)} \\ a_{11}^{(2)} & a_{12}^{(2)} & \ldots & a_{nn}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ a_{11}^{(n^2)} & a_{12}^{(n^2)} & \ldots & a_{nn}^{(n^2)} \end{pmatrix} \begin{pmatrix} a_{11}^{(1)} & a_{11}^{(2)} & \ldots & a_{11}^{(n^2)} \\ a_{21}^{(1)} & a_{21}^{(2)} & \ldots & a_{21}^{(n^2)} \\ \vdots & \vdots & \cdots & \vdots \\ a_{nn}^{(1)} & a_{nn}^{(2)} & \ldots & a_{nn}^{(n^2)} \end{pmatrix}.$$

□

Remark: $\det(\mathrm{tr}(A_iA_j))_{1 \leq i,j \leq n^2}$ may be more suitable to be called the discriminant than $\Delta(A_1, A_2, \ldots, A_{n^2})$. However in the present paper we define the discriminant as $\Delta(A_1, A_2, \ldots, A_{n^2})$.

Next we prepare the following lemma for Reconstruction of Matrices.

Lemma 3.2. *Assume that* $A_1, A_2, \ldots, A_{n^2} \in \mathrm{M}_n(R)$. *We consider the following linear map* f, g.

$$\begin{array}{rccc} f: & \mathrm{M}_n(R) & \to & R^{n^2} \\ & C & \mapsto & (\mathrm{tr}(A_1C), \ldots, \mathrm{tr}(A_{n^2}C)) \end{array}$$

$$\begin{array}{rccc} g: & R^{n^2} & \to & \mathrm{M}_n(R) \\ & (\alpha_1, \ldots, \alpha_{n^2}) & \mapsto & \alpha_1A_1 + \cdots + \alpha_{n^2}A_{n^2}. \end{array}$$

Then $\det(g \circ f) = (-1)^{\frac{n(n-1)}{2}} \Delta(A_1, \ldots, A_{n^2})^2$.

Proof: Easy. □

Now we introduce Reconstruction of Matrices.

subspace is either $\{0\}$ or some $\oplus_{j=1}^{i} k \cdot w_j$ $(i = 1, 2, \ldots, n)$. Therefore $\rho : \mathrm{F}_2 \to \mathrm{GL}_n(k)$ is irreducible. Moreover this is true for any extension field of k. We have thus proved the proposition. □

Corollary 2.12. *Let k be a field. For any positive integer n and the free group F_m of rank $m \geq 2$, there exists an absolutely irreducible representation $\rho : \mathrm{F}_m \to \mathrm{GL}_n(k)$.*

Proof: We get an absolutely irreducible representation by composing a surjective morphism $\mathrm{F}_m \twoheadrightarrow \mathrm{F}_2$ and an absolutely irreducible representation $\mathrm{F}_2 \to \mathrm{GL}_n(k)$ given in Prop 2.11. □

We have proved that the absolutely irreducible representation part is nonempty for the case of the free group of rank $m \geq 2$.

Corollary 2.13. *Let k be a field. For $m \geq 2$ there exist n^2 elements $\alpha_1, \alpha_2, \ldots, \alpha_{n^2} \in \mathrm{F}_m$ such that $\Delta(\alpha_1, \alpha_2, \ldots, \alpha_{n^2}) \neq 0 \in \mathrm{A}_n(\mathrm{F}_m) \otimes_{\mathbf{Z}} k$. In particular $\mathrm{Rep}_n(\mathrm{F}_m)_{\mathrm{a.i.r.}}$ is nonempty.*

In Definition 2.9 we defined the a.i.r. part of the representation variety. Now we define the a.i.r. part of $\mathrm{Ch}_n(\Gamma)$ and $\mathrm{Ch}_n^0(\Gamma)$.

Definition 2.14. We define the a.i.r. part in the following way.

$$\mathrm{Ch}_n(\Gamma)_{\mathrm{a.i.r.}} := \bigcup_{\alpha_1, \alpha_2, \ldots, \alpha_{n^2} \in \Gamma} \mathrm{Spec}\, \mathrm{A}_n(\Gamma)^{\mathrm{PGL}_n}{}_{\Delta(\alpha_1, \alpha_2, \ldots, \alpha_{n^2})}$$

$$\mathrm{Ch}_n^0(\Gamma)_{\mathrm{a.i.r.}} := \bigcup_{\alpha_1, \alpha_2, \ldots, \alpha_{n^2} \in \Gamma} \mathrm{Spec}\, \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_{\Delta(\alpha_1, \alpha_2, \ldots, \alpha_{n^2})}.$$

There exists a natural morphism.

$$\mathrm{Rep}_n(\Gamma)_{\mathrm{a.i.r.}} \to \mathrm{Ch}_n(\Gamma)_{\mathrm{a.i.r.}} \to \mathrm{Ch}_n^0(\Gamma)_{\mathrm{a.i.r.}}.$$

3. Reconstruction of Matrices

In this section we reconstruct matrices from their invariants as traces, determinants, and so on. We have many applications of this reconstruction. For example, we can prove that the localization of $\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}$ by a discriminant is a finitely generated algebra over $\mathbf{Z}$ if Γ is a finitely generated group. The proof of this fact in the present paper has the advantage that generators of the algebra can be taken explicitly.

We begin with an easy lemma.

Proposition 2.10. *Under the same notation as above,*

$$\mathrm{Rep}_n(\Gamma)_{\mathrm{a.i.r.}} = \bigcup_{\alpha_1,\alpha_2,\dots,\alpha_{n^2}\in\Gamma} \mathrm{Spec}\, \mathrm{A}_n(\Gamma)_{\Delta(\alpha_1,\alpha_2,\dots,\alpha_{n^2})}.$$

Proof: We now use Proposition 2.3.

$\sigma_\wp : \Gamma \to GL_n(k(\wp))$ is absolutely irreducible
$\Leftrightarrow$ $\mathrm{M}_n(k(\wp))$ is generated by $\{\sigma_\wp(\gamma) \mid \gamma \in \Gamma\}$ as a $k(\wp)$ vector space
$\Leftrightarrow$ $\exists \alpha_1, \alpha_2, \dots, \alpha_{n^2} \in \Gamma$ such that $\{\sigma_\wp(\alpha_1), \dots, \sigma_\wp(\alpha_{n^2})\}$ is a basis
$\Leftrightarrow$ $\exists \alpha_1, \alpha_2, \dots, \alpha_{n^2} \in \Gamma$ such that $\Delta(\alpha_1, \alpha_2, \dots, \alpha_{n^2}) \neq 0$ *in* $k(\wp)$
$\Leftrightarrow$ $\exists \alpha_1, \alpha_2, \dots, \alpha_{n^2} \in \Gamma$ such that $\Delta(\alpha_1, \alpha_2, \dots, \alpha_{n^2}) \notin \wp$.

This completes the proof of this proposition. □

The previous proposition allows us to see that the absolutely irreducible representation part is a Zariski open set of the representation variety. Next we will see that the absolutely irreducible representation part is a non-empty open set for the case of free groups of rank ≥ 2.

Proposition 2.11. *Let k be a field. For any positive integer n there exists an absolutely irreducible representation $\rho : \mathrm{F}_2 \to \mathrm{GL}_n(k)$.*

Proof: Let α, β be a free generator of F_2. $\rho(\alpha)$ is defined by

$$\rho(\alpha) := \begin{pmatrix} c & 1 & 0 & \cdots & 0 \\ 0 & c & 1 & \cdots & 0 \\ 0 & 0 & c & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}, \quad \text{where } c \neq 0,\ c \in k.$$

We define $\rho(\beta)$ in the following:
Let $V := k^{\oplus n}$ be a n-dimensional vector space over k and $\mathrm{GL}_n(k)$ act naturally on V. Take a basis $\{v_1, v_2, \dots, v_n\}$ of V such that

$$\begin{aligned} \rho(\alpha)v_1 &= cv_1 \\ \rho(\alpha)v_i &= cv_i + v_{i-1} \quad (i = 2, \dots, n). \end{aligned}$$

Put $V_i := \oplus_{j=1}^{i} k \cdot v_j$. Then any $\rho(\alpha)$ invariant subspace of V is either $\{0\}$ or some V_i $(i = 1, 2, \dots, n)$. Now we choose a base $\{w_1, w_2, \dots, w_n\}$ of V such that $w_1 \notin V_{n-1}$. With respect to the basis $\{w_1, w_2, \dots, w_n\}$ $\rho(\beta)$ is defined by

$$\begin{aligned} \rho(\beta)w_1 &= c'w_1 \\ \rho(\beta)w_i &= c'w_i + w_{i-1} \qquad (i = 2, \dots, n), \\ &\quad \text{where } c' \neq 0,\ c' \in k. \end{aligned}$$

Then V has no nontrivial subspace which is invariant under $\rho(\alpha)$ and $\rho(\beta)$, since w_1 is not contained in V_{n-1} and any $\rho(\beta)$ invariant

sending entries of D by $\mathrm{Ad} : \mathrm{A}_n(\mathrm{F}_{n^2}) \to \mathrm{A}_n(\mathrm{F}_{n^2}) \otimes_{\mathbf{Z}} \mathrm{A}(\mathrm{PGL}_n)$. Then $\mathrm{Ad}(D) = D^t C$ holds.

$$\begin{aligned} \det(\mathrm{Ad}(D)) &= \det(D^t C) \\ &= \det(D)\det(C) \\ &= \det(D)\det(P)^n \det(P^{-1})^n \\ &= \det(D) \\ &= \Delta. \end{aligned}$$

Therefore we get $\mathrm{Ad}(\Delta) = \Delta$. This completes the proof of this proposition. □

The natural morphism is induced from the above proposition.

$$\phi_\Delta : \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_\Delta \to \mathrm{A}_n(\Gamma)_\Delta$$

$$\pi_\Delta : \mathrm{Rep}_n(\Gamma)_\Delta \to \mathrm{Ch}^0_n(\Gamma)_\Delta,$$

where we put

$$\mathrm{Rep}_n(\Gamma)_\Delta := \mathrm{Spec}\, \mathrm{A}_n(\Gamma)_\Delta$$

$$\mathrm{Ch}^0_n(\Gamma)_\Delta := \mathrm{Spec}\, \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}{}_\Delta.$$

From now on we explain what the discriminant means. The absolutely irreducible representation part of the representations variety is described by discriminants.

For $\wp \in \mathrm{Spec}\, \mathrm{A}_n(\Gamma)$, we denote the residue field of $\wp$ by $k(\wp)$. Then the representations $\sigma_\wp$ is given by composing of the universal representation $\sigma_\Gamma : \Gamma \to \mathrm{GL}_n(\mathrm{A}_n(\Gamma))$ and the natural group hom $\mathrm{GL}_n(R) \to \mathrm{GL}_n(k(\wp))$.

$$\sigma_\wp : \Gamma \to \mathrm{GL}_n(\mathrm{A}_n(\Gamma)) \to \mathrm{GL}_n(k(\wp)).$$

Now we define the absolutely irreducible representation part of the representation variety. Later we will see that the action of PGL_n on this part is free and that there is a universally geometric quotient.

Definition 2.9 (The absolutely irreducible representation part)**.** We define the *absolutely irreducible representation part* of the representation variety (or simply the *a.i.r. part*) in the following way.

$$\mathrm{Rep}_n(\Gamma)_{\mathrm{a.i.r.}} := \left\{ \wp \in \mathrm{Spec}\, \mathrm{A}_n(\Gamma) \;\middle|\; \begin{array}{c} \sigma_\wp : \Gamma \to \mathrm{GL}_n(k(\wp)) \\ \text{is an absolutely irreducible} \\ \text{representation} \end{array} \right\}.$$

The next proposition allows us to understand the relation between discriminants and $\mathrm{Rep}_n(\Gamma)_{\mathrm{a.i.r.}}$.

Let Γ be a group and let $\beta_1, \ldots, \beta_{n^2}$ be elements of Γ. Assume that $\phi : \mathrm{F}_{n^2} \to \Gamma$ is a group homomorphism sending α_i to β_i for $1 \leq i \leq n^2$. Then by the *standard morphism* with respect to $\beta_1, \ldots, \beta_{n^2}$ we mean the induced map $\phi_* : \mathrm{A}_n(\mathrm{F}_{n^2}) \to \mathrm{A}_n(\Gamma)$, $\phi^* : \mathrm{Rep}_n(\Gamma) \to \mathrm{Rep}_n(\mathrm{F}_{n^2})$, $\phi_* : \mathrm{A}_n(\mathrm{F}_{n^2})_\Delta \to \mathrm{A}_n(\Gamma)_{\Delta(\beta_1,\ldots,\beta_{n^2})}$ or $\phi_* : \mathrm{A}_n(\mathrm{F}_{n^2})^{\mathrm{Ch}} \to \mathrm{A}_n(\Gamma)^{\mathrm{Ch}} \ldots$ etc.

For $\alpha_1, \alpha_2, \ldots, \alpha_{n^2} \in \Gamma$ we define the discriminant $\Delta(\alpha_1, \alpha_2, \ldots, \alpha_{n^2}) \in \mathrm{A}_n(\Gamma)$. Moreover $\Delta(\alpha_1, \alpha_2, \ldots, \alpha_{n^2}) \in \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}$.

Proposition 2.8. *Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_{n^2} \in \Gamma$. Then we have $\Delta(\alpha_1, \alpha_2, \ldots, \alpha_{n^2}) \in \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}$.*

Proof: Let F_{n^2} be the free group of rank n^2. Suppose that $\{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_{n^2}\}$ is a free generator of F_{n^2}. We consider the standard morphism with respect to $\alpha_1, \alpha_2, \ldots, \alpha_{n^2}$. Then we get the following commutative diagram.

$$\begin{array}{ccccc} \mathrm{A}_n(\mathrm{F}_{n^2})^{\mathrm{Ch}} & \to & \mathrm{A}_n(\mathrm{F}_{n^2})^{\mathrm{PGL}_n} & \to & \mathrm{A}_n(\mathrm{F}_{n^2}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{A}_n(\Gamma)^{\mathrm{Ch}} & \to & \mathrm{A}_n(\Gamma)^{\mathrm{PGL}_n} & \to & \mathrm{A}_n(\Gamma). \end{array}$$

Since the discriminant $\Delta(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_{n^2})$ is sent to $\Delta(\alpha_1, \alpha_2, \ldots, \alpha_{n^2})$ by φ, it suffices to show that $\Delta(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_{n^2}) \in \mathrm{A}_n(\mathrm{F}_{n^2})^{\mathrm{Ch}}$. Moreover by Donkin's theorem we have reduced to proving that $\Delta(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_{n^2}) \in \mathrm{A}_n(\mathrm{F}_{n^2})^{\mathrm{PGL}_n}$.

Before proving it, we need some preliminaries. Let R be a commutative ring. We consider the full matrix ring $\mathrm{M}_n(R)$ as a free module of rank n^2 over R. For $Q \in \mathrm{M}_n(R)$ we define the following linear mappings.

$$\begin{array}{rccc} L_Q : & \mathrm{M}_n(R) & \to & \mathrm{M}_n(R) \\ & A & \mapsto & QA \end{array} \qquad \begin{array}{rccc} R_Q : & \mathrm{M}_n(R) & \to & \mathrm{M}_n(R) \\ & A & \mapsto & AQ. \end{array}$$

Then $\det(L_Q) = \det(R_Q) = \det(Q)^n$.

Now we put $R = \mathrm{A}_n(\mathrm{F}_{n^2}) \otimes_{\mathbf{Z}} \mathrm{A}(\mathrm{GL}_n)$ and $P = (X_{ij})_{1 \leq i,j \leq n}$, where $\mathrm{A}(\mathrm{GL}_n)$ is the affine ring of GL_n. Remark that R contains the subring $\mathrm{A}_n(\mathrm{F}_{n^2}) \otimes_{\mathbf{Z}} \mathrm{A}(\mathrm{PGL}_n)$. We consider the next linear mapping.

$$\begin{array}{rccc} \mathrm{Ad}(P) : & \mathrm{M}_n(R) & \to & \mathrm{M}_n(R) \\ & A & \mapsto & P^{-1}AP. \end{array}$$

Let C be the matrix expression of $\mathrm{Ad}(P)$ with respect to the basis $\{E_{ij}\}_{1 \leq i,j \leq n}$, where E_{ij} is the matrix unit.

We define $D(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_{n^2}) \in \mathrm{M}_n(\mathrm{A}_n(\mathrm{F}_{n^2}))$ to be the matrix same as in Definition 2.4. We denote by $\mathrm{Ad}(D)$ the matrix which we get by

representation if and only if the subalgebra $k[\rho(\Gamma)]$ generated by the image of ρ over k coincides with the full matrix ring $\mathrm{M}_n(k)$.

Here we introduce the discriminant. The definition of the discriminant seems to be artificial, but it will later turn out to be a natural notion.

Definition 2.4 (Definition of the discriminant)**.** Let R be a commutative ring. For n^2 matrices $A_1, A_2, \ldots, A_{n^2} \in \mathrm{M}_n(R)$ we define the *discriminant* $\Delta(A_1, \ldots, A_{n^2}) \in R$ for $A_1, \ldots, A_{n^2}$ as follows. Put $A_k := (a_{ij}^{(k)})_{1 \leq i,j \leq n}$. The matrix $D(A_1, A_2, \ldots, A_{n^2}) \in \mathrm{M}_{n^2}(R)$ is defined as

$$D(A_1, A_2, \ldots, A_{n^2})$$

$$:= \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \ldots & a_{1n}^{(1)} & a_{21}^{(1)} & \ldots & a_{2n}^{(1)} & \ldots & a_{n1}^{(1)} & \ldots & a_{nn}^{(1)} \\ a_{11}^{(2)} & a_{12}^{(2)} & \ldots & a_{1n}^{(2)} & a_{21}^{(2)} & \ldots & a_{2n}^{(2)} & \ldots & a_{n1}^{(2)} & \ldots & a_{nn}^{(2)} \\ \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots \\ a_{11}^{(n^2)} & a_{12}^{(n^2)} & \ldots & a_{1n}^{(n^2)} & a_{21}^{(n^2)} & \ldots & a_{2n}^{(n^2)} & \ldots & a_{n1}^{(n^2)} & \ldots & a_{nn}^{(n^2)} \end{pmatrix}.$$

Then

$$\Delta(A_1, A_2, \cdots, A_{n^2}) := \det(D(A_1, A_2, \cdots, A_{n^2})).$$

The *discriminant* $\Delta(\alpha_1, \alpha_2, \ldots, \alpha_{n^2})$ for $\alpha_1, \alpha_2, \ldots, \alpha_{n^2} \in \Gamma$ is defined as $\Delta(\alpha_1, \alpha_2, \ldots, \alpha_{n^2}) := \Delta(\sigma_\Gamma(\alpha_1), \sigma_\Gamma(\alpha_2), \ldots, \sigma_\Gamma(\alpha_{n^2})) \in \mathrm{A}_n(\Gamma)$, where σ_Γ is the universal representation of degree n for Γ.

Remark 2.5. For $A_1, A_2, \ldots, A_{n^2} \in \mathrm{M}_n(R)$,

$\Delta(A_1, A_2, \ldots, A_{n^2}) \in R^\times \Leftrightarrow \{A_1, A_2, \ldots, A_{n^2}\}$ is a R-basis of $\mathrm{M}_n(R)$.

This is the reason why the discriminant is a natural notion.

We remark on the relation between representation varieties for two groups.

Remark 2.6. Let Γ, Γ' be groups and $\varphi : \Gamma \to \Gamma'$ a group homomorphism. Then φ canonically induces a ring homomorphism $\varphi_* : \mathrm{A}_n(\Gamma) \to \mathrm{A}_n(\Gamma')$. Moreover it induces a morphism of schemes $\varphi^* : \mathrm{Rep}_n(\Gamma') \to \mathrm{Rep}_n(\Gamma)$.

We define some terms on the representation variety. These terms are not mathematical, but useful expressions.

Definition 2.7 (The prototype and the standard morphism)**.** Let F_{n^2} be the free group of rank n^2. Take a free generator $\alpha_1, \alpha_2, \ldots, \alpha_{n^2}$ of F_{n^2} and put $\Delta := \Delta(\alpha_1, \alpha_2, \ldots, \alpha_{n^2})$. By the *prototype* of degree n we mean $\mathrm{Spec}\, \mathrm{A}_n(\mathrm{F}_{n^2})$, $\mathrm{Spec}\, \mathrm{A}_n(\mathrm{F}_{n^2})_\Delta$, or $\mathrm{Spec}\, \mathrm{A}_n(\mathrm{F}_{n^2})^{\mathrm{Ch}} \ldots$ etc.

Remark 1.13. Just as the representation variety for a group, the representation variety for a semigroup (resp. a monoid) has the universal property.

Let Υ_m be the free monoid of rank m. Then we have the following isomorphism.

$$\mathrm{A}_n(\Upsilon_m) \cong \mathbf{Z}[\mathrm{M}_n^{\oplus m}]$$

2. Absolutely irreducible representations and discriminants

In this section we introduce an important invariant, the *discriminant*, for describing absolutely irreducible representations. *Absolutely irreducible representations* in general linear group over arbitrary commutative ring can be defined as a natural extension for a field case. We will later see that the set of points corresponding to absolutely irreducible representations in the representation variety is a Zariski open set, which is described by discriminants. We call this open set the *absolutely irreducible representation part.*

Now we introduce the basic concept and a well-known fact on absolutely irreducible representations.

Definition 2.1. Let k be a field. Suppose that $\rho : \Gamma \to \mathrm{GL}_n(k)$ is a representation of a group Γ. We say ρ is an *absolutely irreducible representation* if $\rho : \Gamma \to \mathrm{GL}_n(k) \to \mathrm{GL}_n(\bar{k})$ is irreducible for an algebraic closure $\bar{k}$ of k.

Let R be a commutative ring. Suppose that $\rho : \Gamma \to \mathrm{GL}_n(R)$ is a representation of a group Γ. We say ρ is an *absolutely irreducible representation* (or simply *a.i.r.*) if for any prime ideal $\wp$ of R, $\rho : \Gamma \to \mathrm{GL}_n(R) \to \mathrm{GL}_n(k(\wp))$ is absolutely irreducible, where $k(\wp)$ is the residue field of $\wp$.

Definition 2.2. Let $\rho, \rho' : \Gamma \to \mathrm{GL}_n(R)$ be representations. We say ρ and ρ' are equivalent if there exists an R-algebra automorphism $\sigma : \mathrm{M}_n(R) \to \mathrm{M}_n(R)$ such that $\sigma(\rho(\gamma)) = \rho'(\gamma)$ for any $\gamma \in \Gamma$. The symbol $\rho \sim \rho'$ denotes that ρ and ρ' are equivalent.

In particular by Skolem-Noether Theorem we can see that when R is a field, $\rho \sim \rho'$ if and only if there exists an matrix $P \in \mathrm{GL}_n(R)$ such that $P^{-1}\rho(\gamma)P = \rho'(\gamma)$ for each $\gamma \in \gamma$.

On absolutely irreducible representations we have the following well-known result. Later we will use this result, so here we introduce it without proof.

Proposition 2.3. *Let k be a field. Suppose that $\rho : \Gamma \to \mathrm{GL}_n(k)$ is a representation of a group Γ. Then ρ is an absolutely irreducible*

$$\begin{aligned} \mathrm{Ch}_n(\Gamma) &:= \mathrm{Spec}\, \mathrm{A}_n(\Gamma)^{\mathrm{PGL}_n} \\ \mathrm{Ch}_n^0(\Gamma) &:= \mathrm{Spec}\, \mathrm{A}_n(\Gamma)^{\mathrm{Ch}}. \end{aligned}$$

Then the natural morphism is defined as follows.

$$\pi : \mathrm{Rep}_n(\Gamma) \to \mathrm{Ch}_n(\Gamma) \to \mathrm{Ch}_n^0(\Gamma).$$

In the definition introduced above, we have two subalgebras of the representation ring. When do two subalgebras coincide? We give the partial answer here.

At first, we quote a useful and important theorem from [1].

Theorem 1.10 (S. Donkin [1])**.** *In the same notation as Notation 1.4,* $\mathbf{Z}[\mathrm{M}_n^{\oplus m}]^{\mathrm{PGL}_n}$ *is generated by* $\{c_j(A_{i_1}A_{i_2}\cdots A_{i_r}) \mid 0 \leq j \leq n,$ *and* $i_1, i_2, \ldots, i_r \in \{1, 2, \ldots, m\}\}$ *over* $\mathbf{Z}$*, where the action of* PGL_n *is adjoint. Moreover if* Ω *is arbitrary algebraically closed field, then* $\Omega[\mathrm{M}_n^{\oplus m}]^{\mathrm{PGL}_n} \cong \mathbf{Z}[\mathrm{M}_n^{\oplus m}]^{\mathrm{PGL}_n} \otimes_{\mathbf{Z}} \Omega$ *holds.*

The next corollary immediately follows from Donkin's Theorem.

Corollary 1.11. *Let* F_m *be the free group of rank* m*. Then* $\mathrm{A}_n(\mathrm{F}_m)^{\mathrm{Ch}} = \mathrm{A}_n(\mathrm{F}_m)^{\mathrm{PGL}_n}$ *holds.*

From the above corollary we conclude that two subalgebras coincide for free groups. In general case we will later see that two subalgebra coincide on the absolutely irreducible representation part.

Before we finish this section, we introduce the representation ring and the representation variety for another algebraic systems, here semigroups and monoids.

Definition 1.12. Let Υ be a semigroup (resp. a monoid). $\rho : \Upsilon \to \mathrm{M}_n(R)$ is a representation of Υ if $\rho(\upsilon_1\upsilon_2) = \rho(\upsilon_1)\rho(\upsilon_2)$ holds for $\forall \upsilon_1, \upsilon_2 \in \Upsilon$ (resp. and $\rho(e) = 1$).

We define the *representation ring* $\mathrm{A}_n(\Upsilon)$ of degree n for Υ as follows. For each $\upsilon \in \Upsilon$, consider a $n \times n$ matrix

$$\sigma(\upsilon) := (a_{ij}(\upsilon))_{1 \leq i,j \leq n}$$

whose entries $a_{ij}(\upsilon)$ are regarded as indeterminates.

$$\mathrm{A}_n(\Upsilon) := \mathbf{Z}[a_{ij}(\upsilon) \mid 1 \leq i, j \leq n, \upsilon \in \Upsilon]/I(\Upsilon),$$

where $I(\Upsilon)$ is the ideal generated by all entries of the matrices $\sigma(\gamma\delta) - \sigma(\gamma)\sigma(\delta)$, *for* $\gamma, \delta \in \Upsilon$ (resp. and $\sigma(e) - I_n$).

We call $\mathrm{Spec}\, \mathrm{A}_n(\Upsilon)$ the representation variety of degree n for a semigroup (resp. a monoid) Υ.

Definition 1.7 (The adjoint action of PGL_n (cf.[6]))**.** We now define the adjoint action of the group scheme PGL_n on the representation variety $\mathrm{Rep}_n(\Gamma)$. The ring homomorphism associated to the action $\mathrm{Rep}_n(\Gamma) \times_{\mathrm{Spec}\,\mathbf{Z}} \mathrm{PGL}_n \to \mathrm{Rep}_n(\Gamma)$ is defined in the following way:

$$\begin{array}{rccc} \mathrm{Ad} & : & \mathrm{A}_n(\Gamma) & \to & \mathrm{A}_n(\Gamma) \otimes_{\mathbf{Z}} \mathrm{A}(\mathrm{PGL}_n) \\ & & \sigma_\Gamma(\gamma) & \mapsto & P^{-1}\sigma_\Gamma(\gamma)P \end{array} ,$$

where $\mathrm{A}(\mathrm{PGL}_n)$ is the affine ring of PGL_n. The symbol P denote the matrix $(X_{ij})_{1\le i,j\le n}$ in $\mathrm{GL}_n(\mathrm{A}(\mathrm{GL}_n))$, where $\mathrm{A}(\mathrm{GL}_n)$ is the affine ring of GL_n. In other words, $\mathrm{A}(\mathrm{GL}_n) = \mathbf{Z}[X_{ij}]_{\det(X_{ij})}$. $\mathrm{A}_n(\Gamma) \otimes_{\mathbf{Z}} \mathrm{A}(\mathrm{GL}_n)$ is considered as a graded ring. $\mathrm{A}_n(\Gamma) \otimes_{\mathbf{Z}} \mathrm{A}(\mathrm{PGL}_n)$ is the subring consisting of elements of degree 0 in $\mathrm{A}_n(\Gamma) \otimes_{\mathbf{Z}} \mathrm{A}(\mathrm{GL}_n)$.

The ring homomorphism is defined by sending an entry of $\sigma_\Gamma(\gamma)$ to the corresponding entry of the matrix $P^{-1}\sigma_\Gamma(\gamma)P$. Since any entry of $P^{-1}\sigma_\Gamma(\gamma)P$ has degree 0, so the ring homomorphism is well-defined.

We introduce the notation about PGL_n invariants.

Notation 1.8. Let R be a commutative ring. If $A \in \mathrm{M}_n(R)$, then we will write $\mathrm{c}_i(A)$ for $(-1)^i\times$ the coefficients of x^{n-i} of the characteristic polynomials $\det(xI_n - A)$ of A. In particular, the trace $\mathrm{c}_1(A)$ will be also denoted by $\mathrm{s}(A)$, and the determinant $\mathrm{c}_n(A)$ will be also denoted by $\mathrm{d}(A)$. From Cayley-Hamilton Theorem we get

$$A^n - \mathrm{s}(A)A^{n-1} + \mathrm{c}_2(A)A^{n-2} - \cdots + (-1)^n\mathrm{d}(A)I_n = 0.$$

Let Γ be a group. If $\sigma_\Gamma : \Gamma \to \mathrm{GL}_n(\mathrm{A}_n(\Gamma))$ is the universal representation of degree n for Γ, then for an element γ of Γ the coefficients of the characteristic polynomial of $\sigma_\Gamma(\gamma)$ will be denoted by $\mathrm{s}(\gamma), \mathrm{c}_2(\gamma), ..., \mathrm{c}_{n-1}(\gamma), \mathrm{d}(\gamma)$.

We define two subalgebras of the representation ring, which turn out to be almost the coordinate ring of the quotient of the representation variety.

Definition 1.9 (Definition of $\mathrm{A}_n(\Gamma)^{\mathrm{PGL}_n}$ and $\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}$)**.** The subalgebra $\mathrm{A}_n(\Gamma)^{\mathrm{PGL}_n}$ of the representation ring $\mathrm{A}_n(\Gamma)$ is defined to be $\{x \in \mathrm{A}_n(\Gamma) \mid \mathrm{Ad}(x) = x \otimes 1 \in \mathrm{A}_n(\Gamma) \otimes_{\mathbf{Z}} \mathrm{A}(\mathrm{PGL}_n)\}$.

Let σ_Γ be the universal representation of degree n for Γ and let $\mathrm{c}_j(\gamma)$ be $\mathrm{c}_j(\sigma_\Gamma(\gamma))$. The subalgebra $\mathrm{A}_n(\Gamma)^{\mathrm{Ch}}$ of $\mathrm{A}_n(\Gamma)$ is defined as the subalgebra generated by $\{\mathrm{c}_j(\gamma) \mid 1 \le j \le n,\ \gamma \in \Gamma\}$ over $\mathbf{Z}$.

Clearly $\mathrm{A}_n(\Gamma)^{\mathrm{Ch}} \subseteq \mathrm{A}_n(\Gamma)^{\mathrm{PGL}_n} \subseteq \mathrm{A}_n(\Gamma)$. We now introduce notation as below.

in $\mathrm{GL}_n(R)$. It is easy to see that the above natural morphism is an isomorphism.

Definition 1.3 (Definition of the representation variety (cf. [6]))**.** The *representation variety* $\mathrm{Rep}_n(\Gamma)$ of degree n for Γ is defined as $\mathrm{Spec}\,\mathrm{A}_n(\Gamma)$. $\mathrm{Rep}_n(\Gamma)$ represents the contravariant functor from the category of schemes to the category of sets defined as follows:

$$\begin{array}{rccl} \mathrm{Rep}_n(\Gamma) & : & \textbf{(Sch)} & \to \quad \textbf{(Sets)} \\ & & X & \mapsto \quad \{\rho \,:\, \Gamma \to \mathrm{GL}_n(\Gamma(X, \mathcal{O}_X)) \mid \\ & & & \qquad\qquad \rho \text{ is a representation } \}. \end{array}$$

Here we consider the structure of representation varieties. For free groups we can describe representation varieties.

Notation 1.4. Let $\mathbf{Z}[\mathrm{M}_n^{\oplus m}]$ be the coordinate ring for the space of m-tuple $n \times n$ matrices $(A_1, A_2, \ldots, A_m)$, that is

$$\mathbf{Z}[\mathrm{M}_n^{\oplus m}] := \mathbf{Z}[a_{ij}^k \mid 1 \le i, j \le n, 1 \le k \le m],$$

$$\textit{where } A_k := (a_{ij}^k)_{1 \le i,j \le n}.$$

$\mathbf{Z}[\mathrm{M}_n^{\oplus m}]$ is isomorphic to the polynomial ring in $n^2 m$ variables over $\mathbf{Z}$. For a commutative ring R, define $R[\mathrm{M}_n^{\oplus m}]$ by $\mathbf{Z}[\mathrm{M}_n^{\oplus m}] \otimes_{\mathbf{Z}} R$.

The symbol $\det A$ denotes $\Pi_{k=1}^n \det A_k$. We denote by $R[\mathrm{M}_n^{\oplus m}]_{\det A}$ the localization of $R[\mathrm{M}_n^{\oplus m}]$ by $\det A$.

Proposition 1.5. (cf.[5]). *Let* F_m *be the free group of rank* m*. Then the representation ring* $\mathrm{A}_n(\mathrm{F}_m)$ *is isomorphic to* $\mathbf{Z}[\mathrm{M}_n^{\oplus m}]_{\det A}$.

Proof: Giving a representation of the free group of rank m corresponds to giving m nonsingular matrices. It is clear that $\mathrm{Spec}\,\mathbf{Z}[\mathrm{M}_n^{\oplus m}]_{\det A}$ represents the functor giving m nonsingular matrices. Therefore $\mathrm{A}_n(\mathrm{F}_m)$ is isomorphic to $\mathbf{Z}[\mathrm{M}_n^{\oplus m}]_{\det A}$. □

The group scheme PGL_n acts on the representation variety by conjugating. Now we define the definition of PGL_n and the adjoint action of PGL_n.

Definition 1.6 (Definition of the group scheme PGL_n)**.** The group scheme PGL_n over $\mathbf{Z}$ is defined as the open set $\{\det(X_{ij}) \neq 0\}$ of $\mathbf{P}_{\mathbf{Z}}^{n^2-1} := \mathrm{Proj}\,\mathbf{Z}[\, X_{ij} \mid 1 \le i, j \le n\,]$. In other words, $\mathrm{PGL}_n := \mathrm{Spec}\,\mathbf{Z}[\, X_{ij} \mid 1 \le i, j \le n\,]_{(\det(X_{ij}))}$, where $\mathbf{Z}[\, X_{ij} \mid 1 \le i, j \le n\,]_{(\det(X_{ij}))}$ is the subring of elements of degree 0 in the localized ring $\mathbf{Z}[\, X_{ij} \mid 1 \le i, j \le n\,]_{\det(X_{ij})}$. The group structure of PGL_n is defined in the natural way.

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1. Preliminaries

In this section, we introduce the notion of the representation variety. The representation variety is a scheme parameterizing representations, so it is worth calling its name. The group scheme PGL_n naturally acts on the representation variety by conjugation. Here we explain the situation we discuss in this paper.

Definition 1.1 (Definition of the representation ring (cf. [6]))**.** Let Γ be a group and n a positive integer. For each $\gamma \in \Gamma$, consider a $n \times n$ matrix

$$\sigma(\gamma) := (a_{ij}(\gamma))_{1 \leq i,j \leq n}$$

whose entries $a_{ij}(\gamma)$ are regarded as indeterminates. We consider the polynomial ring $\mathbf{Z}[a_{ij}(\gamma) \mid 1 \leq i, j \leq n, \gamma \in \Gamma]$, which is in infinite variables in general. Then the *representation ring* $\mathrm{A}_n(\Gamma)$ of degree n for Γ is defined as

$$\mathrm{A}_n(\Gamma) := \mathbf{Z}[a_{ij}(\gamma) \mid 1 \leq i, j \leq n, \gamma \in \Gamma]/I(\Gamma),$$

where $I(\Gamma)$ is the ideal generated by all entries of the matrices $\sigma(e) - I_n$ and $\sigma(\gamma\delta) - \sigma(\gamma)\sigma(\delta)$, that is $a_{ij}(e) - \delta_{ij}$ and $a_{ij}(\gamma\delta) - \sum_{k=1}^{n} a_{ik}(\gamma)a_{kj}(\delta)$, for $1 \leq i, j \leq n$ and $\gamma, \delta \in \Gamma$.

The *universal representation* σ_Γ of degree n for Γ is defined as

$$\begin{array}{ccccc} \sigma_\Gamma & : & \Gamma & \to & \mathrm{GL}_n(\mathrm{A}_n(\Gamma)) \\ & & \gamma & \mapsto & \sigma(\gamma). \end{array}$$

Remark 1.2. (cf. [6]). Let R be a commutative ring. Then there is a natural isomorphism between the set of ring homomorphisms from $\mathrm{A}_n(\Gamma)$ to R and the set of representations of Γ in $\mathrm{GL}_n(R)$.

$$\begin{array}{ccc} \mathrm{Hom}_{ring}(\mathrm{A}_n(\Gamma), R) & \to & \mathrm{Hom}_{group}(\Gamma, \mathrm{GL}_n(R)) \\ \phi & \mapsto & \phi \circ \sigma_\Gamma. \end{array}$$

For each $\gamma \in \Gamma$ we define $(\phi \circ \sigma_\Gamma)(\gamma)$ as the matrix obtained by sending each entry of $\sigma_\Gamma(\gamma)$ by ϕ. Hence we get a representation $\phi \circ \sigma_\Gamma$ of Γ

In [6] Kyoji Saito showed that the character variety exists in case of degree 2. He proved that the universal quotient morphism is a PGL_2 principal fiber bundle on the absolutely irreducible representations part in that case. Here we will prove that this fact holds in any degree case. More precisely, we show that there is a universal geometric quotient of the a.i.r. part by PGL_n and that the quotient morphism is a PGL_n principal fiber bundle for any degree n. This universal geometric quotient is nothing but the coarse moduli scheme.

There are two key points to prove the existence of the universal geometric quotient. The first is "Reconstruction of Matrices and Representations". These theorems allow us to reconstruct matrices and representations from their invariants, for example, traces or determinants. This fact reduce the existence problems for arbitrary groups cases to them for free groups cases.

The second is to study the adjoint action of PGL_n. The group scheme PGL_n represents the group of automorphisms of the full matrix ring. For any absolutely irreducible representation the full matrix ring is generated by the image of the representation as an algebra, so the stabilizer in PGL_n is trivial at each a.i.r. point. That is the essence of the proof.

The representation variety and the character variety are defined over $\mathbf{Z}$, and they are universal objects independent of coefficients. These geometric objects have the possibility to be bridges from representation theory to algebraic geometry. Some Problems in representation theory can be transformed into them in algebraic geometry. For example they connect ordinary representations with modular representations.

The representation variety and the character variety have relation to the moduli of vector bundles, Teichmüller spaces, and quiver varieties. Moreover they may describe embeddings of discrete subgroups in GL_n and SL_n related with rigidity theorem. In this way we construct a general framework from the viewpoint of group representations. These object may have the possibility more than we consider in the present.

We denote a group by Γ and a commutative ring R throughout this paper. e denotes the identity element of a group Γ. I_n denotes the identity matrix of degree n. If Z is a scheme, then h_Z denotes the functor $\mathrm{Hom}\,(\cdot, Z)$ from the category of schemes to the category of sets.

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Representation varieties and character varieties

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ABSTRACT. In this paper we construct the coarse moduli scheme over $\mathbf{Z}$ of equivalence classes of absolutely irreducible representations. It is obtained as the universal geometric quotient of the absolutely irreducible representation part of the representation variety by the adjoint action of PGL_n. The quotient morphism is a PGL_n principal fiber bundle.

The purpose of this paper is to construct the scheme over $\mathbf{Z}$ which parameterizes equivalent classes of absolutely irreducible representations of fixed degree of a given group. In other words, we construct the coarse moduli scheme of equivalent classes of absolutely irreducible representations. We call this coarse moduli scheme the character variety.

We define the representation variety of degree n to be the scheme over $\mathbf{Z}$ which represents representations of a given group in GL_n. The theory of representation varieties has been investigated by many mathematicians. There are many definitions of representation varieties, although the essential part of notions of representation varieties is invariant. In this paper we follow Kyoji Saito's definition [6].

There is a natural action of PGL_n on the representation variety by conjugation. The character variety is obtained as the quotient of the Zariski open subset consisting of absolutely irreducible representations of the representation variety by PGL_n. The notion of absolutely irreducible representations is introduced as an extension of the usual notion for a field case. The absolutely irreducible representation part (or simply the a.i.r. part) coincides with the open subset of the representation variety consisting of properly stable points with respect to the action of PGL_n.

The open subset is described by discriminants, which are important invariants. The discriminant is defined for n^2 matrices of size $n \times n$. It is an indicator whether n^2 matrices is a basis of the full matrix ring. The a.i.r. part is described as the Zariski open subset on which discriminants do not vanish.