Homology ring mod 2 of Free Loop Groups of Spinor Groups

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1 Introduction

Let G, ΩG be a compact connected Lie group and its based loop space, whose multiplication maps are μ, λ respectively. We denote the free loop group of G by $LG(G)$, which is the topological group of all continuous maps from S^1 to G. Then $LG(G)$ contains ΩG as its normal subgroup and $LG(G)$ is the semi-direct product of ΩG and G. Thus $LG(G)$ is homeomorphic to $\Omega G \times G$ and we have the following commutative diagram:

$$\begin{array}{ccccc} \Omega G \times G \times \Omega G \times G & \xrightarrow{\omega} & \Omega G \times \Omega G \times G \times G & \xrightarrow{\lambda \times \mu} & \Omega G \times G \\ \downarrow \Phi \times \Phi & & & & \downarrow \Phi \\ LG\,(G) \times LG\,(G) & & \xrightarrow{\qquad\qquad \lambda' \qquad\qquad} & & LG\,(G) \end{array}$$

where $\Phi : \Omega G \times G \to LG\,(G)$ is a map defined by $\Phi(l, g)(t) = l(t) \cdot g$, λ' is the multiplication map of $LG(G)$ and ω is the composition

$$(1_{\Omega G} \times T \times 1_G) \circ (1_{\Omega G} \times \mathrm{Ad} \times 1_G) \circ (1_{\Omega G} \times \Delta_{*G} \times 1_{\Omega G \times G}).$$

Here Ad is the point-wise adjoint action of G on ΩG and T is the transposition. This shows that the knowledge of the Hopf algebra structures of the

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homology rings of G, ΩG and the induced homology map of Ad enable us to know the Hopf algebra structure of the homology ring of $LG(G)$.

Moreover, in [8], it is showed that provided G is simply connected, $H_*(Ad; Z/p)$ is equal to the induced homology map of second projection if and only if $H^*(G; Z)$ is p-torsion free. This implies that, if we assume G is simple, non-trivial cases are $G = E_8$ for $p = 5$, $G = F_4, E_6, E_7, E_8$ for $p = 3$ and $G = G_2, F_4, E_6, E_7, E_8, Spin(N)$ for $p = 2$. The induced homology map of Ad is in all these cases except for $(G, p) = (Spin(N), 2)$ were investigated in [1],[2],[3] and [4].

In this paper we compute Ad_*, the induced homology map of Ad for the case of $(G, p) = (Spin(N), p)$ and consider the mod 2 homology ring of $LG\,(Spin(N))$. In the rest of this paper we use $\mathbf{Z}/2\mathbf{Z}$ as the coefficient ring of cohomology and homology unless mentioned.

2 Generating variety

$Spin(N)$ is the universal covering group of $SO(N)$ and the mod 2 cohomology of them are well known as follows.

Theorem 2.1. *We denote the covering map $Spin(N) \to SO(N)$ by p. Then*

$$\begin{aligned} \mathrm{H}^*(SO(N)) &= \Delta(x_1, x_2, x_3, \cdots x_{N-1}), \\ \mathrm{H}^*(Spin(N)) &= \Delta(\{x_\alpha | \alpha \in A\}) \otimes \bigwedge u \end{aligned}$$

where

$$A = \{1, 2, \cdots, N-1\} \backslash \{\textit{powers of 2}\},$$

$$|x_i| = i,\ |w| = 2^k - 1,\ 2^{k-1} < N \leq 2^k,$$

x_i's are primitive and the coproduct of u is

$$u \otimes 1 + 1 \otimes u + \sum_{\substack{i,j \neq 2^l, i<j \\ i+j=2^k-1}} x_i \otimes x_j.$$

Also $p^*(x_\alpha) = x_\alpha$ for $\alpha \in A$.

Here we have the following diagram:

$$\begin{array}{ccc} Spin(N) \times \Omega Spin(N) & \xrightarrow{\mathrm{Ad}} & \Omega Spin(N) \\ \downarrow p \times \Omega p & & \| \\ SO(N) \times \Omega_0 SO(N) & \xrightarrow{\mathrm{Ad}_0} & \Omega_0 SO(N) \end{array}$$

where $\Omega_0 SO(N)$ is the connected component of $\Omega SO(N)$ containing null homotopic loops and Ad_0 is the pointwise adjoint action of $SO(N)$ on $\Omega_0 SO(N)$.

This diagram and the above theorem tells that

$$\mathrm{Ad}_* = \mathrm{Ad}_{0*} \circ p_* \tag{1}$$

and what we have to compute is Ad_{0*}.

Also, to set the generators of $\mathrm{H}_*(\Omega_0 SO(N))$ we remember the R.Bott's theorem of generating variety. (See [7].)

Let G be a compact connected Lie group and $s : S^1 \to G$ be a injective homomorphism. We denote the centralizer of s in G by G_s and the quotient space G/G_s by G^s. Then there is a map $g^s : G^s \to \Omega_0 G$ defined by

$$g^s([x])(t) = [x, s(t)]$$

where $x \in G$, $t \in S^1$. Moreover, let T be a maximal torus containing s and $W(G,T)$ be the Weil group of G. Let $\boldsymbol{s} \in \mathrm{H}_1(T; \mathbf{Z})$ be the homology class determined by s and Λ_s be the submodule of $\mathrm{H}_*(T; \mathbf{Z})$ generated by $W(G,T)\boldsymbol{s}$.

We call $s \in T \subset G$ a generating circle for G if each root $\theta \in \mathrm{H}^1(T; \mathbf{Z})$ takes on the value 1 on some element of $\bigwedge_s$. Also we call G^s a generating variety and g^s a generating map.

Theorem 2.2. *(R.Bott) If s is a generating circle for G, then the image of $g^s_* : \mathrm{H}_*(G^s; \mathbf{Z}) \to \mathrm{H}_*(\Omega_0 G; \mathbf{Z})$ generates the Pontryagin ring $\mathrm{H}_*(\Omega_0 G; \mathbf{Z})$.*

R.Bott applied this theorem to the case of $G = SO(N)$. It is known that $s : SO(2) \hookrightarrow SO(N)$ is a generating circle and the mod 2 cohomology ring of generating variety $G^s = SO(N)/(SO(2) \times SO(N-2))$ is as follows. See [5] for detail.

- If N is even and $N = 2m + 2$,

$$\mathrm{H}^*\left(\frac{SO(N)}{SO(2) \times SO(N-2)}\right) = \mathbf{Z}/2\mathbf{Z}[t, s]/(t^{m+1}, s^2 - \delta_m t^m s)$$

where $|t| = 2$, $|s| = 2m$ and

$$\delta_m = \begin{cases} 0 & m \text{ is odd,} \\ 1 & m \text{ is even.} \end{cases}$$

- If N is odd and $N = 2m + 1$,

$$\mathrm{H}^*\left(\frac{SO(N)}{SO(2)\times SO(N-2)}\right) = \mathbf{Z}/2\mathbf{Z}[t,s]/(t^m, s^2)$$

where $|t| = 2$, $|s| = 2m$.

We take the dual element of t^i, st^i with respect to the monomial basis and denote them by $b_i, b'_i \in \mathrm{H}_*(G^s)$ respectively. (We put $b_0 = 1$.) Then the generators of $\mathrm{H}_*(G^s)$ as module are

$$\begin{array}{ll} \{b_1, b_2, \cdots, b_m, b'_m, b'_{m+1}, \cdots b'_{2m}\} & \text{if } N = 2m+2, \\ \{b_1, b_2, \cdots, b_{m-1}, b'_m, b'_{m+1}, \cdots b'_{2m-1}\} & \text{if } N = 2m+1. \end{array}$$

We also denote $g^s_* b_i, g^s_* b'_i \in \mathrm{H}_*(\Omega_0 SO(N))$ by b_i, b'_i and regard $\mathrm{H}_*(G^s)$ as a submodule of $\mathrm{H}_*(\Omega SO(N))$ by g^s_*, because g^s_* is injective. Here by Theorem 2.2 b_i, b'_i's are generators of $\mathrm{H}_*(\Omega_0 SO(N))$ and R.Bott shows their relations:

Theorem 2.3. *(R.Bott)*

1. *If $N = 2m+2$, $\mathrm{H}_*(\Omega_0 SO(N))$ is generated by b_i, b'_{m+i} $(i = 0, 1, \cdots, m)$ and the relations are*

$$\begin{cases} {b_j}^2 = 0 & j < \frac{m}{2} \\ {b_j}^2 = \sum_{i=0}^{2j-m} (b_i b'_{2j-i}) & j \geq \frac{m}{2}. \end{cases}$$

2. *If $N = 2m+1$, $\mathrm{H}_*(\Omega_0 SO(N))$ is generated by b_i, b'_{m+i} $(i = 0, 1, \cdots, m-1)$ and the relations are the same as the case of 1.*

In other words, by the induced homology map of the natural inclusion $\Omega_0 SO(2m+1) \to \Omega_0 SO(2m+2)$, $\mathrm{H}_*(\Omega_0 SO(2m+1))$ is a subring of $\mathrm{H}_*(\Omega_0 SO(2m+2))$.

Remark In [7], all the computations are done with its coefficient ring Z and notations are different from those in this paper. If $N = 2n+1$, b_j, b'_{n+j} $(0 \leq j \leq n-1)$ in this paper are mod 2 reduction of σ_j, $2\sigma_{n+j}$ in [7] respectively and if $N = 2n+2$, b_j, $(1 \leq j \leq m-1)$, $b_{n+j'}$ $(1 \leq j' \leq m)$, b_n and b'_n in this paper are mod 2 reduction of σ_j, $2\sigma_{n+j'}$, $\sigma_n + \epsilon$ and 2ϵ in [7] respectively. Here remark that σ_j's and ϵ are elements of $\mathrm{H}_*(\Omega_0 SO(N); \mathbf{Q})$.

Moreover the coproduct can be determined easily since we know the algebra structure of the cohomology ring of the generating variety.

Theorem 2.4. *(R.Bott) In Theorem 2.3 the coproduct is given by*

$$\begin{aligned}
\Delta_* b_j &= \sum_{\alpha+\beta=j} b_\alpha \otimes b_\beta \qquad (1 \le j \le m) \\
\Delta_* b'_{m+j} &= \sum_{\alpha+\beta=j} (b'_{m+\alpha} \otimes b_\beta + b_\alpha \otimes b'_{\beta+m}) \qquad (0 \le j \le m-1) \\
\Delta_* b'_{2m} &= \sum_{\alpha+\beta=m} (b'_{m+\alpha} \otimes b_\beta + b_\alpha \otimes b'_{\beta+m}) + \delta_m (b'_m \otimes b'_m)
\end{aligned}$$

From now, we consider the even case, because, by the above theorem, one can deduce the result for odd case from that for even case.

3 Action of $SO(N-2)$ on $\Omega_0 SO(N)$

In this section we set $N = 2m+2$ and $G = SO(N)$. We consider a map $\alpha : SO(2m+2) \times G^s \to G^s$ defined by

$$\alpha(g, [x]) = [gx] \quad \text{for } g \in SO(2m+2) \text{ and } [x] \in G^s \ .$$

Lemma 3.1. *The following diagram commutes.*

$$\begin{array}{ccc}
SO(2m) \times G^s & \overset{\alpha|_{SO(2m)\times G^s}}{\longrightarrow} & G^s \\
\downarrow 1 \times g^s & & \downarrow g^s \\
SO(2m) \times \Omega_0 G & \overset{Ad_0}{\longrightarrow} & \Omega_0 G
\end{array}$$

Proof. By the definition, for $g \in SO(2m)$, $x \in SO(2m)$ and $t \in S^1$,

$$\begin{aligned}
g^s(\alpha(g,[x]))(t) &= g^s([gx])(t) \\
&= gxs(t)x^{-1}g^{-1}s(t)^{-1} \\
&= gxs(t)x^{-1}s(t)^{-1}g^{-1} \\
&= \mathrm{Ad}_0(g, g^s(x))(t).
\end{aligned}$$

∎

Now we denote $\mathrm{Ad}_{0*}(y \otimes b)$ by $y * b$ for $y \in \mathrm{H}_*(SO(N))$ and $b \in \mathrm{H}_*(\Omega_0 SO(N))$. Then above diagram says:

Theorem 3.2. *If* $y \in \mathrm{H}_*(SO(2m)) \subset \mathrm{H}_*(SO(2m+2))$ *then*

$$y * b = \alpha_*(y \otimes b)$$

for $b \in \mathrm{H}_*(G^s) \subset \mathrm{H}_*(\Omega_0 G)$.

To compute α_* we start with the computation in cohomology.

Lemma 3.3.

$$\alpha^*(t) = 1 \otimes t$$

Proof. We set $H = SO(2m) \times SO(2)$. In the following diagram $p_2 : SO(2m+2) \times (EG \times_G G/H) \to EG \times_G G/H$ and $p_2' : EG \times G/H \to G/H$ are second projections, $q : EG \times G/H \to EG \times_G G/H$ is the natural projection and $\beta : SO(2m+2) \times EG \times G/H \to EG \times G/H$ is defined as

$$\beta(h, g, x) = (gh^{-1}, hx)$$

for $h \in G, g \in EG$ and $x \in G/H$.

$$\begin{array}{ccccc}
SO(2m+2) \times G/H & \xrightarrow{\alpha} & G/H & & \\
\uparrow 1_G \times p_2' & & \uparrow p_2' & & \\
SO(2m+2) \times EG \times G/H & \xrightarrow{\beta} & EG \times G/H & & \\
\downarrow 1_G \times q & & \downarrow q & & \\
SO(2m+2) \times (EG \underset{G}{\times} G/H) & \xrightarrow{p_2} & EG \underset{G}{\times} G/H & \cong & BH
\end{array}$$

Then this diagram commutes. Here remark that $EG \times_G G/H = BH$ and p_2' is a homotopy equivalence. Thus if we set $\phi = q \circ {p_2'}^{-1}$, ϕ is a classifying map $G/H \to BH$ of the fibration $H \to G \to G/H$.

Also, since t is in the transgression image with regard to the fibration $H \to G \to G/H$, t is in the image of ϕ^*, that is, there exists $t' \in \mathrm{H}^*(BH)$ such that $\phi^*(t') = t$. Thus

$$\begin{aligned}
\alpha^*(t) &= \alpha^* \phi^*(t') \\
&= (1_G \times {p_2'}^{-1})^* \circ (1_G \times q)^* \circ p_2^*(t') \\
&= 1 \otimes \phi^*(t') \\
&= 1 \otimes t
\end{aligned}$$

To compute $\alpha^*(s)$, we consider the commutative diagram below:

$$\begin{array}{ccc}
SO(2m+2) \times \dfrac{SO(2m+2)}{SO(2)\times SO(2m)} & \xrightarrow{\alpha} & \dfrac{SO(2m+2)}{SO(2)\times SO(2m)} \\
\uparrow 1 \times p' & & \uparrow p' \\
SO(2m+2) \times \dfrac{SO(2m+2)}{U(1)\times U(m)} & \xrightarrow{\alpha'} & \dfrac{SO(2m+2)}{U(1)\times U(m)} \\
\downarrow 1 \times p'' & & \downarrow p'' \\
SO(2m+2) \times \dfrac{SO(2m+2)}{U(m+1)} & \xrightarrow{\alpha''} & \dfrac{SO(2m+2)}{U(m+1)} \\
\uparrow 1 \times p''' & & \uparrow p''' \\
SO(2m+2) \times SO(2m+2) & \xrightarrow{\mathrm{ad}} & SO(2m+2)
\end{array}$$

Here α', α'' are actions of $SO(2m+2)$ by the left translation and ad is the usual adjoint action and p', p'', p''' are natural projections. The cohomology rings of spaces in the diagram are shown as follows. See [5] for detail.

$$\begin{aligned}
\mathrm{H}^*\left(\frac{SO(2m+2)}{U(1)\times U(m)}\right) &\cong \mathrm{H}^*\left(\frac{SO(2m+2)}{U(m+1)}\right)[t]/(t^{m+1}), \\
\mathrm{H}^*\left(\frac{SO(2m+2)}{U(m+1)}\right) &\cong \Delta(x_2, x_4, \cdots x_{2m}).
\end{aligned}$$

Here by the induced cohomology map of the natural projection $SO(2m+2) \to SO(2m+2)/U(m+1)$, $x_{2i} \in \mathrm{H}^*\left(SO(2m+2)/U(m+1)\right)$ is mapped to $x_{2i} \in \mathrm{H}^*\left(SO(2m+2)\right)$ and also, in [5] it was shown that

$$\begin{aligned}
p'^*(s) &= \delta_m t^m + e, \\
p'^*(t) &= t
\end{aligned}$$

where $e = \sum_{i=1}^{m} (-1)^i x_{2i} t^{m-i}$. This implies p'^* is injective.

We can easily show that

$$\mathrm{ad}^*(x_{2i}) = 1 \otimes x_{2i} + x_{2i} \otimes 1$$

and this implies

$$\alpha''^*(x_{2i}) = 1 \otimes x_{2i} + x_{2i} \otimes 1$$

and

$$\alpha'^*(x_{2i}) = 1 \otimes x_{2i} + x_{2i} \otimes 1.$$

Thus we can proceed the computation as:

$$\begin{aligned}(1 \times p')^* \circ \alpha^*(s) &= \alpha'^* \circ p'^*(s) \\ &= \alpha'^*(\delta_m t^m + e) \\ &= \delta_m(1 \otimes t^m) + 1 \otimes e + \sum_{i=1}^{m} x_{2i} \otimes t^{m-i} \\ &= (1 \times p')^*(1 \otimes s + \sum_{i=1}^{m} x_{2i} \otimes t^{m-i}).\end{aligned}$$

By the injectivity of p'^*, we have :

Lemma 3.4.

$$\alpha^*(s) = 1 \otimes s + \sum_{i=1}^{m} e_{2i} \otimes t^{m-i}.$$

Now, we dualize the results of Lemma 3.3 and Lemma 3.4. We denote the dual element of x_i with regard to the monomial basis by y_i. Then we obtain

$$\begin{aligned}\alpha_*(y_{\mathrm{odd}} \otimes b) &= 0 \quad \text{for any } b \in \mathrm{H}_*(G/H), \\ \alpha_*(y_{2i} \otimes b_j) &= \begin{cases} 0 & i+j < m, \\ b_i & i+j \geq m, \end{cases} \\ \alpha_*(y_{2i} \otimes b'_{m+j}) &= 0\end{aligned}$$

for $1 \leq i \leq m$ and $0 \leq j \leq m$. By Theorem 3.2 it immediately follows that:

Proposition 3.5. *For* $y_{2i} \in \mathrm{H}_*(SO(2m+2))$ $(i \leq m-1)$, $b_j, b'_{m+j} \in \mathrm{H}_*(\Omega_0 SO(2m+2))$ $(0 \leq j \leq m)$ *and any* $b \in \mathrm{H}_*(\Omega_0 SO(2m+2))$,

$$\begin{aligned}y_{\mathrm{odd}} * b &= 0, \\ y_{2i} * b_j &= \begin{cases} 0 & i+j < m, \\ b_i & i+j \geq m, \end{cases} \\ y_{2i} * b'_{m+j} &= 0.\end{aligned}$$

4 Action of $SO(N)$ on $\Omega_0 SO(N)$

Concerning the computation of Ad_{0*}, all we have to do is to compute $y_{2m} * b_j$ and $y_{2m} * b'_{m+j}$ for $0 \leq j \leq m$. To do this we use the diagram below:

$$\begin{array}{ccc} SO(2m+2) \times \Omega_0 SO(2m+2) & \stackrel{\mathrm{Ad}_{2m+2}}{\longrightarrow} & \Omega_0 SO(2m+2) \\ i \times \Omega_0 i \downarrow & & \downarrow \Omega_0 i \\ SO(2m+4) \times \Omega_0 SO(2m+4) & \stackrel{\mathrm{Ad}_{2m+4}}{\longrightarrow} & \Omega_0 SO(2m+4) \end{array}$$

To avoid confusion, we denote $\mathrm{Ad} : SO(n) \times \Omega_0 SO(n) \to \Omega_0 SO(n)$ by Ad_n. By this diagram and Proposition 3.5, we obtain

$$\left.\begin{array}{rcl} \Omega_0 i_*(y_{2m} * b_j) & = & \mathrm{Ad}_{2m+4*} \circ (i \times \Omega_0 i)_*(y_{2m} \otimes b_j) = b'_{m+j}, \\ \Omega_0 i_*(y_{2m} * b'_{m+j}) & = & \mathrm{Ad}_{2m+4*} \circ (i \times \Omega_0 i)_*(y_{2m} \otimes b'_{m+j}) = 0. \end{array}\right\} \quad (2)$$

These tell that $y_{2m} * b_j$, $y_{2m} * b'_j$ was determined up to $\mathrm{Ker}(\Omega_0 i_*)$.

Lemma 4.1. $\mathrm{Ker}(\Omega_0 i_*) = (\mathrm{b}'_{\mathrm{m}})$.

Proof. This can be checked by the fact

$$\left.\begin{array}{rcl} \Omega_0 i_* b_j & = & b_j \ (\ 1 \leq j \leq m\ ), \\ \Omega_0 i_* b'_m & = & 0, \\ \Omega_0 i_* b'_j & = & b'_j \ (\ m+1 \leq j \leq 2m\ ) \end{array}\right\} \quad (3)$$

and the relations in Theorem 2.3.

Also (3) can be obtained by dualizing the induced cohomology map $SO(2m+2)/SO(2) \times SO(2m) \to SO(2m+4)/SO(2) \times SO(2m+2)$ since the generating maps commute with the respective inclusion maps. ∎

By (2) and the above lemma, we can say

$$\begin{array}{rcl} y_{2m} * b_j & = & b'_{m+j} + b'_m c_j, \\ y_{2m} * b'_j & = & b'_m c'_{m+j} \end{array}$$

for some $c_j \in \mathrm{H}_j(\Omega_0 SO(2m+2))$, $c'_{m+j} \in \mathrm{H}_{m+j}(\Omega_0 SO(2m+2))$. To determine c_j and c'_{m+j} we use some properties of 'adjoint action on homology' stated in the following theorem. See [1], [6] for its proof.

Theorem 4.2. *For y, y', $y'' \in \mathrm{H}_*(G)$ and b, $b' \in \mathrm{H}_*(\Omega G)$*

(i) $1 * b = b$. $y * 1 = 0$, *if* $|y| > 0$.

(ii) $\Delta_*(y * b) = \sum (y' * b') \otimes (y'' * b'')$ *where* $\Delta_* y = \sum y' \otimes y''$ *and* $\Delta_* b = \sum b' \otimes b''$. *And also* $\overline{\Delta}_*(y * b) = (\Delta_* y) * (\overline{\Delta}_* b)$.

(iii) $(yy') * b = y * (y' * b)$.

(iv) $y * (bb') = \sum (y' * b)(y'' * b')$ *where* $\Delta_* y = \sum y' \otimes y''$.

(v) *If* b *is primitive then* $y * b$ *is primitive.*

Proposition 4.3.

$$\begin{array}{rcll} y_{2m} * b_j & = & b'_{m+j} + b'_m b_j & j \leq m-1 \\ y_{2m} * b_m & = & b'_{2m} + b'_m b_m + \rho {b'_m}^2 & j = m \end{array} \tag{4}$$

for some $\rho \in \mathbf{Z}/2\mathbf{Z}$.

Proof. By Theorem 2.4 and Theorem 4.2 (ii), we can compute the coproduct of $y_{2m} * b_j$ for $1 \leq j \leq m$ as:

$$\begin{aligned} \Delta_*(y_{2m} * b_j) &= \sum_{\alpha+\beta=j} (y_{2m} * b_\alpha) \otimes b_\beta + \sum_{\alpha+\beta=j} (y_m * b_\alpha) \otimes (y_m * b_\beta) \\ &\quad + \sum_{\alpha+\beta=j} b_\alpha \otimes (y_{2m} * b_\beta). \end{aligned} \tag{5}$$

Here remark that by Proposition 3.5 $y_m * b_\alpha \neq 0$ if and only if $\alpha \geq \frac{m}{2}$. This implies $\sum_{\alpha+\beta=j} (y_m * b_\alpha) \otimes (y_m * b_\beta)$ vanishes if $j \neq m$.

Now we use induction. When $j = 1$, the above formula says

$$\overline{\Delta}_*(y_{2m} * b_1) = 0$$

where $\overline{\Delta}_*(w) = \Delta_*(w) - 1 \otimes w - w \otimes 1$. Thus $b'_{m+1} + b'_m c_1$ is primitive and $c_1 = b_1$.

For $1 < j \leq m-1$, if we assume that $c_k = b_k$ for $1 \leq k < j$, then

$$\begin{aligned}
\overline{\Delta}_*(y_{2m} * b_j) &= \sum_{\substack{\alpha+\beta=j \\ \alpha>0,\beta>0}} \left\{(b'_{m+\alpha} + b'_m b_\alpha) \otimes b_\beta + b_\alpha \otimes (b'_{m+\beta} + b'_m b_\beta)\right\} \\
&= \sum_{\substack{\alpha+\beta=j \\ \alpha>0,\beta>0}} (b'_{m+\alpha} \otimes b_\beta + b_\alpha \otimes b'_{m+\beta}) \\
&\qquad +(b'_m \otimes 1 + 1 \otimes b'_m) \sum_{\substack{\alpha+\beta=j \\ \alpha>0,\beta>0}} (b_\alpha \otimes b_\beta) \\
&= \overline{\Delta}_*(b'_{m+j} + b'_m b_j). \qquad (6)
\end{aligned}$$

Since there is no primitive element of the form $b'_m c_j$ in $\mathrm{H}_{m+j}(\Omega_0 SO(2m+2))$, $c_j = b_j$.

Next we consider the case $j = m$ and assume $c_j = b_j$ for $1 \leq j \leq m-1$. If m is odd, since $\sum_{\alpha+\beta=j}(y_m * b_\alpha) \otimes (y_m * b_\beta) = 0$, (6) is true for $j = m$. Also if m is even,

$$\begin{aligned}
\overline{\Delta}_*(y_{2m} * b_m) &= \sum_{\substack{\alpha+\beta=m \\ \alpha>0,\beta>0}} \left\{(b'_{m+\alpha} + b'_m b_\alpha) \otimes b_\beta + b_\alpha \otimes (b'_{m+\beta} + b'_m b_\beta)\right\} \\
&\qquad +(y_m * b_{m/2}) \otimes (y_m * b_{m/2}) \\
&= \sum_{\substack{\alpha+\beta=m \\ \alpha>0,\beta>0}} (b'_{m+\alpha} \otimes b_\beta + b_\alpha \otimes b'_{m+\beta}) \\
&\qquad +(b'_m \otimes 1 + 1 \otimes b'_m) \sum_{\substack{\alpha+\beta=m \\ \alpha>0,\beta>0}} (b_\alpha \otimes b_\beta) + b'_m \otimes b'_m \\
&= \overline{\Delta}_*(b'_{2m} + b'_m b_m).
\end{aligned}$$

and (6) remains true for this case. Thus $y_{2m} * b_m$ is determined up to primitive elements:

$$y_{2m} * b_m = b'_{2m} + b'_m b_m + \rho {b'_m}^2 \quad \rho \in \mathbf{Z}/2\mathbf{Z}.$$

∎

Proposition 4.4.

$$\begin{aligned}
y_{2m} * b'_{m+j} &= b'_m b'_{m+j} & 0 \leq j \leq m-1 \\
y_{2m} * b'_{2m} &= b'_m b'_{2m} + \rho {b'_m}^3 & j = m
\end{aligned}$$

where ρ is the same as in Proposition 4.3.

Proof. By Theorem 4.2 (iii),(iv),

$$\begin{aligned} y_{2m} * b'_m &= y_{2m} * (y_2 * b_{m-1}) \\ &= y_2 * (y_{2m} * b_{m-1}) \\ &= y_2 * (b'_{2m-1} + b'_m b_{m-1}) \\ &= {b'_m}^2. \end{aligned}$$

For $1 \leq j \leq m-1$, in the similar way we have

$$\begin{aligned} y_{2m} * b'_{m+j} &= y_{2m} * (y_{2j} * b_m) \\ &= y_{2j} * (y_{2m} * b_m) \\ &= y_{2j} * (b'_{2m} + b'_m b_m + \rho {b'_m}^2) \\ &= b'_m b'_{m+j}. \end{aligned}$$

Finally, it follows in the same way that

$$\begin{aligned} y_{2m} * b'_{2m} &= y_{2m} * (y_{2m} * b_m + b'_m b_m + \rho {b'_m}^2) \\ &= (y_{2m} * b'_m) b_m + b'_m (y_{2m} * b_m) \\ &= {b'_m}^2 b_m + b'_m (b'_{2m} + b'_m b_m + \rho {b'_m}^2) \\ &= b'_m b'_{2m} + \rho {b'_m}^3. \end{aligned}$$

∎

Now we summarize the above results.

Theorem 4.5. *For the generators y_k of $\mathrm{H}_*(SO(2m+2))$, b_j, b'_{m+j} of $\mathrm{H}_*(\Omega SO(2m+2))$ $(1 \leq k \leq 2m+1,\ 0 \leq j \leq m)$,*

$$\begin{aligned} &\textit{if } k \textit{ is odd} && y_{odd} * b_j &&= 0, \\ & && y_{odd} * b'_{m+j} &&= 0, \\ &\textit{if } i < m && y_{2i} * b_j &&= \begin{cases} 0 & i+j < m, \\ b'_{i+j} & i+j \geq m, \end{cases} \\ & && y_{2i} * b'_{m+j} &&= 0, \\ &\textit{if } k = 2m \textit{ and } j \neq m && y_{2m} * b_j &&= b'_{m+j} + b'_m b_j, \\ & && y_{2m} * b'_{m+j} &&= b'_m b'_{m+j}, \\ &\textit{if } k = 2m \textit{ and } j = m && y_{2m} * b_m &&= b'_{2m} + b'_m b_m, \\ & && y_{2m} * b_m &&= b'_m b'_{2m}. \end{aligned}$$

Above equations remain true for $y_k \in \mathrm{H}_*(SO(2m+1))$ and b_j, $b'_{m+j} \in \mathrm{H}_*(\Omega SO(2m+1))$. $(1 \leq k \leq 2m,\ 0 \leq j \leq m-1)$

Proof. We prove in §5 that ρ in Proposition 4.3 is zero. Then the statement for even m is true.

Because $\mathrm{H}_*(\Omega SO(2m+1))$ is a subalgebra of $\mathrm{H}_*(\Omega SO(2m+2))$ and adjoint actions commute with the respective inclusions, the whole statement follows. ∎

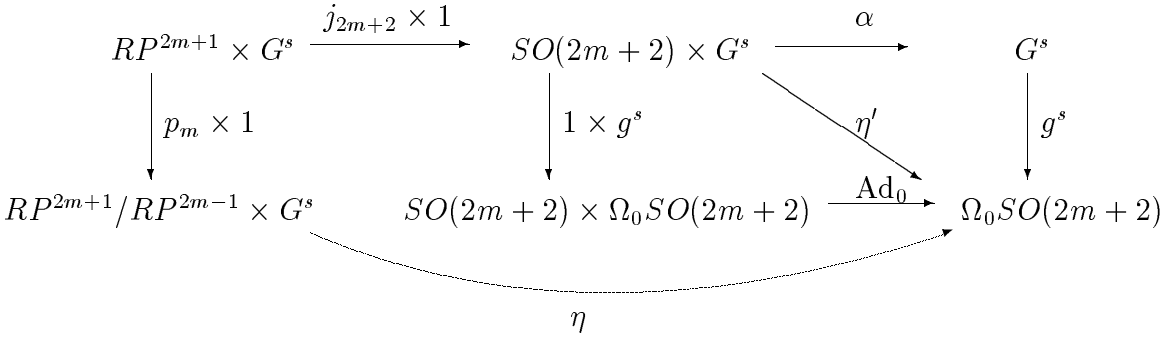
5 Determination of ρ in Proposition 4.3

To determine ρ we introduce a map η from $SO(2m+2)\otimes G^s$ to $\Omega_0 SO(2m+2)$ defined as follows.

Let f and g be the compositions of maps $\alpha \circ g^s$ and $(1 \times g^s) \circ \mathrm{Ad}_0$ respectively. We put η' as

$$\eta'(x,y) = f(x,y)g(x,y)^{-1}.$$

Refer to the diagram below.



Here remark that $\eta'|_{SO(2m)\times G^s}$ is the constant map by Lemma 3.1 and also $\eta' \circ \Omega_0 i$, where i is the inclusion of $SO(2m+2)$ into $SO(2m+4)$, is homotopic to the constant map.

There is an inclusion of RP^{n-1} into $SO(n)$ denoted by j_n, which is defined by attaching a line $l \in RP^{n-1}$ with the composition of a fixed orientation reversing self linear map A_n of $\mathbf{R}^n$ and the reflection in the direction of l. We set

$$A_n = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Since $\eta'|_{SO(2m)\times G^s} \simeq *$ and $j_{2m+2}|_{RP^{2m-1}} \subset SO(2m)$, there is a map $\eta : RP^{2m+1}/RP^{2m-1} \times G^s \to \Omega_0 SO(2m+2)$ where $\eta \circ (p_m \times 1) \simeq \eta' \circ (j_{2m+2} \times 1)$

First, we consider the next lemma.

Lemma 5.1. *In* $\mathbf{Z}/2\mathbf{Z}$ *homology,*

$$\eta'_*(y_{2m} \otimes b_m) = \rho {b'_m}^2.$$

Proof. By the definition, $\eta' = \lambda \circ (1 \times \iota) \circ (f \times g) \circ \Delta_{SO(2m+2)\times G^s}$ where ι is the map of taking the inverse in $\Omega_0 SO(2m+2)$. Thus we can proceed the calculation as:

$$\begin{aligned}
& \lambda_* \circ (1 \times \iota)_* \circ (f \times g)_* \circ \Delta_{SO(2m+2)\times G^s *}(y_{2m} \otimes b_m) \\
= \; & \lambda_* \circ (1 \times \iota)_* \circ (f \times g)_* \{ \sum_{j=0}^{m} (y_{2m} \otimes b_j) \otimes (1 \otimes b_{m-j}) \\
& \qquad + \sum_{j=0}^{m} (y_m \otimes b_j) \otimes (y_m \otimes b_{m-j}) + \sum_{j=0}^{m} (1 \otimes b_j) \otimes (y_{2m} \otimes b_{m-j}) \} \\
= \; & \lambda_* \circ (1 \times \iota)_* \{ \sum_{j=0}^{m} (b'_{m+j} + b'_m b_j) \otimes b_{m-j} + \rho {b'_m}^2 \otimes 1 \\
& + \delta_m b'_m \otimes b'_m + \sum_{j=0}^{m} b_{m-j} \otimes b'_{m+j} \} \\
= \; & (\rho + \delta_m) {b'_m}^2 + \sum_{j=0}^{m} b'_m b_j b_{m-j}.
\end{aligned}$$

(As defined in §2, $\delta_{\text{even}} = 1$ and $\delta_{\text{odd}} = 0$.)

Here remark that, when m is even, $b'_m = {b_{m/2}}^2$ and we obtain

$$\begin{aligned}
\eta'_*(y_{2m} \otimes b_m) &= (\rho + \delta_m) {b'_m}^2 + \sum_{j=0}^{m} b'_m b_j b_{m-j} \\
&= (\rho + \delta_m) {b'_m}^2 + 2 \sum_{0 \le j < m/2} b'_m b_j b_{m-j} + \delta_m b'_m {b_{m/2}}^2 \\
&= \rho {b'_m}^2.
\end{aligned}$$

∎

In the following we use integral homology. By the result of R.Bott ([7]), $\mathrm{H}_*(\Omega_0 SO(2m+2);\mathbf{Z})$ has its generators $\overline{b_j}$, $\overline{b'_{m+j}}$ $(0 \le j \le m)$, whose mod 2 reduction is b_j, b'_{m+j} in $\mathrm{H}_*(\Omega_0 SO(2m+2);\mathbf{Z}/2\mathbf{Z})$, and its relations are:

$$\begin{aligned} 0 &= \overline{b_j}^2 + 2\sum_{k=1}^{j}(-1)^k \overline{b_{j-k}b_{j+k}}, \qquad (1 \le j < m/2) \\ 0 &= \overline{b_j}^2 + 2\sum_{k=1}^{m-j}(-1)^k \overline{b_{j-k}b_{j+k}} + \sum_{k=m-j}^{j} \overline{b_{j-k}b'_{j+k}}. \qquad (m/2 \le j \le m) \end{aligned}$$

Also, it can be easily checked that

$$\mathrm{H}^*(RP^{2m+1}/RP^{2m-1};\mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}\tau_{2m} \oplus \mathbf{Z}\tau_{2m+1}$$

where $|\tau_i| = i$. And if we denote the generator of $\mathrm{H}^1(RP^{2m+1};\mathbf{Z}/2\mathbf{Z})$ by τ, the mod 2 reduction of $p_m^*\tau_i$ is τ^i.

Theorem 5.2. *In Proposition 4.3,* $\rho = 0$.

Proof. First we calculate $\eta_*(\tau_{2m} \otimes \overline{b_m}) \in \mathrm{H}_*(\Omega_0 SO(2m+2);\mathbf{Z})$ in integral homology. Since we know $\Delta_*\overline{b_m} = \sum_{i+j=m} \overline{b_i} \otimes \overline{b_j}$ (See [7].), we can compute its coproduct as

$$\begin{aligned} & \Delta_* \circ \eta_*(\tau_{2m} \otimes \overline{b_m}) \\ = \; & (\eta \times \eta)_* \circ \Delta_*(\tau_{2m} \otimes \overline{b_m}) \\ = \; & (\eta \times \eta)_*\Big(\sum_{i+j=m} \tau_{2m} \otimes \overline{b_i} \otimes 1 \otimes \overline{b_j} + \sum_{i+j=m} 1 \otimes \overline{b_i} \otimes \tau_{2m} \otimes \overline{b_j}\Big) \\ = \; & \sum_{i+j=m} \{\eta_*(\tau_{2m} \otimes \overline{b_i}) \otimes \eta_*(1 \otimes \overline{b_j})\} + \sum_{i+j=m} \{\eta_*(1 \otimes \overline{b_i}) \otimes \eta_*(\tau_{2m} \otimes \overline{b_j})\}. \end{aligned}$$

Here remark that, since $\eta'|_{SO(2m) \times G^s} \simeq *$, $\eta|_{* \times G^s}$ is null homotopic, too. Thus $\eta_*(1 \otimes \overline{b_j}) = 0$, if $j > 0$ and we obtain that $\eta_*(\tau_{2m} \otimes \overline{b_m})$ is primitive. But there exists no primitive element in $\mathrm{H}_{4m}(\Omega_0 SO(2m+2);\mathbf{Z})$. (See Proposition 10.1 in [7].) Therefore $\eta_*(\tau_{2m} \otimes \overline{b_m}) = 0$.

On the other hand, in mod 2 homology, $\eta_*(\tau_{2m} \otimes b_m) = \eta_* \circ (p_m \times 1)_*(\tau^{2m} \otimes b_m) = \eta'_* \circ (j_{2m+2} \times 1)_*(\tau^{2m} \otimes b_m) = \rho {b'_m}^2$.

Since $\mathrm{H}_*(\Omega_0 SO(2m+2);\mathbf{Z})$ is free, it follows that $\rho = 0$. ∎

6 $\mathrm{H}_*(Spin(N))$ as Hopf algebra

In this section, using Theorem 4.5, we consider the homology ring of $LG\,(Spin(N))$. First, we see the homology ring of $Spin(N)$. Theorem 6.1 and 6.2 are the dual results of Theorem 2.1.

Theorem 6.1. *There are generators y and y_i ($i \in A_N$) of $\mathrm{H}_*(Spin(N))$ as algebra where $A_N = \{1, 2, 3, \cdots, N-1\} \backslash \{$powers of 2$\}$, $|y_i| = i$ and $|y| = 2^{s+1} - 1$. (s is the integer such that $2^s < N \leq 2^{s+1}$.) And their fundamental relations are*

$$y_i{}^2 = 0, \quad y^2 = 0, \tag{7}$$

$$[y_i, y_j] = \begin{cases} y & \text{if } i + j = 2^{s+1} - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

For $1 \leq k \leq 2^p$, we set $k_i \in \{0, 1\}$ $(p = 0, 1, \cdots, p)$ as

$$k = \sum_{i=0}^{p} k_i 2^i.$$

And for $a, k \in Z$ such that a is odd, $a > 1$, $ak < N - 1$, we set $Y_{(a,k)} \in \mathrm{H}_{ak}(Spin(N))$ as $Y_{(a,k)} = \prod_{i=1}^{p} k_i y_{a2^i}$.

Theorem 6.2. *The coproducts of generators in $\mathrm{H}_*(Spin(N))$ are given as*

$$\begin{aligned} \Delta_* y_i &= \sum_{k+k'=2^p} Y_{(a,k)} \otimes Y_{(a,k')}, \\ \Delta_* y &= 0 \end{aligned}$$

where $i = a2^p$, a is odd and $a > 1$.

By the diagram in §1, we can calculate $\mathrm{H}_*(LG\,(Spin(N)))$ directly.

Theorem 6.3. *The isomorphism $\mathrm{H}_*(LG\,(Spin(N))) \cong \mathrm{H}_*(\Omega Spin(N)) \otimes \mathrm{H}_*(Spin(N))$ as $\mathbf{Z}/2\,\mathbf{Z}$ module allows us to identify $b_i, b'_{m+i} \in \mathrm{H}_*(\Omega Spin(N))$ and $y_i, y \in \mathrm{H}_*(Spin(N))$ with elements of $\mathrm{H}_*(LG\,(Spin(N)))$.*

1. *When N is even, we set $N = 2m + 2$. Then $\mathrm{H}_*(LG\,(Spin(N)))$ is the non-commutative algebra generated by b_i, b'_{m+i} ($i = 0, 1, \cdots, m$), y_j ($j \in A_N$) and y.*

2. *When N is odd, we set $N = 2m+1$. Then $\mathrm{H}_*(LG(Spin(N)))$ is the non-commutative algebra generated by b_i, b'_{m+i} $(i = 0, 1, \cdots, m-1)$, y_i $(j \in A_N)$ and y.*

This generators satisfy the fundamental relations in $\mathrm{H}_*(\Omega Spin(N))$ and $\mathrm{H}_*(Spin(N))$ and also for $y_j \in \mathrm{H}_*(Spin(N))$ and $b \in \mathrm{H}_*(\Omega Spin(N))$

$$y_j b = \sum (y' * b) y''$$

where $\Delta_* y_j = \sum y' \otimes y''$. Also above relations are the fundamental relations of the generators b_i, b'_{m+i}, y_j, y in $\mathrm{H}_*(LG(Spin(N)))$.

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