A SHARP SLOPE INEQUALITY FOR GENERAL STABLE FIBRATIONS OF CURVES

ATSUSHI MORIWAKI

ABSTRACT. Let $\overline{\mathcal{M}}_g$ be the moduli space of stable curves of genus $g \geq 2$. Let Δ_i be the irreducible component of the boundary of $\overline{\mathcal{M}}_g$ such that general points of Δ_i correspond to stable curves with one node of type i. Let $\overline{\mathcal{M}}_g^0$ be the set of stable curves that have at most one node of type $i > 0$. Let δ_i be the class of Δ_i in $\mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q}$ and λ the Hodge class on $\overline{\mathcal{M}}_g$. In this paper, we will prove a sharp slope inequality for general stable fibrations. Namely, if C is a complete curve on $\overline{\mathcal{M}}_g^0$, then $\left((8g+4)\lambda - g\delta_0 - \sum_{i=1}^{[g/2]} 4i(g-i)\delta_i \ \cdot C\right) \geq 0$. As an application, we can prove effective Bogomolov's conjecture for general stable fibrations.

CONTENTS

1. Introduction 1
2. Preliminaries 3
2.1. Canonical linear systems of reduced curves with only node singularities 3
2.2. Bogomolov inequality for fiber spaces 6
3. Proof of Theorem 1.1 6
4. Calculation of invariants arising from Green functions 12
5. Bogomolov conjecture over function fields 16
References 18

1. INTRODUCTION

Let k be an algebraically closed field. Throughout this paper, we will fix this field k.

Let X be a smooth projective surface over k, Y a smooth projective curve over k, and $f : X \to Y$ a generically smooth semistable curve over Y of genus $g \geq 2$. Let P be a node of a singular fiber X_y over y. We can assign a number i to the node P in the following way. Let $\iota : X_y' \to X_y$ be the partial normalization of X_y at P. If X_y' is connected, then $i = 0$. Otherwise, i is the minimum of arithmetic genera of two connected components of X_y'. We say the node P of the singular fiber X_y is of type i. We denote by δ_i the number of nodes of type i in singular fibers. If $\mathrm{char}(k) = 0$, we know the following inequality due to Cornalba-Harris [2] and Xiao [11]:

$$(8g+4)\deg(f_*(\omega_{X/Y})) \geq g\delta,$$

where $\delta = \delta_0 + \delta_1 + \cdots + \delta_{[g/2]}$. (By virtue of [8], this holds even if $\mathrm{char}(k) > 0$.) This inequality is actually sharp because we know an example which attains equality of the above

Date: 1995, December (Version 1.0).

inequality. When we consider a fibration with reducible fibers, we can however observe that $(8g+4)\deg(f_*(\omega_{X/Y}))$ is rather larger than $g\delta$. According to the exact formula [2, Proposition 4.7] for hyperelliptic fibrations, we can guess a sharper inequality:

$$(8g+4)\deg(f_*(\omega_{X/Y})) \geq g\delta_0 + \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i.$$

In this paper, we would like to prove this sharper inequality for general stable fibrations.

Theorem 1.1 ($\mathrm{char}(k)=0$)**.** *Let $\bar{f}:\overline{X}\to Y$ be the stable model of $f:X\to Y$. If every singular fiber of $\bar{f}:\overline{X}\to Y$ has at most one node of type $i>0$, then*

$$(8g+4)\deg(f_*(\omega_{X/Y})) \geq g\delta_0 + \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i.$$

In other words, we have the following. Let $\overline{\mathcal{M}}_g$ be the moduli space of stable curves of genus $g\geq 2$ over k. Let Δ_i be the irreducible component of the boundary of $\overline{\mathcal{M}}_g$ such that general points of Δ_i correspond to stable curves with one node of type i. Let $\overline{\mathcal{M}}_g^0$ be the set of stable curves that have at most one node of type $i>0$. Let δ_i be the class of Δ_i in $\mathrm{Pic}(\overline{\mathcal{M}}_g)\otimes\mathbb{Q}$ and λ the Hodge class on $\overline{\mathcal{M}}_g$. Then, for all complete curves C on $\overline{\mathcal{M}}_g^0$,

$$\left((8g+4)\lambda - g\delta_0 - \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i \ \cdot\ C\right) \geq 0.$$

An idea of the proof of Theorem 1.1 is as follow. Let E be the kernel of the natural homomorphism $f^*f_*(\omega_{X/Y})\to\omega_{X/Y}$. By [4], E is semistable on the geometric generic fiber $X_{\bar{\eta}}$ of f. By the same idea as in [7], we can apply Bogomolov inequality (Theorem 2.2.1) to E. This is however insufficient to get the sharper inequality. Actually we have only Cornalba-Harris-Xiao inequality (cf. Remark 3.10). For the sharper inequality, we need to modify E along singular fibers, namely, we change a compactification of $E_{\bar{\eta}}$ on $X_{\bar{\eta}}$. This modification can be done by a special elementary transformation.

As an application of Theorem 1.1, we can show the following answer for Bogomolov conjecture over function fields (cf. Theorem 5.2). First of all, we fix a notation. Let $\bar{f}:\overline{X}\to Y$ be the stable model of $f:X\to Y$. Let X_y (resp. $\overline{X}_y$) be a singular fiber of f (resp. $\bar{f}$) over y, and S_y the set of nodes P of $\overline{X}_y$ such that P is not an intersection of two irreducible components of $\overline{X}_y$. Let $\pi:Z_y\to\overline{X}_y$ be the partial normalization of $\overline{X}_y$ at each node in S_y. We say X_y is a chain of stable components if the dual graph of Z_y is homeomorphic to the interval $[0,1]$.

Theorem 1.2 ($\mathrm{char}(k)=0$)**.** *We assume that f is not smooth, every singular fiber of f is a chain of stable components, and one of the following conditions:*

(a) *the generic fiber of f is hyperelliptic, or*
(b) *every singular fiber of the stable model $\bar{f}:\overline{X}\to Y$ has at most one node of type $i>0$.*

Then Bogomolov conjecture holds for the generic fiber of f, i.e., we have the following. Let K be the function field of Y, C the generic fiber of f, $\mathrm{Jac}(C)$ the Jacobian of C, and let $j : C(\overline{K}) \to \mathrm{Jac}(C)(\overline{K})$ be a morphism given by $j(P) = \omega_C - P$. Then, $j(C(\overline{K}))$ is discrete in terms of the semi-norm arising from the Neron-Tate height paring on $\mathrm{Jac}(C)(\overline{K})$. More precisely, the maximal radius of ball in which we have only finitely many points coming from $C(\overline{K})$ via j is greater than or equal to

$$\sqrt{\frac{(g-1)^2}{g(2g+1)}\left(\frac{g-1}{3}\delta_0 + \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i\right)}.$$

This is a generalization of the main result of [8] in the case where $\mathrm{char}(k) = 0$. For the proof of the above corollary, we will calculate an invariant $e(G, D)$ for a metrized graph G with a polarization D. Using this, we have the following inequality, which is rather weaker than the sharper inequality (cf. Corollary 4.6).

Proposition 1.3 ($\mathrm{char}(k) \geq 0$)**.** *If every singular fiber of f is a chain of stable components, then*

$$\left(\omega_{X/Y} \cdot \omega_{X/Y}\right) \geq \frac{g-1}{3g}\delta_0 + \sum_{i=1}^{\left[\frac{g}{2}\right]} \left(\frac{4i(g-i)}{g} - 1\right)\delta_i.$$

Moreover, if the above inequality is strict, then Bogomolov conjecture holds for the generic fiber of f.

2. Preliminaries

In this section, we will recall base loci of canonical linear systems of reduced curves with only node singularities, and Bogomolov inequality for semistable vector bundles.

2.1. **Canonical linear systems of reduced curves with only node singularities.** Let C be a reduced projective curve over k with only node singularities. Let $\{z_1, \ldots, z_s\}$ be the set of singularities of C, and $\pi : \widetilde{C} \to C$ the normalization of C. We set $\pi^{-1}(z_i) = \{x_i, y_i\}$ for each i. The dualizing sheaf ω_C of C is defined by properties: (1) $\omega_C \subset \pi_*(\Omega^1_{\widetilde{C}}(\sum_i x_i + y_i))$ and (2) for an open set U of C, $t \in \omega_C|_U$ if and only if $\mathrm{Res}_{x_i}(t) + \mathrm{Res}_{y_i}(t) = 0$ for all i with $z_i \in U$. Note that ω_C is an invertible sheaf on C, i.e., a local section $t \in \pi_*(\Omega^1_{\widetilde{C}}(x_i + y_i))_{z_i}$ with $\mathrm{Res}_{x_i}(t) + \mathrm{Res}_{y_i}(t) = 0$ and $\mathrm{Res}_{x_i}(t) \neq 0$ forms a local frame of $(\omega_C)_{z_i}$. The arithmetic genus of C, denoted by $p_a(C)$, is given by $\dim_k H^0(C, \omega_C)$ $(= \dim_k H^1(C, \mathcal{O}_C))$.

Lemma 2.1.1 ($\mathrm{char}(k) \geq 0$)**.** *For smooth points $P_1, \ldots, P_n$ of C, let us consider a homomorphism*

$$\phi : H^0(C, \omega_C(P_1 + \cdots + P_n)) \to k^n$$

defined by $\phi(t) = (\mathrm{Res}_{P_1}(t), \ldots, \mathrm{Res}_{P_n}(t))$. If C is connected, then the kernel of ϕ is $H^0(C, \omega_C)$ and the image of ϕ is the subspace given by $\{(a_1, \ldots, a_n) \in k^n \mid a_1 + \cdots + a_n = 0\}$.

Proof. Let $\pi : \widetilde{C} \to C$, x_i, y_i and z_i be the same as before. Let t be an element of $H^0(C, \omega_C(P_1 + \cdots + P_n))$ with $\phi(t) = 0$. Then, since $\mathrm{Res}_{P_j}(t) = 0$ for all j, t has no pole at each P_j, which implies $t \in H^0(C, \omega_C)$. Thus, $\mathrm{Ker}(\phi) = H^0(C, \omega_C)$.

Next we would like to see $\sum_j \mathrm{Res}_{P_j}(t) = 0$ for all $t \in H^0(C, \omega_C(P_1 + \cdots + P_n))$. By the definition of ω_C, $t \in H^0(\widetilde{C}, \Omega^1_{\widetilde{C}}(\sum_i (x_i + y_i) + P_1 + \cdots + P_n))$ and $\mathrm{Res}_{x_i}(t) + \mathrm{Res}_{y_i}(t) = 0$ for all i. On the other hand, by Residue formula,

$$\sum_{i=1}^{s} (\mathrm{Res}_{x_i}(t) + \mathrm{Res}_{y_i}(t)) + \sum_{j=1}^{n} \mathrm{Res}_{P_j}(t) = 0.$$

Thus, $\sum_{j=1}^{n} \mathrm{Res}_{P_j}(t) = 0$. Therefore, the image of ϕ is contained in $\{(a_1, \ldots, a_n) \in k^n \mid a_1 + \cdots + a_n = 0\}$.

Since C is connected, by Serre duality,

$$\dim_k H^1(C, \omega_C(P_1 + \cdots + P_n)) = \dim_k H^0(C, \mathcal{O}_C(-P_1 - \cdots - P_n)) = 0.$$

Thus, using Riemann-Roch theorem, we have

$$\dim_k H^0(C, \omega_C(P_1 + \cdots + P_n)) = \dim_k H^0(C, \omega_C) + n - 1.$$

Since the kernel of ϕ is $H^0(C, \omega_C)$, the above formula says us that the dimension of the image of ϕ is $n - 1$. Therefore, we get the last assertion. □

Corollary 2.1.2 ($\mathrm{char}(k) \geq 0$)**.** *Let C be a reduced and irreducible projective curve over k with only node singularities. If $p_a(C) > 0$, then ω_C is generated by global sections.*

Proof. Let s be the number of singularities of C and g the genus of the normalization of C. If $s = 0$ and $g > 0$, or $s = 1$ and $g = 0$, then our assertion is trivial. We will prove it by induction on s. Let P be a node of C, $h : C' \to C$ the partial normalization at P, and $h^{-1}(P) = \{Q, R\}$. Then, by hypothesis of induction, $\omega_{C'}$ is generated by global sections and $H^0(C'\omega_{C'}) \subset H^0(C, \omega_C)$. Thus, $H^0(C, \omega_C) \otimes \mathcal{O}_C \to \omega_C$ is surjective outside P. On the other hand, by Lemma 2.1.1, there is a section t of $H^0(C', \omega_{C'}(Q + R))$ such that $\mathrm{Res}_Q(t) + \mathrm{Res}_R(t) = 0$ and $\mathrm{Res}_Q(t) \neq 0$. By the definition of ω_C, $t \in H^0(C, \omega_C)$ and t generates $(\omega_C)_P$. □

Let C be a connected reduced projective curve over k with only node singularities. Let P be a node of C and $h : C' \to C$ the partial normalization of C at P. We say P is a disconnecting node if C' is not connected. Note that, if C is semistable curve, then a disconnected node is nothing more than a node of type $i > 0$. An irreducible component D of C is said to be of socket type if D is smooth and rational, and all nodes of C on D (i.e. intersections on D with other components) are disconnecting nodes. The base locus of $|\omega_C|$, denoted by $\mathrm{Bs}(\omega_C)$, is defined by the support of $\mathrm{Coker}(H^0(C, \omega_C) \otimes \mathcal{O}_C \to \omega_C)$.

Proposition 2.1.3 ($\mathrm{char}(k) \geq 0$)**.** *Let C be a connected reduced projective curve over k with only node singularities, $D_1, \cdots, D_r$ irreducible components of socket type, and DN_C the set of all disconnecting nodes of C. Then we have the following.*

(1) $\mathrm{Bs}(\omega_C) = D_1 \cup \cdots \cup D_r \cup DN_C$.

(2) *Let P be a disconnecting node of C, $h : C' \to C$ the partial normalization of C at P, C_1 and C_2 two connected components of C'. Then,*

$$H^0(C_1, \omega_{C_1}) \oplus H^0(C_2, \omega_{C_2}) \xrightarrow{\sim} H^0(C, \omega_C).$$

(3) *If P is a disconnecting node not lying on irreducible components of socket type, then the image of $H^0(C, \omega_C) \otimes \mathcal{O}_{C,P} \to (\omega_C)_P$ coincides with the image of $(\Omega^1_C)_P \to (\omega_C)_P$.*

Proof. In the following, we denote by N_C the set of all nodes of C. In order to see (1), it is sufficient to show the following facts:

(a) If $P \in N_C$ and P is a disconnecting node, then $P \in \mathrm{Bs}(\omega_C)$.
(b) If $P \in N_C$ and P is not a disconnecting node, then $P \notin \mathrm{Bs}(\omega_C)$.
(c) If D is an irreducible component of socket type, then $D \subset \mathrm{Bs}(\omega_C)$.
(d) If D is an irreducible component not of socket type, then $H^0(C, \omega_C) \otimes \mathcal{O}_C \to \omega_C$ is surjective on $D \setminus N_C$.

(a) Let $h : C' \to C$ be the partial normalization of C at P, C_1 and C_2 two connected components of C', and $h^{-1}(P) = \{Q, R\}$ with $Q \in C_1$ and $R \in C_2$. Let t be any section of $H^0(C, \omega_C)$. Then, $t|_{C_1}$ (resp. $t|_{C_2}$) is a section of $H^0(C_1, \omega_{C_1}(Q))$ (resp. $H^0(C_2, \omega_{C_2}(R))$). Then, by Lemma 2.1.1, $\mathrm{Res}_Q(t|_{C_1}) = \mathrm{Res}_R(t|_{C_2}) = 0$. Thus, $t|_{C_1} \in H^0(C_1, \omega_{C_1})$ and $t|_{C_2} \in H^0(C_2, \omega_{C_2})$. Hence, $t(P) = 0$. Therefore, $P \in \mathrm{Bs}(\omega_C)$.

(b) Let $h : C' \to C$ be the partial normalization of C at P, and $h^{-1}(P) = \{Q, R\}$. By Lemma 2.1.1, there is a section t of $H^0(C', \omega_{C'}(Q + R))$ such that $\mathrm{Res}_Q(t) + \mathrm{Res}_R(t) = 0$ and $\mathrm{Res}_Q(t) \neq 0$. Thus, t is a section of $H^0(C, \omega_C)$ and t generates $(\omega_C)_P$.

(c) Let $D \cap N_C = \{Q_1, \ldots, Q_r\}$. Let t be any section of $H^0(C, \omega_C)$. Then, $t|_D \in H^0(D, \omega_D(Q_1 + \cdots + Q_r))$. Since Q_i is a disconnecting node, in the same way as in (a), we can see that $\mathrm{Res}_{Q_i}(t|_D) = 0$ for all i. Thus, $t|_D$ has no pole on D. Therefore, $t|_D = 0$ because $D \simeq \mathbb{P}^1$. Hence, $D \subset \mathrm{Bs}(\omega_C)$.

(d) If $p_a(D) > 0$, then our assertion is a consequence of Corollary 2.1.2. Thus, we may assume that D is smooth and rational. Let E be the closure of $C \setminus D$. Since D is not of socket type, there is a connected component D' of E with $\#(D \cap D') \geq 2$. We set $D \cap D' = \{Q_1, \ldots, Q_r\}$ $(r \geq 2)$ and $D'' = E \setminus D'$. Let R be any point of $D \setminus N_C$. We would like to see $R \notin \mathrm{Bs}(\omega_C)$. Since $\omega_D(Q_1 + \cdots + Q_r)$ is generated by global sections, there is a section $t \in H^0(D, \omega_D(Q_1 + \cdots + Q_r))$ with $t(R) \neq 0$. Since $\mathrm{Res}_{Q_1}(t) + \cdots + \mathrm{Res}_{Q_r}(t) = 0$, by Lemma 2.1.1, there is a section t' of $H^0(D', \omega_{D'}(Q_1 + \cdots + Q_r))$ such that $\mathrm{Res}_{Q_i}(t') + \mathrm{Res}_{Q_i}(t) = 0$ for all i. Moreover, let t'' be the zero form on D''. Then, t, t' and t'' give a section $s \in H^0(C, \omega_C)$ with $s|_D = t$, $s|_{D'} = t'$ and $s|_{D''} = t''$. Here $s(R) \neq 0$. Thus, s generates $(\omega_C)_R$. Therefore, $R \notin \mathrm{Bs}(\omega_C)$.

(2) is obvious by the proof of (a).

Finally, let us consider (3). Let $h : C' \to C$ be the partial normalization of C at P, C_1 and C_2 two connected components of C', and $h^{-1}(P) = \{Q, R\}$ with $Q \in C_1$ and $R \in C_2$. Let x (resp. y) be a local parameter of C_1 at Q (resp. C_2 at R). The image of $H^0(C, \omega_C) \otimes \mathcal{O}_{C,P} \to (\omega_C)_P$ is contained in the image of $(\Omega^1_C)_P \to (\omega_C)_P$ because $H^0(C, \omega_C) = H^0(C_1, \omega_{C_1}) \oplus H^0(C_2, \omega_{C_2})$. On the other hand, since P is not lying on irreducible components

of socket type, there are $t_1 \in H^0(C_1, \omega_{C_1})$ and $t_2 \in H^0(C_2, \omega_{C_2})$ with $t_1(Q) \neq 0$ and $t_2(R) \neq 0$. Locally, $t_1 = u_1 dx$ and $t_2 = u_2 dy$, where $u_1(Q) \neq 0$ and $u_2(R) \neq 0$. Thus, we have (3). □

2.2. **Bogomolov inequality for fiber spaces.** Let X be a smooth projective surface over k and E a torsion free sheaf of rank r on X. We set

$$\delta(E) = 2rc_2(E) - (r-1)c_1(E)^2.$$

Then we have the following version of Bogomolov inequality for fiber spaces.

Theorem 2.2.1 ($\mathrm{char}(k) = 0$)**.** *Let X be a smooth projective surface over k, Y a smooth projective curve over k, and $f : X \to Y$ a surjective morphism with $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. Let E be a torsion free sheaf on X. If E is semistable on the geometric generic fiber $X_{\bar{\eta}}$ of f, then $\delta(E) \geq 0$.*

Proof. Let $E^{\vee\vee}$ be the double dual of E. Then, $E^{\vee\vee}$ is locally free, $c_1(E^{\vee\vee}) = c_1(E)$, and $c_2(E^{\vee\vee}) = c_2(E) - \mathrm{length}(E^{\vee\vee}/E)$. Thus, we may assume E is locally free.

We assume $\delta(E) < 0$. Then, by Bogomolov instability theorem (cf. [1] or [6]), there is a non-zero saturated subsheaf G of E such that, if we set $D = (\mathrm{rk}\, E)c_1(G) - (\mathrm{rk}\, G)c_1(E)$, then $(D^2) > 0$ and $(D \cdot H) > 0$ for some ample divisor H. By Riemann-Roch theorem, we can see that there are a positive number n and an effective divisor L such that nD is linearly equivalent to L. Let F be a general fiber of f. Then, $(L \cdot F) \geq 0$, which implies that $(D \cdot F) \geq 0$. Here we claim that $(D \cdot F) > 0$. To see this, we assume $(D \cdot F) = 0$. Then, $(L \cdot F) = 0$. Thus, L is a linear combination of irreducible components of fibers. By Zariski lemma, this implies $(L^2) \leq 0$, which contradicts to $(D^2) > 0$. Therefore, we get $(D \cdot F) > 0$. On the other hand,

$$(D \cdot F) = (\mathrm{rk}\, E)\deg(G|_F) - (\mathrm{rk}\, G)\deg(E|_F).$$

Hence, $(D \cdot F) > 0$ means that $G|_F$ is a destabilizing subsheaf of $E|_F$. This contradicts to the assumption that E is semistable on the geometric generic fiber $X_{\bar{\eta}}$ of f. □

3. Proof of Theorem 1.1

To start the proof of Theorem 1.1, we need a lot of preparations.

Fix an integer $g \geq 2$ and a polynomial $P_g(n) = (6n-1)(g-1)$. Let $H_g \subset \mathrm{Hilb}^{P_g}_{\mathbb{P}^{5g-6}}$ be a subscheme of all tri-canonically embededded stable curves over k, $Z_g \subset H_g \times \mathbb{P}^{5g-6}$ the universal tri-canonically embededded stable curves over k, and $\pi : Z_g \to H_g$ the natural projection. Let S be the set of all points $x \in Z_g$ such that π is not smooth at x, and $\Delta = \pi(S)$. Then, by [3, Theorem (1.6) and Corollary (1.9)], Z_g, H_g, and S are smooth over k, Δ and $\pi^*(\Delta)$ are divisors with only normal crossings, and $\pi|_S : S \to \Delta$ is the normalization of Δ. Let $\Delta = \Delta_0 \cup \cdots \cup \Delta_{[g/2]}$ be the irreducible decomposition of Δ such that, if $x \in \Delta_i \setminus \mathrm{Sing}(\Delta)$, then $\pi^{-1}(x)$ is a stable curve with one node of type i. Let $\delta_i(x)$ be the number of nodes of type i on a fiber $\pi^{-1}(x)$ over $x \in Z_g$. Since $\delta_i(x) = \mathrm{mult}_x(\Delta_i)$, $\delta_i : H_g \to \mathbb{Z}$ is upper-semicontinuous. Therefore, if we set

$$H_g^0 = \{x \in H_g \mid \pi^{-1}(x) \text{ has at most one node of type } i > 0\},$$

then H_g^0 is an open set. In other words, $H_g^0 = H_g \setminus \mathrm{Sing}(\Delta_1 + \cdots + \Delta_{[g/2]})$. We set $Z_g^0 = \pi^{-1}(H_g^0)$, $\Delta^0 = \Delta \cap H_g^0$, $\Delta_i^0 = \Delta_i \cap H_g^0$, $S^0 = (\pi|_S)^{-1}(\Delta^0)$, and $S_i^0 = (\pi|_S)^{-1}(\Delta_i^0)$.

Then, for all $1 \leq i < j \leq [g/2]$, $\Delta_i^0 \cap \Delta_j^0 = \emptyset$. For $1 \leq i \leq [g/2]$, let $\pi^{-1}(\Delta_i^0) = C_i^1 \cup C_i^2$ be the irreducible decomposition such that the generic fiber of $\pi|_{C_i^1} : C_i^1 \to \Delta_i^0$ (resp. $\pi|_{C_i^2} : C_i^2 \to \Delta_i^0$) is of genus i (resp. $g-i$). Then, $C_i^1 \cap C_i^2 = S_i^0$. Moreover, we denote $S_1^0 \cup \cdots \cup S_{[g/2]}^0$ by S_+^0.

Claim 3.1. $\mathrm{Supp}(\mathrm{Coker}(\pi^*\pi_*(\omega_{Z_g^0/H_g^0}) \to \omega_{Z_g^0/H_g^0})) = S_+^0$ *and, for* $z \in S_+^0$, *the image of* $\left(\pi^*\pi_*(\omega_{Z_g^0/H_g^0})\right)_z \to \left(\omega_{Z_g^0/H_g^0}\right)_z$ *is* $\left(\Omega^1_{Z_g^0/H_g^0}\right)_z$.

Proof. The first assertion is a consequence of (1) of Proposition 2.1.3. Let F be the image of $\pi^*\pi_*(\omega_{Z_g^0/H_g^0}) \to \omega_{Z_g^0/H_g^0}$ and $x = \pi(z)$. As we see in (2) of Proposition 2.1.3, any section s of $\left(\pi_*(\omega_{Z_g^0/H_g^0})\right)_x$ has no pole on irreducible components of fibers around z, which means $F_z \subseteq \left(\Omega^1_{Z_g^0/H_g^0}\right)_z$. Therefore, by (3) of Proposition 2.1.3, we can conclude $F_z = \left(\Omega^1_{Z_g^0/H_g^0}\right)_z$. □

We set

$$E = \mathrm{Ker}\left(\pi^*\pi_*(\omega_{Z_g^0/H_g^0}) \longrightarrow \omega_{Z_g^0/H_g^0}\right).$$

Claim 3.2. *E is locally free.*

Proof. Let I_{S^0} (resp. $I_{S_+^0}$) be the defining ideal of S^0 (resp. S_+^0). It is well known that $\Omega^1_{Z_g^0/H_g^0} = I_{S^0} \cdot \omega_{Z_g^0/H_g^0}$ (cf. [10, Proof of Theorem 5.10]). Thus, by Claim 3.1, the image of $\pi^*\pi_*(\omega_{Z_g^0/H_g^0}) \to \omega_{Z_g^0/H_g^0}$ is $I_{S_+^0} \cdot \omega_{Z_g^0/H_g^0}$. Since S_+^0 is smooth and of codimension 2, $\mathcal{E}xt^i_{\mathcal{O}_{Z_g^0}}(I_{S_+^0}, \mathcal{O}_{Z_g^0}) = 0$ for $i \geq 2$. Therefore, we can see that $\mathcal{E}xt^i_{\mathcal{O}_{Z_g^0}}(E, \mathcal{O}_{Z_g^0}) = 0$ for $i \geq 1$. Hence, we have our claim. □

Claim 3.3. *For* $1 \leq i \leq [g/2]$, $\pi_*(\omega_{C_i^1/\Delta_i^0}) \oplus \pi_*(\omega_{C_i^2/\Delta_i^0}) \xrightarrow{\sim} \pi_*(\omega_{C_i^1 \cup C_i^2/\Delta_i^0})$.

Proof. Clearly we have the natural injection

$$\pi_*(\omega_{C_i^1/\Delta_i^0}) \oplus \pi_*(\omega_{C_i^2/\Delta_i^0}) \longrightarrow \pi_*(\omega_{C_i^1 \cup C_i^2/\Delta_i^0}).$$

For $x \in \Delta_i^0$, if we set $C_1 = \pi^{-1}(x) \cap C_i^1$ and $C_2 = \pi^{-1}(x) \cap C_i^2$, then by (2) of Proposition 2.1.3

$$H^0(C_1, \omega_{C_1}) \oplus H^0(C_2, \omega_{C_2}) = H^0(\pi^{-1}(x), \omega_{\pi^{-1}(x)}).$$

Thus, the above injection is bijective. □

For $1 \leq i \leq [g/2]$ and $j = 1, 2$, let

$$Q_i^j = \mathrm{Ker}\left(\left(\pi|_{C_i^j}\right)^* \left(\pi|_{C_i^j}\right)_* (\omega_{C_i^j/\Delta_i^0}) \longrightarrow \omega_{C_i^j/\Delta_i^0}\right).$$

Claim 3.4. *Q_i^j is a locally free sheaf on C_i^j.*

Proof. The homomorphism $\left(\pi|_{C_i^j}\right)^* \left(\pi|_{C_i^j}\right)_* (\omega_{C_i^j/\Delta_i^0}) \to \omega_{C_i^j/\Delta_i^0}$ is surjective by (1) of Proposition 2.1.3. Therefore, we can see our claim. □

Using projection $\pi_*(\omega_{C_i^1 \cup C_i^2/\Delta_i^0}) \to \pi_*(\omega_{C_i^j/\Delta_i^0})$, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E|_{C_i^j} & \longrightarrow & \pi^*\pi_*(\omega_{Z_g^0/H_g^0})\Big|_{C_i^j} & \longrightarrow & \Omega^1_{Z_g^0/H_g^0}\Big|_{C_i^j} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_i^j} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_i^j & \longrightarrow & \left(\pi|_{C_i^j}\right)^* \left(\pi|_{C_i^j}\right)_* (\omega_{C_i^j/\Delta_i^0}) & \longrightarrow & \omega_{C_i^j/\Delta_i^0} & \longrightarrow & 0,
\end{array}$$

where $\alpha_i^j : E|_{C_i^j} \to Q_i^j$ is the induced homomorphism. Thus, we can give a homomorphism

$$\phi_i : E \to E|_{C_i^1} \oplus E|_{C_i^2} \stackrel{\alpha_i^1 \oplus \alpha_i^2}{\longrightarrow} Q_i^1 \oplus Q_i^2.$$

Claim 3.5. *ϕ_i is surjective.*

Proof. To show this claim, it is sufficient to see it on each fiber. Let C be a fiber of π over $x \in \Delta_i^0$, and $C = C^1 \cup C^2$ the irreducible decomposition with $C^1 = C \cap C_i^1$ and $C^2 = C \cap C_i^2$. For simplicity, we set $E_C = E|_C$, $Q^1 = Q_i^1|_{C^1}$, $Q^2 = Q_i^2|_{C^2}$, i.e., E_C is the kernel of $H^0(C, \omega_C) \otimes \mathcal{O}_C \to \omega_C$ and Q^j is the kernel of $H^0(C^j, \omega_{C^j}) \otimes \mathcal{O}_{C^j} \to \omega_{C^j}$. Since $H^0(C, \omega_C) = H^0(C^1, \omega_{C^1}) \oplus H^0(C^2, \omega_{C^2})$, $Q^1 \oplus Q^2 \subset H^0(C, \omega_C) \otimes \mathcal{O}_C$. On the other hand, $Q^1, Q^2 \subset E_C$. Thus, we get $Q^1 \oplus Q^2 \subset E_C$, which shows us that $E_C \to Q^1 \oplus Q^2$ is surjective. □

Let

$$F = \mathrm{Ker}\left(\bigoplus_{i=1}^{\left[\frac{g}{2}\right]} \phi_i \ : \ E \longrightarrow \bigoplus_{i=1}^{\left[\frac{g}{2}\right]} Q_i^1 \oplus Q_i^2 \right).$$

Claim 3.6. *F is locally free.*

Proof. We set $F_1 = \mathrm{Ker}\left(E \to \bigoplus_{i=1}^{[g/2]} Q_i^1\right)$. Then, $F = \mathrm{Ker}\left(F_1 \to \bigoplus_{i=1}^{[g/2]} Q_i^2\right)$. F_1 is an elementary transformation of E. Thus, by [5], F_1 is locally free. Since ϕ_i is surjective, so is $F_1|_{C_i^2} \to Q_i^2$. Therefore, F is locally free because F is an elementary transformation of F_1. □

For a vector bundle G on Z_g^0, we define $\delta(G) \in A_{\dim H_g^0 - 1}(H_g^0)$ by

$$\delta(G) = \pi_* \left(2\,\mathrm{rk}(G) c_2(G) - (\mathrm{rk}(G) - 1) c_1(G)^2\right).$$

Claim 3.7. $\delta(E) = (8g+4) c_1(\pi_*(\omega_{Z_g^0/H_g^0})) - g\Delta_0^0 - (3g-2) \sum_{i=1}^{\left[\frac{g}{2}\right]} \Delta_i^0$.

Proof. First of all, let us recall the Grothendieck-Riemann-Roch theorem. Let $f : X \to Y$ be a proper morphism of smooth varieties over k. The Grothendieck-Riemann-Roch theorem says that, for any coherent sheaf $\mathcal{F}$ on X,

$$\mathrm{ch}\left(\sum_i (-1)^i R^i f_*(\mathcal{F})\right) = f_*(\mathrm{ch}(\mathcal{F})\,\mathrm{td}(T_{X/Y})).$$

We denote $\mathrm{ch}(\sum_i R^i f_*(\mathcal{F}))$ by $\chi(X/Y,\mathcal{F})$, and $A_i(X)$-component of $f_*(\mathrm{ch}(\mathcal{F})\,\mathrm{td}(T_{X/Y}))$ by $f_*(\mathrm{ch}(\mathcal{F})\,\mathrm{td}(T_{X/Y}))_{(i)}$. Note that if $\dim(X/Y)=1$, then

$$f_*(\mathrm{ch}(\mathcal{F})\,\mathrm{td}(T_{X/Y}))_{(\dim Y)} = f_*\left(c_1(\mathcal{F}) - \mathrm{rk}(\mathcal{F})\frac{\omega_{X/Y}}{2}\right)$$

and

$$f_*(\mathrm{ch}(\mathcal{F})\,\mathrm{td}(T_{X/Y}))_{(\dim Y-1)} = f_*\left(\frac{c_1(\mathcal{F})\cdot(c_1(\mathcal{F})-\omega_{X/Y})}{2} - c_2(\mathcal{F}) + \mathrm{rk}(\mathcal{F})\frac{\omega_{X/Y}^2 + c_2(\Omega^1_{X/Y})}{12}\right).$$

Let us go back to the proof of our claim. For simplicity, we denote $c_1(\pi_*(\omega_{Z_g^0/H_g^0}))$ and $\omega_{Z_g^0/H_g^0}$ by λ and ω respectively. Consider an exact sequence

$$0 \to E \to \pi^*(\pi_*(\omega)) \to I_{S_+^0}\cdot\omega \to 0.$$

First of all, we have $c_1(E) = \pi^*(\lambda) - \omega$. Moreover,

$$\chi(Z_g^0/H_g^0, \pi^*(\pi_*(\omega))) = \chi(Z_g^0/H_g^0, E) + \chi(Z_g^0/H_g^0, I_{S_+^0}\cdot\omega).$$

Thus, using the Grothendieck-Riemann-Roch theorem, we can see

$$\pi_*(c_2(E)) = \pi_*\left(\omega\cdot\omega - \pi^*(\lambda)\cdot\omega - S_+^0 + c_2(\pi^*(\pi_*(\omega)))\right).$$

Noting that $\pi_*(\pi^*(\lambda)\cdot\omega) = (2g-2)\lambda$, $\pi_*(c_2(\pi^*(\pi_*(\omega)))) = 0$ and $\pi_*(S_i^0) = \Delta_i^0$, the above implies

$$\pi_*(c_2(E)) = \pi_*(\omega\cdot\omega) - (2g-2)\lambda - \sum_{i=1}^{\left[\frac{g}{2}\right]}\Delta_i^0.$$

Therefore, by virtue of Noether formula: $\pi_*(\omega\cdot\omega) = 12\lambda - \Delta^0$, we can conclude our claim. □

Claim 3.8. $\delta(F) = (8g+4)c_1(\pi_*(\omega_{Z_g^0/H_g^0})) - g\Delta_0^0 - \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\Delta_i^0$.

Proof. Consider an exact sequence:

$$0 \to F \to E \to \bigoplus_{i=1}^{\left[\frac{g}{2}\right]}\left(Q_i^1 \oplus Q_i^2\right) \to 0.$$

First of all, we have

$$\begin{aligned} c_1(F) &= c_1(E) - \sum_{i=1}^{\left[\frac{g}{2}\right]}(\mathrm{rk}(Q_i^1)C_i^1 + \mathrm{rk}(Q_i^2)C_i^2) \\ &= c_1(E) - \sum_{i=1}^{\left[\frac{g}{2}\right]}((i-1)C_i^1 + (i'-1)C_i^2), \end{aligned}$$

where $i' = g - i$. Moreover, the above exact sequence gives rise to

$$\chi(Z_g^0/H_g^0, E) = \chi(Z_g^0/H_g^0, F) + \sum_{i=1}^{\left[\frac{g}{2}\right]} (\chi(C_i^1/\Delta_i^0, Q_i^1) + \chi(C_i^2/\Delta_i^0, Q_i^2)),$$

which, by Grothendieck-Riemann-Roch theorem, implies

$$\pi_*(c_2(F)) = \pi_* \left(c_2(E) + \frac{c_1(F) \cdot (c_1(F) - \omega)}{2} - \frac{c_1(E) \cdot (c_1(E) - \omega)}{2} \right) + \sum_{i=1}^{\left[\frac{g}{2}\right]} \left\{ \left(\pi|_{C_i^1}\right)_* \left(c_1(Q_i^1) - \frac{(i-1)\omega_{C_i^1/\Delta_i^0}}{2} \right) + \left(\pi|_{C_i^2}\right)_* \left(c_1(Q_i^2) - \frac{(i'-1)\omega_{C_i^2/\Delta_i^0}}{2} \right) \right\}.$$

Thus, we have

$$\begin{aligned} \delta(F) = \delta(E) &+ \sum_{i=1}^{\left[\frac{g}{2}\right]} \pi_* \left(((i-1)C_i^1 + (i'-1)C_i^2)^2 \right) \\ &+ \sum_{i=1}^{\left[\frac{g}{2}\right]} \pi_* \left(((g-1)\omega - 2c_1(E)) \cdot ((i-1)C_i^1 + (i'-1)C_i^2) \right) \\ &+ 2(g-1) \sum_{i=1}^{\left[\frac{g}{2}\right]} \left(\pi|_{C_i^1}\right)_* \left(c_1(Q_i^1) - \frac{(i-1)\omega_{C_i^1/\Delta_i^0}}{2} \right) \\ &+ 2(g-1) \sum_{i=1}^{\left[\frac{g}{2}\right]} \left(\pi|_{C_i^2}\right)_* \left(c_1(Q_i^2) - \frac{(i'-1)\omega_{C_i^2/\Delta_i^0}}{2} \right). \end{aligned}$$

Therefore, using formulae:

$$\begin{cases} \left(\pi|_{C_i^1}\right)_* (c_1(Q_i^1)) = -2(i-1)\Delta_i^0, & \left(\pi|_{C_i^2}\right)_* (c_1(Q_i^2)) = -2(i'-1)\Delta_i^0, \\ \left(\pi|_{C_i^1}\right)_* (\omega_{C_i^1/\Delta_i^0}) = 2(i-1)\Delta_i^0, & \left(\pi|_{C_i^2}\right)_* (\omega_{C_i^2/\Delta_i^0}) = 2(i'-1)\Delta_i^0, \\ \pi_*(c_1(E) \cdot C_i^1) = -(2(i-1)+1)\Delta_i^0, & \pi_*(c_1(E) \cdot C_i^2) = -(2(i'-1)+1)\Delta_i^0, \\ \pi_*(\omega \cdot C_i^1) = (2(i-1)+1)\Delta_i^0, & \pi_*(\omega \cdot C_i^2) = (2(i'-1)+1)\Delta_i^0, \\ \pi_*(C_i^1 \cdot C_i^2) = \Delta_i^0, & \pi_*(C_i^j \cdot C_i^j) = -\Delta_i^0 \quad \text{for } j = 1, 2, \end{cases}$$

we can see that

$$\delta(F) = \delta(E) + \sum_{i=1}^{\left[\frac{g}{2}\right]} (3g - 2 - 4ii')\Delta_i^0.$$

Hence, by Claim 3.7, we get

$$\delta(F) = (8g+4)c_1(\pi_*(\omega_{Z_g^0/H_g^0})) - g\Delta_0^0 - \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\Delta_i^0.$$

Proof of Theorem 1.1: Now we obtain everything to prove Theorem 1.1. Let $f : X \to Y$ be a semistable curve as in Theorem 1.1. Then, there is a morphism $h : Y \to H_g^0$ with $Z_g^0 \times_{H_g^0} Y \simeq \overline{X}$. Let $h' : X \to Z_g^0$ be the induced morphism. Let us consider a vector bundle $h'^*(F)$ on Y. By [4], $h'^*(F)$ is semistable on the generic fiber of f. Thus, $\delta(h'^*(F)) \geq 0$ by Theorem 2.2.1. On the other hand, by Claim 3.8,

$$\begin{aligned}\delta(h'^*(F)) &= \deg(h^*((8g+4)c_1(\pi_*(\omega_{Z_g^0/H_g^0})) - g\Delta_0^0 - \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\Delta_i^0)) \\ &= (8g+4)\deg(f_*(\omega_{X/Y})) - g\delta_0 - \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i.\end{aligned}$$

Thus, we have

$$(8g+4)\deg(f_*(\omega_{X/Y})) \geq g\delta_0 + \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i.$$

□

Let $\iota : Z_g^0 \to Z_g$ be the inclusion map. If we set $\overline{F} = \iota_*(F)$, then $\overline{F}$ is a reflexive coherent sheaf on Z_g because $\mathrm{codim}(Z_g \setminus Z_g^0) = 2$. Using $\overline{F}$, we can slightly generalize Theorem 1.1.

Theorem 3.9 ($\mathrm{char}(k) = 0$)**.** *Let X be a smooth projective surface over k, Y a smooth projective curve over k, and $f : X \to Y$ a semistable curve of genus $g \geq 2$ over Y. Let $h : Y \to H_g$ and $h' : X \to Z_g$ be the induced morphisms such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{h'} & Z_g \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow[h]{} & H_g \end{array}$$

If $\overline{F}$ is locally free along $h'(X)$, then

$$(8g+4)\deg(f_*(\omega_{X/Y})) \geq g\delta_0 + \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i.$$

Proof. Since $\mathrm{codim}(H_g \setminus H_g^0) = 2$, we have

$$\delta(\overline{F}) = (8g+4)c_1(\pi_*(\omega_{Z_g/H_g})) - g\Delta_0 - \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\Delta_i.$$

Thus, we can conclude our theorem in the same way as before. □

Remark 3.10. Let X be a smooth projective surface over k, Y a smooth projective curve over k, and $f : X \to Y$ a surjective morphism with $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. Then, in the same idea of Claim 3.7 or [7], we can see that, if E is the kernel of $f^*f_*(\omega_{X/Y}) \to \omega_{X/Y}$ and $\omega_{X/Y}$ is f-nef,

$$\delta(E) \leq g(\omega_{X/Y} \cdot \omega_{X/Y}) - 4(g-1)\deg(f_*(\omega_{X/Y})),$$

where $g \geq 2$ is the genus of the generic fiber. Thus, by [4] and Theorem 2.2.1, we can recover Cornalba-Harris-Xiao inequality:

$$g(\omega_{X/Y} \cdot \omega_{X/Y}) \geq 4(g-1)\deg(f_*(\omega_{X/Y})).$$

4. Calculation of invariants arising from Green functions

In this section, we would like to calculate an invariant $e(G, D)$ for a metrized graph G with a polarization D. For details of metrized graphs, see [12].

Let G be a connected metrized graph and D an $\mathbb{R}$-divisor on G. If $\deg(D) \neq -2$, then there are a unique measure $\mu_{(G,D)}$ on G and a unique function $g_{(G,D)}$ on $G \times G$ with the following properties.

(a) $\displaystyle\int_G \mu_{(G,D)} = 1$.
(b) $g_{(G,D)}(x, y)$ is symmetric and continuous on $G \times G$.
(c) For a fixed $x \in G$, $\Delta_y(g_{(G,D)}(x, y)) = \delta_x - \mu_{(G,D)}$.
(d) For a fixed $x \in G$, $\displaystyle\int_G g_{(G,D)}(x, y)\mu_{(G,D)}(y) = 0$.
(e) $g_{(G,D)}(D, y) + g_{(G,D)}(y, y)$ is a constant for all $y \in G$.

The constant $g_{(G,D)}(D, y) + g_{(G,D)}(y, y)$ is denoted by $c(G, D)$. Further we set

$$e(G, D) = 2\deg(D)c(G, D) - g_{(G,D)}(D, D).$$

First of all, let's consider another expression of $e(G, D)$.

Lemma 4.1. *Let G be a connected metrized graph and D an $\mathbb{R}$-divisor on G with $\deg(D) \neq -2$. Then, for any point O of G,*

$$e(G, D) = (\deg(D) + 2)g_{(G,D)}(O, D) + r_G(O, D),$$

where $r_G(P, Q)$ is the resistance between P and Q on G.

Proof. We set $D = \sum_i a_i P_i$. Then,

$$\begin{aligned} e(G, D) &= 2\deg(D)c(G, D) - \sum_i a_i g_{(G,D)}(D, P_i) \\ &= 2\deg(D)c(G, D) - \sum_i a_i \left(c(G, D) - g_{(G,D)}(P_i, P_i)\right) \\ &= \deg(D)c(G, D) + \sum_i a_i g_{(G,D)}(P_i, P_i) \end{aligned}$$

Since we know

$$r_G(P,Q) = g_{(G,D)}(P,P) - 2g_{(G,D)}(P,Q) + g_{(G,D)}(Q,Q)$$

for all points $P, Q \in G$, the above implies

$$\begin{aligned} e(G,D) &= \deg(D)\left(g_{(G,D)}(O,O) + g_{(G,D)}(O,D)\right) \\ &\quad + \sum_i a_i \left(r_G(O,P_i) + 2g_{(G,D)}(O,P_i) - g_{(G,D)}(O,O)\right) \\ &= (\deg(D)+2)g_{(G,D)}(O,D) + r_G(O,D). \end{aligned}$$

□

Let G_1 and G_2 be metrized graphs. Fix points $x_1 \in G_1$ and $x_2 \in G_2$. The one point sum $G_1 \vee G_2$ with respect to x_1 and x_2, defined by $G_1 \times \{x_2\} \cup \{x_1\} \times G_2$ in $G_1 \times G_2$, is a metrized graph obtained by joining $x_1 \in G_1$ and $x_2 \in G_2$. The joining point, which is $\{x_1\} \times \{x_2\}$ in $G_1 \times G_2$, is denoted by $j(G_1 \vee G_2)$. Any $\mathbb{R}$-divisor on G_i $(i = 1, 2)$ can be viewed as an $\mathbb{R}$-divisor on $G_1 \vee G_2$.

Proposition 4.2. *Let G_1 and G_2 be connected metrized graphs, and D_1 and D_2 $\mathbb{R}$-divisors on G_1 and G_2 respectively with $\deg(D_i) \neq -2$ $(i = 1, 2)$. Let $G = G_1 \vee G_2$, $O = j(G_1 \vee G_2)$, and $D = D_1 + D_2$ on $G_1 \vee G_2$. If $\deg(D_1 + D_2) \neq -2$, then*

$$\begin{aligned} e(G,D) &= e(G_1,D_1) + e(G_2,D_2) \\ &\quad + \frac{2\deg(D_2)(\deg(D_1)+2)g_{(G_1,D_1)}(O,O) + 2\deg(D_1)(\deg(D_2)+2)g_{(G_2,D_2)}(O,O)}{\deg(D_1)+\deg(D_2)+2}. \end{aligned}$$

Moreover, if $P \in G_2$, then

$$\begin{aligned} g_{(G,D)}(P,P) &= \frac{\deg(D_1)}{\deg(D_1)+\deg(D_2)+2} r_{G_2}(O,P) \\ &\quad + \frac{\deg(D_2)+2}{\deg(D_1)+\deg(D_2)+2} g_{(G_2,D_2)}(P,P) \\ &\quad - \frac{\deg(D_1)(\deg(D_2)+2)}{(\deg(D_1)+\deg(D_2)+2)^2} g_{(G_2,D_2)}(O,O) \\ &\quad + \frac{(\deg(D_1)+2)^2}{(\deg(D_1)+\deg(D_2)+2)^2} g_{(G_1,D_1)}(O,O). \end{aligned}$$

Proof. For simplicity, we set $d_i = \deg(D_i)$ and $g_i = g_{(G_i,D_i)}(O,O)$ for $i = 1, 2$. By [12, Lemma 3.7], we have

$$\mu_{(G,D)} = \frac{d_1+2}{d_1+d_2+2}\mu_{(G_1,D_1)} + \frac{d_2+2}{d_1+d_2+2}\mu_{(G_2,D_2)} - \frac{2}{d_1+d_2+2}\delta_O.$$

Consider the following function on G:

$$g(x) = \begin{cases} \dfrac{d_1+2}{d_1+d_2+2} g_{(G_1,D_1)}(O,x) + \dfrac{(d_2+2)^2 g_2 - d_2(d_1+2)g_1}{(d_1+d_2+2)^2} & \text{if } x \in G_1, \\ \dfrac{d_2+2}{d_1+d_2+2} g_{(G_2,D_2)}(O,x) + \dfrac{(d_1+2)^2 g_1 - d_1(d_2+2)g_2}{(d_1+d_2+2)^2} & \text{if } x \in G_2. \end{cases}$$

Then, we can easily check that g is continuous on G, $\Delta(g) = \delta_O - \mu_{(G,D)}$ and $\int_G g\mu_{(G,D)} = 0$. Thus, $g_{(G,D)}(O,x) = g(x)$. Therefore, by Lemma 4.1, we get the first formula. Moreover, using

$$g_{(G,D)}(P,P) - 2g_{(G,D)}(O,P) + g_{(G,D)}(O,O) = r_G(O,P) = r_{G_2}(O,P)$$

and

$$g_{(G_2,D_2)}(P,P) - 2g_{(G_2,D_2)}(O,P) + g_{(G_2,D_2)}(O,O) = r_{G_2}(O,P),$$

we obtain the second formula. □

Corollary 4.3. *Let G be a connected metrized graph and D an $\mathbb{R}$-divisor on G with $\deg(D) \neq -2$. Let C be a circle of length l. Then,*

$$e(G \vee C, D) = e(G,D) + \frac{\deg(D)}{3(\deg D + 2)} l.$$

Proof. Let $O = j(G \vee C)$ and $t : C \to [0,l)$ a coordinate of C with $t(O) = 0$. Then, it is easy to see that

$$\mu_{(C,0)} = \frac{dt}{l} \quad \text{and} \quad g_{(C,0)}(O,x) = \frac{t(x)^2}{2l} - \frac{t(x)}{2} + \frac{l}{12}.$$

Thus, we have this formula by Proposition 4.2. □

Next, let's consider $e(G,D)$ for a segment.

Lemma 4.4. *Let G be a segment of length l, and P and Q terminal points of G. Let a and b be real numbers with $a+b \neq 0$, and D an $\mathbb{R}$-divisor on G given by $D = (2a-1)P + (2b-1)Q$. Then,*

$$e(G,D) = \left(\frac{4ab}{a+b} - 1\right) l, \quad g_{(G,D)}(P,P) = \frac{b^2}{(a+b)^2} l \quad \textit{and} \quad g_{(G,D)}(Q,Q) = \frac{a^2}{(a+b)^2} l.$$

Proof. First of all, by [12, Lemma 3.7],

$$\mu_{(G,D)} = \frac{1}{a+b}(a\delta_P + b\delta_Q).$$

Let $t : G \to [0,l]$ be a coordinate of G. We set

$$f(x) = -\frac{b}{a+b} t(x) + \frac{b^2}{(a+b)^2} l.$$

Then, $\Delta(f) = \delta_P - \mu$ and $\int_G f\mu = 0$. Thus, $f(x) = g_{(G,D)}(x, P)$. Therefore,

$$g_{(G,D)}(P,P) = \frac{b^2}{(a+b)^2}l \qquad \text{and} \qquad g_{(G,D)}(P,Q) = g_{(G,D)}(Q,P) = -\frac{ab}{(a+b)^2}l.$$

In the same way, we can see that

$$g_{(G,D)}(Q,Q) = \frac{a^2}{(a+b)^2}l.$$

Therefore, we have

$$e(G,D) = \left(\frac{4ab}{a+b} - 1\right) l.$$

□

This lemma can be generalized as follows.

Proposition 4.5. *Let G_n be a metrized graph given by the following figure.*

Let l_i be the length between P_{i-1} and P_i. Let

$$D_n = (2a_0 - 1)P_0 + (2a_n - 1)P_n + \sum_{i=1}^{n-1} 2a_i P_i$$

be an $\mathbb{R}$-divisor on G with $a_i > 0$ for all i. Then, we have

$$e(G_n, D_n) = \sum_{i=1}^{n} \left(\frac{4(a_0 + \cdots + a_{i-1})(a_i + \cdots + a_n)}{a_0 + \cdots + a_n} - 1\right) l_i.$$

Proof. We set $e_n = e(G_n, D_n)$ and $t_n = g_{(G_n,D_n)}(P_n, P_n)$. We would like to prove

$$e_n = \sum_{i=1}^{n} \left(\frac{4(a_0 + \cdots + a_{i-1})(a_i + \cdots + a_n)}{a_0 + \cdots + a_n} - 1\right) l_i$$

and

$$t_n = \frac{\displaystyle\sum_{i=1}^{n} (a_0 + \cdots + a_{i-1})^2 l_i}{(a_0 + \cdots + a_n)^2}.$$

For this purpose, it is sufficient to show that

$$t_{n+1} = \frac{(a_0 + \cdots + a_n)^2}{(a_0 + \cdots + a_n + a_{n+1})^2}(t_n + l_{n+1})$$

and

$$e_{n+1} = e_n + \frac{4a_{n+1}(a_0 + \cdots + a_n)}{a_0 + \cdots + a_n + a_{n+1}} t_n + \left(\frac{4a_{n+1}(a_0 + \cdots + a_n)}{a_0 + \cdots + a_n + a_{n+1}} - 1 \right) l_{n+1}.$$

Let L be a segment of length l_{n+1}, and Q and P terminal points of L. Let E be an $\mathbb{R}$-divisor on L given by $E = Q + (2a_{n+1} - 1)P$. Let's consider a one point sum $G_n \vee L$ obtained by joining P_n and Q. Then, $G_{n+1} = G_n \vee L$ and $D_{n+1} = D_n + E$. Thus, by Proposition 4.2 and Lemma 4.4, we have the above recursive equations. □

Corollary 4.6 ($\mathrm{char}(k) \geq 0$)**.** *Let X be a smooth projective surface over k, Y a smooth projective curve over k, and $f : X \to Y$ a generically smooth semistable curve over Y of genus $g \geq 2$. Let X_y be a singular fiber of f over $y \in Y$. Let $X_y = C_1 + \cdots + C_n$ be the irreducible decomposition of X_y. Let G_y be the metrized graph given by the configuration of X_y, v_i the vertex of G_y corresponding to C_i, and ω_y the divisor on G_y defined by $\omega_y = \sum_i (\omega_{X/Y} \cdot C_i) v_i$. If X_y is a chain of stable components, then*

$$e_y = e(G_y, \omega_y) = \frac{g-1}{3g} \delta_{0,y} + \sum_{i=1}^{\left[\frac{g}{2}\right]} \left(\frac{4i(g-i)}{g} - 1 \right) \delta_{i,y},$$

where $\delta_{i,y}$ is the number of nodes of type i in X_y. In particular, if every singular fiber of f is a chain of stable components, then

$$(\omega_{X/Y} \cdot \omega_{X/Y}) \geq \frac{g-1}{3g} \delta_0 + \sum_{i=1}^{\left[\frac{g}{2}\right]} \left(\frac{4i(g-i)}{g} - 1 \right) \delta_i.$$

Moreover, if the above inequality is strict, then Bogomolov conjecture holds for the generic fiber of f.

Proof. Under our assumption, G_y can be obtained by performing one sum of one segment and many circles. Thus, using Corollary 4.3 and Proposition 4.5, we can see the formula for e_y. For the last inequality, note that $\left(\omega^a_{X/Y} \cdot \omega^a_{X/Y}\right)_a = (\omega_{X/Y} \cdot \omega_{X/Y}) - \sum_y e_y$ and $\left(\omega^a_{X/Y} \cdot \omega^a_{X/Y}\right)_a \geq 0$ (cf. [12] or [8]). Furthermore, if $\left(\omega^a_{X/Y} \cdot \omega^a_{X/Y}\right)_a > 0$, then Bogomolov conjecture holds for the generic fiber of f. (cf. [12]) □

5. Bogomolov conjecture over function fields

Let X be a smooth projective surface over k, Y a smooth projective curve over k, and $f : X \to Y$ a generically smooth semistable curve of genus $g \geq 2$ over Y. Let K be the function field of Y, $\overline{K}$ the algebraic closure of K, and C the generic fiber of f. Let $j : C(\overline{K}) \to \mathrm{Jac}(C)(\overline{K})$ be a morphism given by $j(x) = (2g-2)x - \omega_C$ and $\| \ \|_{NT}$ the semi-norm arising from the Neron-Tate height pairing on $\mathrm{Jac}(C)(\overline{K})$. We set

$$B_C(P; r) = \left\{ x \in C(\overline{K}) \mid \|j(x) - P\|_{NT} \leq r \right\}$$

for $P \in \mathrm{Jac}(C)(\overline{K})$ and $r \geq 0$, and

$$r_C(P) = \begin{cases} -\infty & \text{if } \#\left(B_C(P;0)\right) = \infty, \\ \sup\left\{r \geq 0 \mid \#\left(B_C(P;r)\right) < \infty\right\} & \text{otherwise.} \end{cases}$$

An effective version of Bogomolov conjecture claims the following.

Conjecture 5.1 (Effective Bogomolov conjecture)**.** If f is non-isotrivial, then there is an effectively calculated positive number r_0 with

$$\inf_{P \in \mathrm{Jac}(C)(\overline{K})} r_C(P) \geq r_0.$$

In [8] and [9], we proved the following results.

(A) ($\mathrm{char}(k) \geq 0$) If f is non-isotrivial and the stable model of $f : X \to Y$ has only irreducible fibers, then

$$\inf_{P \in \mathrm{Jac}(C)(\overline{K})} r_C(P) \geq \begin{cases} \sqrt{12(g-1)} & \text{if } f \text{ is smooth,} \\ \sqrt{\dfrac{(g-1)^3}{3g(2g+1)}\delta_0} & \text{otherwise.} \end{cases}$$

(B) ($\mathrm{char}(k) \geq 0$) If f is non-isotrivial and $g = 2$, then f is not smooth and

$$\inf_{P \in \mathrm{Jac}(C)(\overline{K})} r_C(P) \geq \sqrt{\frac{2}{135}\delta_0 + \frac{2}{5}\delta_1}.$$

(According to the exact calculations in [9], $(\omega_{X/Y} \cdot \omega_{X/Y}) = \frac{1}{5}\delta_0 + \frac{7}{5}\delta_1$ and $\sum_y e_y \leq \frac{5}{27}\delta_0 + \delta_1$.)

In this section, we would like to prove the following answer as an application of our slope inequality.

Theorem 5.2 ($\mathrm{char}(k) = 0$)**.** *We assume that f is not smooth, every singular fiber of f is a chain of stable components, and one of the following conditions:*

(a) *the generic fiber of f is hyperelliptic, or*

(b) *every singular fiber of the stable model of $f : X \to Y$ has at most one node of type $i > 0$.*

Then we have

$$\inf_{P \in \mathrm{Jac}(C)(\overline{K})} r_C(P) \geq \sqrt{\frac{(g-1)^2}{g(2g+1)}\left(\frac{g-1}{3}\delta_0 + \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i\right)}.$$

Proof. First of all, note the following fact (cf. [12, Theorem 5.6], [8, orollary 2.3] or [9, Theorem 2.1]). If $(\omega_{X/Y}^a \cdot \omega_{X/Y}^a)_a > 0$, then

$$\inf_{P \in \mathrm{Jac}(C)(\overline{K})} r_C(P) \geq \sqrt{(g-1)(\omega_{X/Y}^a \cdot \omega_{X/Y}^a)_a},$$

where $(\ \cdot\ )_a$ is the admissible pairing.

By the definition of admissible pairing, we can set

$$\left(\omega^a_{X/Y}\cdot\omega^a_{X/Y}\right)_a = \left(\omega_{X/Y}\cdot\omega_{X/Y}\right) - \sum_{y\in Y} e_y,$$

where e_y is $e(G_y,\omega_y)$ treated in §4. Under our assumption, by [2, Proposition 4.7] and Theorem 1.1, we have

$$(8g+4)\deg(f_*(\omega_{X/Y})) \geq g\delta_0 + \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i.$$

Thus, using Noether formula, the above inequality implies

$$(\omega_{X/Y}\cdot\omega_{X/Y}) \geq \frac{g-1}{2g+1}\delta_0 + \sum_{i=1}^{\left[\frac{g}{2}\right]}\left(\frac{12i(g-i)}{2g+1}-1\right)\delta_i.$$

Moreover, by Corollary 4.6, we get

$$\sum_y e_y = \frac{g-1}{3g}\delta_0 + \sum_{i=1}^{\left[\frac{g}{2}\right]}\left(\frac{4i(g-i)}{g}-1\right)\delta_i.$$

Thus, we have our theorem. □

References

[1] F. A. Bogomolov, Holomorphic tensors and vector bundles on projective varieties, Math. USSR-Izv., 13 (1978), 499–555

[2] M. Cornalba and J. Harris, Divisor classes associated to families of stable varieties, with application to the moduli space of curves, Ann. Scient. Ec. Norm. Sup., 21 (1988), 455–475.

[3] P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus, Publ. Math. IHES, 36 (1969), 75–110.

[4] A. Paranjape and S. Ramanan, On the canonical ring of an algebraic curve, Algebraic Geometry and Commutative Algebra in Honor of Masayoshi NAGATA, vol. II (1987), 501–516.

[5] M. Maruyama, On a family of algebraic vector bundles, Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki, Kinokuniya, Tokyo, (1973), 95–146.

[6] A. Moriwaki, Arithmetic Bogomolov-Gieseker's inequality, Amer. J. Math. 117 (1995), 1325–1347.

[7] A. Moriwaki, Faltings modular height and self-intersection of dualizing sheaf, Math. Z., 220 (1995), 273–278.

[8] A. Moriwaki, Bogomolov conjecture over function fields for stable curves with only irreducible fibers, to appear in Comp. Math..

[9] A. Moriwaki, Bogomolov conjecture for curves of genus 2 over function fields, preprint (1995), Kyoto-Math 95-13.

[10] D. Mumford, Stability of projective varieties, L'Ens. Math. 23 (1997), 39-110.

[11] G. Xiao, Fibered algebraic surfaces with low slope, Math. Ann., 276 (1987), 449–466.

[12] S. Zhang, Admissible pairing on a curve, Invent. Math., 112 (1993), 171–193.

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto, 606-01, Japan