Cycles on the Moduli Space of Abelian Varieties

Gerard van der Geer

Introduction

In this paper I present a number of results on cycles on the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g. The results on the tautological ring are my own work, the results on the torsion of λ_g and on the cycle classes of the Ekedahl-Oort stratification are joint work with Torsten Ekedahl and some of the results on curves are joint work with Carel Faber. Our results include:

- A description of the tautological subring – generated by the Chern classes λ_i of the Hodge bundle $\mathbb{E}$ – of the Chow ring of $\mathcal{A}_g$ and its compactifications.
- As a corollary we find the Hirzebruch-Mumford proportionality theorem for $\mathcal{A}_g$ and bounds for the dimension of complete subvarieties of $\mathcal{A}_g$.
- A bound for the order of the torsion of the top Chern class λ_g of the Hodge bundle.
- A description of the Ekedahl-Oort stratification of $\mathcal{A}_g \otimes \mathbb{F}_p$ in terms of degeneracy loci of a map between flag bundles.
- The description of the Chow classes of the strata of this stratification. This includes as special cases formulas for the classes of loci like p-rank $\leq f$ locus or a-number $\geq a$ locus.
- The irreducibility of the locus T_a of abelian varieties of a-number $\geq a$ for $a < g$.
- A computation of this stratification for hyperelliptic curves of 2-rank 0 in characteristic 2.
- A formula for the class of the supersingular locus for low genera.

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§1. The Tautological Subring of $\mathcal{A}_g$ and of $\tilde{\mathcal{A}}_g$.

Let $\mathcal{A}_g/\mathbb{Z}$ denote the moduli stack of principally polarized abelian varieties of dimension g. This is an irreducible algebraic stack of relative dimension $g(g+1)/2$. All the fibres $\mathcal{A}_g \otimes \mathbb{F}_p$ and $\mathcal{A}_g \otimes \mathbb{Q}$ are irreducible. This stack carries a locally free sheaf $\mathbb{E}$ of rank g ("the Hodge bundle") defined by giving for every morphism $S \to \mathcal{A}_g$ a locally free

sheaf of rank g which is compatible with pull backs. It is defined as $s^*\Omega_{A/S}$, where A/S is the principally polarized abelian variety corresponding to $S \to \mathcal{A}_g$ and s is the zero section of A/S. If $\pi : \mathcal{X}_g \to \mathcal{A}_g$ is the universal abelian variety we have $\Omega_{\mathcal{X}_g/\mathcal{A}_g} = \pi^*(\mathbb{E})$.

Let $\tilde{\mathcal{A}}_g$ be a smooth toroidal compactification of $\mathcal{A}_g$. The Hodge bundle can be extended to a locally free sheaf on $\tilde{\mathcal{A}}_g$.

The Chern classes λ_i of the Hodge bundle $\mathbb{E}$ are defined over $\mathbb{Z}$ and give rise to classes λ_i in $CH^*(\mathcal{A}_g)$, and in $CH^*(\tilde{\mathcal{A}}_g)$. They generate subrings ($\mathbb{Q}$-subalgebras) of $CH^*_{\mathbb{Q}}(\mathcal{A}_g)$ and of $CH^*_{\mathbb{Q}}(\tilde{\mathcal{A}}_g)$ which we shall call the *tautological subrings*.

We shall first describe these tautological subrings. It will turn out that the tautological subring of $CH^*_{\mathbb{Q}}(\mathcal{A}_g)$ is isomorphic to the cohomology ring R_{g-1} of the compact dual of the Siegel upper half space of degree $g-1$, while the tautological subring of $CH^*_{\mathbb{Q}}(\tilde{\mathcal{A}}_g)$ is isomorphic to R_g. This cohomology ring is of the form

$$R_g = \mathbb{Q}[u_1, \ldots, u_g]/((1 + u_1 + \ldots + u_g)(1 - u_1 + u_2 - \ldots + (-1)^g u_g) - 1)$$

and is a Gorenstein ring. As a corollary of this we find the Proportionality Principle of Hirzebruch and Mumford for $\mathcal{A}_g$ and its compactifications.

We have the following relation for the Chern classes λ_i on $\mathcal{A}_g$.

(1.1) Theorem. *The Chern classes λ_i in $CH^*_{\mathbb{Q}}(\mathcal{A}_g)$ satisfy the relation*

$$(1 + \lambda_1 + \ldots + \lambda_g)(1 - \lambda_1 + \ldots + (-1)^g \lambda_g) = 1. \tag{1}$$

The idea of the proof is to apply the Grothendieck-Riemann-Roch theorem to the theta divisor on the universal abelian variety $\mathcal{X}_g$ over $\mathcal{A}_g$. We choose this divisor (on a level cover) so that its restriction $s^*(\Theta)$, with s the zero section, is trivial on $\mathcal{A}_g$ and apply Grothendieck-Riemann-Roch to the line bundle $L = O(\Theta)$:

$$\begin{aligned} ch(\pi_! L) &= \pi_*(ch(L) \cdot \mathrm{Td}(\Omega^1_{\mathcal{X}_g/\mathcal{A}_g})^\vee) \\ &= \pi_*(ch(L) \cdot \mathrm{Td}(\pi^*(\mathbb{E}^\vee))) \\ &= \pi_*(ch(L)) \cdot \mathrm{Td}(\mathbb{E}^\vee) \end{aligned}$$

by the projection formula. Since $R^i\pi_*(L) = 0$ for $i > 0$ it follows that $\pi_!(L)$ is a vector bundle and it is of rank one since Θ is a principal polarization. We write $c_1(\pi_!(L)) = \theta$ and find

$$\sum_{k=0}^{\infty} \frac{\theta^k}{k!} = \pi_*(\sum_{k=0}^{\infty} \frac{\Theta^{g+k}}{(g+k)!}) \cdot \mathrm{Td}(\mathbb{E}^\vee). \tag{2}$$

Comparison of the term of degree 1 gives

$$\theta = -\lambda_1/2 + \pi_*(\Theta^{g+1})/(g+1)!.$$

Replace now L by $L^{\otimes n}$. The term of degree k in $\pi_*(\sum \Theta^{g+k}/(g+k)!)$ changes by a factor n^{g+k}. But $\pi_!(L^n)$ is a numerical function of degree $\leq n^g$, cf. the arguments of

Chai-Faltings on p. 26 of [F-C]. Or, alternatively, using the Heisenberg group we find on a suitable cover of $\mathcal{A}_g$

$$\pi_! L^n = \pi_! L \otimes A$$

where A is the standard irreducible representation of dimension n^g of the Heisenberg group. Therefore, up to torsion we have

$$ch(\pi_! L^n) = n^g ch(\pi_! L).$$

This then proves by induction that

$$\pi_*(\sum_{k=0}^{\infty} \frac{\Theta^{g+k}}{(g+k)!}) = 1 \in CH^*_{\mathbb{Q}}(\mathcal{A}_g). \tag{3}$$

In particular we get

$$2\theta = -\lambda_1 \qquad \text{(the "key formula").}$$

Therefore, if we write λ_j as the jth symmetric function of $\alpha_1, \ldots, \alpha_g$ and if we use $\mathrm{Td}(\mathbb{E}^\vee) = \prod(\alpha_i/(e^{\alpha_i} - 1))$ the identity (2) becomes

$$\prod_{i=1}^{g} \frac{e^{\alpha_i/2} - e^{-\alpha_i/2}}{\alpha_i} = 1.$$

and implies $\mathrm{Td}(\mathbb{E} \oplus \mathbb{E}^\vee) = 1$. This is equivalent with $ch_{2k}(\mathbb{E}) = 0$ for $k \geq 1$ and with (1). $\square$

(1.2) Proposition. *In $CH^*_{\mathbb{Q}}(\mathcal{A}_g)$ we have $\lambda_g = 0$.*

Proof. We apply GRR to the structure sheaf $O_{\mathcal{X}}$ of the universal abelian variety $\mathcal{X}$ over the stack $\mathcal{A}_g$. We get

$$\begin{aligned} ch(\pi_! O_{\mathcal{X}}) &= \pi_*(ch(O_{\mathcal{X}}) \cdot \mathrm{Td}(\Omega^1)^\vee) \\ &= \pi_*(1)\mathrm{Td}(\mathbb{E}^\vee), \end{aligned}$$

which gives

$$ch(1 - \mathbb{E}^\vee + \wedge^2 \mathbb{E}^\vee - \ldots + (-1)^g \wedge^g \mathbb{E}^\vee) = \pi_*(1)\mathrm{Td}(\mathbb{E})^\vee = 0.$$

Now for a vector bundle B of rank r we have in general the relation (see [B-S])

$$\sum_{j=0}^{r} (-1)^j ch(\wedge^j B^\vee) = c_r(B)\mathrm{Td}(B)^{-1}.$$

This gives: $\lambda_g = 0$ in $CH^*(\mathcal{A}_g)$. $\square$

The ring R_g is the quotient of the graded ring $\mathbb{Q}[u_1, \ldots, u_g]$ with generators u_i of degree i by the relation

$$(1 + u_1 + u_2 + \ldots + u_g)(1 - u_1 + u_2 - \ldots + (-1)^g u_g) - 1 = 0.$$

This relation implies (by induction)

$$u_g u_{g-1} \cdots u_{k+1} u_k^2 = 0 \qquad \text{for} \quad k = 1, \ldots, g. \tag{4}$$

This ring has additive generators

$$u_1^{\epsilon_1} u_2^{\epsilon_2} \ldots u_g^{\epsilon_g} \qquad \text{with} \quad \epsilon_j \in \{0, 1\}.$$

Obviously, we have $R_g/(u_g) \cong R_{g-1}$.

It follows that the tautological subring of $CH^*_{\mathbb{Q}}(\mathcal{A}_g)$ is a homomorphic image of the ring $R_g/(u_g) \cong R_{g-1}$.

Now consider the moduli space $\mathcal{A}_g \otimes \mathbb{F}_p$. It contains the loci $V_f = V_f(p)$ of abelian varieties with p-rank $\leq f$. Their closures in $\mathcal{A}_g^*$ and in $\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$ define loci again denoted V_f. These loci have been studied by Oort and Norman (cf. [O1 , N-O]).

(1.3) Lemma. *The subvariety V_f is complete in the moduli space $\tilde{\mathcal{A}}_g^{(f)} \subset \tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$ of rank$\leq f$ degenerations; in particular V_0 is complete in $\mathcal{A}_g$.*

(1.4) Corollary. *We have $\lambda_1^{g(g-1)/2+f} \neq 0$ on $\tilde{\mathcal{A}}_g^{(f)}$.*

Proof. Observe that $\det(\mathbb{E})$ is an ample line bundle (its sections are modular forms) and so λ_1 is ample on $\mathcal{A}_g^*$, see [M-B]. On a complete variety of dimension d the d-th power of an ample divisor is non-zero. □

(1.5) Theorem. *The tautological subring of $CH^*_{\mathbb{Q}}(\mathcal{A}_g)$ generated by the λ_i is isomorphic to R_{g-1}.*

Proof. By the relation $(1 + \lambda_1 + \ldots + \lambda_g)(1 - \lambda_1 + \ldots + (-1)^g \lambda_g) = 1$ we get a quotient ring of R_g. Since moreover $\lambda_g = 0$ we get a quotient of $R_g/(u_g) \cong R_{g-1}$. This ring R_{g-1} is Gorenstein and its top degree elements are proportional. We now use the fact that $\mathcal{A}_g \otimes \mathbb{F}_p$ has a complete subvariety of codimension g, namely the p-rank zero locus. The existence of of a complete subvariety of dimension $g(g-1)/2$ in $\mathcal{A}_g \otimes \mathbb{F}_p$ for every prime p and the ampleness of λ_1 on $\mathcal{A}_g \otimes \mathbb{F}_p$ imply $\lambda_1^{g(g-1)/2} \neq 0$. This implies that in the quotient of R_{g-1} the one-dimensional socle does not map to zero, hence that the quotient is isomorphic to R_{g-1}. Or more explicitly, consider the set of 2^{g-1} generators of the form

$$\lambda_1^{\epsilon_1} \lambda_2^{\epsilon_2} \cdots \lambda_{g-1}^{\epsilon_{g-1}} \quad \text{with} \quad \epsilon_i \in \{0, 1\}.$$

Order these elements λ_ϵ with $\epsilon \in \{0, 1\}^{g-1}$ lexicographically. Suppose we have a relation

$$\sum a_\epsilon \lambda_\epsilon = 0$$

in $CH^*_{\mathbb{Q}}(\mathcal{A}_g)$. Suppose that ϵ' is the 'smallest' exponent. Let ϵ'' be the complementary exponent with $\epsilon' + \epsilon'' = (1, \ldots, 1)$. Then we have

$$\lambda_{\epsilon''}(\sum a_\epsilon \lambda_\epsilon) = a_{\epsilon'} \lambda_{g-1} \cdots \lambda_1 = 0,$$

and this implies that $a_{\epsilon'} = 0$. By induction the theorem follows. □

(1.6) Corollary. *In $CH^*_{\mathbb{Q}}(\mathcal{A}_g)$ we have: $\lambda_1^{g(g-1)/2} \neq 0$ and $\lambda_1^{1+g(g-1)/2} = 0$.*

Proof. The first statement was given in (1.4). In the ring R_{g-1} all top monomials (of degree $g(g-1)/2$) are proportional, hence $\lambda_1^{g(g-1)/2}$ is a non-zero multiple of the top monomial $\lambda_{g-1}\lambda_{g-2}\cdots\lambda_1$, so $\lambda_1^{g(g-1)/2+1}$ is a non-zero multiple of $\lambda_{g-1}\lambda_{g-2}\cdots\lambda_2\lambda_1^2$, which is zero by (4). □

(1.7) Corollary. *Let $\mathbb{F}$ be a field. A complete subvariety of $\mathcal{A}_g \otimes \mathbb{F}$ has codimension $\geq g$.*

Proof. If Z is a complete subvariety of dimension m then $\lambda_1^m \neq 0$ on Z. Since $g(g-1)/2$ is the highest power of λ_1 which is not zero on $\mathcal{A}_g$ the result follows. □

For a discussion of questions concerning complete subvarieties of $\mathcal{A}_g$ we refer to [O2].

Now we come to the tautological ring of a suitable toroidal compactification of $\mathcal{A}_g$. We shall consider various compactifications of $\mathcal{A}_g$. We choose a suitable compactification $\tilde{\mathcal{A}}_g$ as constructed in [F-C]. We let $\mathcal{A}_g^*$ be the minimal ('Satake') compactification as in [F-C]. Furthermore, we let $\tilde{\mathcal{A}}_g^{(1)}$ be the moduli space of rank 1-degenerations, i.e. the inverse image of $\mathcal{A}_g \cup \mathcal{A}_{g-1} \subset \mathcal{A}_g^*$ under the natural map $q : \tilde{\mathcal{A}}_g \to \mathcal{A}_g^*$. This space $\mathcal{A}'_g = \tilde{\mathcal{A}}_g^{(1)}$ does *not* depend on a choice $\tilde{\mathcal{A}}_g$ of compactification of $\mathcal{A}_g$. See [M3].

Let $\mathcal{G}$ be the 'universal' semi-abelian variety over $\tilde{\mathcal{A}}_g$ with zero section $s : \tilde{\mathcal{A}}_g \to \mathcal{G}$. Then we have $\mathbb{E} = s^*(\mathrm{Lie}\mathcal{G})^\vee$. Moreover, let $D = \tilde{\mathcal{A}}_g - \mathcal{A}_g$ be the divisor at infinity. Then we have the isomorphism (cf. [F-C]. p. 117):

$$\mathrm{Sym}^2(\mathbb{E}) \cong \Omega^1(\log D).$$

Since the class λ_g vanishes on $\mathcal{A}_g$ it can be represented in the form $\lambda_g = i_*(x)$ with x a class in $CH^{g-1}_{\mathbb{Q}}(D)$ and $i\colon D \to \tilde{\mathcal{A}}_g$ the embedding of the boundary. By applying Grothendieck-Riemann-Roch to the structure sheaf on the semi-abelian variety over $\tilde{\mathcal{A}}_g$ (as in (1.2)) one gets an expression for x in $CH^*_{\mathbb{Q}}(D)$, see [EFG].

As we shall see later in characteristic p a multiple of the class λ_g is represented in the Chow ring by the class of a complete subvariety V_0 of $\mathcal{A}_g$. More generally, a multiple of λ_{g-f} is represented by a complete subvariety of $\mathcal{A}_g^{(f)}$, the moduli space of rank $\leq f$ degenerations.

(1.8) Lemma. *The class $q_*(\lambda_g)$ in $CH^g_{\mathbb{Q}}(\mathcal{A}_g^*)$ is represented by a multiple of the fundamental class of the boundary $B_g^* = \mathcal{A}_g^* - \mathcal{A}_g$.*

Indeed, λ_g is zero on $\mathcal{A}_g$; for dimension reasons $q_*\lambda_g$ is represented by a multiple of the fundamental class of B_g^*.

(1.9) Proposition. *The cycle class $[B_g^*]$ of the boundary is the same in the Chow group $CH^g_{\mathbb{Q}}(\mathcal{A}_g^*)$ as a multiple of the class of the image of $\mathcal{A}_{g-1}$ in $\mathcal{A}_g$ under the map $[X] \mapsto [X \times E]$, with E a generic elliptic curve.*

Proof. Consider (for $g > 2$) the space $\mathcal{A}_{g-1,1} \sim \mathcal{A}_{g-1} \times \mathcal{A}_1$ in $\mathcal{A}_g$. Since $\mathcal{A}_1$ is the affine j-line we find a rational equivalence between the cycle class of a generic fibre $\mathcal{A}_{g-1} \times \{j\}$ and a multiple of the fundamental class of the boundary B_g^*. $\square$

Let B_g be the cycle on $\mathcal{A}_g^{(1)}$ defined by the semi-abelian varieties which are trivial extensions

$$1 \to \mathbb{G}_m \to G \to X_{g-1} \to 0,$$

where X_{g-1} is a $(g-1)$-dimensional abelian variety. The cycle $B_g \sim \mathcal{A}_{g-1}$ is of codimension g in $\mathcal{A}_g^{(1)}$ and extends to $\tilde{\mathcal{A}}_g$.

The following proposition describes the class of B_g in *cohomology*.

(1.10) Proposition. *The cohomology class of B_g and of $\mathcal{A}_{g-1} \times \{j\}$ are both equal to $(-1)^g \lambda_g / \zeta(1-2g)$. (Here $\zeta(s)$ denotes the Riemann zeta function.)*

Proof. The intersection of the closure of $\mathcal{A}_{g-1,1}$ with the boundary divisor in $\tilde{\mathcal{A}}_g$ is the closure of B_g. Using the rational equivalence of B_g and $\mathcal{A}_{g-1} \times \{j\}$ for generic j and the fact that the class of $\mathcal{A}_{g-1} \times \{j\}$ on $\mathcal{A}_g^*$ was a multiple of $q_* \lambda_g$ the result follows by pull back. To find the multiple we look at characteristic zero and integrate the $Sp(2g, \mathbb{R})$-invariant forms representing the λ_i. $\square$

Examples. We have in cohomology:

$g = 1 \qquad [B_1] = 12\lambda_1;$

$g = 2 \qquad [B_2] = 120\lambda_2.$

Consider the moduli space $\mathcal{A}'_g = \tilde{\mathcal{A}}_g^{(1)}$ of rank ≤ 1 degenerations. Let D^0 be the closed subset corresponding to rank 1 degenerations. The divisor D^0 has a morphism to $\phi : D^0 \to \mathcal{A}_{g-1}$ which exhibits D^0 as the universal abelian variety over $\mathcal{A}_{g-1}$. The fibre over $x \in \mathcal{A}_{g-1}$ is the dual $\hat{X}_{g-1}$ of the abelian variety X corresponding to x. The 'universal' semi-abelian variety G over $\hat{\mathcal{X}}_{g-1}$ is the $\mathbb{G}_m$-bundle obtained from the Poincaré bundle $P \to \mathcal{X}_{g-1} \times \hat{\mathcal{X}}_{g-1}$ by deleting the zero-section. We have the maps

$$G = P - \{(0)\} \to \mathcal{X}_{g-1} \times \hat{\mathcal{X}}_{g-1} \to \mathcal{A}_{g-1}.$$

The divisor D^0 contains the subvariety B_g corresponding to the trivial extensions

$$1 \to \mathbb{G}_m \to G \to X_{g-1} \to 0$$

and this is a codimension g cycle $B_g \sim \mathcal{A}_{g-1}$ in $\mathcal{A}'_g$.

Consider now the cotangent bundle to G at the zero section t of $G \to \hat{X}_{g-1}$. We have an exact sequence

$$0 \to q^* \mathbb{E}_{g-1} \to \mathbb{E}_{g\,|D^0} \to U \to 0,$$

with U a pull back of a line bundle on $\mathcal{A}^*_{g-1}$. Now U is trivial since the restriction of $\mathbb{E}_g$ to B_g is a direct sum of $\mathbb{E}_{g-1}$ and a trivial line bundle.

Consider $\mathcal{X}'$, the compactified family of semi-abelian varieties over the moduli stack $\mathcal{A}'_g$ of rank ≤ 1 degenerations. A semi-abelian variety which is a $\mathbb{G}_m$-bundle over an abelian variety X_{g-1} is compactified by taking the $\mathbb{P}^1$-bundle associated to the $\mathbb{G}_m$-bundle and then identifying the 0- and the ∞-section under a shift $\xi \in X_{g-1}$. The image of the zero section of the $\mathbb{P}^1$-bundle maps to a codimension 2 cycle Δ, the locus of singular points of the fibres of $\mathcal{X}'_g \to \mathcal{A}'_g$. We have

$$\Delta \cong \mathcal{X}_{g-1} \times_{\mathcal{A}_{g-1}} \hat{\mathcal{X}}_{g-1}.$$

We analyze $\Omega^1 = \Omega_{\mathcal{X}'_g/\mathcal{A}'_g}$. We have an exact sequence

$$0 \to \Omega^1 \to \pi^*(\mathbb{E}) \to \mathcal{F} \to 0, \tag{5}$$

Here $\mathcal{F}$ is a sheaf with support on Δ, the codimension 2 cycle. Let u be a fibre coordinate on a $\mathbb{G}_m$ bundle over $\mathcal{X}_{g-1}$. A section of $\pi^*(\mathbb{E})$ is given locally by du/u. Pull a section back to the $\mathbb{P}^1$-bundle and take the residue along the 0- and ∞-section. The residue map yields an isomorphism on $\mathcal{A}'_g$

$$\mathcal{F} \cong O_{\tilde{\Delta}},$$

where $\tilde{\Delta}$ is the etale double cover of Δ corresponding to choosing the branches (0 and ∞).

(1.11) Main Theorem. *The tautological subring of $\tilde{\mathcal{A}}_g$ in $CH^*_{\mathbb{Q}}(\tilde{\mathcal{A}}_g)$ is isomorphic to R_g.*

In order to prove this we shall apply GRR to the Θ-divisor again. We can do this on a level cover of $\tilde{\mathcal{A}}_g$ for a line bundle $L = O(T)$ trivialized along the zero section. But we start in codimension 1 and therefore we work on $\mathcal{A}'_g$. There we have:

$$ch(\pi_!(L^{\otimes n})) = \pi_*(e^{nT} \cdot \mathrm{Td}^\vee(O_\Delta)^{-1})\mathrm{Td}^\vee(\mathbb{E}). \tag{6}$$

In particular, for $n = 1$ we have

$$ch(\pi_!(L)) = \pi_*(e^{T} \cdot \mathrm{Td}^\vee(O_\Delta)^{-1})\mathrm{Td}^\vee(\mathbb{E}). \tag{7}$$

We write

$$ch(\pi_!(L^{\otimes n})) = 1 + \theta_1^{(n)} + \theta_2^{(n)} + \ldots$$

and set $\theta_1^{(1)} = \theta$. In equation (7) we compare terms of codimension 1:

$$\theta_1^{(n)} = -\frac{\lambda_1}{2} \cdot \pi_*[e^{nT} \cdot \mathrm{Td}^\vee(O_\Delta)^{-1})]_0 + \pi_*[e^{nT} \cdot \mathrm{Td}^\vee(O_\Delta)^{-1})]_1$$

(1.12) Lemma. *We have $\theta = -\lambda_1/2 + \delta/8$, where δ is the class of the 'boundary' D.*

Proof. First we do the case $g = 1$. We have

$$1 + \theta = (\pi_*(T + T^2/2) + \pi_*(\Delta/12))(1 - \lambda_1/2).$$

Let S be the zero-section. By Kodaira's results on elliptic surfaces we have $S^2 = -\lambda_1$; so the normalized T is $T = S + \pi^*(\lambda_1)$ with $T^2 = S \cdot \pi^*(\lambda_1)$, i.e. $\pi_*(T^2/2) = \lambda_1/2$. We find

$$\theta = -\lambda_1/2 + \lambda_1/2 + \delta/12.$$

We can rewrite this as

$$\theta = -\lambda_1/2 + \delta/8.$$

For general g we have a priori $\theta = -\lambda_1/2 + a\delta$. Restriction to the space of products of elliptic curves $\mathcal{A}_{1,\ldots,1}$ gives $a = 1/8$. $\square$

As a corollary we find

$$\theta_1^{(n)} = n^g(-\lambda_1/2) + (n^{g+1} + 2n^{g-1})\delta/24.$$

To finish the proof we now work on the whole space $\tilde{\mathcal{A}}_g$ and we also introduce a (suitable) level ℓ structure and then apply Grothendieck-Riemann-Roch on the moduli space $\tilde{\mathcal{A}}_g[\ell]$. The Hodge bundle is a pull back, but since the natural maps $\tilde{\mathcal{A}}_g[\ell] \to \tilde{\mathcal{A}}_g$ are ramified over the divisor D one can separate the contributions from the 'interior' and the boundary in the analogue of (6) by their dependence on ℓ. This leads to the fact that the pull back of the formula $e^{-\lambda_1/2} = \mathrm{Td}^\vee(\mathbb{E})$ holds on the spaces $\tilde{\mathcal{A}}_g[\ell]$. We refer to [EFG] for the details of the proof.

A top monomial u^α in R_g can be written as a multiple $m_\alpha u_1 u_2 \ldots u_g$ with $m_\alpha \in \mathbb{Z}$ using the relation (1). Therefore, the degree $\deg \lambda^\alpha$ equals $m_\alpha \deg \lambda_1 \lambda_2 \ldots \lambda_g([\tilde{\mathcal{A}}_g])$. The ring R_g is also the Chow ring (and cohomology ring) of the Lagrangian Grassmann variety Y_g of maximal isotropic subspaces of a symplectic complex vector space of dimension $2g$, see [P] and the references there. If we identify R_g with $CH^*_{\mathbb{Q}}(Y_g)$ then we find as a corollary:

(1.13) Theorem. *(Proportionality Principle of Hirzebruch-Mumford) The characteristic numbers of the Hodge bundle are proportional to those of the tautological bundle on Y_g:*

$$\lambda^\alpha([\tilde{\mathcal{A}}_g]) = (-1)^G \frac{1}{2^g} \prod_{k=1}^{g} \zeta(1-2k) \cdot u^\alpha([Y_g])$$

with $G = g(g+1)/2$.

Proof. We have $\lambda^\alpha([\tilde{\mathcal{A}}_g]) = c(g) \times u^\alpha([Y_g]$ for some constant $c(g)$. To evaluate the constant one can compare the Euler number of $\mathcal{A}_g$ (in a suitable sense; actually for

$\Omega(\log D) = \mathrm{Sym}^2(\mathbb{E}))$ and the Euler number of Y_g. The Euler number of Y_g equals 2^g. We know the Euler number of $\mathcal{A}_g$ by the work of Siegel and Harder:

$$\begin{aligned}\chi(Sp(2g,\mathbb{Z})) &= \frac{\#W_{Sp(\mathbb{R})}}{\#W_{U(\mathbb{R})}} \prod_{j=1}^{g} \frac{\zeta(1-2j)}{2} = \frac{2^g g!}{g!} \prod_{j=1}^{g} \frac{\zeta(1-2j)}{2} \\ &= \zeta(-1)\zeta(-3)\cdots\zeta(-2g+1).\end{aligned}$$

This proves the result. □

Define a proportionality factor by

$$p(g) = (-1)^G \prod_{j=1}^{g} \frac{\zeta(1-2j)}{2}.$$

Since $\deg u_1 u_2 \ldots u_g = 1$ we find $\deg \lambda_1 \lambda_2 \cdots \lambda_g = p(g)$. Similarly, using the structure of R_g we have

$$\frac{\lambda_1^G}{\prod_{k=1}^{g} \zeta(1-2k)} = (-1)^G \frac{u_1^G}{2^g}.$$

and thus get

$$\lambda_1^G = p(g) G! \prod_{k=1}^{g} \frac{1}{(2k-1)!!}.$$

This has to be interpreted with care (in the orbifold sense).

Some examples. We have

$$\begin{aligned} p(0) &= 1 \\ p(1) = 1/24 \qquad & \deg \lambda_1 = 1/24, \\ p(2) = 1/5760 \qquad & \deg \lambda_1^3 = 1/2880, \\ p(3) = 1/2903040 \qquad & \deg \lambda_1^6 = 1/181440. \end{aligned}$$

For the classical Proportionality principle of Hirzebruch we refer to his collected works, [Hi 1]. In a letter to Atiyah Hirzebruch sketches how one obtains a copy of R_g in the cohomology of suitable compact quotients of the Siegel upper half space, cf. [Hi 2].

Proposition (1.2) shows that the top Chern class λ_g of $\mathbb{E}$ vanishes in the rational Chow group $CH^g_{\mathbb{Q}}(\mathcal{A}_g)$. So λ_g is a torsion class on $\mathcal{A}_g$. Mumford proved in [M1] that the order of λ_1 in $CH^1(\mathcal{A}_1)$ is 12. In [EFG] we shall prove the following bound for the order of the torsion class λ_g.

(1.14) Definition. *Let n_g be the greatest common divisor of all $p^{2g}-1$ where p runs through all primes greater than $2g+1$.*

We have a little lemma.

(1.15) Lemma. *We have*

$$\prod_{i=1}^{g} n_i = \prod_{p \text{ prime}} ([\frac{2gp}{p-1}]!)_p \quad (= \text{multiple of denominator of } p(g)\,).$$

(1.16) Theorem. *On $\mathcal{A}_g$ we have:* $(g-1)!\,(\prod_{i=1}^{g} n_i)\,\lambda_g = 0$.

Example i) For $g = 1$ we get $24\lambda_1 = 0$ which is off by a factor 2. ii) For $g = 2$ we get $24 \cdot (16 \cdot 3 \cdot 5)\lambda_2 = 0$. iii) For $g = 3$ we get $2 \cdot 24 \cdot (16 \cdot 3 \cdot 5) \cdot (8 \cdot 9 \cdot 7)\lambda_3 = 0$.

We refer to [EFG] for other cycle relations in the Chow ring of $\tilde{\mathcal{A}}_g$.

§2. The Cycle Classes of the Ekedahl-Oort Strata.

Ekedahl and Oort introduced a stratification of the moduli space $\mathcal{A}_g \otimes \mathbb{F}_p$, cf. Oort's paper [O3] in this volume. It is defined by analyzing for abelian varieties X the action of Frobenius and Verschiebung on the group scheme $X[p]$, the kernel of multiplication by p. The strata include the well-known loci V_f of abelian varieties of p-rank $\leq f$ (for $0 \leq f \leq g$) and the loci T_a of abelian varieties with a-number $\geq a$ (for $0 \leq a \leq g$).

We shall describe these strata in a somewhat different way using the Hodge bundle $\mathbb{E}$. We then can apply theorems of Porteous type to calculate the cycle classes of these loci. Such Porteous type formulas are obtained by applying results of Fulton and of Pragacz on degeneracy maps between symplectic bundles. The cycle classes all lie in the tautological ring. We get explicit formulas for the loci V_f and T_a generalizing Deuring's formula for the number of supersingular elliptic curves and the formula for the number of superspecial abelian varieties (with $a = g$).

In the following we shall study the moduli space $\mathcal{A}_g \otimes \mathbb{F}_p$ of principally polarized abelian varieties in characteristic p. For simplicity we shall write $\mathcal{A}_g$ instead of $\mathcal{A}_g \otimes \mathbb{F}_p$.

Recall the *canonical filtration* on the de Rham cohomology of a principally polarized abelian variety X as defined by Ekedahl and Oort. We write $G = H^1_{dR}(X)$ on which we have a σ-homomorphism F and a σ^{-1}-homomorphism V. For a moment we shall ignore the $\sigma^{\pm}$-linearity. We have $FV = VF = 0$. The $2g$-dimensional space is provided with a symplectic form $\langle\ ,\ \rangle$ and F and V are adjoints: $\langle Vg, g' \rangle = \langle g, Fg' \rangle$. For any subspace H of G we have $(VH)^{\perp} = F^{-1}(H^{\perp})$. The spaces $V(G) = F^{-1}(0)$ and $F(G) = V^{-1}(0)$ are maximally isotropic subspaces of dimension g.

To construct the canonical filtration one starts with $0 \subset G$ and constructs finer filtrations by adding the images of V and the orthogonal complements of the spaces present. This process stops. The filtration obtained is stable under V and under $^{\perp}$, hence under F^{-1} as well and is called the *canonical filtration*. The canonical filtration

$$0 \subset C_1 \subset \ldots \subset C_r \subset C_{r+1} \subset \ldots \subset C_{2r}$$

satisfies $C_r = V(C_{2r})$ and $C_{r-i}^{\perp} = C_{r+i}$. This filtration can be refined to a so-called *final filtration* by choosing a V- and $^{\perp}$-stable filtration of length $2g$ which refines the canonical one. In general, there is no unique choice for a final filtration. We thus get a filtration

$$0 \subset G_1 \subset \ldots \subset G_g \subset G_{g+1} \subset \ldots \subset G_{2g}$$

which satisfies $V(G_{2g}) = G_g$, $G_{g-i}^{\perp} = G_{g+i}$ and now also $\dim(G_i) = i$. The associated *final type* is the increasing and surjective map

$$\nu : \{0, 1, 2, \ldots, 2g\} \to \{0, 1, 2, \ldots, g\}$$

satisfying

$$\nu(2g - i) = \nu(i) - i + g \qquad \text{for} \quad 0 \leq i \leq g \tag{9}$$

obtained by $\nu(i) = \dim(V(G_i))$ and $\nu(0) = 0$. The *canonical type* is the restriction of ν to the integers which arise as dimensions of the C_i. Although the final filtration is not unique, the final type is. (The dimensions between two steps of the canonical filtration either remain constant or grow each step by 1.)

Example. Let X be an abelian variety with p-rank f and $a(X) = 1$ (equivalently, on G_g the operator V has rank $g-1$ and semi-simple rank $g-f$). Then the canonical type is given by the numbers $\{\text{rank}(C_i)\}$

$$= \{0 < f < f+1 < \ldots < g-1 < g < g+1 < \ldots < 2g-f-1 < 2g-f < 2g\}$$

and

$$\nu(f) = f,\ \nu(f+1) = f,$$
$$\nu(f+2) = f+1, \ldots, \nu(g) = g-1, \ldots, \nu(2g-f-1) = g-1,$$
$$\nu(2g-f) = g,\ \nu(2g) = g.$$

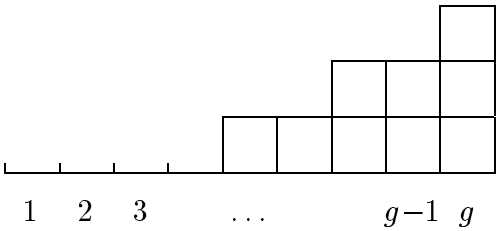
It is not difficult to see that there is a bijection between the set of final types and the set of canonical types and we have 2^g of them. Note that in view of (9) the function ν is determined by its restriction to $\{1, 2, \ldots, g\}$. This restriction is again denoted by ν.

The Ekedahl-Oort stratification of $\mathcal{A}_g$ is obtained by looking for each geometric point of $\mathcal{A}_g$ what the canonical type (or final type) is. The set Z_ν of all abelian varieties which have given final type ν is locally closed and these Z_ν define a stratification, cf. [O3].

I prefer to describe the combinatorial datum of a final type ν by a *partition* $\mu = \{g \geq \mu_1 > \mu_2 > \ldots > \mu_r\}$ as follows:

$$\mu_j = \#\{i : 1 \leq i \leq g, \nu(i) \leq i - j\};$$

equivalently, we can visualize it by the associated Young-type diagram with μ_j squares in the j-th layer (i.e. by putting a stack of $i - \nu(i)$ squares in position i):



This example corresponds to $\{\nu(i) : i = 1, ..., g\} = \{1, 2, \ldots, g-5, g-5, g-4, g-4, g-3, g-3\}$ and to $\mu = \{5, 3, 1\}$.

(2.1) Definition. We call a partition μ *admissible* if $g \geq \mu_1 > \mu_2 > \ldots > \mu_r > 0$. We call the function $\nu\colon \{1, \ldots, g\} \to \{1, \ldots, g\}$ *admissible* if

$$0 \leq \nu(i) \leq \nu(i+1) \leq \nu(i) + 1 \leq g + 1$$

for $1 \leq i \leq g$. The number $|\mu| := \sum_i \mu_i$ is called the *area of the diagram*.

The notions 'admissible' diagram (or partition) and 'admissible function' ν and final type are all equivalent. The set of admissible diagrams carries a partial ordering in the obvious way: $\mu \geq \mu'$ if $\mu_i \geq \mu'_i$ for all i.

We shall now give another approach to these strata by defining them globally on a flag space over $\mathcal{A}_g$. Our starting point is the observation that if we define for

a final filtration E_i $(1 \leq i \leq g)$ another filtration by $F_g = \ker(V) = V^{-1}(0)$, by $F_{g+i} = V^{-1}(E_i)$ for $i = 1, ..., g$ and by $F_{g-i} = F_{g+i}^{\perp}$ then we have

$$V(E_i) \subseteq E_{\nu(i)} \Longleftrightarrow E_i \subseteq V^{-1}(E_{\nu(i)}) = F_{g+\nu(i)} \Longleftrightarrow \dim(E_i \cap F_{g+\nu(i)}) \geq i. \qquad (10)$$

Working now globally, we let S be a scheme in characteristic p and let $\mathcal{X} \to S$ be an abelian variety over S with principal polarization. Then we consider the de Rham cohomology sheaf $\mathcal{H}^1_{dR}(\mathcal{X}/S)$. It is defined as the hyper-direct image $\mathcal{R}^1\pi_*(O_{\mathcal{X}} \to \Omega^1_{\mathcal{X}/S})$. It is a locally free sheaf of rank $2g$ on S equipped with a non-degenerate alternating form (cf. [O])

$$\langle\ ,\ \rangle \colon \mathcal{H}^1_{dR}(\mathcal{X}/S) \times \mathcal{H}^1_{dR}(\mathcal{X}/S) \to O_S.$$

Indeed, the polarization (locally in the étale topology given by a relatively ample line bundle on $\mathcal{X}/S$) provides us with a symmetric homomorphism $\rho \colon \mathcal{X} \to \hat{\mathcal{X}}$ and the Poincaré bundle defines a perfect pairing between $\mathcal{H}^1_{dR}(\mathcal{X}/S)$ and $\mathcal{H}^1_{dR}(\hat{\mathcal{X}}/S)$. Moreover, we have an exact sequence of locally free sheaves

$$0 \to \pi_*(\Omega^1) \to \mathcal{H}^1_{dR}(\mathcal{X}/S) \to R^1\pi_* O_{\mathcal{X}} \to 0.$$

We shall write $\mathbb{H}$ for the sheaf $\mathcal{H}^1_{dR}(\mathcal{X}_g/\mathcal{A}_g)$ and $\mathcal{X}$ for $\mathcal{X}_g$. We thus have an exact sequence

$$0 \to \mathbb{E} \to \mathbb{H} \to \mathbb{E}^{\vee} \to 0.$$

The relative Frobenius $F : \mathcal{X} \to \mathcal{X}^{(p)}$ and the Verschiebung $V : \mathcal{X}^{(p)} \to \mathcal{X}$ satisfy $F \cdot V = p \cdot \mathrm{id}_{\mathcal{X}^{(p)}}$ and $V \cdot F = p \cdot \mathrm{id}_{\mathcal{X}}$ and they induce maps in cohomology, again denoted by F and V:

$$F : \mathbb{H}^{(p)} \to \mathbb{H} \qquad \text{and} \qquad V : \mathbb{H} \to \mathbb{H}^{(p)}.$$

Of course, we have $FV = 0$ and $VF = 0$ and F and V are adjoints. This implies that $\mathrm{Im}(F) = \ker(V)$ and $\mathrm{Im}(V) = \ker(F)$ are maximally isotropic subbundles of $\mathbb{H}$ and $\mathbb{H}^{(p)}$. Moreover, since $dF = 0$ on $\mathrm{Lie}(\mathcal{X})$ it follows that $F = 0$ on $\mathbb{E}$ and thus $\mathrm{Im}(V) = \mathbb{E}^{(p)}$. Verschiebung thus provides us with a bundle map (again denoted by V): $V : \mathbb{H} \to \mathbb{E}^{(p)}$.

Consider the space of symplectic flags $\mathcal{F} = \mathrm{Flag}(\mathbb{H})$ on the bundle $\mathbb{H}$. This space is fibred by the spaces $\mathcal{F}^{(i)}$ of partial flags

$$\mathbb{E}_i \subsetneq \mathbb{E}_{i+1} \subsetneq \ldots \subsetneq \mathbb{E}_g.$$

So $\mathcal{F}^{(1)} = \mathrm{Flag}(\mathbb{H})$ and $\mathcal{F}^{(g)} = \mathcal{A}_g$ and there are natural maps

$$\pi_{i,i+1} \colon \mathcal{F}^{(i)} \to \mathcal{F}^{(i+1)}.$$

The fibres are Grassmann varieties of dimension i. The space $\mathcal{F}^i$ is equipped with a universal flag. On $\mathcal{F}$ the Chern classes of the bundle $\mathbb{E}$ decompose into their roots:

$$\lambda_i = \sigma_i(\ell_1, \ldots, \ell_g) \qquad \text{with} \qquad \ell_i = c_1(\mathbb{E}_i/\mathbb{E}_{i-1}).$$

Given an arbitrary flag of subbundles

$$0 \subsetneq \mathbb{E}_1 \subsetneq \ldots \subsetneq \mathbb{E}_g = \mathbb{E} \qquad (11)$$

with $\mathrm{rank}(\mathbb{E}_i) = i$ we can extend this to a symplectic filtration on $\mathbb{H}$ by putting

$$\mathbb{E}_{g+i} = (\mathbb{E}_{g-i})^{\perp}.$$

By base change we can transport this filtration to $\mathbb{H}^{(p)}$.

We introduce a second filtration by starting with the isotropic subbundle

$$\mathbb{F}_g = \ker(V) = V^{-1}(0) \subset \mathbb{H}$$

and continuing with

$$\mathbb{F}_{g+i} = V^{-1}(\mathbb{E}_i^{(p)}) \qquad \text{for} \quad 1 \leq i \leq g.$$

We extend it to a symplectic filtration by setting $\mathbb{F}_{g-i} = (\mathbb{F}_{g+i})^{\perp}$. We thus have two filtrations $\mathbb{E}_\bullet$ and $\mathbb{F}_\bullet$ on $\mathbb{H}$.

Example. i) Let X be an ordinary abelian variety. Then $\mathrm{Lie}(X) = \mathrm{Lie}(\mu)$ with μ the multiplicative subgroup scheme of $X[p]$ of rank p^g. It follows that V is invertible on $\mathrm{Lie}(X)^\vee = \omega(X)$, i.e. $\mathbb{F}_g \cap \mathbb{E}_g = (0)$. ii) If X is a superspecial abelian variety (i.e. X without its polarization is a product of supersingular elliptic curves) then $V = 0$ on $\omega(X)$ so that $\mathrm{rk}(\mathbb{E}_i \cap \mathbb{F}_g) = i$.

These two (extreme) examples show that the respective position of the two filtrations $\mathbb{E}_\bullet$ and $\mathbb{F}_\bullet$ for an abelian variety X gives information on the structure of the kernel of multiplication by p on X. These respective positions are encoded by a combinatorial datum, e.g. ν or to be more precise, by an element of a Weyl group. We shall associate strata to such data.

To either ν or μ we now associate an element of the Weyl group of the symplectic group. The Weyl group W_g of type C_g in Cartan's terminology is isomorphic to the semi-direct product $S_g \ltimes (\mathbb{Z}/2\mathbb{Z})^g$, where S_g acts on $(\mathbb{Z}/2\mathbb{Z})^g$ by permuting the g factors. Another description of this group is as the subgroup of S_{2g} of elements which map any symmetric 2-element subset of the form $\{i, 2g+1-i\}$ of $\{1, \ldots, 2g\}$ to a subset of the same type:

$$W_g = \{\sigma \in S_{2g} \colon \sigma(i) + \sigma(2g+1-i) = 2g+1 \text{ for } i = 1, \ldots, g\}.$$

An element in this Weyl group has a *length* and a *codimension*:

$$\ell(w) = \#\{i < j : w(i) > w(j)\} + \#\{i < j : w(i) + w(j) > 2g+1\}$$

and

$$\mathrm{codim}(w) = \#\{i < j : w(i) < w(j)\} + \#\{i < j : w(i) + w(j) < 2g+1\}.$$

We have the equality

$$\ell(w) + \mathrm{codim}(w) = g^2.$$

To a function ν we associate the following element of the Weyl group, a the permutation of $\{1, 2, \ldots, 2g\}$: let

$$S = \{i_1, i_2, \ldots\} = \{1 \leq i \leq g : \nu(i) = \nu(i-1)\}$$

with $i_1 < i_2 < \ldots$ given in increasing order. Let

$$S^c = \{j_1, j_2, \ldots\}$$

be the elements of $\{1, 2, \ldots, g\}$ not in S, in increasing order. Then one gets the permutation of S_{2g} defined by ν by writing $g + k$ at position j_k for $k = 1, 2, \ldots$ and k at position i_k for $k = 1, 2, \ldots$. We finish off by putting $2g + 1 - k$ at position $2g + 1 - i$ if k is written at position i. We obtain a sequence s which is a permutation of $\{1, 2, \ldots, 2g\}$.

Alternatively, using diagrams, we can describe the element $w = w_\mu$ as follows. Let t_i be the operator on the set of diagrams which is 'remove the top box of the i-th column'. We let the t_i act from the right. Given an admissible diagram μ we consider the complementary diagram

$$\mu^c = \{g, g-1, \ldots, 1\} - \mu$$

which is also an admissible diagram (partition). We can successively apply operators t_i to it such that after every step we obtain an admissible diagram and such that after $|\mu^c|$ steps we obtain the empty diagram $(\mu^c) \cdot t_{i_1} \cdots t_{i_\ell} = \emptyset$. We remove first the top layer, then the next one and so on. For $1 \leq i < g$ let $s_i \in S_{2g}$ be the permutation $(i, i+1)(2g - i, 2g + 1 - i)$ and let $s_g = (g, g+1) \in S_{2g}$. Now associate to the diagram μ the element of the Weyl group:

$$\mu \mapsto w_\mu = s_{i_1} \cdots s_{i_\ell}.$$

Each diagram thus yields an element in the Weyl group. The admissible partitions yield 2^g elements of the $2^g g!$ elements of W_g.

Example. $g = 3$

μ	ν	$[1,2,3,4,5,6]$ maps to	ℓ	w_μ
$\emptyset$	$\{1,2,3\}$	$[4,5,6,1,2,3]$	6	$s_3 s_2 s_3 s_1 s_2 s_3$
$\{1\}$	$\{1,2,2\}$	$[4,5,1,6,2,3]$	5	$s_2 s_3 s_1 s_2 s_3$
$\{2\}$	$\{1,1,2\}$	$[4,1,5,2,6,3]$	4	$s_3 s_1 s_2 s_3$
$\{3\}$	$\{0,1,2\}$	$[1,4,5,2,3,6]$	3	$s_3 s_2 s_3$
$\{2,1\}$	$\{1,1,1\}$	$[4,1,2,5,6,3]$	3	$s_1 s_2 s_3$
$\{3\}$	$\{0,1,2\}$	$[1,4,5,2,3,6]$	3	$s_3 s_2 s_3$
$\{3,1\}$	$\{0,1,1\}$	$[1,4,2,5,3,6]$	2	$s_2 s_3$
$\{3,2\}$	$\{0,0,1\}$	$[1,2,4,3,5,6]$	1	s_3
$\{3,2,1\}$	$\{0,0,0\}$	$[1,2,3,4,5,6]$	0	1

We can associate to the map $V : \mathbb{H} \to \mathbb{E}^{(p)}$ and an element w a degeneracy locus $\mathcal{U}_w$ and in particular to an admissible diagram μ a degeneracy locus $\mathcal{U}_\mu = \mathcal{U}_{w_\mu}$ in $\mathcal{F} = \mathrm{Flag}(\mathbb{H})$. Intuitively, $\mathcal{U}_w$ is defined as the locus of points x such that at x we have

$$\dim(\mathbb{E}_i \cap \mathbb{F}_j) \geq \#\{a \leq i : w(a) \leq j\} \quad \text{for all} \quad 1 \leq i, j \leq g$$

or equivalently, that $\ker(V) \cap \mathbb{E}_i \geq i - \nu(i)$ (= the number of squares in the diagram μ in position i). For the precise definition we refer to Fulton [F]. Note that for the admissible diagrams we do not use the full filtration of $\mathbb{F}$, but only $\mathbb{F}_g$.

We first look at the case of the empty diagram $\mu = \emptyset$ or equivalently, that $\nu = \{1, 2, 3, \ldots, g\}$. The degeneracy conditions say that (cf. (10))

$$V(\mathbb{E}_i) \subseteq \mathbb{E}_i^{(p)} \qquad i = 1, \ldots, g.$$

i.e. we are looking at the space $\mathcal{U}_{\{\emptyset\}}$ of symplectic filtrations on $\mathbb{E}$ which are compatible with the action of V. The codimension of this space in $\mathcal{F}$ is $g(g-1)/2$, hence $\dim(U_{\{\emptyset\}}) = g(g+1)/2 = \dim(\mathcal{A}_g)$. We have a finite map $\pi : U_{\{\emptyset\}} \to \mathcal{A}_g$ of degree

$$\deg(\pi) = (1+p)(1+p+p^2)\ldots(1+p+p^2+\ldots+p^{g-1}).$$

This space $\mathcal{U} = U_{\{\emptyset\}}$ can be seen as a component of the moduli space $\Gamma_0(p)$. Indeed, a filtration of the subgroup scheme $X[p]$ corresponds to a filtration on H^1_{dR}. Fix g and let $\Gamma_0(p)'$ be the functor that associates to a scheme S the set of isomorphism classes of principally polarized abelian schemes X over S plus a V-stable symplectic filtration on $X[p]$.

(2.2) Proposition. *The degeneracy cycle $\mathcal{U}_\emptyset$ is the algebraic stack representing the functor $\Gamma_0(p)'$. The stack $\mathcal{U}_\emptyset$ is fibred by finite morphisms*

$$\mathcal{U}_\emptyset = \mathcal{U}^{(1)} \xrightarrow{\pi_{1,2}} \mathcal{U}^{(2)} \xrightarrow{\pi_{2,3}} \ldots \xrightarrow{\pi_{g-1,g}} \mathcal{U}^{(g)} = \mathcal{A}_g.$$

with $\deg(\pi_{i,i+1}) = 1 + p + \ldots + p^i$. It contains the degeracy loci $\mathcal{U}_\mu$ for all 2^g partitions μ.

The stacks $\mathcal{U}^{(i)}$ come with universal (partial) flags $\mathbb{E}_i \subsetneq \ldots \subsetneq \mathbb{E}_g$. We denote by $\lambda_j(i) = c_j(\mathbb{E}_i)$ the Chern class in $CH^j_{\mathbb{Q}}(\mathcal{U}^{(i)})$. We also have tautological quotient bundles $L_i = \mathbb{E}_i/\mathbb{E}_{i-1}$. We denote by l_i the Chern class of L_i. We have $(\lambda_j)(i) = \sigma_j(l_1, \ldots, l_i)$, the j-th elementary symmetric function of the $l_1, \ldots, l_i$.

Next we look at the cases where μ is a partition of the form $\mu = \{\mu_1 = g - f\}$. The corresponding degeneracy loci classify the loci of p-rank $\leq f$. It is well-known by Oort (cf. [O-N]) that the codimension of V_f in $\mathcal{A}_g$ equals $g - f$. They admit the following explicit description. The pullback of V_{g-1} to $\mathcal{U}_\emptyset$ consists of g components, say $Z_1, ..., Z_g$. An abelian variety of p-rank $g-1$ and $a = 1$ has a unique subgroup scheme α_p. The index i of Z_i indicates where for the generic point of Z_i the α_p can be found: $\alpha_p \subset G_i$, $\alpha_p \not\subset G_{i-1}$.

Then the pull back of V_f consists of

$$\pi^{-1}(V_f) = \sum_{\#S = g-f} Z^S,$$

where for a subset $S \subset \{1, \ldots, g\}$ the cycle Z^S is defined as

$$Z^S = \cap_{i \in S} Z_i.$$

Then one sees easily that the cycle V_{g-f} on $\mathcal{A}_g$ is obtained as the push forward of $Z_g \cap \ldots \cap Z_{g-f+1}$.

(2.3) Lemma. *The cycle class of Z_i is equal to $(p-1)\ell_i$.*

Proof. It is described as the locus where $\det(V) : L_i \to L_i^{(p)}$ vanishes. By viewing $\det(V)$ as a section of $L_i^{(p)} \otimes L_i^{-1}$ the result follows. □

We find after a calculation:

$$\pi^*([V_f]) = (p-1)^{g-f}\lambda_{g-f}$$

on $\mathcal{U}_\emptyset$. Using the push forward of the $\pi_{i,i+1}$ on the classes ℓ_i and extending the result to a compactification $\tilde{\mathcal{A}}_g$ we get:

(2.4) Theorem. *The cycle class of V_f, the p-rank $\leq f$ locus, in the Chow ring $CH^*_{\mathbb{Q}}(\tilde{\mathcal{A}}_g)$, is given by $[V_f] = (p-1)(p^2-1)\ldots(p^{g-f}-1)\lambda_{g-f}$.*

(2.5) Corollary. *(Deuring Mass Formula) Let $g = 1$. We have*

$$\sum_E \frac{1}{\#\mathrm{Aut}(E)} = \frac{p-1}{24},$$

where the sum is over the isomorphism classes over $\bar{\mathbb{F}}_p$ of supersingular elliptic curves.

Proof. The formula gives $(p-1)\lambda_1$ and by (1.10) this equals $\delta/12$. The class of δ is equivalent to $1/2$ times the class of a 'physical' point of the j-line because the degenerate elliptic curve corresponding to δ has 2 automorphisms. □

Another case where we can find an explicit formula is the case of the locus T_a. This corresponds to the case $\mu = \{a, a-1, \ldots, 2, 1\}$. But here we can work directly on $\mathcal{A}_g$. The locus T_a on $\mathcal{A}_g$ may be defined as the locus

$$\{x \in \mathcal{A}_g : \mathrm{rank}(V)|_{\mathbb{E}_g} \leq g - a\}.$$

We have $T_{a+1} \subset T_a$ and $\dim(T_g) = 0$.

Pragacz and Ratajski, cf. [P-R], have developed formulas for the degeneracy locus for the rank of a self-adjoint bundle map of symplectic bundles globalizing the results in isotropic Schubert calculus from [P]. Before we apply their result to our case we have to introduce some notation.

Define for a vectorbundle A with Chern classes a_i the expression

$$Q_{ij}(A) := a_i a_j + 2\sum_{k=1}^{j}(-1)^k a_{i+k}a_{j-k} \quad \text{for} \quad i > j.$$

For a admissible partition $\beta = (\beta_1, \ldots, \beta_r)$ (with r even, β_r may be zero) we set

$$Q_\beta = \mathrm{Pfaffian}(x_{ij}),$$

where the (x_{ij}) is an anti-symmetric matrix with $x_{ij} = Q_{\beta_i,\beta_j}$. Applying the Pragacz-Ratajski formula to our situation gives the following result:

(2.6) Theorem. *The class of the reduced locus T_a of abelian varieties with a-number $\geq a$ is given by*

$$\sum Q_\beta(\mathbb{E}^{(p)}) \cdot Q_{\rho(a)-\beta}(\mathbb{E}^*),$$

where the sum is over the admissible partitions β contained in the partition $\rho(a) = (a, a-1, a-2, ..., 1)$.

Example:

$$\begin{aligned}
[T_1] =& p\lambda_1 - \lambda_1 \\
[T_2] =& (p-1)(p^2+1)(\lambda_1\lambda_2) - (p^3-1)2\lambda_3 \\
& \cdots \\
[T_g] =& (p-1)(p^2+1)\ldots(p^g+(-1)^g)\lambda_1\lambda_2\cdots\lambda_g.
\end{aligned}$$

As a corollary we find a classical result of Ekedahl (cf. [E]) on the number of principally polarized abelian varieties with $a = g$:

(2.7) Corollary. *We have*

$$\sum_X \frac{1}{\#\mathrm{Aut}(X)} = (-1)^G 2^{-g}\big[\prod_{j=1}^{g}(p^j+(-1)^j)\big]\cdot\zeta(-1)\zeta(-3)\ldots\zeta(1-2g),$$

where the sum is over the isomorphism classes (over $\bar{\mathbb{F}}_p$) of principally polarized abelian varieties of dimension g with $a = g$.

Proof. Combine the formula for T_g with the Proportionality Theorem. □

For each element w we now find a degeneracy locus $\mathcal{U}_w$. In particular we find such a locus $\mathcal{U}_\mu$ for each partition μ in $\mathcal{F}$ and the $\mathcal{U}_\mu$ actually lie in $\mathcal{U}_\emptyset$. It is known that T_g is zero-dimensional, cf. (2.7). This implies that the codimension of each $\mathcal{U}_\mu$ equals $\mathrm{codim}(w(\mu)) = |\mu|$. We can apply a theorem of Fulton to determine the class of the degeneracy locus U_μ in $CH^*_{\mathbb{Q}}(\mathcal{F})$. For an admissible diagram $\mu = \{\mu_1 > \ldots > \mu_r > 0\}$ Fulton defines a determinant ('Schur function')

$$\Delta_\mu(x_i) = \det(x_{\mu_i+j-i})_{1\leq i,j\leq r};$$

this is a polynomial with integer coefficients in the commuting variables $x_1, x_2, \ldots$. We define a 'double Schubert function' by putting

$$\Delta(x,y) = \Delta_{g,g-1,\ldots,1}\big(\sigma_i(x_1,\ldots,x_g)+\sigma_i(y_1,\ldots,y_g)\big).$$

The operators ∂_i ('divided difference operators') on the ring $\mathbb{Z}[x_1, x_2, \ldots,]$ are defined by setting for $F(x) \in \mathbb{Z}[x_1, \ldots, x_g]$

$$\begin{cases} \partial_i(F(x)) = \frac{F(x)-F(s_i(x))}{x_i-x_{i+1}} & \text{if } i < g \\ \partial_g(F(x)) = \frac{F(x)-F(s_g(x))}{2x_g} & \text{if } i = g. \end{cases}$$

We put

$$\Delta = \Delta(x,y) = \Delta_{g,g-1,\ldots,1}\big(\sigma_i(x_1,\ldots,x_g)+\sigma_i(y_1,\ldots,y_g)\big).$$

We shall apply the result of Fulton and state an abstract formula for our cycle classes. In principle one then can calculate the push forward algorithmically. But it seems difficult to get closed formulas for the cycle classes of these push forwards.

By applying a theorem of Fulton [F] we find:

(2.8) Theorem. *Let μ be an admissible diagram whose corresponding element w_μ in the Weyl group is written as $s_{i_\ell} \cdots s_{i_1}$. The cycle class of the degeneracy locus U_μ in $CH^*_{\mathbb{Q}}(\mathcal{F})$ is given by*

$$u_\mu = \partial_{i_\ell} \cdots \partial_{i_1} \big((\prod_{i+j \le g} (x_i - y_i)) \cdot \Delta(x,y) \big) \Big|_{\{x_i = p l_i, y_i = -l_i\}}$$

(2.9) Corollary. *The push forward of the class of $[\mathcal{U}_\mu]$ under π is given by $\pi_*(u_\mu)$ and is a multiple of the class of the reduced cycle Z_μ, the multiple being equal to the number of final filtrations refining the canonical filtration associated to μ. The class belongs to the tautological ring.*

We can calculate these classes from this formula in an algorithmic way. But is seems difficult to get closed formulas in the general case. We list here the formulas for $g = 3$. The multiplicity is given by the factor in square brackets.

Formulas for $g = 3$.

$$\pi_*(\{\emptyset\}) = [(1+p)(1+p+p^2)]$$
$$\pi_*(\{1\}) = [(1+p)] \times (1-p)\,\lambda_1$$
$$\pi_*(\{2\}) = (p-1)(p^2-1)\,\lambda_2$$
$$\pi_*(\{1,2\}) = [(1+p)] \times \{(1-p)(1+p^2)\,\lambda_1\,\lambda_2 - 2(-1+p^3)\,\lambda_3\}$$
$$\pi_*(\{3\}) = (p-1)(p^2-1)(p^3-1)\,\lambda_3$$
$$\pi_*(\{1,3\}) = [(1+p)] \times (-1+p)^2\,(1-p+p^2)\,\lambda_1\,\lambda_3$$
$$\pi_*(\{2,3\}) = (-1+p)^3\,(1+p)\,(-1+p-p^2)\,(1+p+p^2)\,\lambda_2\,\lambda_3$$
$$\pi_*(\{1,2,3\}) = [(1+p^3)] \times (p-1)\,(p^2+1)(p^3-1)\,\lambda_1\,\lambda_2\,\lambda_3.$$

The canonical filtration on $H^1_{dR}(X)$ is the most economical one with respect to the operation V. The types of this filtration correspond to the elements of $(\mathbb{Z}/2\mathbb{Z})^g$ in the Weyl group $(\mathbb{Z}/2\mathbb{Z})^g \rtimes S_g$. The other types of filtrations are obtained by applying an element of S_g to the right. Therefore, in the closure of a stratum $\mathcal{U}_\mu$ we find also $\mathcal{U}_w$ with $w \notin (\mathbb{Z}/2\mathbb{Z})^g$. This explains the phenomenon observed by Oort in [O3], 4.6. We can use Pieri-type formulas to get results on the boundary of strata. In particular one obtains the result of Ekedahl-Oort that the class λ_1 is torsion on the open strata. We refer to [EFG].

Some strata can be described in detail. Consider the locus associated to the partition

$$\mu = \{g, g-1, \ldots, 3, 2\}.$$

This is the penultimate stratum and it is of dimension 1. It is proved by Ekedahl and Oort that this locus is connected.

Define $h(g)$ by $h(1) = 1$ and $h(g) = h(g-1) \times \frac{p^g + (-1)^g}{p + (-1)^g}$. Then we have

$$\pi_*(U_{[g,g-1,\ldots,1]}) = h(g)[T_g]$$

with $[T_g]$ given as above.

(2.10) Theorem. *The cycle class of $\pi_*(\mathcal{U}_\mu)$ for $\mu = \{g, g-1, \ldots, 3, 2\}$ is given by*

$$\frac{(p-1)}{(p^2+1)} \times h(g) \times \left(\prod_{i=1}^{g} (p^i + (-1)^i) \right) \times \lambda_2 \lambda_3 \ldots \lambda_g.$$

In $\mathcal{U}_\emptyset$ this locus consists of a configuration of $\mathbb{P}^1$'s.

Since the degree of λ_1 on each copy of $\mathbb{P}^1$ is $p-1$ one can compute the number of components. This locus is highly reducible. But some loci are irreducible:

(2.11) Theorem. *For $a < g$ the locus T_a is irreducible. In particular, the locus $T_1 = V_{g-1}$ is irreducible.*

Proof. By Ekedahl-Oort (cf. [O3]) we know that the locus corresponding to the diagram $\{g, g-1, \ldots, 3, 2\}$ is connected. This implies that T_a is connected for $a < g$. We know that $\mathrm{Sing}(T_a) \subseteq T_{a+1}$, hence of codimension > 1. Actually, one can describe the normal bundle to $T_a - T_{a+1}$. By a theorem of Hartshorne (cf. [H]) this implies that $T_a - \mathrm{Sing}(T_a)$ is connected. □

The irreducibility of V_{g-1} was also observed by Oort, cf. [O3].

§3 Some additional results.

We describe as an example the canonical type for hyperelliptic curves of 2-rank 0 in characteristic $p = 2$.

(3.1) Lemma. *A hyperelliptic curve C of genus g and of 2-rank 0 over $k = \bar{k}$ of characteristic $p = 2$ can be written as*

$$y^2 + y = xP(x^2) = x(a_1 x^2 + a_2 x^4 + \ldots + a_{g-1} x^{2g-2} + x^{2g}). \tag{13}$$

(3.2) Theorem. *For a hyperelliptic curve as in (13) the canonical type is described by the corresponding partition $\mu = [g, g-2, g-4, \ldots]$. In particular, the canonical filtration is independent of the coefficients a_i.*

We refer for the proof to [EFG].

For $g = 1$ and $g = 2$ the supersingular locus coincides with V_0 and thus occurs as a stratum in the Ekedahl-Oort stratification. For $g \geq 3$ this is no longer true. The definition of supersingular locus involves also the higher filtrations, i.e. it uses not only $X[p]$, but the higher group schemes $X[p^i]$ as well. But in a number of cases we can determine the class of this locus. We state here the result for the case $g = 3$.

(3.3) Theorem. *The class of the supersingular locus S_3 in $\mathcal{A}_3^* \otimes \mathbb{F}_p$ is*

$$[S_3] = (p-1)(p^2-1)(p^3-1)(p-1)(p^2+1)\lambda_1\lambda_3.$$

The proof uses the explicit description by Li and Oort of the moduli of principally polarized supersingular abelian varieties, cf. [L-O]. We refer the reader to [EFG] for the proof and formulas for other genera.

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Gerard van der Geer

Faculteit WINS

Universiteit van Amsterdam

Plantage Muidergracht 24

1018 TV Amsterdam

The Netherlands

e-mail: `geer@fwi.uva.nl`