ON UPPERBOUNDS OF VIRTUAL MORDELL-WEIL RANKS

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Abstract. We study relatively minimal fibrations of curves of genus $g \geq 1$ $f : X \to C$ over a smooth projective curve C of genus b defined over the field of complex numbers. A new concept of the virtual Mordell-Weil rank is introduced. We give an upper bound of the virtual Mordell-Weil rank r in term of g, b, q and p_g. Moreover the case of equality in the bound implies that f is a hyperelliptic fibration and (if f admits a global section) r is precisely the real Mordell-Weil rank. In the non-hyperelliptic case with $f_*\omega_{X/C}$ semi-stable (in particular if $p_g = q = 0$) a sharper bound can be given. We consider a similar problem for the number s of singular fibres. Pencils with $\chi(\mathcal{O}_X) = 1$ and $c_1^2(X) = -4(g-1) < 0$ are treated in detail.

§0 Introduction.

(0.0) Let $f : X \to C$ be a relatively minimal fibration of curves of genus $g \geq 1$ over a smooth projective curve C of genus b defined over an algebraically closed field k of characteristic zero. Let $K = k(C)$ be the field of rational functions on C. In the theory of Mordell-Weil lattices due to Shioda (*cf.* [Sh1], [Sh2]) the following conditions are assumed:
(0.1) (i)*f admits a global section (O) as zero-section,*

(ii) *K/k-trace of the Jacobian J_F of the generic fibre F/K of f is trivial, or equivalently $q(X) = b$.*

Under these conditions the Mordell-Weil group $J(K)$ of K-rational points of J is finitely generated. The rank r of its free part is called the Mordell-Weil rank. In [S, Theorem 1.3] an upperbound of r via the invariants of f is given. In particular, for the case of rational surfaces X it was shown in a joint paper ([SS]) that $r \leq 4g + 4$. Moreover the structure of fibrations with maximal rank $r = 4g + 4$ and the structure of corresponding Mordell-Weil lattices are completely determined in [SS] (a such fibration is obtained as a blowing up of a linear pencil of hyperelliptic curves on a Hirzebruch surface Σ_e with $0 \leq e \leq g$ $(g \geq 2)$).

In this note we consider a similar problem for locally non-trivial fibrations, not necessarily satisfying conditions (0.1). Let $\mathrm{NS}(X)$ be the Néron-Severi group of X. Then

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NS$(X)/$*torsion* admits the lattice structure with intersection pairing. Hodge's index theorem asserts that its signature is $(1, \rho - 1)$, where $\rho :=$rank NS(X) is the Picard number of X.

(0.2) Definition (*cf.* [V]) *The virtual Mordell-Weil rank r of f is defined to be the rank of the essential sublattice of the Néron-Severi lattice (cf. [Sh1], [Sh2]), i.e., as*

$$r = \rho - 2 - \sum_{t \in C}(n_t - 1), \tag{0.3}$$

where n_t is the number of irreducible components of $X_t := f^{-1}(t)$.

If f satisfies conditions (0.1) then this is nothing but the well known formula for the Mordell-Weil rank r (*loc. cit.*). This justifies our definition.

(0.4) A natural question arising here is to give a best possible upperbound of virtual Mordell-Weil rank and we are interested in knowing when it becomes the real Mordell-Weil rank. In a similar way as in [S, Theorem 1.3] by using Xiao's inequality ([X]), one can have the following bound for a locally non-trivial fibration $f : X \to C$

$$r \leq (6 + 4/g)d + 2(q - b) + 2g(b - 1), \tag{0.5}$$

where $d =$deg $(f_*\omega_{X/C})$, $q = q(X)$. Moreover we can show that the equality holds only if f is a hyperelliptic fibration, all fibres of f are irreducible and $q = b$.

In the non-hyperelliptic case with $f_*\omega_{X/C}$ semi-stable we have a sharper bound for r due to Konno's stronger version of the slope inequality ([K, Lemma 2.5]). In the light of new results of Konno (personal communication) we know that the case of equality implies that Cliff$(f) = 1$, i.e., genaral fibres of f are trigonal or plane quintic (see also his forthcoming paper in the Inter. J. of Math. where he treates the non-semistable case with Cliff$(f) = 1$). From the point of view of Mordell-Weil lattices the "computable" case $p_g = q = 0$ is most interesting. In this case $r \leq 3g+6$ (also for the number of singular fibres s: $s \leq 7g+6$). We give two examples showing that bounds actually are sharp. It should be very interesting to get a complete description as in [SS] for the maximal case.*

We can also have more precise structure theorem for the following pencils ($b = 0$):

(I) *Pencils with $\chi(\mathcal{O}_X) = 1$.* In this case bound (0.5) can be read as $r \leq 4g + 4 + 2q$. We remark that the equality $r = 4g + 4 + 2q$ leads us to the maximal case studied in [SS] by using [V, Lemma 3.1.2]. Also $s \leq 8g + 4$ and the equality $s = 8g + 4$ gives us Lefschetz pencils in the constructions of [SS].

(II) *Pencils with $c_1^2(X) = -4(g - 1) < 0$.* Here we have $r \leq 4g + 4 - 8q$ (*resp.* $s \leq 8g + 4 - 12q$). The maximal case is obtained as a blowing up of a pencil (*resp.* a

*A full account of this situation is now in preparation ([NS]).

Lefschetz pencil, except for $g = 2q$) on a ruled surface $\Sigma_e^q \to E$, $g(E) = q$ $(-q \leq e \leq g-2q)$ whose general members are double coverings of curve E. In the maximal case the structure of the essential sublattice in the Néron-Severi lattice is uniquely determined. The proof uses the fact that in this case X is a double covering of $E \times \mathbb{P}^1$ whose branch locus is a smooth irreducible curve of numerical type $(2g + 2 - 4q, 2)$ (*cf.* [V, Theorem 3.1 and Lemma 3.1.2]).

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§1 Bounds of virtual Mordell-Weil ranks.

(1.1) We use the following notation:

$X(Y, \gamma, B)$:=Spec$(\mathcal{O}_Y \oplus \mathcal{O}_Y(\gamma)^{\vee})$: the double covering of a smooth surface Y with branch locus $B \sim 2\gamma$;

$f : X \to C$: a relatively minimal fibration, not locally trivial, of curves of genus $g \geq 1$;

C: a smooth projective curve with genus b.

S: the finite set of critical points on C, $s =$ the number of S.

$\omega_{X/C}$: the relative dualizing sheaf, $d := deg(f_*\omega_{X/C})$.

ω: the relative canonical class $K_{X/C}$.

$\lambda(f) := \omega^2/d$: the slope of f.

$\rho' := h^{1,1} - \rho$: the difference of the middle Hodge number $h^{1,1}$ and Picard number ρ of X. Note that ρ' is a non-negative number by virtue of Lefschetz' theorem on algebraic cycles.

$e_t(X) := \chi(X_t) - (2 - 2g)$: the Euler local number over $t \in C, X_t := f^{-1}(t)$.

n_t: the number of irreducible components of X_t.

$g(\widetilde{X}_t)$: the genus of the normalization of X_t.

Recall that the ground field k is the field of complex numbers $\mathbb{C}$.

(1.2) Proposition. *Let $f : X \to C$ be a relatively minimal fibration as in (1.1) and r denotes the virtual Mordell-Weil rank of f. Then we have*

$$r \leq (4 + \frac{4}{g})\big[\chi(\mathcal{O}_X) - (g-1)(b-1)\big] + 2\big[\chi(\mathcal{O}_X) + q - 1\big], \tag{1.2.1}$$

or equivalently,

$$r \leq (6 + \frac{4}{g})d + 2(q-b) + 2g(b-1). \tag{1.2.2}$$

Moreover the equality in the bounds above holds if and only if

1) f is a hyperelliptic fibration with lowest slope $\lambda(f) = 4 - 4/g$,

2) all fibres of f are irreducible,

3) $\rho' = 0$.

Proof (cf. [S, Theorem 1.3]). First in view of Xiao's inequality $\lambda(f) \geq 4 - 4/g$ ([X, Theorem 2]) one can put

$$\omega^2 = \omega_f + (4 - \frac{4}{g})d \tag{1.2.3}$$

with non-negative ω_f.

By an easy calculation using Leray's spectral sequence and Riemann-Roch we have

$$\chi(\mathcal{O}_X) = d + (g-1)(b-1). \tag{1.2.4}$$

Next since $\chi(X) = c_2(X) = 2 - 2b_1 + b_2$ and $b_1 = 2q$, where b_i is the i-th Betti number we infer from Noether's formula

$$h^{1,1} = b_2 - 2p_g = (4 + \frac{4}{g})\big[\chi(\mathcal{O}_X) - (g-1)(b-1)\big] + 2\big[\chi(\mathcal{O}_X) + q\big] - \omega_f.$$

Taking into account (0.3) one obtains

$$r = (4 + \frac{4}{g})\big[\chi(\mathcal{O}_X) - (g-1)(b-1)\big] + 2\big[\chi(\mathcal{O}_X) + q - 1\big] - \omega_f - \sum_{t\in C}(n_t - 1) - \rho', \tag{1.2.5}$$

or equivalently (in view of (1.2.4))

$$r = (6 + \frac{4}{g})d + 2(q-b) + 2g(b-1) - \omega_f - \sum_{t\in C}(n_t - 1) - \rho'. \tag{1.2.6}$$

Bounds (1.2.1)–(1.2.2) follow directly from (1.2.5)-(1.2.6). Moreover the equality holds if and only if

1) $\omega_f = 0$, or equivalently, $\lambda(f) = 4 - 4/g$,

2) $n_t = 1, \forall t \in C$, i.e., all fibres of f are irreducible,

3) $\rho' = 0$.

It remains to use Konno's result stating that fibrations with lowest slope $\lambda(f) = 4 - 4/g$ are hyperelliptic ([K, Proposition 2.6]).

(1.3) Remark. *As consequence we obtain $q = b$ in the case of equality of bounds above (*[X, Corollary 1]*). If we assume moreover that f admits a global section then this is equivalent to the triviality of K/k-trace of the Jacobian J_F. So r becomes the real Mordell-Weil rank and it makes sense to study the structure of Mordell-Weil lattices in this case.*

(1.4) Corollary. *Let $f : X \to C$ be as in Proposition 1.2. Assume that r is maximal, i.e., $= (6 + 4/g)d + 2(q - b) + 2g(b - 1)$. Then X is a double covering of a ruled surface $\Sigma_e^b \to C$ with smooth branch locus B and*

$$\omega^2 = 2(g-1)\big[2m - (g+1)e\big], \qquad d = \frac{g}{2}\big[2m - (g+1)e\big], \tag{1.4.1}$$

where $B \equiv 2(g+1)C_0 + 2mF_0$ with C_0, F_0 denoting the minimal section and a fibre on Σ_e^b.

Proof. From Proposition 1.2 and Horikawa's theory it follows that X is the canonical resolution of a double cover of a ruled surface over C with simple singularities. It remains to use standard calculations with double coverings (see, *e.g.*, [Ho], or [BPV,V, 22], *cf.* also §3). The fact that B is smooth follows from the irreducibility of fibres of f.

(1.5) Consider non-hyperelliptic case and assume that $f_*\omega_{X/C}$ is semi-stable then one can have a sharper bound thanks to Konno's stronger version of the slope inequality ([K, Lemma 2.5]). In particular if $p_g = q = 0$ then it is easy to see that $f_*\omega_{X/\mathbb{P}^1}$ is semi-stable (*cf.* [V, A.4.4]). So we have $r \leq 3g + 6$. We give here examples which show that this bound is sharp. Take a Lefschetz pencil of curves of degree m in the projective plane $\mathbb{P}^2$, considered in [Sh3]. By blowing up m^2 distinct base points from $\mathbb{P}^2$ one obtains a smooth rational surface X with natural morphism $f : X \to \mathbb{P}^1$. The fact that f is non-hyperelliptic if $m > 3$ is obvious. An easy computation shows that we have the following invariants:

1) $g = (m-1)(m-2)/2$,
2) $\omega^2 = 3m^2 - 12m + 9$, $\omega_f = (m-3)^2$,
3) $r = m^2 - 1$, $s = 3(m-1)^2$.

Thus the case of $m = 4, 5$ gives us the equality in the bound above. It should be very interesting to describe all such fibrations (see footnote in the Introduction).

§2 Pencils with $\chi(\mathcal{O}_X) = 1$.

(2.1) Proposition. *Let $f : X \to \mathbb{P}^1$ be a relatively minimal fibration of curves of genus $g \geq 1$ (possessing a section if $g = 1$). Assume that $\chi(\mathcal{O}_X) = 1$. Then we have*

$$r \leq 4g + 4 + 2q. \tag{2.1.1}$$

Moreover the equality $r = 4g + 4 + 2q$ implies that X is a rational surface and f possesses a section. In particular, r gives actually the Mordell-Weil rank and if $g \geq 2$ we obtain the known constructions with Hirzebruch surfaces Σ_e, $0 \leq e \leq g$, as in [SS].

Proof. In fact bound (2.2.1) can be easily followed from (1.2.1)–(1.2.2). In our special case we have $d = g$ and

$$\omega_f = \omega^2 - 4(g-1), \tag{2.1.2}$$

is a non-negative integer.

Next (1.2.5)–(1.2.6) can be rewritten as

$$r = 4g + 4 + 2q - \omega_f - \sum_{t\in S}(n_t - 1) - \rho'. \tag{2.1.3}$$

Consequently $r \leq 4g + 4 + 2q$ and the equality holds if and only if $\omega_f = 0$, all singular fibres are irreducible, and $\rho' = 0$. Since the elliptic case ($g = 1$) is obvious we can assume $g \geq 2$. Then the condition $\omega_f = 0$ implies that X is a ruled surface ([V, Theorem 3.1 and Lemma 3.1.2]), in particular $p_g = q = 0$, and by the same token the rationality of X. (Note that the fact $p_g = q = 0$ also follows from Remark 1.3). It remains to refer to [SS, Theorem 4.1] for the rest of the Proposition.

(2.2) Lemma. *(i) For $t \in S$ one has*

$$e_t(X) > 0 \tag{2.2.1}$$

except X_t is a nonsingular elliptic curve with some multiplicity (the case $e_t(X) = 0$).
(ii) Moreover $e_t(X) = 1$ if and only if either X_t X_t is irreducible with at most one node as its singularity , or X_t is a curve with two smooth irreducible components C_1, C_2 meeting at one point transversally such that $g(C_1) + g(C_2) = g$.

The first statement is nothing but Theorem 7 in [Sal, IV]. The proof of the second statement is purely technical and can be followed from the arguments in the proof of that Theorem and Lemma 4 (*loc. cit.*)

(2.3) Corollary. *In the assumptions of Proposition 1.2 we have*

$$s \leq (8 + 4/g)d. \tag{2.3.1}$$

In the case of equality we also have (1.4.1).

Proof. First note that

$$\chi(X) = c_2(X) = \sum_{t\in S} e_t(X) + 4(g-1)(b-1) \tag{2.3.2}$$

(see, *e.g.*, [Sal, IV, §4] or [BPV, III, 11.4]). Furthermore from this , (1.2.3) and Noether's formula we have

$$\sum_{t\in S} e_t(X) = (8+\frac{4}{g})d - \omega_f. \tag{2.3.3}$$

It remains to use (2.2.1) to get (2.3.1). The case of equality implies that $\omega_f = 0$. The same arguments as in the proofs of Proposition 1.2 and Corollary 1.4 show that one has (1.4.1).

(2.3.4) Corollary. $e_t(X) \leq 2\big(g - g(\widetilde{X}_t)\big) + 2(n_t - 1)$.

(2.4) Proposition. *In the situation of Proposition 2.1 we have*

$$s \leq 8g + 4 \tag{2.4.1}$$

Furthermore fibrations with maximal number $s = 8g+4$ are rational hyperelliptic Lefschetz pencils possessing a section and with $\omega^2 = 4(g-1)$.

Proof. Since $d = g$ it follows from (2.3.3) that

$$\sum_{t\in S} e_t(X) + \omega_f = 8g + 4. \tag{2.4.2}$$

So (2.4.2) together with (2.2.1) implies bound (2.4.1). Moreover $s = 8g + 4$ holds if and only if:

1) $\omega_f = 0$,

2) $e_t(X) = 1,\ \forall t \in S$.

As in the proof of Proposition 2.1 $\omega_f = 0$ implies that f is a rational hyperelliptic pencil. Furthermore since X is the canonical resolution of a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ with simple singularities, degeneracies with two smooth irreducible components in the second statement of Lemma 2.2 can not occur. Thus we have a Lefschetz pencil.

(2.4.3) Corollary. *Let $f : X \to \mathbb{P}^1$ be as in Proposition 2.1. If $p_g = q > 0$ then $r \leq 4g + 2q$ (resp. $s \leq 8g$) and the equality is possible only in case $p_g = q = 1$.*

Proof. 1)From [X, Corollary 1] and the assumption $q > 0$ it follows that $\lambda(f) \geq 4$. Furthermore $\lambda(f) = 4$ implies $q = 1$ ([X, Theorem3]). It remains to use (2.1.3) and (2.4.2).

(2.4.4) Remark. *For the detailed construction of Lefschetz pencils in Proposition 2.4 we refer to* [SS]. *Note that those fibrations are irregular in the sense of* [V, §3] *if $g \geq 2$.*

§3 Pencils with $c_1^2(X) = -4(g-1) < 0$.

(3.1). In this section we consider the class of pencils with $c_1^2(X) = -4(g-1) < 0$. First recall some facts from the theory of double covering of surfaces. Let B be an even reduced effective divisor on a smooth surface Y. Consider the double covering $X(Y, \gamma, B)$ with branch locus B, $B \sim 2\gamma$. Let $X_{CR} = X_n \to X_{n-1} \to ... \to X_1 \to X_0 = X(Y, \gamma, B)$ be the canonical resolution of X_0. Putting $Y_0 = Y$, $B_0 = B$ then

1) each X_i is a double covering of Y_i with branch locus B_i,

2) Y_i is a blowing up of Y_{i-1} at a singular point of B_i with multiplicity m_i, $i \leq n-1$,

3) B_n is non-singular.

Recall that X_0 has at most rational double points as its singularities if and only if all m_i are less than 4 ([Ho, Lemma 5]).

(3.2) Lemma. *In the notation above we have*

$$\chi(\mathcal{O}_{X_{CR}}) = 2\chi(\mathcal{O}_Y) + \frac{1}{2}\gamma(K_Y + \gamma) - \frac{1}{2}\sum_i \big[\frac{m_i}{2}\big]\big(\big[\frac{m_i}{2}\big] - 1\big), \tag{3.2.1}$$

$$c_1^2(X_{CR}) = 2(K_Y + \gamma)^2 - 2\sum_i \big(\big[\frac{m_i}{2}\big] - 1\big)^2. \tag{3.2.2}$$

Proof. See [Ho, Lemma 6].

(3.3) Theorem. *Let $f : X \to \mathbb{P}^1$ be a relatively minimal fibration with $g \geq 2$. Assume that*

1) $c_1^2(X) = -4(g-1)$,

2) $\chi(\mathcal{O}_X) \geq 3 - g$.

Then X is a ruled surface defined as a family $\pi : X \to E$, $g(E) = q$. Families f and π define a 2-to-1 map from X to $Y = E \times \mathbb{P}^1$ with branch locus, say B. The second projection of Y induces f and B is of numerical type $(2g + 2 - 4q, 2)$. Moreover X is the canonical resolution of $X(Y, \gamma, B)$ with rational double singularities. In particular, if $q = 0$ then X is a rational surface and f is hyperelliptic.

Proof. From [V, Theorem 3.1 and Lemma 3.1.2] we have known that X is a ruled surface $\pi : X \to E$, $g(E) = q$. Families f and π define a 2-to-1 map from X to Y. So it can be easily seen that the branch locus has the desired numerical type. It remains to show that X is the canonical resolution of $X(Y, \gamma, B)$ with rational double singularities. Indeed in a similar way the arguments of [Mat, §2] show that the natural morphism $h : X_{CR} \to X$ is

a contraction of all (-1)-curves on X_{CR}. Now calculating $\chi(\mathcal{O}_{X_{CR}})$, $c_1^2(X_{CR})$ by (3.2.1)–(3.2.2) gives us:

$$\chi(\mathcal{O}_{X_{CR}}) = 1 - q - \frac{1}{2}\sum_i \big[\frac{m_i}{2}\big]\big(\big[\frac{m_i}{2}\big] - 1\big),$$

$$c_1^2(X_{CR}) = -4(g-1) - 2\sum_i \big(\big[\frac{m_i}{2}\big] - 1\big)^2.$$

Therefore one obtains

$$\sum_i \big[\frac{m_i}{2}\big]\big(\big[\frac{m_i}{2}\big] - 1\big) = 0,$$

$$\sum_i \big(\big[\frac{m_i}{2}\big] - 1\big)^2 = 0.$$

That is $X = X_{CR}$ and all m_i are less than 4 as desired.

(3.3.1) Corollary. *In the assumptions of Theorem 3.3 let m be the number of critical points of π. We have*

$$\big(m = 4g + 4 - 8q\big) \iff \big(B \text{ is a smooth irreducible curve}\big)$$

(3.4) Theorem. *Let $f : X \to \mathbb{P}^1$ be as in Theorem 3.3. Then we have*

(3.4.1) $$r \le 4g + 4 - 8q \quad (\textit{resp.} \quad s \le 8g + 4 - 12q).$$

Moreover the equality implies that it is obtained by blowing up from a pencil (resp. a Lefschetz pencil, except for $2q = g$) on a ruled surface $\Sigma_e^q \to E$, $g(E) = q$ $(-q \le e \le g - 2q)$ whose general members are double coverings of curve E. In the maximal case the structure of the essential sublattice in the Néron-Severi lattice is uniquely determined.

Proof. 1) The first bound in (3.4.1) is obvious since $\rho = b_2 = 4g + 6 - 8q$ and by using (0.3). The second one follows immediately from Lemma 2 and

(3.4.2) $$\sum_{t \in S} e_t(X) = 8g + 4 - 12q$$

(*cf.* (2.4.2)). As a consequence we obtain $g + 1 \ge 2q$.

2) Now assume $r = 4g + 4 - 8q$. Then all singular fibres are irreducible. Since by Theorem 3.3, X is the canonical resolution of $X(Y, \gamma, B)$ with rational double singularities it follows that B is a smooth irreducible curve, so that $m = 4g + 4 - 8q$. Let us denote by $\{E_i^{\pm}\}_{i=1}^{4g+4-8q}$ the irreducible components of corresponding singular fibres of π. Obviously $E_i^{\pm}$s are sections of f. After a succession of blowings down (each time one of $E_i^{\pm}$ which we denote simply by E_i) we obtain a standard ruled surface $\pi' : \Sigma_e^q \to E$. Surface Σ_e^q has

degree $-e$ and a section C_0 such that $C_0^2 = -e$. For invariant e we know that $e \geq -q$ ([N], [Mar]).

Thus we have a birational morphism $\varphi : X \to \Sigma_e^q$. Setting $F' = \varphi(F)$ the image of a smooth general fibre we may assume that F' is also smooth and birational to F. An easy computation shows that

$$F' \equiv 2C_0 + aF_0,$$

where $a = g + 1 + e - 2q$ and $F_0 \simeq \mathbb{P}^1$ is a fibre of π'. It means that we obtain a linear pencil of a linear system $|2C_0 + \bar{a}F_0|$ on Σ_e^q with $\deg \bar{a} = a$ and X is obtained as a blowing up of the base points of this linear pencil. We have to consider two cases.

(+) If $e \geq 0$ then from [Ha1, V, 2.20] it follows that $a \geq 2e$, or equivalently, $e \leq g + 1 - 2q$. Assume that $e = g + 1 - 2q$ then $F'.C_0 = 0$. Let C_0' be the proper transform of C_0 by φ, one can see that $F_0.C_0' = 0$ and $C_0'^2 = C_0^2 = -e = 2q - g - 1$. In view of the irreducibility of fibres of f it is possible only if $e = 2q - g - 1 = 0$. Hence π is a smooth fibering, that is, $X = \Sigma_e^q$. On one hand C_0 is a fibre of f by the above. On the other hand C_0 is a section of π. This contradicts the fact that $F.F_0 = 2$. We have proved $e \leq g - 2q$.

(++) If $e < 0$ then from [Ha1, V, 2.21] we have known that $a \geq e$, or equivalently, $g + 1 \geq 2q$.

3) The Néron-Severi group $\mathrm{NS}(X)$ in the maximal case is as follows.

$$\mathrm{N}S(X) \simeq \mathbb{Z}.C_0 \oplus \mathbb{Z}.F_0 \oplus \left(\oplus_{i=1}^{4g+4-8q} \mathbb{Z}.E_i \right) \tag{3.4.3}$$

where we denote total transforms of C_0, F_0 under φ by the same letters. We have a relation

$$F \sim 2C_0 + \bar{a}F_0 - \sum_{i=1}^{4g+4-8q} E_i. \tag{3.4.4}$$

4) Similar assertions for s could be followed easily with adding the Lefschetz property to the pencinls. In fact from $e_t(X) = 1$ and $g \neq 2q$ we claim that degeneracies with two smooth irreducible componnents C_1, C_2 with $g(C_1) + g(C_2) = g$ (*cf.* Lemma 2.2) do not occur. Since X is the canonical resoluton of $X(Y, \gamma, B)$, one obtains $g(C_1) = g(C_2) = q$, that is impossible by the assumption $g \neq 2q$. Thus arguing as above we get decomposition (3.4.3). Therefore in the case $s = 8g + 4 - 12q$ (even without the condition $g \neq 2q$) the structure of the essential sublattice in the Néron-Severi lattice is uniquely determined.

(3.4.5) Corollary. *Let $f : X \to \mathbb{P}^1$ be as in Theorem 3.3. Assume that all singular fibres are irreducible, then*

$$s \geq 8 + \frac{4q - 4}{g + 1 - 2q}.$$

In particular we have $s \geq 7$ *and if either* $q \geq 1$, *or* $q = 0$ *and* $g \geq 4$, *then* $s \geq 8$.

Proof. By virtue of the Riemann-Hurwitz formula one sees that $g(\widetilde{X}_t) \geq 2q-1$ (Note that since S is not empty we get another proof of estimate $g+1 > 2q$). The corollary now follows easily from (3.4.2) and Corollary 2.3.4.

(3.5) The maximal case with rational base $q = 0$ leads us to known constructions with Hirzebruch surfaces Σ_e. As a rule for constructing examples with maximal numbers s, r we need the very ampleness of linear system $|2C_0 + \bar{a}F_0|$on Σ_e^q (*cf.* [SS]). In general one can construct certain examples with maximal numbers $r = 4g + 4 - 8q$, $s = 8g + 4 - 12q$ under some conditions with respect to e.

Note that linear system $|2E + \bar{a}\mathbb{P}^1|$ on $\mathbb{P}^1 \times E$ with $\deg\bar{a} = a = g+1-2q$ is very ample if $a \geq 2q+1$. This gives an example with $e = 0$. In fact one can prove the following proposition.

(3.5.1) Proposition. *If* $4q \leq g$ *then for* $-q \leq e \leq \dfrac{(g+1)}{2} - 2q$ *linear systems* $|2C_0 + \bar{a}F_0|$ *with* $\deg\ \bar{a} = a = g + 1 + e - 2q$ *are very ample on* Σ_e^q.

Proof. Denote by $\mathcal{L} = \mathcal{O}(2C_0 + \bar{a}F_0)$ and consider any two points P_1, P_2 on Σ_e^q being contained in fibres F_1, F_2 respectively. From standard exact sequences

$$0 \to m_1 \otimes \mathcal{L} \to \mathcal{L} \to \mathcal{O}_{P_1} \to 0$$

$$0 \to m_1 m_2 \otimes \mathcal{L} \to m_1 \otimes \mathcal{L} \to \mathcal{O}_{P_2} \to 0$$

where m_i is the ideal sheaf at P_i, we know that it is sufficient to establish:

$$(*) \qquad H^1(m_i \otimes \mathcal{L}) = H^1(m_1 m_2 \otimes \mathcal{L}) = 0.$$

Due to two exact sequences

$$0 \to \mathcal{L}(-F_1) \to m_1 \otimes \mathcal{L} \to m_1 \otimes \mathcal{L}/\mathcal{L}(-F_1) \to 0$$

$$0 \to \mathcal{L}(-F_1 - F_2) \to m_1 m_2 \otimes \mathcal{L} \to m_1 m_2 \otimes \mathcal{L}/\mathcal{L}(-F_1 - F_2) \to 0$$

one needs to show

1) $H^1(\mathcal{L}(-\bar{b}F_0)) = 0$, $\deg\ \bar{b} \leq 2$,

2) the corresponding to (*) assertion on fibres: it is easy since $\deg\mathcal{L}_{|F_0} = 2$ and $F_0 \simeq \mathbb{P}^1$ so that $\mathcal{L}_{|F_0}$ is very ample on F_0.

For the first statement by virtue of Ramanujam's vanishing theorem ([R]) it suffices to verify the numerical positivity of divisor $D = 2C_0 + (a-2)F_0 - K_{\Sigma_e^q}$. Here are standard calculations using [Ha1, V, 2.20–2.21].

(i) Since $g \geq 4q$ by the assumption we have $D^2 = 8(g+1-4q) > 0$.

(ii) Case $e \geq 0$: let $C' \equiv bC_0 + cF_0$ be an irreducible curve ($b > 0,\ c \geq be$) then from the condition for e we have

$$D.C_0 = g + 1 - 4q - 2e \geq 0,$$

$$D.C' = -4be + b(g+1+2e-4q) + 4c \geq -4be + b(g+1+2e-4q) + 4be > 0.$$

(iii) Case $e < 0$: for an irreducible curve $C' \equiv bC_0 + cF_0$

(+) either $b = 1,\ c \geq 0$ then

$$D.C' = -4e + (g+1+2e-4q) + 4c > -4e + 2e = -2e > 0,$$

(+) or $b \geq 2,\ 2c \geq be$ then

$$D.C' = -4be + b(g+1+2e-4q) + 4c > -4be + 2be + 2be = 0.$$

This completes the proof.

(3.5.2) Remark. *If $q = 1$ then using* [Ha1, V, exer.2.12] *it is easy to verify that the linear systems above are very ample if $-1 \leq e \leq g-6$.*

(3.6) From the very ampleness of $|2C_0 + \bar{a}F_0|$ in a similar way as in [SS] one can find a Lefschetz pencil of this linear system such that $4g+4-8q$ distinct base points $p_1, \ldots, p_{4g+4-8q}$ do not lie on C_0 and any two of them are not on the same fibre of π'. By blowing up the base points we obtain the desired fibration $f : X \to \mathbb{P}^1$ with maximal numbers r and s. Indeed if we denote by $\{E_i\}_{i=1}^{4g+4-8q}$ the exceptional curves dominating points $\{p_i\}_{i=1}^{4g+4-8q}$ respectively then for the Néron-Severi group $\mathrm{NS}(X)$ we have decomposition (3.4.3) with relation (3.4.4). Therefore $\rho = 4g+6-8q$, and by (0.3) $r = 4g+4-8q$. Furthermore it is easy to see that $c_1^2(X) = -4(g-1)$. So (3.4.2) implies $s = 8g+4-12q$.

(3.6.1) Remark. *Examples with $q = 1$ give the best upper bound of self-intersection of curves on irrational ruled surfaces (cf.* [Ha2]*).*

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