NOTES ON THE GROUP OF S^1 EQUIVARIANT HOMEOMORPHISMS

Shuichi Tsukuda

Department of Mathematics, Kyoto University, Kyoto 606-01, Japan

1.Introduction

Let P be a principal S^1 bundle over a compact smooth manifold X with the 1-st betti number $b_1(X) = 0$, $\mathcal{G}$ its gauge group and $\mathcal{G}^0$ its based gauge group. Recall that $\mathcal{G} = \mathrm{Map}(X, S^1)$ and $\mathcal{G}^0 = \mathrm{Map}^*(X, S^1)$, moreover, since S^1 is abelian, $\mathcal{G} \cong \mathcal{G}^0 \times S^1$ as group.

In this paper, we study the topology of $\mathrm{Homeo}_{S^1}(P)$, the group of S^1 equivariant homeomorphisms of P, in paticular in §3, we discus S^1 bundles over $\mathbb{C}P^n$.

It is known ([2],[4]) that there is a fibration

$$\mathcal{G} \to \mathrm{Map}_{S^1}(P, P) \to \mathrm{Map}(X, X) \tag{1.1}$$

and it is shown in [8] that (1.1) restricts to

$$\mathcal{G} \to \mathrm{Homeo}_{S^1}(P) \xrightarrow{\pi} \mathrm{Homeo}_P(X) \tag{1.2}$$

where

$$\mathrm{Homeo}_P(X) = \{\varphi \in \mathrm{Homeo}(X) | \varphi^* P = P\}.$$

Fix a base point $x_0 \in X$ and define a map

$$\mu_{x_0} : \mathrm{Homeo}_P(X) \to X$$

by $\mu_{x_0}(\varphi) = \varphi(x_0)$.

Then we have

Proposition 1.1. *There exits a weak homotopy equivalence*

$$\mathrm{Homeo}_{S^1}(P) \underset{w}{\simeq} \mu_{x_0}^* P$$

Let $\mathrm{Homeo}_P^*(X)$ be the subgroup of $\mathrm{Homeo}_P(X)$ consisting of base point preserving maps. Then $\mathrm{Homeo}_{S^1,x_0}(P) = \pi^{-1}(\mathrm{Homeo}_P^*(X))$ is consisting of maps preserving the fibre over x_0. Let $\mathrm{Homeo}_{S^1}^*(P) \subset \mathrm{Homeo}_{S^1,x_0}(P)$ be the subgroup acting as identity on the fibre over x_0.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Proposition 1.2. *There exist weak homotopy equivalences*

$$B\mathrm{Homeo}_{S^1,x_0}(P) \underset{w}{\simeq} B(\mathrm{Homeo}^*_P(X) \times S^1)$$
$$B\mathrm{Homeo}^*_{S^1}(P) \underset{w}{\simeq} B\mathrm{Homeo}^*_P(X).$$

Moreover for any paracompact subgroup $G \subset \mathrm{Homeo}_P(X)$ with homotopy type of CW complex, since $\mathrm{Map}(X,X)$ is locally contractible and $\mathcal{G}$ acts $\mathrm{Homeo}_{S^1}(P)$ freely on the right, (1.2) restricted to G is a principal bundle. We have

Proposition 1.3. $\mathrm{Homeo}_{S^1}(P)|_G \cong \mu^*_{x_0} P \times \mathcal{G}^0$ *as principal* $\mathcal{G}$ *bundle over* G.

Let P_k be the principal S^1 bundle over $\mathbb{C}P^n$ with $c_1(P_k) = k$, then, by using [6], we obtain the following theorem.

Theorem 1.4. $\pi_{2i+1}(\mathrm{Homeo}_{S^1}(P_k))$ *has a free part for* $i = 0, 1, \cdots, n$.

In fact the similar result holds for S^3 bundles over $\mathbb{H}P^n$, and discussed in the appendix.

The author is grateful to Profesor Akira Kono for his advices, especially to reffer [6][7], and encouragement.

2. Proofs

By [1], there are universal bundles

$$\mathcal{G} \to \mathrm{Map}_{S^1}(P, ES^1) \to \mathrm{Map}_P(X, BS^1)$$
$$\mathcal{G}^0 \to \mathrm{Map}^*_{S^1}(P, ES^1) \to \mathrm{Map}^*_P(X, BS^1).$$

Since $b_1(X) = 0$, $\mathcal{G}^0$ is contractible, and $B\mathcal{G}^0 \simeq \mathrm{Map}^*_P(X, BS^1)$ has homotopy type of CW complex, $B\mathcal{G}^0$ is contractible and

$$B\mathcal{G} \simeq \mathrm{Map}_P(X, BS^1) \underset{ev}{\simeq} BS^1 \simeq \mathbb{C}P^\infty.$$

Then the following diagram commutes up to homotopy,

$$\begin{array}{ccc}
\mathrm{Map}_{S^1}(P, ES^1) & \xrightarrow{\tilde{-f}} & \mathrm{Map}_{S^1}(X \times S^1, ES^1) \\
\downarrow & & \downarrow \\
\mathrm{Map}_P(X, BS^1) & \underset{-f}{\xrightarrow{\simeq}} & \mathrm{Map}_0(X, BS^1) \\
{\scriptstyle ev}\downarrow & & \uparrow{\scriptstyle i} \\
BS^1 & = & BS^1
\end{array}$$

and it can be easily shown that

$$i^*\mathrm{Map}_{S^1}(X \times S^1, ES^1) \cong ES^1 \times \mathcal{G}^0$$

as $\mathcal{G}$ bundle.

Proof of Proposition 1.3. Fix a base point preserving classifying map f for P, and we identify P and f^*ES^1. Then we have following commutative diagram

$$\begin{array}{ccc} \mathrm{Homeo}_{S^1}(P)|_G & \xrightarrow{\tilde{f}_*} & \mathrm{Map}_{S^1}(P, ES^1) \\ \downarrow & & \downarrow \\ G & \xrightarrow{f_*} & \mathrm{Map}_P(X, BS^1) \\ {\scriptstyle \mu_{x_0}}\downarrow & & \downarrow{\scriptstyle ev} \\ X & \xrightarrow{f} & BS^1 \end{array}$$

and $\tilde{f}_*$ is $\mathcal{G}$ equivariant, hence

$$\begin{aligned} \mathrm{Homeo}_{S^1}(P)|_G &\cong (f_*)^*\mathrm{Map}_{S^1}(P, ES^1) \\ &\cong (f_*)^*(-f)^*\mathrm{Map}_{S^1}(X \times S^1, ES^1) \\ &\cong (f_*)^* ev^* i^* \mathrm{Map}_{S^1}(X \times S^1, ES^1) \\ &\cong (f_*)^* ev^* ES^1 \times \mathcal{G}^0 \\ &= \mu_{x_0}^* f^* ES^1 \times \mathcal{G}^0 \\ &= \mu_{x_0}^* P \times \mathcal{G}^0. \end{aligned}$$

□

Proof of Proposition 1.1. Fix a base point p_0 in the fibre over x_0, then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Homeo}_{S^1}(P) & \xrightarrow{\mu_{p_0}} & P \\ \downarrow & & \downarrow \\ \mathrm{Homeo}_P(X) & \xrightarrow{\mu_{x_0}} & X \end{array}$$

and so a map of fibration

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\simeq} & S^1 \\ \downarrow & & \downarrow \\ \mathrm{Homeo}_{S^1}(P) & \longrightarrow & \mu_{x_0}^* P \\ \downarrow & & \downarrow \\ \mathrm{Homeo}_P(X) & = & \mathrm{Homeo}_P(X). \end{array}$$

By the homotopy exact sequence, we have $\mathrm{Homeo}_{S^1}(P) \underset{w}{\simeq} \mu_{x_0}^* P$. □

Of course when restricted to the subgroup G as in Proposition 1.3, this weak homotopy equivalence is the homotopy equivalence given by Proposition 1.3.

Proof of Proposition 1.2. Identify P_{x_0} with S^1 by

$$S^1 \ni z \to p_0 \cdot z \in P_{x_0},$$

then there is a group isomorphism

$$\Phi\colon \mathrm{Homeo}_{S^1,x_0}(P) \to \mathrm{Homeo}^*_{S^1}(P) \times S^1$$

given by $\Phi(\tilde{\varphi}) = (\tilde{\varphi} \cdot \mu_{p_0}(\tilde{\varphi})^{-1}, \mu_{p_0}(\tilde{\varphi}))$.

From (1.2), we have a fibration

$$\mathcal{G}^0 \times S^1 \to \mathrm{Homeo}^*_{S^1}(P) \times S^1 \xrightarrow{\pi} \mathrm{Homeo}^*_P(X)$$

where $\pi(\tilde{\varphi}, z) = \pi(\tilde{\varphi})$. Since $\mathcal{G}^0$ is contractible, we easily deduce the results. □

3. S^1 Bundle over $\mathbb{C}P^n$

Let P_k be the principal S^1 bundle over $\mathbb{C}P^n$ with $c_1(P_k) = k$ and consider the subgroup $PU(n+1) \subset \mathrm{Homeo}_{P_k}(\mathbb{C}P^n)$.

By an easy computation, $\mu^*_{x_0}$ is mod $n+1$ reduction,

$$\begin{array}{ccccc} 0 & \longleftarrow & H^2(PU(n+1);\mathbb{Z}) & \xleftarrow{\mu^*_{x_0}} & H^2(\mathbb{C}P^n;\mathbb{Z}) \\ & & \| & & \| \\ & & \mathbb{Z}/n+1 & & \mathbb{Z}. \end{array}$$

The bundle P_1 is given by the following (for simplicity we orient $\mathbb{C}P^n$ like this)

$$S^1 \to S^{2n+1} \to \mathbb{C}P^n$$

and since $P_k = P_1^{\otimes k}$ we have

$$P_k = S^{2n+1} \times_{\rho_k} S^1$$

where

$$\rho_k\colon S^1 \to S^1$$

is defined by $\rho_k(z) = z^k$.

Since there is a commutative diagram

$$\begin{array}{ccc} S^1 & = & S^1 \\ \downarrow & & \downarrow \\ U(n+1) & \longrightarrow & S^{2n+1} \\ \downarrow & & \downarrow \\ PU(n+1) & \xrightarrow{\mu_{x_0}} & \mathbb{C}P^n, \end{array}$$

we have

$$\mu^*_{x_0} P_k = U(n+1) \times_{\rho_k} S^1 = SU(n+1) \times_k S^1$$

where

$$k = k\times\colon \mathbb{Z}/n+1 \to \mathbb{Z}/n+1 \subset S^1.$$

Note that if $k \equiv k' \pmod{n+1}$, then $U(n+1) \times_{\rho_k} S^1 \cong U(n+1) \times_{\rho_{k'}} S^1$ as group.

By Proposition 1.3, we have

Proposition 3.1. $\mathrm{Homeo}_{S^1}(P_k)|_{PU(n+1)} \cong (U(n+1) \times_{\rho_k} S^1) \times \mathcal{G}^0$ *as* $\mathcal{G}$ *bundle over* $PU(n+1)$.

In fact these are isomorphic as group. We construct the isomorphism explicitly. Consider the following action

$$\rho\colon (U(n+1) \times S^1) \times (S^{2n+1} \times S^1) \to (S^{2n+1} \times S^1)$$

given by $\rho((g,z),(x,z')) = (g \cdot x, z'z)$. This induces the action

$$\rho\colon (U(n+1) \times_{\rho_k} S^1) \times P_k \to P_k.$$

This gives desired isomorphism

$$F\colon (U(n+1) \times_{\rho_k} S^1) \times \mathcal{G}^0 \to \mathrm{Homeo}_{S^1}(P_k)|_{PU(n+1)}$$

by

$$F(g,u)(p) = \rho(g,p) \cdot u(x)$$

where $p \in P_k$ and $\pi(p) = x \in \mathbb{C}P^n$.

Theorem 3.2. $\mathrm{Homeo}_{S^1}(P_k)|_{PU(n+1)} \cong (U(n+1) \times_{\rho_k} S^1) \times \mathcal{G}^0$ *as group. In paticular, there exists a homotopy equivalence*

$$B(\mathrm{Homeo}_{S^1}(P_k)|_{PU(n+1)}) \simeq B(U(n+1) \times_{\rho_k} S^1).$$

Note that for $n = 1$, $PU(2) = SO(3)$ and $\mathbb{C}P^1 = S^2$. By [5], the inclusion $SO(3) \hookrightarrow \mathrm{Homeo}^+(S^2)$ is a homotopy equivalence.

Moreover there are group isomorphisms

$$U(2) \times_{\rho_{2k}} S^1 = SO(3) \times S^1$$
$$U(2) \times_{\rho_{2k+1}} S^1 = Spin^c(3)$$

and we have

Proposition 3.3. *There exist weak homotopy equivalences*

$$B\mathrm{Homeo}_{S^1}(P_k) \underset{w}{\simeq} \begin{cases} B(SO(3) \times S^1) & k\text{:even} \\ BSpin^c(3) & k\text{:odd.} \end{cases}$$

The cases for $k = 0, 1$ are studied in [8].

Finally we consider the homotopy groups.

Proposition 3.4. *The following map is injective for all* i.

$$\pi_i(U(n+1) \times_{\rho_k} S^1) \otimes \mathbb{Q} \to \pi_i(\mathrm{Homeo}_{S^1}(P_k)) \otimes \mathbb{Q}.$$

Proof. Consider the following diagram

$$\begin{array}{ccccc} S^1 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{G} \\ \downarrow & & \downarrow & & \downarrow \\ U(n+1)\times_{\rho_k} S^1 & \xrightarrow{i} & \mathrm{Homeo}_{S^1}(P_k) & \xrightarrow{j} & \mathrm{Map}_{S^1}(P_k,P_k) \\ \downarrow & & \downarrow & & \downarrow \\ PU(n+1) & \xrightarrow{i} & \mathrm{Homeo}_{P_k}(\mathbb{C}P^n) & \xrightarrow{j} & \mathrm{Map}(\mathbb{C}P^n,\mathbb{C}P^n). \end{array}$$

In [6], Sasao proved that

$$(ji)_*\colon \pi_i(PU(n+1))\otimes\mathbb{Q}\to\pi_i(\mathrm{Map}(\mathbb{C}P^n,\mathbb{C}P^n))\otimes\mathbb{Q}$$

is an isomorphism for all i, and

$$\pi_1(PU(n+1))\cong\pi_1(\mathrm{Map}(\mathbb{C}P^n,\mathbb{C}P^n))\cong\mathbb{Z}/n+1$$
$$\pi_2(\mathrm{Map}(\mathbb{C}P^n,\mathbb{C}P^n))\cong\mathbb{Z}/2$$

The result follows from the commutatuvity of the diagram. □

Theorem 1.4 immediately follows from this proposition.

Appendix

In this appendix, we study S^3 bundles over $\mathbb{H}P^n$.

Let P be a principal S^3 bundle over $\mathbb{H}P^n$. Then we have

Theorem A. *$\pi_{4i+3}(\mathrm{Homeo}_{S^3}(P))$ has a free part for $i=1,2,\cdots,n$.*

The proof is similar to that of Proposition 3.4. Here we use the result of [7]. Consider the natural action of $Sp(n+1)$ on $\mathbb{H}P^n$, which induces the map

$$Sp(n+1)/\mathbb{Z}_2 \xrightarrow{i} \mathrm{Homeo}_P(\mathbb{H}P^n) \xrightarrow{j} \mathrm{Map}(\mathbb{H}P^n,\mathbb{H}P^n).$$

Then we have

Theorem [7]. *The induced homomorphism*

$$(ji)_*\otimes 1\colon \pi_i(Sp(n+1)/\mathbb{Z}_2)\otimes\mathbb{Q}\to\pi_i(\mathrm{Map}(\mathbb{H}P^n,\mathbb{H}P^n))\otimes\mathbb{Q}$$

is an isomorphism for all $i>4$.

More over for $1\le i\le 6$,

$$\pi_i(\mathrm{Map}(\mathbb{H}P^n,\mathbb{H}P^n))\otimes\mathbb{Q}=0.$$

Proof of Theorem A. For a space Y, denote the space Y localized at 0 by $Y_{(0)}$. Consider the following commutative diagram

$$\begin{array}{ccccc} \mathcal{G} & = & \mathcal{G} & = & \mathcal{G} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Homeo}_{S^3}(P)|_{Sp} & \longrightarrow & \mathrm{Homeo}_{S^3}(P) & \longrightarrow & \mathrm{Map}_{S^3}(P,P) \\ \downarrow & & \downarrow & & \downarrow \\ Sp(n+1)/\mathbb{Z}_2 & \longrightarrow & \mathrm{Homeo}_P(\mathbb{H}P^n) & \longrightarrow & \mathrm{Map}(\mathbb{H}P^n,\mathbb{H}P^n) \end{array}$$

which induces

$$\begin{array}{ccc} \mathcal{G}_{(0)} & = & \mathcal{G}_{(0)} \\ \downarrow & & \downarrow \\ (\mathrm{Homeo}_{S^3}(P)|_{Sp})_{(0)} & \longrightarrow & \mathrm{Map}_{S^3}(P,P)_{(0)} \\ \downarrow & & \downarrow \\ (Sp(n+1)/\mathbb{Z}_2)_{(0)} & \longrightarrow & \mathrm{Map}(\mathbb{H}P^n, \mathbb{H}P^n)_{(0)} \end{array}$$

where $(\mathrm{Homeo}_{S^3}(P)|_{Sp})_{(0)} = \mathrm{Map}_{S^3}(P,P)_{(0)}|_{(Sp(n+1)/\mathbb{Z}_2)_{(0)}}$ and $\mathcal{G}_{(0)} = \Omega(B\mathcal{G}_{(0)})$.

Recall that ([1],[3])

$$\begin{aligned} B\mathcal{G}_{(0)} &\simeq \mathrm{Map}_P(\mathbb{H}P^n, BS^3)_{(0)} \\ &\simeq \mathrm{Map}_P(\mathbb{H}P^n, BS^3_{(0)}) \\ &\simeq \mathrm{Map}_P(\mathbb{H}P^n, K(\mathbb{Q},4)) \\ &\simeq K(\mathbb{Q},4) \end{aligned}$$

where the last line is due to the R. Thom's famous result, hence

$$\mathcal{G}_{(0)} \simeq K(\mathbb{Q},3).$$

Therefore the following map is an isomorphism for all $i \neq 3$

$$\pi_i((\mathrm{Homeo}_{S^3}(P)|_{Sp})_{(0)}) \to \pi_i(\mathrm{Map}_{S^3}(P,P)_{(0)}),$$

hence

$$\pi_i(\mathrm{Homeo}_{S^3}(P)|_{Sp}) \otimes \mathbb{Q} \to \pi_i(\mathrm{Homeo}_{S^3}(P)) \otimes \mathbb{Q}$$

is injective for $i \neq 3$ and we complete the proof. □

References

1. M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Proc. Royal Soc. London **A308** (1982), 523–615.
2. D. H. Gottlieb, *Applications of bundle map theory*, Trans. Amer. Math. Soc. **171** (1972), 23–50.
3. P. Hilton, G. Mislin and J. Roitberg, *Localization of nilpotent groups and spaces*, North Holland, 1975.
4. I. M. James, *The space of bundle maps*, Topology **2** (1963), 45–59.
5. H. Kneser, *Die DEformationssätze der eifach zusammenhängenden Flächen*, Math. Zeit. **25** (1926), 362–372.
6. S. Sasao, *The homotopy of Map(CP^m, CP^n)*, J. London Math. Soc. **8** (1974), 193–197.
7. K. Yamaguchi, *On the rational homotopy of Map(HP^m, HP^n)*, Kodai Math. J. **6** (1983), 279–288.
8. T. Yamanoshita, *On the Group of S^1-equivariant Homeomorphisms of the 3-Sphere*, preprint.