AN EXTENSION OF THE STABILITY INDEX FOR TRAVELLING WAVE SOLUTIONS AND ITS APPLICATION FOR BIFURCATIONS

SHUNSAKU NII

ABSTRACT. We treat the stability index for travelling wave solutions of one-dimensional reaction diffusion equations due to Alexander, Gardner and Jones [1]. An extension of the stability index which makes the index robust is given and an additive formula for gluing bifurcation of travelling waves is proven using the extension. We, also, consider some types of heteroclinic bifurcations as an application and some examples are dealt with.

1. INTRODUCTION

We often encounter following type of equations as model equations in chemistry, mathematical biology and so on, concerning formation of spatial patterns. This kind of equations are called reaction diffusion equations.

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \Delta u_1 + f_1(u_1, \cdots, u_n) \\ \qquad\qquad \vdots \\ \dfrac{\partial u_n}{\partial t} = d_n \Delta u_n + f_n(u_1, \cdots, u_n), \end{cases} \tag{1.1}$$

where $d_i \geq 0$ represent, as a model equation of chemical reaction for instance, the spatial diffusion rate of chemicals and f_i model production of them.

Stable steady state constant solutions of such systems correspond to homogeneous equilibrium states of described phenomena. On the other hand, pattern formations which appear in transition processes are attracting interest and much research has been devoted to them. For systems with one-dimensional space variable, travelling wave solutions are especially important from this point of view. In this paper we deal with stability and bifurcations of them.

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-01, Japan
E-mail:shunsaku@kusm.kyoto-u.ac.jp
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Consider a one-dimensional reaction diffusion equation

$$u_t = Bu_{xx} + F(u), \tag{1.2}$$

where $x, t \in \mathbb{R}$, $u(x,t) \in \mathbb{R}^n$, B: $n \times n$ positive diagonal matrix and $F\colon \mathbb{R}^n \to \mathbb{R}^n$. Put $\xi = x - \theta t$ and rewrite this equation on a moving frame (ξ, t), then we have

$$u_t = Bu_{\xi\xi} + \theta u_\xi + F(u). \tag{1.3}$$

In the x-coordinate, a steady state solution of the equation (1.3) is a solution which translates with constant velocity θ preserving its shape. This is a travelling wave solution of the system (1.2). In this paper we restrict our attention to the travelling waves satisfying the boundary condition

$$\lim_{x \to \pm\infty} u(x,t) = u_\pm, \tag{1.4}$$

where we assume that $u \equiv u_\pm$ are stable steady state constant solutions of (1.2). *i.e.* $u(x,t) = u(\xi)$ is a solution of the "O.D.E."

$$Bu_{\xi\xi} + \theta u_\xi + F(u) = 0, \tag{1.5}$$

with $\lim_{\xi \to \pm\infty} u(\xi) = u_\pm$. This means $(u(\xi), u'(\xi))$ is a heteroclinic solution connecting $(u_-, 0)$ and $(u_+, 0)$ of the first order system:

$$\begin{cases} u' = v \\ v' = -B^{-1}F(u) - \theta B^{-1} v. \end{cases} \qquad (\ ' = \tfrac{d}{d\xi}) \tag{1.6}$$

This kind of wave corresponds to movement of a transition layer between two equilibrium states when u_- is different from u_+ whereas it corresponds to propagation of a pulse when u_- equals to u_+.

The existence problem of such travelling waves or, equivalently, heteroclinic (homoclinic) orbits of (1.6) already commands a large body of literature, and it is also one of the promoters of development of bifurcation theory for homoclinic or heteroclinic orbits of vector fields.

Not all travelling waves, however, correspond to observed phenomena which are described by the equations. Physical systems always suffer from noise which comes from out of the system. Therefore waves represented by unstable solutions, which are destroyed by small perturbation for initial conditions, can not sustain themselves in the systems in the real world. That is why we need stability analysis, in other words, we have to know stable solutions which, even if they are perturbed slightly, recover to original solutions as time goes on.

We usually consider linear stability, that is to say, we linearize the equation about the wave under study and investigate whether this linearized operator on the appropriate space has any eigenvalue with positive real part. This is because, in the context of semi-linear parabolic equations, it is well known that stability for the linear

problem implies the same for the nonlinear problem under an appropriate setting. So the eigenvalue problem is the main issue for the stability analysis for travelling waves. See Alexander, Gardner and Jones [1], and for more details see Bates and Jones [3] and Henry [8].

One way to approach the eigenvalue problem is to investigate the Evans function which is an analytic function on the complex plane derived from the eigenvalue problem and vanishes at the eigenvalues (Evans [5][6], Jones [10], Yanagida [17]). This function is the prototype of the stability index due to Alexander, Gardner and Jones [1] which is the main subject of this paper. In fact it is an algebraically dressed refinement of the Evans function.

The aim of this paper is to determine the stability of travelling waves which are generated by certain bifurcations.

First, we make an extension of the stability index.

Consider a system depending on a parameter $\mu \in \mathbb{R}$,

$$u_t = Bu_{xx} + F(u;\mu). \tag{1.7}$$

Assume that at $\mu = 0$ this system has a travelling wave solution $u_0(\xi)$ with velocity θ, *i.e.* $u_0(\xi)$ satisfies

$$Bu_{\xi\xi} + \theta u_\xi + F(u;\mu) = 0 \tag{1.8}$$

for $\xi = x - \theta t$. The stability index is defined for u_0. For $\mu \neq 0$, generically (1.7) does not have a travelling wave so that the stability index is not defined. The equation (1.8) or the following system corresponding to it, however, should retain some information for $\mu \approx 0$, and it is important to get it when (1.8) (or (1.9)) undergoes various bifurcations.

$$\begin{cases} u' = v \\ v' = -B^{-1}F(u;\mu) - \theta B^{-1}v. \end{cases} \qquad (' = \tfrac{d}{d\xi}) \tag{1.9}$$

In fact, for each μ, (1.9) has a solution $(u(\xi;\mu), u'(\xi;\mu))$ which is "near" $(u_0(\xi), u_0'(\xi))$ and we can make use of it. The stability index is originally defined as the first Chern number of certain vector bundle over the two-dimensional sphere, but we regard it as an element of the two-dimensional homotopy group of the complex projective space (Theorem 2.2). We prove this definition has a kind of robustness, namely we can define the index for $u(\xi;\mu)$ as an element of the relative homotopy group and it inherits the information of the index defined for $u_0(\xi)$ (Theorem 3.1).

Next, an additive formula (Theorem 3.2) of the index for the gluing bifurcation([7]) of heteroclinic orbits of (1.9) is proven as an application of the extension of the stability index.

Sometimes we see gluing bifurcations of travelling waves — two travelling waves are glued with each other at the ends of $\xi \to \pm\infty$ and a new travelling wave is produced —. At first, Yanagida and Maginu [18] treated this kind of problem for

pulses when they had oscillating tails and the equilibrium of (1.9) to which the pulses are associated had one-dimensional unstable manifold. In this case the definition and calculation of the Evans function was relatively easy. Later Alexander and Jones [2] dealt with the same problem when unstable manifold is two-dimensional. In [2] additive formula for the bifurcation is proven using cobordism. Our result gives another way of proving it.

The additive formula tells us that for a wave generated by a gluing bifurcation of two waves, the linearized operator possesses two eigenvalues near the origin; one is at the origin corresponding to the translation of the wave, so that the sign of the other one determines the stability. Alexander and Jones [2] showed that the direction of the intersection of stable and unstable manifolds of equilibria of (1.9) determines the sign (Proposition 4.1). This is also a generalization of the works of Evans [5][6], Jones [10], Yanagida [17] and Yanagida and Maginu [18], in those works the geometrical meaning of direction was clear. In the general situation, however, we have to define the direction properly.

The remainder of the paper is devoted to an application for a heteroclinic bifurcation. A generic two parameter family of vector fields with two heteroclinic orbits in a row undergoes a gluing bifurcation which produces a heteroclinic orbit that stays in some neighborhood of the concatenation of original two heteroclinic orbits (Kokubu [15]). This type of bifurcation is also observed in some reaction diffusion equations. We apply the preceding argument to this kind of bifurcations. In this case we have to determine the direction of the intersections to apply for actual examples (Theorem 4.3). This is not an easy task in general situations, but we can determine the stability for the case of a heteroclinic loop. More precisely, assume that the system (1.9) has two equilibria $(u_1, 0)$ and $(u_2, 0)$ and assume that for a certain parameter value, there exist two heteroclinic orbits from $(u_1, 0)$ to $(u_2, 0)$ and from $(u_2, 0)$ to $(u_1, 0)$ forming a loop called a heteroclinic loop. Then, the system undergoes a gluing bifurcation producing a homoclinic orbit with respect to $(u_1, 0)$. In this situation we show that the stability of the travelling wave corresponding to the homoclinic orbit is determined by the bifurcation diagram (Theorem 4.4). We show the relation between bifurcation diagrams and stability in Figure 14. This is another main theorem of this paper.

This paper is divided into 5 parts. In section 2, we briefly summarize the definition of the stability index and translate it into our way of treating it. In section 3 we prove that the index can be extended and give a proof of the additive formula. Section 4 and 5 are devoted to an application for a heteroclinic bifurcation and its examples.

2. The stability index

First of all, we briefly summarize the construction of stability indices for travelling wave solutions of one-dimensional reaction diffusion equations due to J.Alexander, R.Gardner, and C.Jones. For details and proofs, see [1] and references therein.

2.1. Travelling waves and their stability.

Consider one-dimensional reaction diffusion equations of the following form.

$$u_t = Bu_{xx} + F(u). \tag{2.1}$$

Here $u(x,t) \in \mathbb{R}^n, x,t \in \mathbb{R}$, B : positive diagonal $n \times n$ matrix and $F : C^2$ with derivatives through order 2 bounded on $\mathbb{R}^n$.

We get on the moving frame (ξ, t), where $\xi = x - \theta t$ and the velocity θ is a constant. In this coordinate (2.1) is written as follows.

$$u_t = Bu_{\xi\xi} + \theta u_\xi + F(u). \tag{2.2}$$

A travelling wave solution of (2.1) is a stationary solution $u_0(\xi) = u_0(x - \theta t)$ of (2.2) for some θ, in other words, it is a solution of (2.1) which translate at some constant velocity θ preserving its shape.

In what follows, we assume the existence of a travelling wave solution satisfying the following assumptions.

(T.W.) There exist $C > 0$, $a > 0$, and $u_\pm \in \mathbb{R}^n$ with $F(u_\pm) = 0$ such that

(1) $|u_0(\xi) - u_+| \le Ce^{-a\xi}$ $\quad (\xi > 0)$
(2) $|u_0(\xi) - u_-| \le Ce^{a\xi}$ $\quad (\xi < 0)$
(3) $|u_0'(\xi)| \le Ce^{-a|\xi|}$ $\quad (\xi \in \mathbb{R} \quad \text{where} \quad ' = \frac{d}{d\xi})$.

Next we define stability of travelling waves.

Definition 2.1. *A travelling wave $u_0(\xi)$ is asymptotically stable relative to (2.2) if there exists a neighborhood N of u_0 in $BU(\mathbb{R}, \mathbb{R}^n)$ such that each solution $u(\xi, t)$ of (2.2) which starts in N at $t = 0$ satisfies*

$$\|u(\xi, t) - u_0(\xi + k)\|_\infty \to 0 \quad (t \to +\infty) \tag{2.3}$$

for some $k \in \mathbb{R}$ depending on $u(\xi, t)$, where $BU(\mathbb{R}, \mathbb{R}^n) = \{v : \mathbb{R} \to \mathbb{R}^n |$ bounded uniformly continuous$\}$.

Remark 2.1. *If $u_0(\xi)$ is a travelling wave, then so is $u_0(\xi + k)$.*

To find out the stability of $u_0(\xi)$, we linearize (2.2) about $u_0(\xi)$

$$P_t = LP := BP_{\xi\xi} + \theta P_\xi + DF\left(u_0(\xi)\right) P, \tag{2.4}$$

and consider the eigenvalue problem for L:

$$LP = \lambda P. \tag{2.5}$$

In order to determine the stability of u_0, it suffices to study the eigenvalue problem (2.5), because of the following well-known fact:

Fact ([1],[3] and [8]). *$u_0(\xi)$ is asymptotically stable if the spectrum $\sigma(L)$ of L satisfies that*

(1) *there exists $\beta < 0$ such that $\sigma(L) \setminus \{0\} \subset \{\lambda : Re\lambda < \beta\}$;*

(2) 0 *is a simple eigenvalue* (0 *is an eigenvalue corresponding to translation*).

Here we have the following for $\sigma(L)$.

Proposition 2.1. *If $u_\pm$ are stable for* (2.1) *then there exists a simple closed curve K and a constant $\beta < 0$ such that*

$$\sigma(L) \cap \{\lambda | Re\lambda > \beta\} \subset K^\circ, \tag{2.6}$$

where K° is the interior enclosed by K. Moreover $\sigma(L) \cap K^\circ$ consists of isolated eigenvalues with finite multiplicity.

With this proposition we conclude that $u_0(\xi)$ is stable if L has no eigenvalue in K° other than the simple one at 0.

From now on we construct an index to detect eigenvalues of L inside K°

2.2. Construction of the stability index.

The eigenvalue problem

$$(L-\lambda)P := BP_{\xi\xi} + \theta P_\xi + DF\left(u_0(\xi)\right)P - \lambda P = 0, \quad P(\xi) \in \mathbb{C}^n \tag{2.7}$$

can be taken as a second order linear ordinary differential equation, which we write in a first order system as follows.

$$Y' = A(\lambda, \xi)Y, \tag{2.8}$$

where

$$Y(\xi) = \begin{pmatrix} P(\xi) \\ Q(\xi) \end{pmatrix} \in \mathbb{C}^{2n}, Q = P' \text{ and } A(\lambda,\xi) = \begin{pmatrix} 0 & I \\ B^{-1}\left(\lambda I - DF\left(u_0(\xi)\right)\right) & -\theta B^{-1} \end{pmatrix}.$$

For each λ, (2.8) has at least one non-trivial bounded solution up to multiplication of constants if and only if it is an eigenvalue of L, in which case, the P component of such a solution is an eigenfunction of L.

We introduce a new variable τ via the relation

$$\xi = \frac{1}{\kappa}\log(\frac{1+\tau}{1-\tau}). \tag{2.9}$$

This turns (2.8) into an autonomous system on $\mathbb{C}^{2n} \times (-1,+1)$:

$$\begin{cases} Y' = A(\lambda,\tau)Y := A\left(\lambda, \xi(\tau)\right)Y \\ \tau' = \kappa(1-\tau^2). \end{cases} \qquad (\ ' = \tfrac{d}{d\xi}) \tag{2.10}$$

Letting

$$A(\lambda, \pm 1) = \begin{pmatrix} 0 & I \\ B^{-1}\left(\lambda I - DF\left(u_\pm\right)\right) & -\theta B^{-1} \end{pmatrix}, \tag{2.11}$$

we augment (2.10) into a system on $\mathbb{C}^{2n}\times[-1,+1]$, which is of class C^1 for sufficiently small $\kappa>0$. This system is also denoted as

$$\begin{cases} Y' = A(\lambda,\tau)Y \\ \tau' = \kappa(1-\tau^2). \end{cases} \tag{2.12}$$

$(\mathbf{O},-1)$ and $(\mathbf{O},+1)$ are equilibria of (2.12). The next lemma shows that λ is an eigenvalue of L if and only if the unstable manifold $\mathbf{W}^u_-$ of $(\mathbf{O},-1)$ and the stable manifold $\mathbf{W}^s_+$ of $(\mathbf{O},+1)$ has a non-trivial intersection, namely, $\mathbf{W}^u_-\cap\mathbf{W}^s_+\neq\{\mathbf{O}\}\times(-1,1)$.

Lemma 2.1. *Let $(Y(\xi),\tau(\xi))$ be a non-trivial solution of (2.12).*

(1) *$Y(\xi)$ is bounded as $\xi\to-\infty$ if and only if* $(Y(\xi),\tau(\xi))\in\mathbf{W}^u_-$.
(2) *$Y(\xi)$ is bounded as $\xi\to+\infty$ if and only if* $(Y(\xi),\tau(\xi))\in\mathbf{W}^s_+$.

Now we are at the final stage of constructing the stability index.

Proposition 2.2. *If $u_\pm$ are stable then there is a constant $\beta<0$ so that for all λ with $Re\lambda>\beta$, $A(\lambda,-1)$ (resp.$A(\lambda,+1)$) have n eigenvalues with positive real part and n with negative.*

This means $\mathbf{W}^u_-\big|_{\mathbb{C}^{2n}\times\{\tau\}}$ and $\mathbf{W}^s_+\big|_{\mathbb{C}^{2n}\times\{\tau\}}$ are n-dimensional subspaces of $\mathbb{C}^{2n}\times\{\tau\}$. We regard them as n-dimensional vector bundles over intervals.

For each fixed λ, define a continuous map

$$\Phi_\lambda\colon[-1,1)\to G_n(\mathbb{C}^{2n}):\tau\mapsto\left[\mathbf{W}^u|_{\mathbb{C}^{2n}\times\{\tau\}}\right], \tag{2.13}$$

where $G_n(\mathbb{C}^{2n})$ is the Grassmannian manifold of n-dimensional subspaces in $\mathbb{C}^{2n}$.

Let $U^\lambda_\pm$ ($S^\lambda_\pm$) be the unstable (stable) subspace of $A(\lambda,\pm1)$.

Lemma 2.2. *If λ is not an eigenvalue of L, then*

$$\Phi_\lambda(\tau)\to[U^\lambda_+]\quad(\tau\to+1) \tag{2.14}$$

uniformly in $G_n(\mathbb{C}^{2n})$ in the wider sense.

Put

$$\mathfrak{K}:=(K^\circ\times\{-1\})\cup(K\times[-1,+1])\cup(K^\circ\times\{+1\})\cong\mathbb{S}^2 \tag{2.15}$$

and

$$\mathfrak{G}\colon\mathfrak{K}\to G_n(\mathbb{C}^{2n}):\mathfrak{G}(\lambda,\tau)=\begin{cases}\left[U^\lambda_-\right] & \lambda\in K^\circ,\quad \tau=-1\\ \Phi_\lambda(\tau) & \lambda\in K,\quad \tau\in(-1,1)\\ \left[U^\lambda_+\right] & \lambda\in K^\circ,\quad \tau=+1.\end{cases} \tag{2.16}$$

By Lemma 2.2, $\mathfrak{G}$ is continuous and it induces an n-dimensional vector bundle $\mathfrak{E}(K)$ over $\mathfrak{K}$.

Definition 2.2.

$$\mathfrak{E}(K) := \mathfrak{G}^* \left(\Gamma_n(\mathbb{C}^{2n}) \right) \tag{2.17}$$

where $\Gamma_n(\mathbb{C}^{2n})$ *is the canonical bundle over* $G_n(\mathbb{C}^{2n})$.

Alexander *et al.* showed that this bundle determines stability of the travelling wave $u_0(\xi)$.

Theorem 2.1 (Alexander, Gardner and Jones [1]). *The first Chern number* $c_1(\mathfrak{E}(K))$ *of the bundle* $\mathfrak{E}(K)$ *coincides with the number of eigenvalues of* L *in* K° *counting their multiplicity.*

We slightly modify this theorem for later use.

Consider a system which (2.12) induces on $\overset{n}{\wedge}\mathbb{C}^{2n} \times [-1,+1]$,

$$\begin{cases} Y^{(n)\prime} = A^{(n)}(\lambda,\tau)Y^{(n)} \\ \tau' = \kappa(1-\tau^2). \end{cases} \tag{2.18}$$

Here $Y^{(n)}(\xi) \in \overset{n}{\wedge}\mathbb{C}^{2n}$ and $A^{(n)} = A \otimes I \otimes \cdots \otimes I + \cdots + I \otimes I \otimes \cdots \otimes I \otimes A|_{\overset{n}{\wedge}\mathbb{C}^{2n}}$, where $\overset{n}{\wedge}\mathbb{C}^{2n}$ means the n-th exterior power of $\mathbb{C}$. Let $m = \binom{2n}{n} - 1$ then $\mathbb{CP}^m \cong \mathbb{P}(\overset{n}{\wedge}\mathbb{C}^{2n})$ and (2.18) defines a system of the following form on $\mathbb{CP}^m \times [-1,+1]$ as it is linear in the $Y^{(n)}$ component:

$$\begin{cases} z' = Z(z,\tau:\lambda) \\ \tau' = \kappa(1-\tau^2) \end{cases} \qquad (z \in \mathbb{CP}^m). \tag{2.19}$$

Let $\tilde{U}^\lambda_\pm$ ($\tilde{S}^\lambda_\pm$) be points on $\mathbb{CP}^m$ corresponding to $\overset{n}{\wedge}U^\lambda_\pm$ ($\overset{n}{\wedge}S^\lambda_\pm$), *i.e.* $\tilde{U}^\lambda_\pm := \Pi(\overset{n}{\wedge}U^\lambda_\pm)$ (resp. $\tilde{S}^\lambda_\pm := \Pi(\overset{n}{\wedge}S^\lambda_\pm)$) for the projection $\Pi\colon \mathbb{C}^{m+1} \setminus \{\mathbf{O}\} \to \mathbb{CP}^m$, then $(\tilde{U}^\lambda_-,-1)$ and $(\tilde{U}^\lambda_+,+1)$ (resp. $(\tilde{S}^\lambda_-,-1)$ and $(\tilde{S}^\lambda_+,+1)$) are equilibria of (2.19). These equilibria are attractive in invariant subspaces $\mathbb{CP}^m \times \{\tau = -1\}$ and $\mathbb{CP}^m \times \{\tau = +1\}$, which means that $(\tilde{U}^\lambda_-,-1)$ have one-dimensional unstable direction in the whole space $\mathbb{CP}^m \times [-1,+1]$, namely the τ direction, and $(\tilde{U}^\lambda_+,+1)$ is an attracting equilibrium. By Lemma 2.2 the system (2.19) has a heteroclinic orbit from $(\tilde{U}^\lambda_-,-1)$ to $(\tilde{U}^\lambda_+,+1)$ for $\lambda \notin \sigma(L)$. This heteroclinic orbit is given by $\{\left(\tilde{\Phi}_\lambda(\tau),\tau\right)\}$, where $\tilde{\Phi}_\lambda(\tau) := \Pi\left(\overset{n}{\wedge}\Phi_\lambda(\tau)\right)$. Define a map G by

$$G\colon \mathfrak{K} \to \mathbb{CP}^m : G(\lambda,\tau) = \begin{cases} \tilde{U}^\lambda_- & \lambda \in K^\circ, \quad \tau = -1 \\ \tilde{\Phi}_\lambda(\tau) & \lambda \in K, \quad \tau \in (-1,1) \\ \tilde{U}^\lambda_+ & \lambda \in K^\circ, \quad \tau = +1, \end{cases} \tag{2.20}$$

then Theorem 2.1 can be restated as follows. (See Figure 1.)

Theorem 2.2. $[G] \in \pi_2(\mathbb{CP}^m)) \cong \mathbb{Z}$ *counts the number of eigenvalues of* L *in* K°.

Proof. This interpretation is clear from the proof of Theorem 2.1 in [1]. □

Remark 2.2. *We omit references to distinguished points because the isomorphism corresponding to change of base points is uniquely determined by those points as* $\pi_1(\mathbb{CP}^m) = 0$.

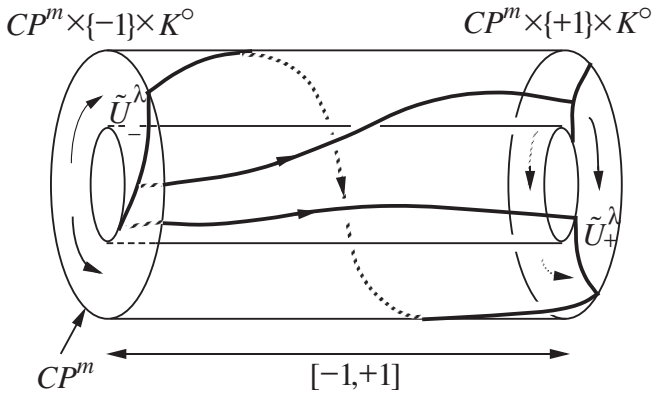

Figure 1

3. An extension of the stability index

In this section we consider an extension of the stability index for perturbed systems and give a proof of one of our main theorem.

3.1. Perturbed systems.

Consider a system of the form (2.2) with a parameter $\mu \in \mathbb{R}$:

$$u_t = Bu_{\xi\xi} + \theta(\mu)u_\xi + F(u;\mu). \tag{3.1}$$

Suppose that this system has a travelling wave $u_0(\xi)$ at $\mu = \mu_0$ which satisfies the condition **(T.W.)**. We assume without loss of generality that $F(u_\pm;\mu) = 0$ for all $\mu \in \mathbb{R}$. We also assume that $u_\pm$ are both stable for (2.1), or in this case for

$$u_t = Bu_{xx} + F(u;\mu). \tag{3.2}$$

We consider the equation of the stationary solutions for (3.1),

$$Bu_{\xi\xi} + \theta(\mu)u_\xi + F(u;\mu) = 0, \tag{3.3}$$

and again we write this as a first order system,

$$\begin{cases} u' &= v \\ v' &= -B^{-1}F(u;\mu) - \theta(\mu)B^{-1}v \end{cases} \qquad (\ \ ' = \tfrac{d}{d\xi}). \tag{3.4}$$

Let $(u_0(\xi), u_0'(\xi))$ be the heteroclinic solution from $(u_-, 0)$ to $(u_+, 0)$ of (3.4) with $\mu = \mu_0$. Choose some point $(u_0^\dagger, v_0^\dagger)$ near $(u_+, 0)$ on the orbit of $(u_0(\xi), u_0'(\xi))$, and take an n-dimensional plane Σ which intersects with the unstable manifold $\mathfrak{W}_-^u$ of $(u_-, 0)$ transversely at $(u_0^\dagger, v_0^\dagger)$.

Lemma 3.1. *When μ is near μ_0, $\mathfrak{W}_-^u \cap \Sigma = \left\{(u_\mu^\dagger, v_\mu^\dagger)\right\}$ (one point). Moreover the point $(u_\mu^\dagger, v_\mu^\dagger)$ depends on μ continuously. (See Figure 2.)*

Proof. Consider (3.4) with a trivial equation $\mu' = 0$:

$$\begin{cases} u' &= v \\ v' &= -B^{-1}F(u;\mu) - \theta(\mu)B^{-1}v \\ \mu' &= 0 \end{cases} \qquad . \tag{3.5}$$

$S := \{(u_-, 0, \mu) | \mu \in \mathbb{R}\}$ is a normally hyperbolic invariant set for (3.5) and let V^u be its unstable manifold. From the definition of Σ, V^u and $\Sigma \times \mathbb{R}$ intersect transversely at $(u_0, v_0, 0)$, and moreover both V^u and $\Sigma \times \mathbb{R}$ are $n+1$-dimensional, so that the intersection $V^u \cap \{\Sigma \times \mathbb{R}\}$ is a one-dimensional curve. The image under projection for (u, v) components is the curve $\{(u_\mu^\dagger, v_\mu^\dagger)\}$ parameterized by μ, and thus is continuous. □

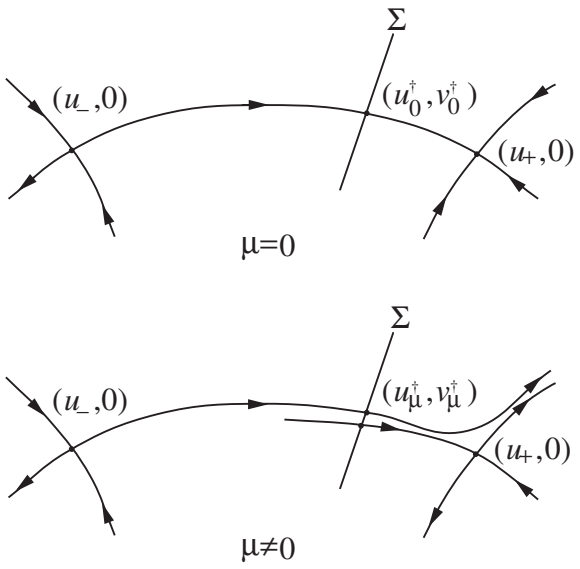

Figure 2

By translation on ξ we can assume $(u_0(0), u_0'(0)) = (u_0^\dagger, v_0^\dagger)$. We denote a solution which starts at $(u_\mu^\dagger, v_\mu^\dagger)$ when $\xi = 0$ by $(u(\xi;\mu), u'(\xi;\mu))$, namely $(u(\xi;0), u'(\xi;0)) = (u_0(\xi), u_0'(\xi))$ and $u(\xi,\mu) \to u_-$ as $\xi \to -\infty$ since $(u_\mu^\dagger, v_\mu^\dagger) \in \mathfrak{W}_-^u$.

For this solution $u(\xi;\mu)$ and

$$\mathfrak{K}' = (K^\circ \times \{-1\}) \cup (K \times [-1,0]) \cong D^2,$$

we construct a map $G_-^\mu \colon \mathfrak{K}' \to \mathbb{CP}^m$ as follows.

Consider a system corresponding to (2.8)

$$Y' = A(\lambda, \xi; \mu) Y \tag{3.6}$$

where

$$Y(\xi) = \begin{pmatrix} P(\xi) \\ Q(\xi) \end{pmatrix} \in \mathbb{C}^{2n}, Q = P' \text{ and } A(\lambda,\xi;\mu) = \begin{pmatrix} 0 & I \\ B^{-1}(\lambda I - DF(u(\xi;\mu))) & -\theta B^{-1} \end{pmatrix}.$$

A similar argument as before leads to a system on $\mathbb{C}^{2n} \times [-1,+1)$,

$$\begin{cases} Y' = A(\lambda, \tau; \mu) Y \\ \tau' = \kappa(1-\tau^2) \end{cases} \qquad (\ ' = \tfrac{d}{d\xi}), \tag{3.7}$$

where

$$A(\lambda, -1; \mu) = \begin{pmatrix} 0 & I \\ B^{-1}(\lambda I - DF(u_-)) & -\theta B^{-1} \end{pmatrix}.$$

Again this equation induces a system on $\mathbb{CP}^m \times [-1,1)$,

$$\begin{cases} z' = Z(z, \tau : \lambda; \mu) \\ \tau' = \kappa(1-\tau^2) \end{cases} \qquad (z \in \mathbb{CP}^m). \tag{3.8}$$

We define a map G_-^μ as

$$G_-^\mu \colon \mathfrak{K}' \to \mathbb{CP}^m : G_-^\mu(\lambda,\tau) = \begin{cases} \tilde{U}_-^\lambda & \lambda \in K^\circ, \quad \tau = -1 \\ \tilde{\Phi}_\lambda^\mu(\tau) & \lambda \in K, \quad \tau \in (-1,0], \end{cases} \tag{3.9}$$

where $\left(\tilde{\Phi}_\lambda^\mu(\tau), \tau\right)$ is the unstable manifold of an equilibrium $(\tilde{U}_-^\lambda, -1)$ of (3.8). Obviously this map is continuous and $G_-^{\mu_0} = G|_{\mathfrak{K}'}$ for G in (2.16).

Take a neighborhood N of $G((K \times [0,+1]) \cup (K^\circ \times \{-1\}))$ in $\mathbb{CP}^m$ satisfying $\pi_1(N) = 0$ and $\pi_2(N) = 0$. Such an N exists, for example, for K small and $(u_0^\dagger, v_0^\dagger)$ near $(u_+, 0)$.

Clearly $G_-^\mu(\lambda,\tau)$ is continuous, also, in μ, so that if μ is near μ_0 then

$$G_-^\mu(K \times \{0\}) \subset N. \tag{3.10}$$

This means G_-^μ defines an element of the relative homotopy group $\pi_2(\mathbb{CP}^m, N)$.

Theorem 3.1.

$$\pi_2(\mathbb{CP}^m, N) \cong \pi_2(\mathbb{CP}^m) \cong \mathbb{Z},$$

and $[G_-^\mu] \in \pi_2(\mathbb{CP}^m, N)$ *corresponds to* $[G] \in \pi_2(\mathbb{CP}^m)$ *through the above isomorphism. Consequently,* $[G_-^\mu]$ *determines stability of the travelling wave* $u_0(\xi)$ *at* $\mu = \mu_0$.

Proof. Consider the following exact sequence:

$$0 = \pi_2(N) \to \pi_2(\mathbb{CP}^m) \xrightarrow{p} \pi_2(\mathbb{CP}^m, N) \to \pi_1(N) = 0. \tag{3.11}$$

By definition of N, $[G] (\in \pi_2(\mathbb{CP}^m)$ corresponds to $[G_-^{\mu_0}] (\in \pi_2(\mathbb{CP}^m, N))$ through the homomorphism p. Moreover (3.10) implies $[G_-^\mu] = [G_-^{\mu_0}]$ in $\pi_2(\mathbb{CP}^m, N)$. Here from the assumption on N, both ends of the above sequence are trivial, which implies that

$$\pi_2(\mathbb{CP}^m, N) \cong \pi_2(\mathbb{CP}^m) \cong \mathbb{Z},$$

and we get the result. □

Corollary 3.1. *The same argument holds true for perturbation of orbits. That is, at* $\mu = \mu_0$*, we can make a similar extension of the stability index for a solution of* (3.4) *on the unstable manifold* $\mathbf{W}_-^u$ *of* $(u_-, 0)$*, if it is near* $(u_0(\xi), u_0'(\xi))$*. (See Figure 3.)*

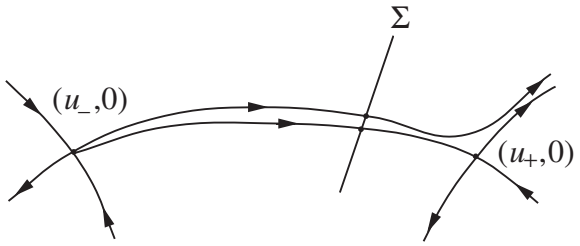

Figure 3

3.2. Additive formula for gluing bifurcation.

In this subsection we give a proof of an additive formula for the stability index under the gluing bifurcation of travelling waves. This theorem has been proven by Alexander and Jones [2] using cobordism invariance of the Chern number. Here we give another proof of additivity based on the above extension of index.

Suppose the system (3.1) has two travelling waves $u_1(\xi)$, $u_2(\xi)$ when $\mu = \mu_0$ which satisfy

$$\begin{aligned} &u_1(\xi) \to u_- \quad (\xi \to -\infty) \qquad u_1(\xi) \to u_* \quad (\xi \to +\infty) \\ &u_2(\xi) \to u_* \quad (\xi \to -\infty) \qquad u_2(\xi) \to u_+ \quad (\xi \to +\infty), \end{aligned}$$

and **(T.W.)** where the stationary solutions $u_\pm, u_*$ are stable. Let $\Gamma_i = \{(u_i(\xi), u_i'(\xi))\}$, $(i = 1, 2)$ be CPthe heteroclinic orbits of (3.4) corresponding to the travelling waves $u_i(\xi)$. Assume that the system (3.4) undergoes a gluing bifurcation, that is to say

that the system has a heteroclinic orbit $\Gamma = \{(u(\xi), u'(\xi))\}$ from $(u_-, 0)$ to $(u_+, 0)$ for $\mu \approx \mu_0$ which stays in a tubular neighborhood of $\Gamma_1 \cup \Gamma_2$. (See Figure 4.)

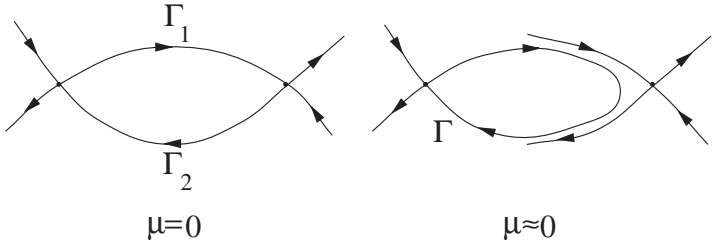

Figure 4

Fix a simple closed curve K on the complex plane $\mathbb{C}$, and let

$$G, G_i : \mathfrak{K} \to \mathbb{CP}^m \qquad i = 1, 2 \tag{3.12}$$

be the maps defined for $u(\xi)$ and $u_i(\xi)$.

Under the above situation we have the following additive formula.

Theorem 3.2.

$$[G] = [G_1] + [G_2] \in \pi_2\left(\mathbb{CP}^m\right). \tag{3.13}$$

Proof. By definition, it holds that

$$G_1|_{(K \cup K^\circ) \times \{\tau = +1\}} = G_2|_{(K \cup K^\circ) \times \{\tau = -1\}} : K \cup K^\circ \to \mathbb{CP}^m. \tag{3.14}$$

Write this map as

$$H : K \cup K^\circ \to \mathbb{CP}^m. \tag{3.15}$$

At first let us assume

$$\pi_1\left(H(K \cup K^\circ)\right) = 0 \qquad \text{and} \qquad \pi_2\left(H(K \cup K^\circ)\right) = 0. \tag{3.16}$$

Take a neighborhood N of $H(K \cup K^\circ)$ in $\mathbb{CP}^m$ which satisfies

$$\pi_1(N) = 0 \qquad \text{and} \qquad \pi_2(N) = 0. \tag{3.17}$$

If μ is close enough to μ_0 and as a result Γ is included in some small neighborhood of $\Gamma_1 \cup \Gamma_2$, we can assume that

$$\begin{cases} u(\xi) \approx u_1(\xi) & (\xi \le 0) \\ u(\xi) \approx u_2(\xi) & (\xi \ge 0) \end{cases} \tag{3.18}$$

through some translations in the ξ variable. (See Figure 5.) This implies $u(0) \approx u_*$ and

$$G_1|_{K \times \{\tau = +1\}} \approx G_1|_{K \times \{\tau = 0\}} \approx G|_{K \times \{\tau = 0\}} : K \to \mathbb{CP}^m, \tag{3.19}$$

which means

$$G(K \times \{0\}) \subset N. \tag{3.20}$$

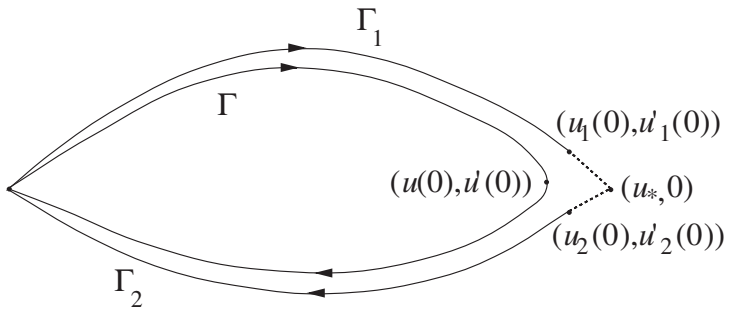

Figure 5

Put

$$G_- := G|_{(K^\circ \times \{\tau=-1\}) \cup (K \times [-1,0])} \tag{3.21}$$
$$G_+ := G|_{(K \times [0,+1]) \cup (K^\circ \times \{\tau=+1\})} \tag{3.22}$$
$$G_{1-} := G_1|_{(K^\circ \times \{\tau=-1\}) \cup (K \times [-1,+1])} \tag{3.23}$$
$$G_{2+} := G_2|_{(K \times [-1,+1]) \cup (K^\circ \times \{\tau=+1\})} \tag{3.24}$$

then we have

$$CPG_- \simeq G_{1-} \colon (D^2, \partial D^2) \to ({}^m, N) \tag{3.25}$$
$$G_+ \simeq G_{2+} \colon (D^2, \partial D^2) \to (\mathbb{CP}^m, N)\,, \tag{3.26}$$

where $\simeq$ means homotopic.

Consider the following exact sequence again, where both ends are trivial:

$$0 = \pi_2(N) \to \pi_2(\mathbb{CP}^m) \xrightarrow{i} \pi_2(\mathbb{CP}^m, N) \to \pi_1(N) = 0 \tag{3.27}$$

From the definitions, we have (see Figure 6)

$$i([G]) = [G_-] + [G_+] \tag{3.28}$$
$$i([G_1]) = [G_{1-}] + [H] \tag{3.29}$$
$$i([G_2]) = -[H] + [G_{2+}]. \tag{3.30}$$

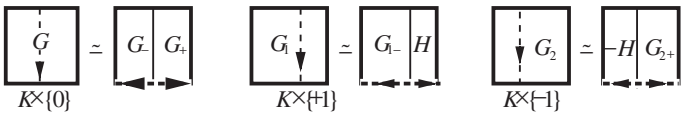

Figure 6

Here (3.25) and (3.26) imply that

$$[G_-] = [G_{1-}], \qquad [G_+] = [G_{2+}] \qquad \text{in } \pi_2(\mathbb{CP}^m, N), \tag{3.31}$$

and by definition of N, $[H] = 0$. This leads to

$$\begin{aligned} i([G]) &= [G_-] + [G_+] \\ &= [G_{1-}] + [G_{2+}] \\ &= i([G_1]) + i([G_2]) \qquad \text{in } \pi_2(^m, N). \end{aligned} \tag{3.32}$$

From this, we get the result, namely,

$$\begin{aligned} [G] &= i^{-1} \circ i([G]) \\ &= i^{-1} \circ i([G_1]) + i^{-1} \circ i([G_2]) \\ &= [G_1] + [G_2] \qquad \text{in } \pi_2(\mathbb{CP}^m). \end{aligned} \tag{3.33}$$

For general K, divide K° into small parts, then each part satisfies the condition (3.16) if the partition is fine enough, and therefore the above argument is applicable for each part. Collecting all parts we get the result for K. □

4. Application

4.1. Direct consequences of the additive formula.

In this section we consider some applications of Theorem 3.2. First of all, the next theorem is its direct consequence.

Theorem 4.1. *Under the assumption of Theorem* 3.2, *$u(\xi)$ is unstable if either $u_1(\xi)$ or $u_2(\xi)$ is unstable.*

Proof. Suppose $u_1(\xi)$ is unstable, that is, the eigenvalue problem

$$(L_1 - \lambda)P := BP_{\xi\xi} + \theta P_\xi + DF(u_1(\xi))P - \lambda P \tag{4.1}$$

has an eigenvalue λ_1 with positive real part. Take a simple closed curve $K \subset \mathbb{C}$ satisfying

$$\lambda_1 \in K^\circ \quad \text{and} \quad K \cup K^\circ \subset \{\lambda \in \mathbb{C} | Re\lambda > 0\}. \tag{4.2}$$

By applying Theorem 3.2 to this K, we have an eigenvalue λ_0 inside K°, of the eigenvalue problem:

$$(L-\lambda)P := BP_{\xi\xi} + \theta P_\xi + DF\left(u(\xi)\right)P - \lambda P, \tag{4.3}$$

which means $Re\lambda_0 > 0$, and hence $u(\xi)$ is unstable. (See Figure 7.)

The case of $u_2(\xi)$ being unstable can be similarly treated. □

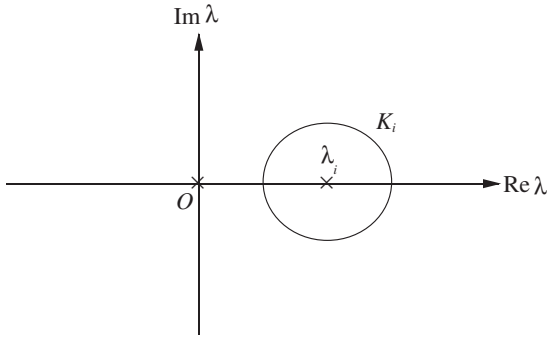

Figure 7

This theorem means that $u(\xi)$ can be stable only if both $u_1(\xi)$ and $u_2(\xi)$ are stable.

In what follows, let us suppose that both $u_1(\xi)$ and $u_2(\xi)$ are stable. In this case, Theorem 3.2 only tells that L has two eigenvalues near the origin, and consequently we cannot decide the stability of $u(\xi)$ only from this theorem. Actually we have examples of both cases. (See Section 5.)

4.2. Orientation index.

Consider the eigenvalue problem for $\lambda \in \mathbb{R}$.

$$(L-\lambda)P := BP_{\xi\xi} + \theta P_\xi + DF\left(u_0(\xi)\right)P - \lambda P = 0, \quad P(\xi) \in \mathbb{R}^n \tag{4.4}$$

Again we rewrite it as

$$\begin{cases} Y' = A(\lambda,\tau)Y \\ \tau' = \kappa(1-\tau^2), \end{cases} \tag{4.5}$$

which defines a system on $\mathbb{RP}^m \times [-1,+1]$ as follows.

$$\begin{cases} z' = Z(z,\tau:\lambda) \\ \tau' = \kappa(1-\tau^2) \end{cases} \qquad (z \in \mathbb{RP}^m). \tag{4.6}$$

Take $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\beta < \lambda_1 < \lambda_2$, where β is as in Proposition 2.1, and assume $\lambda_1, \lambda_2 \notin \sigma(L)$. Of course Lemma 2.2 holds true for $\lambda = \lambda_i$, $i = 1, 2$.

Put

$$\mathfrak{k} := ([\lambda_1,\lambda_2] \times \{-1\}) \cup (\{\lambda_1,\lambda_2\} \times [-1,+1]) \cup ([\lambda_1,\lambda_2] \times \{+1\}) \cong \mathbb{S}^1 \tag{4.7}$$

and let

$$g\colon \mathfrak{k} \to \mathbb{RP}^m \tag{4.8}$$

be defined in a similar manner as before, that is

$$g(\lambda,\tau) = \begin{cases} \tilde{U}_-^\lambda & \lambda \in [\lambda_1,\lambda_2], \quad \tau = -1 \\ \tilde{\Phi}_\lambda(\tau) & \lambda = \lambda_i, \quad i = 1,2 \quad \tau \in (-1,1) \\ \tilde{U}_+^\lambda & \lambda \in [\lambda_1,\lambda_2], \quad \tau = +1, \end{cases} \tag{4.9}$$

where $\tilde{U}_\pm^\lambda$ are the equilibria of (4.6) which correspond to the unstable subspaces $U_\pm^\lambda$ of $A(\lambda,\pm 1)$ and $\left(\tilde{\Phi}_\lambda(\tau),\tau\right)$ is a heteroclinic orbit connecting $(\tilde{U}_-^\lambda,-1)$ and $(\tilde{U}_+^\lambda,+1)$.

For this g we have, next, as a part of Theorem 2.2,

Theorem 4.2. $[g] \in \pi_1(\mathbb{RP}^m) \cong \mathbb{Z}_2$ *counts parity of the number of eigenvalues of* L *on the interval* $[\lambda_1,\lambda_2]$. *That is, if the interval includes an odd number of eigenvalues then* $[g]=1$, *and if even then* $[g]=0$.

Proof. Theorem 2.1 was proven in [1] using the Evans function. First, we briefly summarize the construction of it.

Consider equations (2.18) and (2.19). An orbit $\{(Y(\lambda,\tau),\tau)\}_{-1<\tau<1}$ of (2.18) can be chosen so that $\Pi(Y(\lambda,\tau)) = \tilde{\Phi}_\lambda(\tau)$ for the projection $\Pi\colon \mathbb{C}^{m+1}\setminus\{\mathbf{O}\} \to \mathbb{CP}^m$ and for each τ, $Y(\lambda,\tau)$ is analytic in λ ([1]). Similarly an orbit $\{(Y^*(\lambda,\tau),\tau)\}_{-1<\tau<1}$ of (2.18) can be chosen so that $\Pi(Y^*(\lambda,\tau)) = \tilde{\Phi}^*_\lambda(\tau)$ and for each τ, $Y^*(\lambda,\tau)$ is analytic in λ, where $\{(\tilde{\Phi}^*_\lambda(\tau),\tau)\}_{-1<\tau<1}$ is the unique orbit of (2.19) satisfying

$$\lim_{\tau\to+1}(\tilde{\Phi}^*_\lambda(\tau),\tau) = (\tilde{S}_+^\lambda,+1).$$

Note that $Y(\lambda,\tau) \in \overset{n}{\wedge}\left(\mathbf{W}_-^u\Big|_{\mathbb{C}^{2n}\times\{\tau\}}\right)$ and $Y^*(\lambda,\tau) \in \overset{n}{\wedge}\left(\mathbf{W}_+^s\Big|_{\mathbb{C}^{2n}\times\{\tau\}}\right)$.

We define the Evans function $D(\lambda)$ as

$$D(\lambda) := Y(\lambda,0)\wedge Y^*(\lambda,0) \in \overset{2n}{\wedge}\mathbb{C}^{2n} \cong \mathbb{C}. \tag{4.10}$$

This function is analytic in λ and the number of zeroes of D in K° coincides with the number of eigenvalues of L in K° counting its multiplicity. For λ real, $Y(\lambda,\tau)$ and $Y^*(\lambda,\tau)$ can be taken so that they are both real analytic in λ and thus $D(\lambda)$ is real analytic.

Let $\Pi_s\colon \mathbb{R}^{m+1}\setminus\mathbf{O} \to \mathbb{S}^m$ and $\Pi_p\colon \mathbb{S}^m \to \mathbb{RP}^m$ be projections and put $\hat{\Phi}_\lambda(\tau) = \Pi_s(Y(\lambda,\tau))$ and $\hat{\Phi}^*_\lambda(\tau) = \Pi_s(Y^*(\lambda,\tau))$. Then $\tilde{\Phi}_\lambda(\tau) = \Pi_p(\hat{\Phi}_\lambda(\tau))$ and $\tilde{\Phi}^*_\lambda(\tau) = \Pi_p(\hat{\Phi}^*_\lambda(\tau))$, so it is easy to see that $D(\lambda_1)D(\lambda_2) > 0$ if and only if $[g]=0$ and $D(\lambda_1)D(\lambda_2) < 0$ if and only if $[g]=1$, and thus the theorem holds. □

Alexander and Jones([2]) showed, for $0 > \lambda_1$ close to 0 and λ_2 large, the direction of intersection of the unstable manifold of $(u_-, 0)$ and the stable manifold of $(u_+, 0)$ with respect to the travelling wave speed θ determines the index $[g]$. More precisely, the sign of the derivative $\frac{dD}{d\lambda}|_{\lambda=0}$ of the Evans function $D(\lambda)$ at $\lambda = 0$ is determined as follows.

Assume that $u \equiv u_\pm$ are stationary solutions of the system and for $\theta = \theta_0$ the system has a travelling wave. Let $X^{\theta_0}(\xi)$ be the corresponding heteroclinic solution from $(u_-, 0)$ to $(u_+, 0)$ at $\theta = \theta_0$ of

$$\begin{cases} u' &= v \\ v' &= -B^{-1}F(u) - \theta B^{-1}v \end{cases} \qquad (\quad ' = \tfrac{d}{d\xi}). \tag{4.11}$$

Let $\mathfrak{W}^u_\theta$ be the unstable manifold of $(u_-, 0)$ and $\mathfrak{W}^s_\theta$ be the stable manifold of $(u_+, 0)$, both being n-dimensional. We, also, assume that $\mathfrak{W}^u_\theta \cap \mathfrak{W}^s_\theta = \emptyset$, except for $\theta = \theta_0$. Extend $X^{\theta_0}(\xi)$ smoothly in θ to solutions $X^\theta_u(\xi) \in \mathfrak{W}^u_\theta$ and $X^\theta_s(\xi) \in \mathfrak{W}^s_\theta$ for θ near θ_0.

For the travelling wave at $\theta = \theta_0$, define solutions $(Y(\lambda, \tau), \tau)$ and $(Y^*(\lambda, \tau), \tau)$ of (2.18) so that $D(\lambda) > 0$ holds for large λ, and choose vectors $V^1_u, \cdots, V^{n-1}_u$ in $\mathbf{W}^u_-\Big|_{\mathbb{C}^{2n}\times\{0\}}$ and $V^1_s, \cdots, V^{n-1}_s$ in $\mathbf{W}^s_+\Big|_{\mathbb{C}^{2n}\times\{0\}}$ so that

$$\begin{aligned} Y(0,0) &= X^{\theta_0\prime}(0) \wedge V^1_u \wedge \cdots \wedge V^{n-1}_u \\ Y^*(0,0) &= X^{\theta_0\prime}(0) \wedge V^1_s \wedge \cdots \wedge V^{n-1}_s, \end{aligned}$$

which implies $X^{\theta_0\prime}(0), V^1_u, \cdots, V^{n-1}_u$ $(X^{\theta_0\prime}(0), V^1_s, \cdots, V^{n-1}_s)$ is a basis of $T_{X^{\theta_0}(0)}\mathfrak{W}^u_{\theta_0}$ (resp.$T_{X^{\theta_0}(0)}\mathfrak{W}^s_{\theta_0}$). Then, we have

$$\begin{aligned} D(0) &= Y(0,0) \wedge Y^*(0,0) \\ &= \det\left(X^{\theta_0\prime}_u(0) V^1_u \cdots V^{n-1}_u X^{\theta_0\prime}_s(0) V^1_s \cdots V^{n-1}_s\right). \end{aligned}$$

Proposition 4.1 (Alexander and Jones [2]).

$$\left.\frac{dD(\lambda)}{d\lambda}\right|_{\lambda=0} = \det\left(\left(\left.\frac{dX^\theta_s(0)}{d\theta}\right|_{\theta=\theta_0} - \left.\frac{dX^\theta_u(0)}{d\theta}\right|_{\theta=\theta_0}\right)\ V^1_u \cdots V^{n-1}_u\ X^{\theta_0\prime}(0)\ V^1_s \cdots V^{n-1}_s\right)$$

This proposition implies that the sign of $(\frac{dX^\theta_s(0)}{d\theta}|_{\theta=\theta_0} - \frac{dX^\theta_u(0)}{d\theta}|_{\theta=\theta_0})$ determines $[g]$, because if $\frac{dD}{d\lambda}(0) > 0$ then $D(\lambda_1) < 0$ for $\lambda_1 < 0$ close to 0 which means $[g] = 1$ as $D(\lambda_2) > 0$ for λ_2 large, and if $\frac{dD}{d\lambda}(0) < 0$ then $D(\lambda_1) > 0$ for $\lambda_1 < 0$, so $[g] = 0$.

4.3. Gluing bifurcations and stability.

In this subsection we discuss stability of a travelling wave bifurcating from two travelling waves by a gluing bifurcation.

Consider an O.D.E. on $\mathbb{R}^{2n}$ depending on a parameter $\mu \in \mathbb{R}^k$ $(k \geq 2)$,

$$\dot{x} = f(x) + g(x;\mu) \qquad (\ \dot{} = \tfrac{d}{dt}), \tag{4.12}$$

where f and g are smooth and $g(x;0) = 0$. Assume (4.12) has three equilibria $\mathbf{O}_1(\mu)$, $\mathbf{O}_2(\mu)$ and $\mathbf{O}_3(\mu)$ and the eigenvalues

$$-\eta^i_{n-1}(\mu), \ldots, -\eta^i_1(\mu), -\rho^i(\mu), \nu^i(\mu), \kappa^i_1(\mu), \ldots, \kappa^i_{n-1}(\mu)$$

of linearizations of F at each equilibrium satisfy

$$-Re\eta^i_{n-1}(0) \leq \cdots \leq -Re\eta^i_1(0) < -\rho^i(0) < 0 < \nu^i(0) < Re\kappa^i_1(0) \leq \cdots \leq Re\kappa^i_{n-1}(0).$$

Also assume that for $\mu = 0$ the system

$$\dot{x} = f(x) \tag{4.13}$$

has heteroclinic orbits Γ_i of (4.13) from $\mathbf{O}_i(0)$ to $\mathbf{O}_{i+1}(0)$ $(i = 1, 2)$ simultaneously.

In what follows, we consider bifurcations of these heteroclinic orbits under the following non-degeneracy conditions.

(1) For each i, the heteroclinic orbit $\Gamma_i = \{h_i(t)\}$ is tangent to the eigenspace associated to the eigenvalue $\nu^i(0)$ of linearization of f at $\mathbf{O}_i(0)$ as $t \to -\infty$ and the eigenspace to $-\rho^{i+1}(0)$ of $\mathbf{O}_{i+1}(0)$ as $f \to +\infty$.

(2) For $\mu = 0$, the unstable manifold $\mathfrak{W}^u(\mathbf{O}_i)$ and the stable manifold $\mathfrak{W}^s(\mathbf{O}_{i+1})$ $(i = 1, 2)$ has one-dimensional intersection *i.e.* for all $p \in \Gamma_i$

$$\dim\left(T_p\mathfrak{W}^u(\mathbf{O}_i) \cap T_p\mathfrak{W}^s(\mathbf{O}_{i+1})\right) = 1.$$

(3) $\mathfrak{W}^u(\mathbf{O}_i(0))$ is transverse to the $(n+1)$-dimensional ν-stable manifold $\mathfrak{W}^{\nu,s}(\mathbf{O}_{i+1}(0))$ $(i = 1, 2)$ which is invariant and is tangent to the sum of the eigenspaces corresponding to $\nu^{i+1}(0)$, $-\rho^{i+1}(0)$ and $-\eta^{i+1}_j$ $(1 \geq j \geq n-1)$. Also $\mathfrak{W}^s(\mathbf{O}_{i+1}(0))$ is transversal to the $(n+1)$-dimensional $(-\rho)$-unstable manifold corresponding to the eigenvalues $-\rho^i(0)$, $\nu^i(0)$ and $\kappa^i_k(0)$ $(1 \geq k \geq n-1)$.

(4) For a non-trivial bounded solution $\hat{q}^i(t)$ $(i = 1, 2)$ of the linear O.D.E.

$$\dot{\hat{z}} = -{}^tDf(h_i(t))\hat{z} \qquad (i = 1, 2), \tag{4.14}$$

the vectors given by the integrals

$$\int_{-\infty}^{+\infty} \hat{q}^i(s) \cdot \frac{\partial}{\partial\mu} g(h_i(s);0)ds \tag{4.15}$$

are linearly independent, and hence non-zero.

Remark 4.1. *The bounded solution $\hat{q}^i(t)$ is unique up to multiplication by constants.*

Under these conditions, we have

Proposition 4.2 (Kokubu [15]). *Under the above conditions* $(1)-(4)$*, there exist two hypersurfaces* M_i $(i=1,2)$ *of codimension* 1 *in a sufficiently small neighborhood of* $\mu=0$ *in* $\mathbb{R}^k$*, so that each of* M_i *consists of parameter values* μ *for which the system has a heteroclinic orbit* Γ_i*. Moreover* M_1 *and* M_2 *intersect transversely at* $\mu=0$*.*

In order to investigate the existence of a heteroclinic orbit from $\mathbf{O}_1$ to $\mathbf{O}_3$, we divide our analysis into two cases.

(i) $\nu^2(0) \neq \rho^2(0)$ (the case of non-critical eigenvalues.)
(ii) $\nu^2(0) = \rho^2(0)$ (the case of critical eigenvalues.)

Proposition 4.3 (Kokubu [15]). *Under the condition* $(1)-(4)$ *and for the case of non-critical eigenvalues, there exists a hypersurface* M *of codimension* 1 *with the boundary*

$$\partial M = M_1 \cap M_2$$

in a sufficiently small neighborhood of $\mu=0$ *in* $\mathbb{R}^k$*, so that* M *consists of parameter values* μ *for which the system has a heteroclinic orbit* $\Gamma=\{h(t)\}$ *from* O_1 *to* O_3*. Moreover,*

(a) *if* $\nu^2(0) < \rho^2(0)$*, then* M *is tangent to* M_2 *at* $\mu=0$*;*
(b) *if* $\nu^2(0) > \rho^2(0)$*, then* M *is tangent to* M_1 *at* $\mu=0$*.*

For the case of critical eigenvalues, we impose a further condition as follows.

(5) The set $\{\mu | \nu^2(\mu) = \rho^2(\mu)\}$ forms a surface Π in the parameter space $\mathbb{R}^k$ and is transverse to both of M_1 and M_2 at $\mu=0$.

Proposition 4.4 (Kokubu [15]). *Under the condition* $(1)-(5)$ *and for the case of critical eigenvalues, there exists a hypersurface* M *of codimension* 1 *with the boundary*

$$\partial M = M_1 \cap M_2$$

in a sufficiently small neighborhood of $\mu=0$ *in* $\mathbb{R}^k$*, so that* M *consists of parameter values* μ *for which the system has a heteroclinic orbit* $\Gamma=\{h(t)\}$ *from* O_1 *to* O_3*. Moreover* M *is tangent to neither* M_1 *nor* M_2 *at* $\mu=0$ *in* Π*.*

Proofs of Propositions 4.3 and 4.4 go as follows.

We may assume $\nu^2 < \rho^2$ for otherwise we can apply the same argument by replacing t with $-t$. Take the heteroclinic solution $h_i(t)$ so that $h_i(0)$ is near $\mathbf{O}_2$, and let Σ_i be a properly chosen $(2n-1)$-dimensional plane which passes through $h_i(0)$ and is transverse to $\dot{h}_i(0)$. Let $q^i(t)$ be a solution of the variational equation

$$\dot{z} = DF(h_i(t))z \tag{4.16}$$

along $h_i(t)$, for which the next limits exist and are non-zero, and $q^1(t)$ points to $\mathbf{O}_3$ along the heteroclinic orbit Γ_2 in the limit of $t \to +\infty$ and $q^2(t)$ points to $\mathbf{O}_1$ along the heteroclinic orbit Γ_1 in the limit of $t \to -\infty$:

$$\lim_{t\to-\infty} |q^i(t)|e^{\rho^i(0)t}, \quad \lim_{t\to+\infty} |q^i(t)|e^{-\nu^{i+1}(0)t}. \tag{4.17}$$

We remark that the adjoint $\hat{q}^i(t)$ of $q^i(t)$ is a non-trivial bounded solution of the equation (4.14) for each i. In this situation, there are two points $x_1^u(\mu) \in \mathfrak{W}^u(\mathbf{O}_1) \cap \Sigma_1$ and $x_2^s(\mu) \in \mathfrak{W}^s(\mathbf{O}_2) \cap \Sigma_1$ depending on μ smoothly and the difference of them can be expressed as

$$x_1^u - x_2^s = \alpha(\mu)q^1(0), \tag{4.18}$$

and similarly, there are two points $x_2^u(\mu) \in \mathfrak{W}^u(\mathbf{O}_2) \cap \Sigma_2$ and $x_3^s(\mu) \in \mathfrak{W}^s(\mathbf{O}_3) \cap \Sigma_2$ so that

$$x_3^s - x_2^u = \beta(\mu)q^2(0). \tag{4.19}$$

The function $\alpha(\mu)$ ($\beta(\mu)$) measures the separation of $\mathfrak{W}^u(\mathbf{O}_1)$ and $\mathfrak{W}^s(\mathbf{O}_2)$ in Σ_1 ($\mathfrak{W}^u(\mathbf{O}_2)$ and $\mathfrak{W}^s(\mathbf{O}_3)$ in Σ_2). For $\alpha(\mu) > 0$ and $\beta(\mu) > 0$, $\mathfrak{W}^u(\mathbf{O}_1)$ intersects with Σ_2 and we can, also, find a point $\tilde{x}_1^u(\mu) \in \mathfrak{W}^u(\mathbf{O}_1) \cap \Sigma_2$ so that

$$x_3^s - \tilde{x}_1^u = \chi(\mu)q^2(0), \tag{4.20}$$

where $\chi(\mu)$ measures the separation of $\mathfrak{W}^u(\mathbf{O}_1)$ and $\mathfrak{W}^s(\mathbf{O}_3)$ in Σ_2. $\chi(\mu)$ can be extended to the parameter values for which $\alpha(\mu) = 0$ or $\beta(\mu) = 0$. Proposition 4.2 was proven by showing

$$\frac{d}{d\mu}\alpha(0) = |q^1(0)| \int_{-\infty}^{+\infty} \hat{q}^1(s)\frac{\partial}{\partial\mu}g(h_1(s);0)ds \tag{4.21}$$

and

$$\frac{d}{d\mu}\beta(0) = |q^2(0)| \int_{-\infty}^{+\infty} \hat{q}^2(s)\frac{\partial}{\partial\mu}g(h_2(s);0)ds, \tag{4.22}$$

where they are non-zero and are linearly independent. The proof of Proposition 4.3 is through showing that $\chi(\mu) > 0$ if $\alpha(\mu) = 0$ and $\beta(\mu) > 0$ and $\chi(\mu) < 0$ if $\alpha(\mu) > 0$ and $\beta(\mu) = 0$, and $\frac{d}{d\mu}\chi(0) = \frac{d}{d\mu}\beta(0)$.

For the proof of the existence part of Proposition 4.4, take $a = \alpha(\mu)$, $b = \beta(\mu)$ and $\lambda = \frac{\rho^2(\mu)}{\nu^2(\mu)} - 1$ as parameters and consider in the parameter space (a, b, λ) where $\lambda \geq 0$,then the separation $\chi = \chi(a, b, \lambda)$ is defined in this parameter space and is smooth in b and λ and is Lipschitz continuous in a. For the derivative of χ with respect to b, we have $\frac{\partial\chi}{\partial b}(0) = 1$. By a modification of the implicit function theorem, we get a unique solution $b = b(a, \lambda)$ of $\chi(a, b, \lambda) = 0$ which is smooth in λ and Lipschitz in a. A similar argument for $\lambda \leq 0$ gives a unique solution $a = a(b, \lambda)$ of $\chi(a, b, \lambda) = 0$ and they coincide for $\lambda = 0$.

For some reaction diffusion equations, this kind of bifurcations take place. (See examples below.) In such cases, from stability of the travelling waves corresponding to heteroclinic orbits Γ_1, Γ_2 with some information about the twisting of these orbits, we can determine stability of the travelling wave corresponding to the heteroclinic orbit Γ. We explain it in what follows.

First, we determine $(\frac{dX_s^\theta(0)}{d\theta}|_{\theta=\theta_0} - \frac{dX_u^\theta(0)}{d\theta}|_{\theta=\theta_0})$ in Theorem 4.1 from the above conditions. For the case of $\nu^2 < \rho^2$, take $X_u^\mu(\xi) \in \mathfrak{W}_\mu^u(\mathbf{O}_1)$ of Theorem 4.1 as $X_u^\mu(0) = \tilde{x}_1^u(\mu)$ and $X_s^\mu(\xi) \in \mathfrak{W}_s^\mu(\mathbf{O}_3)$ as $X_s^\mu(0) = x_3^s(\mu)$ so that

$$X_s^\mu(0) - X_u^\mu(0) = \chi(\mu)q^2(0), \tag{4.23}$$

then

$$\left.\frac{\partial X_s^\mu(0)}{\partial \mu}\right|_{\mu=0} - \left.\frac{\partial X_u^\mu(0)}{\partial \mu}\right|_{\mu=0} = \frac{\partial \chi}{\partial \mu}(0)q^2(0) = \frac{\partial \beta}{\partial \mu}(0)q^2(0). \tag{4.24}$$

This means that if μ includes travelling wave speed θ or its translation $\theta - \theta_0$ as one of the parameters, then $(\frac{dX_s^\theta(0)}{d\theta}|_{\theta=\theta_0} - \frac{dX_u^\theta(0)}{d\theta}|_{\theta=\theta_0})$ is near $\frac{\partial \beta}{\partial \theta}(0)q^2(0)$, in particular the sign of

$$\det\left(\left(\left.\frac{dX_s^\theta(0)}{d\theta}\right|_{\theta=\theta_0} - \left.\frac{dX_u^\theta(0)}{d\theta}\right|_{\theta=\theta_0}\right)\ V_u^1 \cdots V_u^{n-1}\ X^{\theta_0\prime}(0)\ V_s^1 \cdots V_s^{n-1}\right)$$

coincides with that of

$$\det\left(\frac{\partial \beta}{\partial \theta}(0)q^2(0)\ V_u^1 \cdots V_u^{n-1}\ X^{\theta_0\prime}(0)\ V_s^1 \cdots V_s^{n-1}\right)$$

for $\mu \approx 0$ if $\frac{\partial \beta}{\partial \theta}(0) \neq 0$.

For the case of $\nu^2 = \rho^2$, for $\alpha > 0$ and $\beta > 0$ it is also smooth in α although χ is not necessarily smooth in α at $\mu = 0$. Because $\frac{\partial \chi}{\partial \beta} = 1$, $d\chi(\mu) \neq 0$ for $\mu \in M$ if $\mu \approx 0$, $\alpha > 0$ and $\beta > 0$, which means $\frac{\partial \chi}{\partial \theta} \neq 0$ if the θ-direction is transversal to M. From this we can easily conclude that if the sign of $\frac{\partial \chi}{\partial \theta}$ coincides with that of $\frac{\partial \chi}{\partial \beta}$ at some $\mu \in M$ then the sign of

$$\det\left(\left(\left.\frac{dX_s^\theta(0)}{d\theta}\right|_{\theta=\theta_0} - \left.\frac{dX_u^\theta(0)}{d\theta}\right|_{\theta=\theta_0}\right)\ V_u^1 \cdots V_u^{n-1}\ X^{\theta_0\prime}(0)\ V_s^1 \cdots V_s^{n-1}\right)$$

coincides with the sign of

$$\det\left(\frac{\partial \beta}{\partial \theta}(0)q^2(0)\ V_u^1 \cdots V_u^{n-1}\ X^{\theta_0\prime}(0)\ V_s^1 \cdots V_s^{n-1}\right)$$

for the same μ, and vice versa. We remark that $\chi(\mu) < 0$ for μ near the α-axis and $\chi(\mu) > 0$ for μ near the β-axis, so that we know the sign of $\frac{\partial \chi}{\partial \theta}$ from the bifurcation diagram. We state the above argument as a theorem.

Theorem 4.3. *Assume that the system* (3.4) *associated to the existence problem of travelling waves for the system* (2.1) *undergoes the bifurcation as described in Propositions* 4.3 *and* 4.4, *then if* $\nu^2 < \rho^2$ *we have the following equality for the derivative of the Evans function which is defined for the travelling wave corresponding to the heteroclinic orbit* Γ

$$sign\left(\left.\frac{dD(\lambda)}{d\lambda}\right|_{\lambda=0}\right) = sign\left(\det\left(\frac{\partial\beta}{\partial\theta}(0)q^2(0)\ V_u^1\cdots V_u^{n-1}\ X^{\theta_0}{}'(0)\ V_s^1\cdots V_s^{n-1}\right)\right).$$

The same equality holds for $\nu^2 = \rho^2$ *if the sign of* $\frac{\partial\chi}{\partial\theta}$ *coincides with that of* $\frac{\partial\chi}{\partial\beta}$, *and if it differs then*

$$sign\left(\left.\frac{dD(\lambda)}{d\lambda}\right|_{\lambda=0}\right) = -sign\left(\det\left(\frac{\partial\beta}{\partial\theta}(0)q^2(0)\ V_u^1\cdots V_u^{n-1}\ X^{\theta_0}{}'(0)\ V_s^1\cdots V_s^{n-1}\right)\right).$$

We also obtain a similar result for $\nu^2 \geq \rho^2$ *by replacing* $\xi(= x - \theta t)$ *with* $-\xi$ *in the system* (3.4).

Of course, Theorem 4.3 is not enough to determine the stability. We have to know the orientation of $(V_u^1, \cdots, V_u^{n-1})$ or equivalently that of $(V_s^1, \cdots, V_s^{n-1})$ to determine the sign of $\frac{dD}{d\lambda}(0)$. To do this we have to use other properties of the system.

In what follows, let us assume that the system (3.4) undergoes the bifurcation as in Propositions 4.3 and 4.4 with $\mathbf{O}_1 = \mathbf{O}_3$. In such a case, the two heteroclinic orbits Γ_1 and Γ_2 for $\mu = 0$ form a loop called a heteroclinic loop and the orbit $\Gamma = \{h(\xi) | \xi \in \mathbb{R}\}$ bifurcating from the loop is a homoclinic orbit. In this situation, the orientation of $(V_u^1, \cdots, V_u^{n-1})$ with respect to the homoclinic orbit is easily determined as follows.

Let $\left(\hat{V}_{u,1}^0, \cdots, \hat{V}_{u,1}^{n-1}\right)$ (resp. $\left(\hat{V}_{s,1}^0, \cdots, \hat{V}_{s,1}^{n-1}\right)$) be a basis of the unstable (resp. stable) eigenspace of $\mathbf{O}_1$ satisfying

$$\hat{V}_{u,1}^0 = \lim_{\xi\to-\infty}\frac{h_1(\xi)}{|h_1(\xi)|} \quad , \qquad \hat{V}_{s,1}^0 = \lim_{\xi\to+\infty}\frac{h_2(\xi)}{|h_2(\xi)|}$$

and

$$\det\left(\hat{V}_{u,1}^0\cdots\hat{V}_{u,1}^{n-1}\ \hat{V}_{s,1}^0\cdots\hat{V}_{s,1}^{n-1}\right) > 0.$$

Then, $\left(\hat{V}_{u,1}^0, \cdots, \hat{V}_{u,1}^{n-1}\right)$ determines the orientation of the local unstable manifold of $\mathbf{O}_1$ and it propagates with the orientation of the global unstable manifold $\mathfrak{W}^u(\mathbf{O}_1)$ of $\mathbf{O}_1$. Let $\left(\hat{V}_{u,2}^0, \cdots, \hat{V}_{u,2}^{n-1}\right)$ be a basis of the unstable eigenspace of $\mathbf{O}_2$ with

$$\hat{V}_{u,2}^0 = \lim_{\xi\to-\infty}\frac{h_2(\xi)}{|h_2(\xi)|} \quad ,$$

and let

$$\hat{V}_{s,2}^0 = \lim_{\xi\to+\infty}\frac{h_1(\xi)}{|h_1(\xi)|} \quad .$$

From the assumption (2) for the heteroclinic orbit Γ_1, the tangent space $T_{h_1(\xi)}\mathfrak{W}^u(\mathbf{O}_1)$ is tangent to the space spanned by $\hat{V}^0_{s,2}, \hat{V}^1_{u,2}, \cdots, \hat{V}^{n-1}_{u,2}$ in the limit of $\xi \to +\infty$. We determine the orientation of $\left(\hat{V}^1_{u,2}, \cdots, \hat{V}^{n-1}_{u,2}\right)$ so that the orientation of $\left(\hat{V}^0_{s,2}, \hat{V}^1_{u,2}, \cdots, \hat{V}^{n-1}_{u,2}\right)$ is compatible with that of $\mathfrak{W}^u(\mathbf{O}_1)$, and then the orientation of $\mathfrak{W}^u(\mathbf{O}_2)$ is naturally determined by that of $\left(\hat{V}^0_{u,2}, \hat{V}^1_{u,2}, \cdots, \hat{V}^{n-1}_{u,2}\right)$. Similarly the orientation of $\mathfrak{W}^u(\mathbf{O}_2)$ again determines the orientation of the unstable eigenspace of $\mathbf{O}_1$, but this orientation is not necessarily compatible with that of $\left(\hat{V}^0_{u,1}, \cdots, \hat{V}^{n-1}_{u,1}\right)$ which we defined at the beginning.

Definition 4.1. *The heteroclinic loop that consists of Γ_1 and Γ_2 is said to be non-twisted with respect to the strong unstable direction if the above orientation is compatible with that of $\left(\hat{V}^0_{u,1}, \cdots, \hat{V}^{n-1}_{u,1}\right)$ defined at the beginning, otherwise it is said to be twisted with respect to the strong unstable direction.*

Twisting with respect to the strong stable direction is similarly defined.

We have another definition of twisting, which is directly related to the structure of bifurcation. (See for example B.Deng [4] and references in it.)

Consider $q^2(\xi)$ which is the solution of the variational equation (4.16) along $h_2(\xi)$ given in the proof of Propositions 4.3 and 4.4. From the requirement for $q^2(\xi)$ (4.17), we may assume

$$\lim_{\xi \to -\infty} q^2(\xi) e^{\rho^2 \xi} = -\hat{V}^0_{s,2}$$

and

$$\lim_{\xi \to +\infty} q^2(\xi) e^{-\nu^1 \xi} = c\hat{V}^0_{u,1}$$

for some non-zero constant c.

Definition 4.2. *The heteroclinic orbit Γ_2 is non-twisted if c is positive, and twisted if negative.*

With these two kinds of twistings, we define the sign σ of $\Gamma = \{h(\xi)\}$ as follows.

Definition 4.3. $\sigma = +1$ *if either of the following holds.*

(1) *The heteroclinic loop is non-twisted with respect to the strong unstable direction and Γ_2 is non-twisted.*

(2) *The heteroclinic loop is twisted with respect to the strong unstable direction and Γ_2 is twisted.*

Otherwise $\sigma = -1$.

Lemma 4.1. *If either of the following is satisfied for the heteroclinic loop which consists of two heteroclinic orbits Γ_1 and Γ_2, then $\sigma = +1$, otherwise $\sigma = -1$.*

(1) *The heteroclinic loop is non-twisted with respect to the strong stable direction and Γ_1 is non-twisted.*

(2) *The heteroclinic loop is twisted with respect to the strong stable direction and* Γ_1 *is twisted.*

Proof. By the definitions of the above two kinds of twistings, the heteroclinic loop consisting of two heteroclinic orbits Γ_1 and Γ_2 which is non-twisted (twisted) with respect to the strong unstable direction is non-twisted (twisted) with respect to the strong stable direction if both heteroclinic orbits Γ_1 and Γ_2 are non-twisted or both are twisted. Otherwise the loop which is non-twisted with respect to the strong unstable direction is twisted with respect to the strong stable direction, and vice versa. The lemma clearly holds. □

Then we can determine the sign of $\frac{dD}{d\lambda}(0)$.

Theorem 4.4.

$$sign\left(\left.\frac{dD(\lambda)}{d\lambda}\right|_{\lambda=0}\right) = \sigma \cdot sign\left(\frac{\partial\beta}{\partial\theta}\right)$$

if $\nu^2 < \rho^2$. *The same equality holds true for* $\nu^2 = \rho^2$ *if the sign of* $\frac{\partial\chi}{\partial\theta}$ *coincides with that of* $\frac{\partial\chi}{\partial\beta}$, *and if it differs then*

$$sign\left(\left.\frac{dD(\lambda)}{d\lambda}\right|_{\lambda=0}\right) = -\sigma \cdot sign\left(\frac{\partial\beta}{\partial\theta}\right).$$

For the case $\nu^2 > \rho^2$, *the following holds.*

$$sign\left(\left.\frac{dD(\lambda)}{d\lambda}\right|_{\lambda=0}\right) = -\sigma \cdot sign\left(\frac{\partial\alpha}{\partial\theta}\right).$$

The same equality holds true for $\nu^2 = \rho^2$ *if the sign of* $\frac{\partial\chi}{\partial\theta}$ *coincides with that of* $\frac{\partial\chi}{\partial\alpha}$ *for* χ *of the time reversed system, and if it differs then*

$$sign\left(\left.\frac{dD(\lambda)}{d\lambda}\right|_{\lambda=0}\right) = \sigma \cdot sign\left(\frac{\partial\alpha}{\partial\theta}\right).$$

Proof. Define the orientation of the unstable manifold $\mathfrak{W}^u(\mathbf{O}_i)$ $(i = 1, 2)$ and the stable manifold $\mathfrak{W}^s(\mathbf{O}_i)$ of $\mathbf{O}_i$ continuously depending on μ by

$$\hat{V}^0_{u,1} = \lim_{\xi\to-\infty} \frac{h(\xi)}{|h(\xi)|}$$

and

$$\hat{V}^0_{s,1} = \lim_{\xi\to+\infty} \frac{h(\xi)}{|h(\xi)|} \quad .$$

Let $V^1_u(\xi), \cdots, V^{n-1}_u(\xi)$ $(V^1_u(\xi), \cdots, V^{n-1}_u(\xi))$ be solutions of the variational equation along $h(\xi)$ such that $h'(\xi), V^1_u(\xi), \cdots, V^{n-1}_u(\xi)$ is a properly ordered basis of $T_{h(\xi)}\mathfrak{W}^u(\mathbf{O}_1)$ and $h'(\xi), V^1_s(\xi), \cdots, V^{n-1}_s(\xi)$ is a basis of $T_{h(\xi)}\mathfrak{W}^s(\mathbf{O}_1)$. Then the orientation of $\mathfrak{W}^u(\mathbf{O}_1)$ propagates along $h(\xi)$ which approaches to $\mathbf{O}_2$ once and comes

back to $\mathbf{O}_1$. By definition, the orientation of $\mathfrak{W}^u(\mathbf{O}_1)$ coincides with that of $\mathfrak{W}^u(\mathbf{O}_2)$ when $h(\xi)$ is near $\mathbf{O}_2$. This implies that the orientation of $\left(\hat{V}^0_{u,1},\ V^1_u(\xi), \cdots, V^{n-1}_u(\xi)\right)$ is compatible with that of $(\hat{V}^0_{u,1},\ \hat{V}^1_{u,1}, \cdots, \hat{V}^{n-1}_{u,1})$ for large ξ if and only if the heteroclinic loop is non-twisted. So the sign of

$$\det\left(\hat{V}^0_{u,1}\ V^1_u(\xi)\cdots V^{n-1}_u(\xi)\ h'(\xi)\ V^1_s(\xi)\cdots V^{n-1}_s(\xi)\right)$$

is positive for large ξ if the heteroclinic loop is non-twisted, and negative if twisted. Therefore by the definition of twisting of Γ_2, it follows that

$$sign\left(\det\left(q^2(\xi)\ V^1_u(\xi)\cdots V^{n-1}_u(\xi)\ h'(\xi)\ V^1_s(\xi)\cdots V^{n-1}_s(\xi)\right)\right)$$

is positive if (1) or (2) of Definition 4.3 holds, and negative otherwise. This proves the theorem.

By replacing ξ with $-\xi$ and taking account of Lemma 4.1 and the definition of χ (4.20) in the proof of Propositions 4.3 and 4.4, we get the result for the case $\nu^2 > \rho^2$. □

Remark 4.2.

(1) *The heteroclinic loop is non-twisted with respect to the strong direction, if the system* (3.4) *has a certain symmetry for* $\mu = 0$. *For example* $h_2(\xi) = -h_1(\xi)$ *or* $h_2(\xi) = (h^u_2(\xi), h^v_2(\xi)) = (h^u_1(-\xi), -h^v_1(\xi))$, *where* h^u_i *stands for the* u*-component of* h_i *and* h^v_i *for the* v*-component of* h_i. *Note that a standing wave (a travelling wave with velocity* $\theta = 0$*) posses the latter type of symmetry.*

(2) *When the dimension of the system* (3.4)*is equal to or less than three, the twisting of the loop is determined only by the twistings of* Γ_1 *and* Γ_2.

Theorem 4.4 means that if σ is given, then we can determine the stability of the travelling wave corresponding to the homoclinic orbit $h(\xi)$ only from the bifurcation diagram. We summarize it in Figure 14.

5. Examples

In this section, we consider two examples of reaction diffusion equations which possesses travelling waves forming a heteroclinic loop, and study the stability of bifurcating waves by applying theorems of this paper. where

Example 1 We treat the following activator–inhibitor system:

$$\begin{cases} \varepsilon\tau u_t = \varepsilon^2 u_{xx} + f(u, v; \gamma, \nu) \\ v_t = \quad v_{xx} + g(u, v; \gamma, \nu), \end{cases} \tag{5.1}$$

where ε and τ are real positive parameters, ε being sufficiently small, and

$$\begin{cases} f(u,v;\gamma,\nu) = -u^3 + u - v \\ g(u,v;\gamma,\nu) = u - \gamma v + \nu. \end{cases} \tag{5.2}$$

The nullcline of f intersects with that of g at three points P, Q and R, where P and Q are stable and R is an unstable constant solution of (5.1) (see Figure 8).

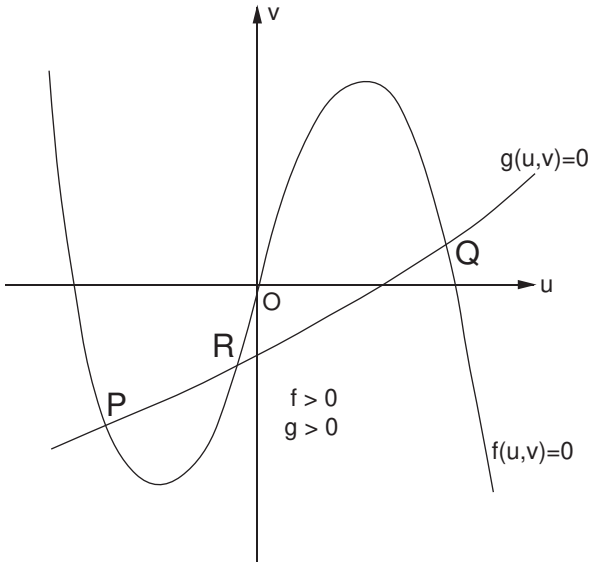

Figure 8

In the sequel, introducing the travelling coordinate $\xi = x - \theta t$, γ, ν and the wave speed θ are regarded as bifurcation parameters. Consider

$$\begin{cases} u' = \dfrac{1}{\varepsilon} u_1 \\ v' = v_1 \\ u_1' = -\dfrac{\theta\tau}{\varepsilon} u_1 - \dfrac{1}{\varepsilon} f(u,v;\gamma,\nu) \\ v_1' = -\theta v_1 - g(u,v;\gamma,\nu) \end{cases} \qquad \left(\ \ ' = \tfrac{d}{d\xi} \right) \tag{5.3}$$

corresponding to the existence of travelling wave for (5.1). This system possesses a heteroclinic orbits $\Gamma_1 = \{h_1(\xi;\gamma,\nu)\}$, from $(P,\mathbf{O})$ to $(Q,\mathbf{O})$, and $\Gamma_2 = \{h_2(\xi;\gamma,\nu)\}$ from $(Q,\mathbf{O})$ to $(P,\mathbf{O})$. The travelling (standing if $\theta = 0$) wave corresponding to Γ_1 is called a travelling (standing) front and Γ_2 a travelling (standing) back. Figure 9 exhibits the situation in the parameter space, where M_1 corresponds to Γ_1 and M_2 to Γ_2. Notice that the thick curve shows coexistence of both heteroclinic orbits or, in other words, existence of what is called a heteroclinic loop.

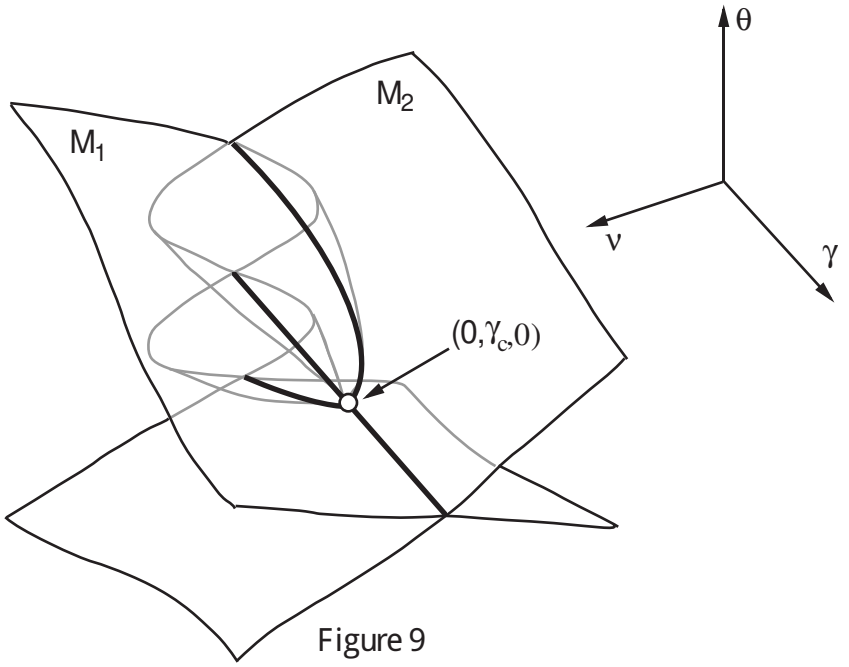

Figure 9

In Figure 10, bifurcation diagrams for ν and θ for fixed γ are shown.

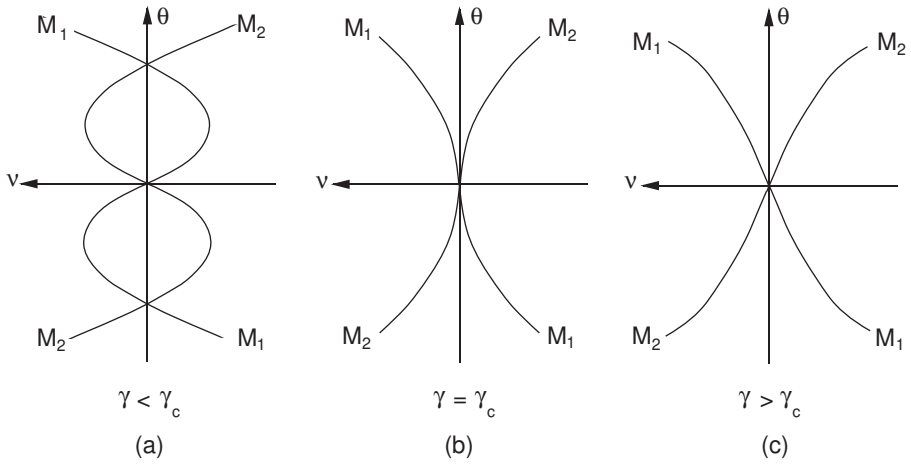

Figure 10

H.Kokubu, Y.Nishiura and H.Oka [16] showed the existence of homoclinic orbits $\Gamma_P = \{h_P(\xi)\}$ associated to $(P, \mathbf{O})$ and $\Gamma_Q = \{h_Q(\xi)\}$ to $(Q, \mathbf{O})$ bifurcating from the heteroclinic loops, these homoclinic orbits corresponding to travelling (standing) pulses. We show bifurcation diagrams in Figures 11 and 12, where M_P and M_Q correspond to these waves.

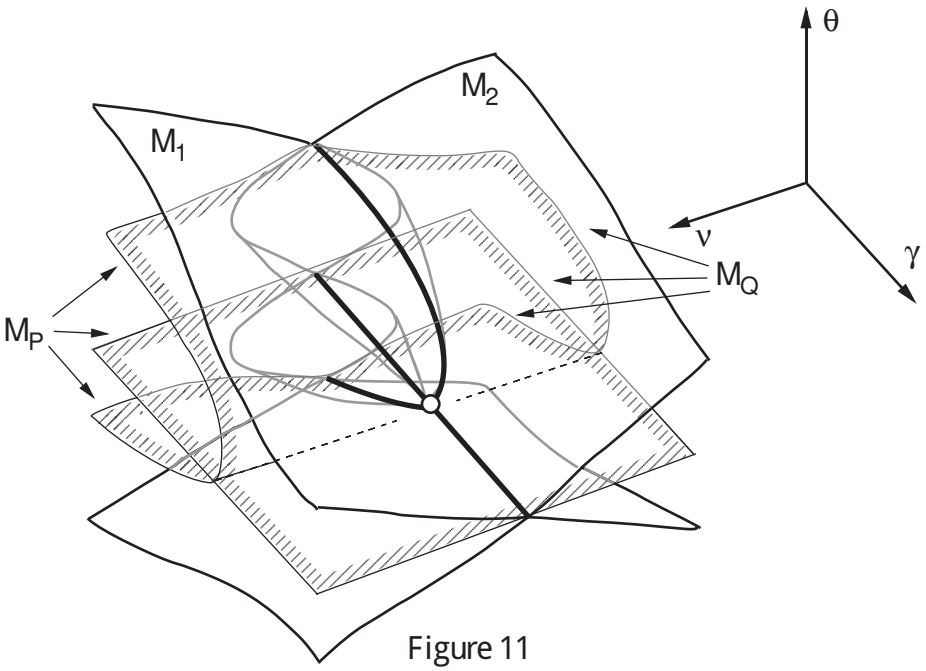

Figure 11

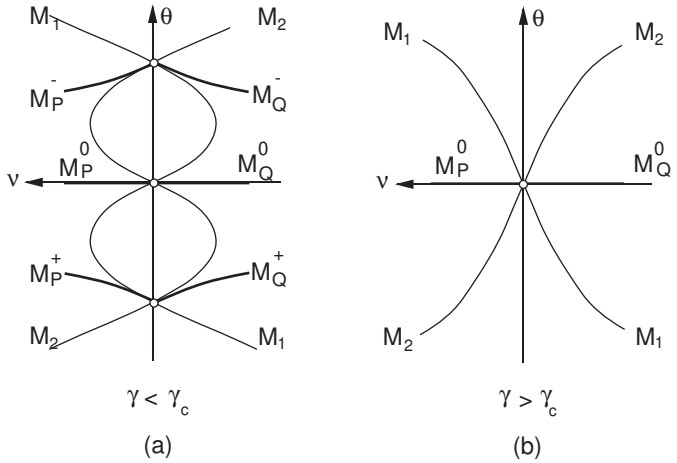

Figure 12

Stability and instability of fronts and backs are known: Ikeda [9] proved stability and instability of pulses in [16] using the singular perturbation method. In this example we prove them applying theorems in this paper without using the information in the singular limit.

First, note that the standing front and back that correspond to the heteroclinic loop for $(\gamma, \nu, \theta) = (\gamma, 0, 0)$ and $\gamma < \gamma_c$ are unstable. From Theorem 4.1, both standing pulses associated to P and Q bifurcating from the heteroclinic loop are unstable. Fronts and backs corresponding to the heteroclinic loops for other parameter values

are known to be stable, and thus the pulses other than the above can be stable, and it is indeed the case. In such cases the eigenvalue problem (2.7) for pulses have one eigenvalue near the origin other than the one at the origin and it is real. We show that the derivative of the Evans function at the origin $\frac{dD}{d\lambda}(0)$ is positive, which means $[g]$ in Theorem 4.2 equals to 1 for $\lambda_1 < 0$ near 0 and λ_2 large *i.e.* 0 is the only eigenvalue whose real part is equal or grater than 0. Thus the pulse is stable.

The heteroclinic orbits Γ_1 and Γ_2 are known to be non-twisted for $\nu = 0$. Notice that the system (5.3) has the symmetry

$$h_2(\xi; \gamma, 0) = -h_1(\xi; \gamma, 0) \tag{5.4}$$

for the heteroclinic loop, and so it is non-twisted with respect to the strong direction. (Remark 4.2.) From Theorem 4.4 and the bifurcation diagram (Figure 12) we have

$$sign\left(\frac{dD}{d\lambda}(0)\right) > 0.$$

and therefore the pulses are stable.

Example 2 Next example is a Lotka–Volterra competition system:

$$\begin{cases} u_t = u_{\xi\xi} + f(u,v) \\ v_t = dv_{\xi\xi} + g(u,v) \qquad\qquad \xi \in \mathbb{R}, \quad t > 0, \\ u(0,\xi) \geq 0, \quad v(0,\xi) \geq 0 \end{cases} \tag{5.5}$$

where

$$\begin{cases} f(u,v) = (1 - u - cv)u \\ g(u,v) = (a - bu - v)v, \end{cases} \tag{5.6}$$

and a, b, c and d are positive constants.

This system has two stable constant solutions $(u,v) \equiv (0,a)$ and $(u,v) \equiv (1,0)$ for $\frac{1}{c} < a < b$. Again consider the next system in search of travelling waves.

$$\begin{cases} u' = u_1 \\ v' = v_1 \\ u_1' = -\theta u_1 - (1 - u - cv)u \qquad (\quad ' = \frac{d}{d\xi}) \\ v_1' = -\dfrac{\theta}{d} v_1 - \dfrac{1}{d}(a - bu - v)v. \end{cases} \tag{5.7}$$

For each fixed d, there exist C^1-families $h_1(\xi; a, b, c)$ and $\theta(a, b, c)$ defined on $\mathfrak{P} = \{(a,b,c) | 0 < \frac{1}{c} < a < b\}$ so that h_1 is a solution of (5.7) with $\theta = \theta(a,b,c)$ satisfying

$$\lim_{\xi \to -\infty} h_1(\xi; a, b, c) = (0, a, 0, 0), \qquad \lim_{\xi \to +\infty} h_1(\xi : a, b, c) = (1, 0, 0, 0), \tag{5.8}$$

and $\theta_a(a,b,c) > 0$, $\theta_b(a,b,c) < 0$ and $\theta_c(a,b,c) > 0$. Moreover there exists a C^1-family $a = a(b,c) \in (\frac{1}{c}, b)$ on $\{(b,c)|0 < \frac{1}{c} < b\}$ such that $\theta(a(b,c),b,c) = 0$ holds (see Kan-on [11]). $\theta = 0$ means this system also has symmetry by which we can get a heteroclinic solution $h_2(\xi;a,b,c)$ satisfying

$$\lim_{\xi\to-\infty} h_2(\xi;a,b,c) = (1,0,0,0), \qquad \lim_{\xi\to+\infty} h_2(\xi:a,b,c) = (0,a,0,0) \tag{5.9}$$

by putting

$$\begin{aligned} &h_2(\xi;a,b,c) \\ &=(h_1^1(-\xi;a,b,c), h_1^2(-\xi;a,b,c), -h_1^3(-\xi;a,b,c), -h_1^4(-\xi;a,b,c)) \end{aligned} \tag{5.10}$$

where

$$h_1(\xi;a,b,c) = (h_1^1(\xi;a,b,c), h_1^2(\xi;a,b,c), h_1^3(\xi;a,b,c), h_1^4(\xi;a,b,c)). \tag{5.11}$$

This means that for $(\theta,a,b,c) = (0,a(b,c),b,c)$, the system possesses a non-twisted heteroclinic loop consisting of h_1 and h_2 and it is known that both h_1 and h_2 are non-twisted. Of course, the system (5.7) has homoclinic orbits bifurcating from the heteroclinic loop (Kan-on [12]). More precisely, there exists a C^1-family of solutions of (5.7) $h(\xi;a,b,c)$ ($\bar{h}(\xi;a,b,c)$) with $\theta = 0$ defined on $\mathfrak{Q} = \{(a,b,c)|0 < \frac{1}{c} < b, \frac{1}{c} < a < a(b,c)\}$ ($\bar{\mathfrak{Q}} = \{(a,b,c)|0 < \frac{1}{c} < b, a(b,c) < a < b\}$) which satisfies

$$\lim_{\xi\to\pm\infty} h(\xi;a,b,c) = (0,a,0,0) \qquad \left(\lim_{\xi\to\pm\infty} \bar{h}(\xi;a,b,c) = (1,0,0,0)\right). \tag{5.12}$$

Again, travelling waves corresponding to the heteroclinic orbits are stable ([14]). Kan-on [13] proved instability of pulses by constructing an eigenfunction that corresponds to a positive eigenvalue. However, by applying Theorem 4.4, we readily see from the bifurcation diagram (Figure 13) that the pulses are unstable.

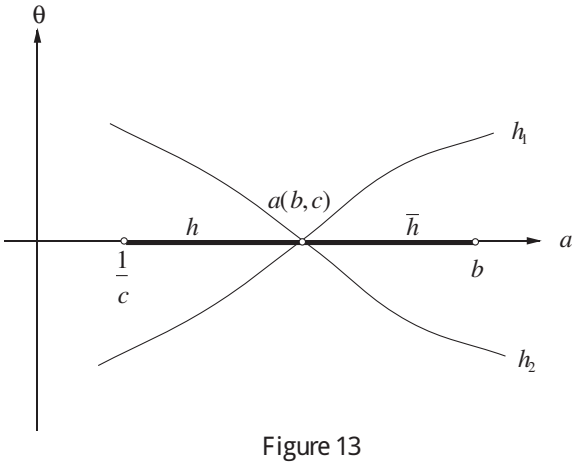

Figure 13

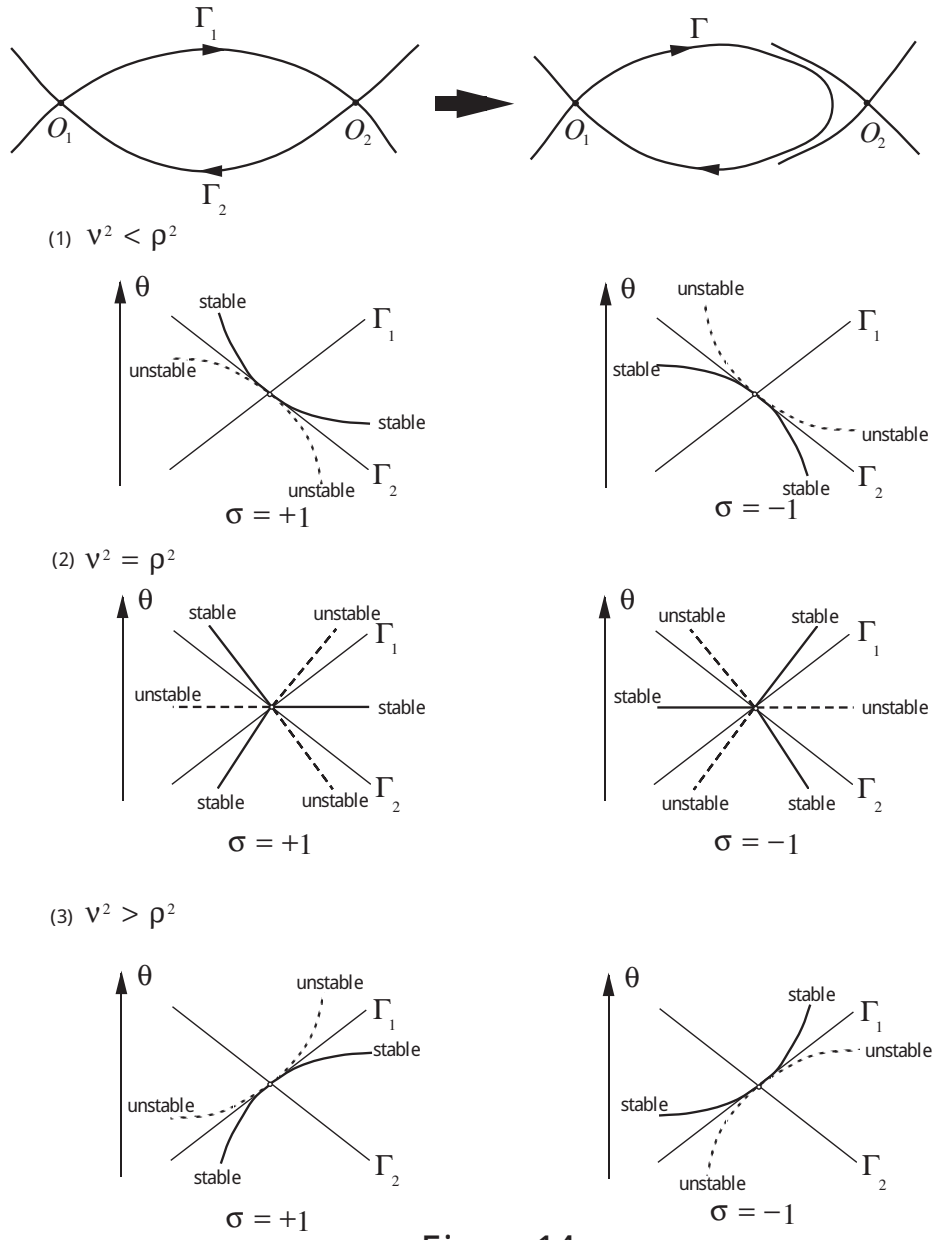

Figure 14

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