REMARKS ON S. LANG'S CONJECTURE OVER FUNCTION FIELDS

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Abstract. In this short note, we will show the following weak evidence of S. Lang conjecture over function fields. Let $f : X \to Y$ be a projective and surjective morphism of algebraic varieties over an algebraically closed field k of characteristic zero, whose generic fiber is geometrically irreducible and of general type. If f is not birationally trivial, then there are countably many proper closed varieties $\{Z_i\}$ of X such that every quasi-section of f is contained in $\bigcup_i Z_i$.

§0. Introduction

In [La], S. Lang conjectured that if X is a projective variety over a number field K and X is of general type, then there is a proper subscheme Z of X with $X(K) \subset Z(K)$. If we consider an analogue over function fields, we must avoid a birationally trivial family, i.e. an algebraic family $f : X \to Y$ of algebraic varieties which is birationally equivalent to a product $W \times Y$ over Y.

Conjecture A. (Analogue of S. Lang's conjecture over function fields) Let $f : X \to Y$ be a projective and surjective morphism of algebraic varieties over an algebraically closed field k, whose generic fiber is geometrically irreducible and of general type. If f is not birationally trivial, then there are a proper subscheme Z of X such that every quasi-section of f is contained in Z.

When $\dim f = 1$, Conjecture A is known as Mordell's conjecture over function fields and was proved by a lot of authors in any characteristic. However, Conjecture A does not hold in this naive form if $\dim f \geq 2$ and the characteristic is positive. For, in this case, there is a family of unirational varieties of general type (For more details, see §4). On the other hand, if the characteristic is zero, from all we know, it is still an open problem. We only know that it holds if $f : X \to Y$ is a hyperbolic fiber space (cf. [No1], [No2] and [Mo]). In this short note, we will prove the following weak evidence of the above Conjecture A in characteristic zero.

Proposition B. *Let $f : X \to Y$ be a projective and surjective morphism of algebraic varieties over an algebraically closed field k of characteristic zero, whose generic fiber is geometrically irreducible and of general type. If f is not birationally trivial, then there*

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are countably many proper closed varieties $\{Z_i\}$ of X such that every quasi-section of f is contained in $\bigcup_i Z_i$.

Our basic tool in this note is the following criterion of birational splitting (cf. Corollary 1.2).

> *Let $f : X \dashrightarrow Y$ be a dominant rational map, whose generic fiber is geometrically irreducible and of general type. Assume that there exists a variety T and a dominant rational map $\phi : T \times Y \dashrightarrow X$. Then X is a birationally equivalent to a product $W \times Y$.*

This is a problem raised in Historical appendix of [La] and essentially was solved by K. Maehara [Ma] earlier than [La].

§1. Birational splitting

In this section, we will consider a criterion for birational splitting. It was essentially due to K. Maehara [Ma]. This is a very important tool for diophantine geometry over function fields, so that we will re-prove it here. Let's us begin with the following lemma, which is an easy application of weak positivity of direct images of n-th relative canonical bundles.

Lemma 1.1. *Let $f : X \to Y$ be a surjective morphism of smooth projective varieties over an algebraically closed field k of characteristic zero. If there are a projective smooth algebraic variety T over k and a dominant rational map $\phi : T \times_k Y \dashrightarrow X$ over Y, then the double dual $f_*(\omega^n_{X/Y})^{\vee\vee}$ of $f_*(\omega^n_{X/Y})$ is a free $\mathcal{O}_Y$-sheaf for all $n \geq 0$.*

Proof. Let A be a very ample line bundle on T. If $\dim T > \dim f$ and T_1 is a general member of $|A|$, then $\phi|_{T_1 \times Y} : T_1 \times Y \dashrightarrow X$ still dominates X. Thus, considering induction on $\dim T$, we may assume that $\dim T = \dim f$.

Let $\mu : Z \to T \times Y$ be a birational morphism of smooth projective varieties such that $\psi = \phi \cdot \mu : Z \to X$ is a morphism. Then, ψ is generically finite. Thus, there is a natural injection $\psi^*(\omega_{X/Y}) \hookrightarrow \omega_{Z/Y}$. Hence, $\psi^*(\omega^n_{X/Y}) \hookrightarrow \omega^n_{Z/Y}$ for all $n > 0$. Therefore,

$$\omega^n_{X/Y} \hookrightarrow \psi_*(\psi^*(\omega^n_{X/Y})) \hookrightarrow \psi_*(\omega^n_{Z/Y}).$$

Applying f_* to the above injection, we have

$$f_*(\omega^n_{X/Y}) \hookrightarrow f_*(\psi_*(\omega^n_{Z/Y})).$$

Further, letting p be the natural projection $p : T \times Y \to Y$,

$$f_*(\psi_*(\omega^n_{Z/Y})) = p_*(\mu_*(\omega^n_{Z/Y})) = p_*(\omega^n_{T\times Y/Y}) = H^0(T, \omega^n_T) \otimes_k \mathcal{O}_Y.$$

Thus, $f_*(\omega^n_{X/Y})^{\vee\vee}$ is a subsheaf of the free sheaf $H^0(T, \omega^n_T) \otimes_k \mathcal{O}_Y$.

Here we claim

$$\left(c_1\left(f_*(\omega^n_{X/Y})^{\vee\vee}\right) \cdot H^{d-1}\right) \geq 0, \tag{1.1.1}$$

where H is an ample line bundle on Y and $d = \dim Y$. This is an immediate consequence of weak positivity of $f_*(\omega^n_{X/Y})^{\vee\vee}$ due to Viehweg [Vi]. We can however conclude our claim by a weaker result of Kawamata [Ka1], namely $\deg(f_*(\omega^n_{X/Y})) \geq 0$ if $\dim Y = 1$. For, considering complete intersections by general members of $|H^m|$ $(m \gg 0)$, we may assume $\dim Y = 1$.

We can find a projection $\alpha : H^0(T, \omega^n_T) \otimes_k \mathcal{O}_Y \to \mathcal{O}_Y^{\oplus r_n}$ such that $r_n = \operatorname{rank} f_*(\omega^n_{X/Y})^{\vee\vee}$ and the composition

$$f_*(\omega^n_{X/Y})^{\vee\vee} \hookrightarrow H^0(T, \omega^n_T) \otimes_k \mathcal{O}_Y \xrightarrow{\alpha} \mathcal{O}_Y^{\oplus r_n}$$

is injective. Therefore, since $f_*(\omega^n_{X/Y})^{\vee\vee}$ is reflexive, the above homomorphism is an isomorphism by (1.1.1). □

As corollary, we have a criterion of birational splitting.

Corollary 1.2. *Let $f : X \dashrightarrow Y$ be a dominant rational map of algebraic varieties over an algebraically closed field k of characteristic zero, whose generic fiber of f is geometrically irreducible and of general type. If there are an algebraic variety T over k and a dominant rational map $\phi : T \times_k Y \dashrightarrow X$ over Y, then there is an algebraic variety W over k such that X is birationally equivalent to a product $W \times_k Y$ over Y.*

Proof. Clearly we may assume that X, Y and T are smooth and projective, and that f is a morphism. Moreover, in the same way as in the proof of Lemma 1.1, we may assume that $\dim T = \dim f$.

Let $\mu : Z \to T \times Y$ be a birational morphism of smooth projective varieties such that $\psi = \phi \cdot \mu : Z \to X$ is a morphism. We take a sufficiently large integer m and an open set U of Y such that

(a) $|\omega^m_{f^{-1}(t)}|$ gives a birational map for all $t \in U$, and
(b) the natural homomorphism $f_*(\omega^m_{X/Y}) \otimes k(t) \to H^0(f^{-1}(t), \omega^m_{f^{-1}(t)})$ is bijective for all $t \in U$, where $k(t)$ is the residue field at t.

Let U_0 be an open set of Y such that $\operatorname{codim}_Y(Y \setminus U_0) \geq 2$ and

$$f_*(\omega^m_{X/Y})^{\vee\vee}\Big|_{U_0} = f_*(\omega^m_{X/Y})\Big|_{U_0}.$$

We set $X_0 = f^{-1}(U_0)$. Then we get

$$H^0(X_0, \omega^m_{X/Y}) = H^0\left(Y, f_*(\omega^m_{X/Y})^{\vee\vee}\right). \tag{1.2.1}$$

Let $h : X_0 \dashrightarrow \mathbb{P}(H^0(X_0, \omega^m_{X/Y}))$ be a rational map induced by $H^0(X_0, \omega^m_{X/Y})$ and W the image of h. Note that $\dim H^0(X_0, \omega^m_{X/Y}) = \operatorname{rank} f_*(\omega^m_{X/Y})$ by Lemma 1.1. In order to see that $h \times f : X_0 \dashrightarrow W \times U_0$ is a birational map over U_0, it is sufficient to show that $h|_{f^{-1}(t)} : f^{-1}(t) \dashrightarrow W$ is a birational map for all $t \in U_0 \cap U$.

Here we claim $\dim W \leq \dim T$. The rational map $h \cdot \psi : \psi^{-1}(X_0) \dashrightarrow W$ is induced by a linear system given by the image $H^0(X_0, \omega_{X/Y}^m) \hookrightarrow H^0(\psi^{-1}(X_0), \omega_{Z/Y}^m)$. Here we have

$$\begin{aligned} H^0(X_0, \omega_{X/Y}^m) &\hookrightarrow H^0(\psi^{-1}(X_0), \omega_{Z/Y}^m) \\ &= H^0(T \times_k U_0, \omega_{T\times U_0/U_0}^m) \\ &= H^0(T, \omega_T^m). \end{aligned}$$

This means that there is a rational map $h' : T \dashrightarrow W$ such that the following diagram of rational maps is commutative:

$$\begin{array}{ccc} T \times_k U_0 & \xrightarrow{\phi} & X_0 \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle h} \\ T & \xrightarrow[h']{} & W \end{array}$$

where q is the natural projection. Since ϕ, h and q are dominant rational maps, so is h'. Therefore $\dim W \leq \dim T$.

By Lemma 1.1, $f_*(\omega_{X/Y}^m)^{\vee\vee}$ is a free sheaf on Y. Hence, by virtue of (b) and (1.2.1),

$$H^0(X_0, \omega_{X/Y}^m) \to H^0(f^{-1}(t), \omega_{f^{-1}(t)}^m)$$

is bijective. This means that $h|_{f^{-1}(t)} : f^{-1}(t) \dashrightarrow W$ is given by the complete linear system $|\omega_{f^{-1}(t)}^m|$. Thus, by (a) and $\dim W \leq \dim f^{-1}(t)$, $h|_{f^{-1}(t)}$ is a birational map. □

§2. Proof of Proposition B

Clearly we may assume that X and Y are projective. Let $\mathrm{Rat}_k(Y, X)$ be a scheme consisting of rational maps from Y to X. We have the natural morphism of schemes $\alpha : \mathrm{Rat}_k(Y, X) \to \mathrm{Rat}_k(Y, Y)$ defined by $\alpha(s) = f \cdot s$ for $s \in \mathrm{Rat}_k(Y, X)$. We set $\mathrm{QSec}(f) = \alpha^{-1}(\mathrm{id}_Y)$. Then, it is easy to check the following properties of $\mathrm{QSec}(f)$.

(a) $\mathrm{QSec}(f)(k)$ is the set of all quasi-sections of $f : X \to Y$.

(b) There is the evaluation map $\phi : \mathrm{QSec}(f) \times Y \dashrightarrow X$, which satisfies $f \cdot \phi = p$, where $p : \mathrm{QSec}(f) \times Y \to Y$ is the natural projection.

(c) $\mathrm{QSec}(f)$ has only countably many connected components.

(a) and (b) are trivial by our construction. Concerning (c), $\mathrm{Rat}_k(Y, X)$ can be realized as an a open subscheme of $\mathrm{Hilb}_{Y\times X}$. If we fix a polarization of $Y \times X$ and a Hilbert polynomial P, then $\mathrm{Hilb}_{Y\times X}^P$ has only finite connected components. On the other hand, the Hilbert polynomials P is a polynomials with coefficients in $\mathbb{Q}$. Thus we have only countably many possibilities of P. Hence we can conclude (c).

Let $\mathrm{QSec}(f) = \bigcup_i \mathrm{QSec}_i(f)$ be the decomposition into connected components. Then, by the criterion of birational splitting, the evaluation map $\mathrm{QSec}_i(f) \times Y \dashrightarrow X$ is not a dominant rational map for every i. Thus, we get our proposition. □

§3. Northcott's theorem over function fields

By the same idea as the proof of Proposition B, we have the following Northcott's theorem over function fields, which is a generalization of Theorem 1.3 of [Mo].

Proposition 3.1. *Let X be a smooth projective variety over an algebraically closed field k of characteristic zero, C a smooth projective curve over k, and $f : X \to C$ a surjective morphism whose generic fiber is geometrically irreducible. Let L be a line bundle on X such that L is ample on the generic fiber of f. Assume that $\deg(f_*(\omega^n_{X/C})) > 0$ for some $n > 0$. Then, for any number A, the set*

$$\{\Delta \mid \Delta \text{ is a section of } f : X \to C \text{ with } (L \cdot \Delta) \leq A\}$$

is not dense in X.

Proof. In the same way as in the proof of Theorem 1.3 of [Mo], we may assume that L is ample on X. Thus, the set

$$S = \{\Delta \mid \Delta \text{ is a section of } f : X \to C \text{ with } (L \cdot \Delta) \leq A\}$$

is a bounded family. Therefore, there is a finite union Y of irreducible components of $\mathrm{QSec}(f)$ such that $Y(k)$ coincides with S. By Lemma 1.1, the evaluation map $\phi|_{Y \times C} : Y \times C \to X$ doesn't dominate X. Thus, we get our proposition. □

§4 Remark for positive characteristic case

Let k be an algebraically closed field of characteristic $p > 0$. In [Sh], T. Shiota constructed a family of unirational hypersurfaces in $\mathbb{P}^3_k$. More precisely, letting $q = p^\nu$ ($\nu \geq 1$), he considered surfaces of degree $q+1$ in $\mathbb{P}^3_k$ defined by

$$x_1^q x_3 + x_2^q x_4 + x_1 f(x_3, x_4) + x_2 g(x_3, x_4) = 0,$$

where f and g are binary forms of degree q in x_3 and x_4 without common factors. By direct calculations, he checked that they are unirational. He also checked that the number of essential parameters of the above type surfaces is $2q-2$. Since they have ample canonical line bundles, a birational map between them is an isomorphism. Therefore this family is not birationally trivial. This observation shows us that Conjecture A does not hold in this example.

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