Adjoint actions on the modulo 5 homology groups of E_8 and ΩE_8

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1 Introduction

Borel proved in [2] that the integral homology group of the exceptional Lie group E_8 is not 5–torsion free and

$$H^*(E_8; \mathrm{Z}/5) \cong \Lambda(x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}) \otimes \mathrm{Z}/5[x_{12}]/({x_{12}}^5), \text{ with } |x_i| = i,$$

as algebra.

Araki showed the non-commutativity of the Pontrjagin ring $H_*(E_8; \mathrm{Z}/5)$ in [1]. The whole Hopf algebra structure and the cohomology operations were determined by Kono in [5]. But it was due to the partial computation of $\mathrm{Cotor}^{H^*(E_8;\mathrm{Z}/5)}(\mathrm{Z}/5, \mathrm{Z}/5)$, which was rather complicated. In [4], using secondary cohomology operations, Kane gave a general theorem to determine the Pontrjagin ring which is non-commutative and determined $H_*(E_8; \mathrm{Z}/5)$ as a Hopf algebra over $\mathcal{A}_5$.

In this paper we offer a more simple method for the determination of the coproduct and the cohomology operations on $H^*(E_8; \mathrm{Z}/5)$ using the adjoint actions of E_8 on ωE_8. We also determine $H_*(\omega E_8; \mathrm{Z}/5)$ as a Hopf algebra over $\mathcal{A}_5$ and the adjoint action of $H_*(E_8; \mathrm{Z}/5)$ on it.

This paper is organized as follows. In the next section we breifly see the algebra structures of $H^*(E_8;\mathrm{Z}/5)$ and $H_*(\Omega E_8;\mathrm{Z}/5)$ using the Serre spectral sequences. In the third section we determine the adjoint action of $H_*(E_8;\mathrm{Z}/5)$ on $H_*(\Omega E_8;\mathrm{Z}/5)$ which was introduced in [7]. It gives a easy computation of the Hopf algebra structures and the cohomology operations on them.

2 Algebra structures

Let $n(j)$, $(1 \leq j \leq 8)$, be the exponent of E_8, i.e.

$$\{n(j)\}_{1\leq j\leq 8} = \{1, 7, 11, 13, 17, 19, 23, 29\}.$$

First we see $H^*(\Omega E_8;\mathrm{Z}/5)$ for low dimensions. Let R be the algebra $\mathrm{Z}/5[a_{2n(j)}|1 \leq j \leq 8]$ with $|a_i| = i$. By Bott ([3]), the Hopf algebra $H^*(\Omega E_8;\mathrm{Z}/5)$ is isomorphic to R as a vector space. There is a map $q : SU(9) \to E_8$ which induces an isomorphism of π_3. Then, $\Omega q : \Omega SU(9) \to \Omega E_8$ induces an isomorhpism of π_2 and, as showed in [6], $(\Omega q)^* a_2 \in H^2(\Omega SU(9);\mathrm{Z}/5)$ is nontrivial and $((\Omega q)^* a_2)^5 \neq 0$ for the generator $a_2 \in H^2(\Omega E_8;\mathrm{Z}/5)$. Thus we have ${a_2}^5 \neq 0$. It follows that $H^*(\Omega E_8;\mathrm{Z}/5)$ is isomorphic to R for $* < 50$ as algebra. Next there is two possibilities (I) : ${a_2}^{25} \neq 0$ and (II) : ${a_2}^{25} = 0$. That is, we can assume it is isomorphic to (I) : R or (II) : $R/({a_2}^{25}) \otimes \mathrm{Z}/5[a_{50}]$, for $* < 10 \cdot n(2) = 70$, where $|a_{50}| = 50$.

Consider the following Serre fibre sequences :

$$\tilde{E}_8 \xrightarrow{k} E_8 \xrightarrow{\iota} K(\mathrm{Z}, 3), \tag{1}$$

$$K(\mathrm{Z}, 1) \longrightarrow \Omega\tilde{E}_8 \xrightarrow{\Omega k} \Omega E_8, \tag{2}$$

$$\Omega\tilde{E}_8 \longrightarrow * \longrightarrow \tilde{E}_8, \tag{3}$$

where ι induces an isomorphism of π_3.

Let $\tilde{R} \equiv \mathrm{Z}/5[\tilde{a}_{2n(i)}|2 \leq i \leq 8]$ with $|\tilde{a}_i| = i$. Computing the Serre spectral sequence associated to (2), we can see that, for $* < 70$, $H^*(\Omega\tilde{E}_8;\mathrm{Z}/5)$ is isomorphic to (I) : $\tilde{R}$ or (II) : $\tilde{R} \otimes \Lambda(\tilde{a}_{49}) \otimes \mathrm{Z}/5[a_{50}]$ according to the case : ${a_2}^{25} \neq 0$ or ${a_2}^{25} = 0$. Let $\tilde{S} \equiv \Lambda(\tilde{x}_{2n(j)+1}|2 \leq j \leq 8)$ with $|\tilde{x}_i| = i$. Again computing the spectral sequence associated to (3), we have, for $* < 71$, $H^*(\tilde{E}_8;\mathrm{Z}/5)$ is isomorphic to (I) : $\tilde{S}$ or (II) : $\tilde{S} \otimes \mathrm{Z}/5[\tilde{x}_{50}] \otimes \Lambda(\tilde{x}_{51})$ where $|\tilde{x}_{50}| = 50, |\tilde{x}_{51}| = 51$.

Recall the fact :

$$H^*(K(\mathrm{Z},3);\mathrm{Z}/5) \cong \Lambda(u_3,u_{11},u_{51},\cdots)\otimes \mathrm{Z}/5[u_{12},u_{52},\cdots],\ |u_i|=i, \qquad (4)$$

where $u_{11}=\mathcal{P}^1u_3, u_{12}=\beta u_{11}, u_{51}=\mathcal{P}^5u_{11}$ and $u_{52}=\beta u_{51}$.

Let $x_i = \iota^*(u_i)$, for $i = 11, 12, 51$ and 52, in $H^*(E_8;\mathrm{Z}/5)$. By the spectral sequence associated to (1), we obtain, for $* < 58$, $H^*(E_8;\mathrm{Z}/5) \cong$ (I) : $S\otimes\Lambda(x_{11},x_{51})\otimes\mathrm{Z}/5[x_{12},x_{52}]$ or (II) : $S\otimes\Lambda(x_{11})\otimes\mathrm{Z}/5[x_{12}]$, where $S\equiv\Lambda(x_{2n(j)+1}|1\le j\le 7)$ with $|x_i|=i$.

As $\dim E_8 = 248$, we can conclude that the possible case is (II) and ${x_{12}}^5=0$. Moreover, the generators $\{x_i\}$ are enough to generate $H^*(E_8;\mathrm{Z}/5)$. We have determined the algebra structure.

Therorem 1 *There is an algebra isomorphism :*

$$H^*(E_8;\mathrm{Z}/5)\cong\Lambda(x_{2n(j)+1}|1\le j\le 7)\otimes\Lambda(x_{11})\otimes\mathrm{Z}/5[x_{12}]/({x_{12}}^5).$$

In $H^*(\tilde{E}_8;\mathrm{Z}/5)$, we can chose $\tilde{x}_{50}$ and $\tilde{x}_{51}$ such that $\tau'\tilde{x}_{50}=u_{51}$ and $\tau'\tilde{x}_{51}=u_{52}$, where τ' is the transgression. Then $\tau'\mathcal{P}^1\tilde{x}_{51}=\mathcal{P}^1u_{52}=\mathcal{P}^1\beta\mathcal{P}^5u_{11}=\mathcal{P}^6\beta u_{11}=\mathcal{P}^6u_{12}={u_{12}}^5$. So we can chose $\tilde{x}_{59}$ as $\mathcal{P}^1\tilde{x}_{51}$. Thus we have

Proposition 2 *There is an isomorphism for $* < 71$:*

$$H^*(\tilde{E}_8;\mathrm{Z}/5)\cong\Lambda(\tilde{x}_{2n(j)+1}|2\le j\le 8)\otimes\mathrm{Z}/5[\tilde{x}_{50}]\otimes\Lambda(\tilde{x}_{51}),$$

and

$$\mathcal{P}^1(\tilde{x}_{51})=\tilde{x}_{59}.$$

Because that $\tilde{a}_i$ is transgressed to $\tilde{x}_{i+1}$ and $(\Omega k)^*a_i=\tilde{a}_i$ for $i=50,58$, the next proposition is obtained.

Proposition 3 *There are isomorphisms for $* < 70$:*

$$H^*(\Omega\tilde{E}_8;\mathrm{Z}/5)\cong\mathrm{Z}/5[\tilde{a}_{2n(j)}|2\le j\le 8]\otimes\Lambda(\tilde{a}_{49})\otimes\mathrm{Z}/5[\tilde{a}_{50}],$$

$$H^*(\Omega E_8;\mathrm{Z}/5)\cong\mathrm{Z}/5[a_{2n(j)}|2\le j\le 8]/({a_2}^{25})\otimes\mathrm{Z}/5[a_{50}],$$

with $\mathcal{P}^1(\tilde{a}_{50})\equiv\tilde{a}_{58}$ *and* $\mathcal{P}^1(a_{50})\equiv a_{58}$ *(modulo decomposable).*

By the use of the Rothenberg-Steenrod spectral sequence ([9]) :

$$E_2\cong H^{**}(H_*(\Omega E_8;\mathrm{Z}/5))\equiv\mathrm{Ext}_{H_*(\Omega E_8:\mathrm{Z}/5)}(\mathrm{Z}/5,\mathrm{Z}/5)\Rightarrow E_\infty=\mathcal{G}r(H^*(E_8;\mathrm{Z}/5)),$$

it is easily seen that

Therorem 4 *There is an algebra isomorphism :*

$$H_*(\Omega E_8; \mathrm{Z}/5) \cong \mathrm{Z}/5[t_{2n(j)} | 1 \leq j \leq 8]/({t_2}^5) \otimes \mathrm{Z}/5[t_{10}].$$

(Remark) The algebra was determined first in [8].

Let σ denote the homology suspension. Examining the spectaral sequence, we have the following.

Proposition 5 *$\sigma(t_{2n(j)}), (1 \leq j \leq 7)$, and $\sigma(t_{10})$ are nontirivial primitive elements in $H_*(E_8; \mathrm{Z}/5)$.*

3 Coproducts, cohomology operations and adjoint actions

Let $(\ )^*$ denote the dual as to the monomial basis of $\{x_i\}$ and put $y_i = (x_i)^*$.

We recall the adjoint action which was mentioned in [7]. Let $ad : G \times G \to G$ and $Ad : G \times \Omega G \to \Omega G$ be the adjoint actions for the Lie group G. Consider the induced maps of homlogy groups :

$$\begin{aligned} ad_* &: H_*(G) \otimes H_*(G) \to H_*(G), \\ Ad_* &: H_*(G) \otimes H_*(\Omega G) \to H_*(\Omega G). \end{aligned}$$

Put $y * y' = ad_*(y \otimes y')$ and $y \cdot t = yt = Ad_*(y \otimes t)$.

Our result is the following.

Therorem 6 *In $H_*(E_8; \mathrm{Z}/5)$, there are $y_{2n(j)+1}, (1 \leq j \leq 7)$, y_{11} and y_{12} satisfying that*

y_i	y_3	y_{11}	y_{12}	y_{15}	y_{23}	y_{27}	y_{35}	y_{39}	y_{47}
$y_{12} * y_i$	y_{15}	y_{23}	0	y_{27}	y_{35}	y_{39}	y_{47}	0	0
$\mathcal{P}^1_* y_i$	0	y_3	0	0	y_{15}	0	y_{27}	0	y_{39}
$\beta_* y_i$	0	0	y_{11}	0	0	0	0	0	0

All y_i are primitive and $y_{12} * y_i = [y_{12}, y_i]$.

(Remark) This result coincides with that of §46 - 2 of [4].

From now on, we prove this theorem combining the adjoint actions on $H^*(E_8; \mathrm{Z}/5)$ and $H^*(\Omega E_8; \mathrm{Z}/5)$.

Dualizing the properties of ad^* and Ad^* stated in [7], we have

Proposition 7 *For $y, y', y'' \in H_*(G)$ and $t, t', t'' \in H_*(\Omega G)$*

(1) $1 * y = y,\ 1 \cdot t = t.$

(2) $y * 1 = 0$ *and* $y \cdot 1 = 0$, *if* $|y| > 0.$

(3) $(yy')t = y(y't).$

(4) $y(tt') = \Sigma(-1)^{|y''||t|}(y't)(y''t')$, *where* $\Delta_* y = \Sigma y' \otimes y''$ *is the coproduct.*

(5) $\phi(y \cdot t) = \Delta_*(y) \cdot \phi(t)$, *where* ϕ *is the coproduct and* $(y' \otimes y'') \cdot (t' \otimes t'') = (-1)^{|y''||t'|}(y't' \otimes y''t'').$

(6) $\sigma(y \cdot t) = y * \sigma(t)$, *where* σ *is the homology suspension.*

(7) If y *is primitive then* $y * y' = [y, y']$, *where* $[y, y'] = yy' - (-1)^{|y||y'|}y'y.$

(8) If t *is primitive then* $y \cdot t$ *is also primitive.*

(9) $\mathcal{P}^n_*(y * y') = \Sigma_i \mathcal{P}^{n-i}_* y * \mathcal{P}^i_* y'$ *and* $\mathcal{P}^n_*(y \cdot t) = \Sigma_i \mathcal{P}^{n-i}_* y \cdot \mathcal{P}^i_* t.$

(Remark) In our case, $|t|$ and $|t'|$ are always even. So $y(tt') = \Sigma(y't)(y''t')$ and $(y' \otimes y'') \cdot (t' \otimes t'') = (y't' \otimes y''t'')$.

To state the non-commutativity of $H_*(E_8; \mathrm{Z}/5)$, we need only the fact :

Lemma 8 $[y_{12}, y_3] \neq 0.$

[Proof] Suppose that $[y_{12}, y_3] = 0$. Then $H_*(E_8; \mathrm{Z}/5) \cong \Lambda(y_3, y_{11}, y_{15}) \otimes \mathrm{Z}/5[y_{12}]$ for $* < 23$. Let $\{E'_r\}$ be the Rothenberg-Steenrod spectral sequece coversing to $H^*(BE_8; \mathrm{Z}/5)$. Then we have

$$E'_2 \cong \mathrm{Z}/5[s(y_3), s(y_{11}), s(y_{15})] \otimes \Lambda(s(y_{12}))$$

for total degree < 24. Since $E'_2 = E'_\infty$ in these degrees, there are indecomposable elements z_4, z_{12}, z_{16} and z_{13} in $H^*(BE_8; \mathrm{Z}/5)$ corresponding to $s(y_3), s(y_{11}), s(y_{15})$ and $s(y_{12})$, respectively. Especially, $z_4 z_{13} \neq 0$. It is a contradiction. (For detail, see Lemma 5.3 and 5.4 of [5].) ∎

Therefore $[y_{12}, y_3]$ is the nontrivial primitive element. So we may define y_{15} by that.

Proposition 9 $[y_{12}, y_3] = y_{15}$.

Since $\sigma(y_{12}t_2) = y_{12} * \sigma(t_2) = y_{12} * y_3 = [y_{12}, y_3] = y_{15}$, $y_{12}t_2$ is the indecomposable element. Thus we may assume that

$$t_{14} = y_{12}t_2. \tag{5}$$

Then t_{14} is primitive and $\sigma(t_{14}) = y_{15}$.

Let ϕ be the coproduct of $H_*(\Omega E_8; \mathrm{Z}/5)$ and $\overline{\phi}(t) = \phi(t) - t \otimes 1 - 1 \otimes t$. $(\ )^*$ denotes the dual as to the monomial basis of $\{t_{2j}\}$. Multiplying a_i and t_i by nonzero scalars or moving them modulo decomposable if we need, we may assume that $a_{2n(j)} = (t_{2n(j)})^*, (1 \leq j \leq 8)$, ${a_2}^5 = (t_{10})^*$ and $a_{50} = ({t_{10}}^5)^*$. As t_{10} is dual to ${a_2}^5$, it is easily verified that

$$\overline{\phi}(t_{10}) = 4{t_2}^4 \otimes t_2 + 3{t_2}^3 \otimes {t_2}^2 + 3{t_2}^2 \otimes {t_2}^3 + 4t_2 \otimes {t_2}^4. \tag{6}$$

$\mathcal{P}^1 a_2 = {a_2}^5$ implies $\mathcal{P}^1_* t_{10} = t_2$. Define t'_{22} by $y_{12}t_{10} - {t_2}^4 t_{14}$. Then by (6) and Proposition 7, $\overline{\phi}(t'_{22}) = \Delta^*(y_{12})\phi(t_{10}) - \phi(t_2)^4\phi(t_{14}) = t'_{22} \otimes 1 + 1 \otimes t'_{22}$. On the other hand, since $\mathcal{P}^1_* y_{12}$ and $\mathcal{P}^1_* t_{14}$ are trivial, $\mathcal{P}^1_* t'_{22} = y_{12}\mathcal{P}^1_* t_{10} = y_{12}t_2 = t_{14}$. So t'_{22} is nontrivial and primitive. Put $t_{22} = t'_{22}$. Now we obtain the following equations.

$$y_{12}t_{10} = t_{22} + {t_2}^4 t_{14}, \tag{7}$$

$$\mathcal{P}^1_* t_{22} = t_{14}. \tag{8}$$

Using Proposition 7 and ${y_{12}}^5 = 0$, we can compute ${y_{12}}^4 t_{22}$, that is,

$$\begin{aligned} {y_{12}}^4 t_{22} &= {y_{12}}^4(y_{12}t_{10} - {t_2}^4 t_{14}) \\ &= {y_{12}}^5 t_{10} - {y_{12}}^4({t_2}^4 t_{14}) \\ &= {y_{12}}^4(t_2^4 t_{14}) \end{aligned}$$

Here, since $y_{12}t_j$ $(j = 14, 26, 38)$ is primitive, there exists $\rho_j \in Z/5$ such that $y_{12}y_j = \rho_j t_{j+12}$, where $t_{50} = t_{10}^5$. Note that $y_{12}(t_{10}^5) = 0$. Therefore modulo the ideal $(t_{26}, t_{38}, t_{10}^5)$, we have

$$y_{12}^4(t_2^4 t_{14}) \equiv 4y_{12}^3(t_2^3 t_{14}^2) \equiv 12y_{12}^2(t_2^2 t_{14}^3) \equiv 24y_{12}(t_2 t_{14}^4) \equiv -t_{14}^5.$$

But, since $y_{12}^4 t_{22}$ is primitive, we obtain $y_{12}t_{22} = -t_{14}^5$. This means that ${y_{12}}^i t_{22}, (1 \leq i \leq 4)$, are nontrivial primitive elements. Therefore we can define the generators so that

$$t_{22+12i} = {y_{12}}^i t_{22}, (1 \leq i \leq 3). \tag{9}$$

Next we will observe ${y_{12}}^i t_{14}, (1 \leq i \leq 3)$. Since $\mathcal{P}^1_* t_{58}$ is primitive, there is $\epsilon \in \mathrm{Z}/5$ such that $\mathcal{P}^1_* t_{58} = \epsilon {t_{10}}^5$. On the other hand, from Proposition 3, $\mathcal{P}^1 a_{50} \equiv a_{58}$ (up to non zero coefficient and modulo decomposable). Dualize it, then we can see $\epsilon \neq 0$. Re-define t_{58} by $\epsilon^{-1} {y_{12}}^3 t_{22}$. We have

Proposition 10

$$ {y_{12}}^3 t_{22} = \epsilon t_{58}, \tag{10}$$

$$\mathcal{P}^1_* t_{58} = {t_{10}}^5. \tag{11}$$

From this, ${y_{12}}^3 t_{14} = {y_{12}}^3 \mathcal{P}^1_* t_{22} = \mathcal{P}^1_*({y_{12}}^3 t_{22}) = \mathcal{P}^1_*(\epsilon t_{58}) = \epsilon {t_{10}}^5$. So we can fix

$$t_{14+12i} = {y_{12}}^i t_{14}, \ (1 \leq i \leq 2). \tag{12}$$

By $\mathcal{P}^1_*({y_{12}}^i t_{2k}) = {y_{12}}^i \mathcal{P}^1_* t_{2k}$, $\mathcal{P}^1_*$ is determined on all t_{2k}.

We summarize the results.

Therorem 11 *In Therem 4, we can chose the generators satisfying the following table :*

t_{2j}	t_2	t_{10}	t_{14}	t_{22}	t_{26}	t_{34}	t_{38}	t_{46}	t_{58}
$y_{12}t_{2j}$	t_{14}	$t_{22} - {t_2}^4 t_{14}$	t_{26}	t_{34}	t_{38}	t_{46}	$\epsilon {t_{10}}^5$	t_{58}	$\epsilon^{-1} {t_{14}}^5$
$\mathcal{P}^1_* t_{2j}$	0	t_2	0	t_{14}	0	t_{26}	0	t_{38}	${t_{10}}^5$

All $t_{2k}, (k \neq 5)$ are primitive and

$$\overline{\phi}(t_{10}) = 4{t_2}^4 \otimes t_2 + 3{t_2}^3 \otimes {t_2}^2 + 3{t_2}^2 \otimes {t_2}^3 + 4t_2 \otimes {t_2}^4.$$

[Proof of Theorem 6] Put $y_{2n(j)+1} = \sigma(t_{2n(j)}), (3 \leq j \leq 7)$. Theorem 6 is an immediate consequence of Theorem 1, Theorem 4 and Proposition 5 with Proposition 7. ∎

Fix the basis of $H_*(E_8; \mathrm{Z}/5)$:

$$\{\Pi_{j=1}^7 {y_{2n(j)+1}}^{\epsilon_{2n(j)+1}} {y_{11}}^{\epsilon_{11}} {y_{12}}^e | 0 \leq \epsilon_i \leq 1, \ \ 0 \leq e < 5\}.$$

Let $(\ )^*$ be the dual with respect to the basis. We may assume that $x_{2n(j)+1} = (y_{2n(j)+1})^*$, $(2 \leq j \leq 7)$. Let φ be the coproduct of $H^*(E_8; \mathrm{Z}/5)$ and $\overline{\varphi}(x) = \varphi(x) - x \otimes 1 - 1 \otimes x$. Then the following theorem is easily obtained by dualizing Theorem 6.

Therorem 12 *In Theorem 1, we can chose the generators satisfying following tables:*

x_i	x_3	x_{11}	x_{12}	x_{15}	x_{23}	x_{27}	x_{35}	x_{39}	x_{47}
$\mathcal{P}^1 x_i$	x_{11}	0	0	x_{23}	0	x_{35}	0	x_{47}	0
βx_i	0	x_{12}	0	0	${x_{12}}^2/2$	0	${x_{12}}^3/3!$	0	${x_{12}}^4/4!$

x_i	$\overline{\varphi} x_i$
x_{15}	$x_{12} \otimes x_3$
x_{23}	$x_{12} \otimes x_{11}$
x_{27}	$x_{12} \otimes x_{15} + {x_{12}}^2/2 \otimes x_3$
x_{35}	$x_{12} \otimes x_{23} + {x_{12}}^2/2 \otimes x_{11}$
x_{39}	$x_{12} \otimes x_{27} + {x_{12}}^2/2 \otimes x_{15} + {x_{12}}^3/3! \otimes x_3$
x_{47}	$x_{12} \otimes x_{35} + {x_{12}}^2/2 \otimes x_{23} + {x_{12}}^3/3! \otimes x_{11}$

(Remark) In [5], $x_{2n(j)+1}$, $(4 \leq j \leq 7)$, are chosen as our $2x_{27}, 2x_{35}, 3!x_{39}$ and $3!x_{47}$ respectively.

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