A q-analog of the sixth Painlevé equation

Michio Jimbo and Hidetaka Sakai
Department of Mathematics, Faculty of Science,
Kyoto University, Kyoto 606, Japan.

July 1995

(Received: October 1995)

Abstract

A q-difference analog of the sixth Painlevé equation is presented. It arises as the condition for preserving the connection matrix of linear q-difference equations, in close analogy with the monodromy preserving deformation of linear differential equations. The continuous limit and special solutions in terms of q-hypergeometric functions are also discussed.

Mathematical Subject Classifications (1991): 17B37, 39A70, 81R50

1 Introduction

Recently the introduction of the singularity confinement method [1] has led to interesting developments in discrete integrable systems. Difference equations are usually given in the form of iterated rational mappings, whose poles may give rise to infinities at particular input values of solutions. The intriguing idea of the singularity confinement is to demand that after a finite iteration steps these infinities disappear and the memory of the input data be recovered. It can be viewed as a discrete counterpart of the Painlevé property. As is well known, the latter was the leading principle in the classification of the Painlevé equations. In the same spirit, the singularity confinement test has led to the discovery of difference analogs of several types of the Painlevé equations [2]. To our knowledge, difference versions of the Painlevé equations are known except for the sixth type P_{VI}.

Another important aspect of the Painlevé equations is their connection to monodromy preserving deformation of linear differential equations. Already in the classic paper of

Birkhoff [3], the generalized Riemann problem was studied for linear differential, difference and q-difference equations in parallel. An obvious next step would be to discuss the difference or q-difference version of the deformation theory. However we have been unable to find such an attempt in the literature. In the present article we report a simple non-trivial example of this problem. Namely we study the deformation of a 2×2 matrix system of linear q-difference equations analogous to the linear differential equaitons underlying P_{VI}. As a result we find a first order system of q-difference equations, which we call q-P_{VI} equation (see (19)–(20)). We shall also discuss some features of q-P_{VI}.

The text is organized as follows. In Section 2 we recall known results concerning the analytic theory of linear q-difference equations. In Section 3 we illustrate their deformations on the particular example mentioned above, and derive a linear q-difference system with respect to the deformation parameter. The compatibility condition between the original and the deformation equations is worked out in Section 4, where we find the q-P_{VI} equation. We show in Section 5 that it reduces in the continuous limit $q \to 1$ to a first order system equivalent to the original P_{VI}. In Section 6 we discuss special solutions given in terms q-hypergeometric functions, which exist for special choice of parameters. The final Section is devoted to discussions.

2 Linear q-difference systems

In this section we recall briefly the classical theory of linear q-difference equations [3] which will be used later. Throughout this article we fix a complex number q such that $0 < |q| < 1$.

Consider an $m \times m$ matrix system with polynomial coefficients

$$Y(qx) = A(x)Y(x), \qquad A(x) = A_0 + A_1 x + \cdots + A_N x^N. \tag{1}$$

More general case of a rational $A(x)$ can be reduced to this case by solving scalar q-difference equations. We assume A_0, A_N are semisimple and invertible. Denoting by θ_j, κ_j $(1 \le j \le m)$ the eigenvalues of A_0 and A_N respectively, we assume further that

$$\frac{\theta_j}{\theta_k}, \frac{\kappa_j}{\kappa_k} \notin \{q, q^2, q^3, \cdots\} \qquad (\forall j, k).$$

Set $A_0 = C_0 q^{D_0} C_0^{-1}$, $A_\infty = C_\infty q^{D_\infty} C_\infty^{-1}$, where $D_0 = \mathrm{diag}(\log \theta_j / \log q)$, $D_\infty = \mathrm{diag}(\log \kappa_j / \log q)$.

Proposition 1 ([3]) *Under the conditions above, there exist unique solutions $Y_0(x)$, $Y_\infty(x)$ of (1) with the following properties:*

$$Y_0(x) = \widehat{Y}_0(x) x^{D_0}, \tag{2}$$

$$Y_\infty(x) = q^{\frac{N}{2}u(u-1)} \widehat{Y}_\infty(x) x^{D_\infty}, \qquad u = \frac{\log x}{\log q}. \tag{3}$$

Here $\widehat{Y}_0(x)$ (resp. $\widehat{Y}_\infty(x)$) is a holomorphic and invertible matrix at $x = 0$ (resp. at $x = \infty$) such that $\widehat{Y}(0) = C_0$ (resp. $\widehat{Y}_\infty(\infty) = C_\infty$).

Let α_j $(j = 1, \cdots, mN)$ denote the zeroes of $\det A(x)$. The q-difference equation (1) entails that $\widehat{Y}_\infty(x)^{\pm 1}$, $\widehat{Y}_0(x)^{\pm 1}$ can be continued meromorphically in the domain $0 < |x| < \infty$. Moreover $\widehat{Y}_\infty(x)$ and $\widehat{Y}_0(x)^{-1}$ have no poles, while $\widehat{Y}_\infty(x)^{-1}$ and $\widehat{Y}_0(x)$ are holomorphic except for possible poles at

$$\widehat{Y}_\infty(x)^{-1} \quad : \quad q\alpha_j, q^2\alpha_j, q^3\alpha_j, \cdots, \tag{4}$$

$$\widehat{Y}_0(x) \quad : \quad \alpha_j, q^{-1}\alpha_j, q^{-2}\alpha_j, \cdots. \tag{5}$$

The connection matrix $P(x)$ is introduced by

$$Y_\infty(x) = Y_0(x)P(x). \tag{6}$$

Clearly $P(qx) = P(x)$. It is known to be expressible in terms of elliptic theta functions. It plays a role analogous to that of the monodromy matrices for differential equations.

3 Connection preserving deformation

In the theory of monodromy preserving deformation of linear differential equations, one introduces extra parameter(s) t in the coefficient matrix and demand that the monodromy stay constant with respect to t. Analogously, in the setting of q-difference equations, one demands that the connection matrix stay pseudo-constant in t, namely that $P(x, qt) = P(x, t)$. The natural candidate for the deformation parameters are the exponents θ_j, κ_j at $x = 0, \infty$ and the zeroes of $\det A(x)$. (Notice that, unlike the case of Fuchsian linear differential equations on $\mathbf{P}^1$, the points $x = 0, \infty$ play distinguished roles in the q-difference systems.) Under appropriate conditions it can be shown that $P(x, t)$ is pseudo-constant in t if and only if the corresponding solutions $Y(x, t) = Y_0(x, t), Y_\infty(x, t)$ satisfy

$$Y(x, qt) = B(x, t)Y(x, t), \tag{7}$$

where $B(x, t)$ is rational in x (see Proposition 2 below).

From now on, we will focus attention to the concrete example of a 2×2 system which is relevant to the q-P_{VI} equation. Recall that the linear system of differential equations associated with P_{VI} has the form [4]

$$\frac{d}{dx}Y(x) = \mathcal{A}(x)Y(x), \qquad \mathcal{A}(x) = \frac{\mathcal{A}_0}{x} + \frac{\mathcal{A}_1}{x-1} + \frac{\mathcal{A}_t}{x-t}.$$

If one naïvely replaces d/dx by the q-differentiation symbol $D_q = (1-q^{\vartheta})/(1-q)x$ ($\vartheta = xd/dx$) and multiplies through the denominator, one obtains a q-difference system (1) with

$$A(x) = (x-1)(x-t)\,(1-\epsilon x\mathcal{A}(x)) = A_0 + A_1 x + A_2 x^2 \qquad (\epsilon = 1-q). \tag{8}$$

Here $A_2 = I + \epsilon\mathcal{A}_\infty$ ($\mathcal{A}_\infty = -\mathcal{A}_0 - \mathcal{A}_1 - \mathcal{A}_t$) is independent of t, whereas $A_0 = t\,(I - \epsilon\mathcal{A}_0)$ is proportional to t. Since $\det A(0)$ is divisible by t^2, it is natural to assume that two of the zeroes of $\det A(x)$ are divisible by t.

Motivated by this observation, we now take $A(x,t)$ to be of the form

$$A(x,t) = A_0(t) + A_1(t)x + A_2 x^2, \tag{9}$$

$$A_2 = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \qquad A_0(t) \text{ has eigenvalues } t\theta_1, t\theta_2, \tag{10}$$

$$\det A(x,t) = \kappa_1\kappa_2(x-ta_1)(x-ta_2)(x-a_3)(x-a_4). \tag{11}$$

Here the parameters κ_j, θ_j, a_j are independent of t. Clearly we have

$$\kappa_1\kappa_2\prod_{j=1}^{4} a_j = \theta_1\theta_2.$$

In what follows we will normalize $Y_\infty(x)$ by $\widehat{Y}_\infty(\infty) = I$.

Proposition 2 *We have $P(x,qt) = P(x,t)$ if and only if (7) holds for $Y = Y_0, Y_\infty$, where $B(x,t)$ is a rational function of the form*

$$B(x,t) = \frac{x\,(xI + B_0(t))}{(x-qta_1)(x-qta_2)}. \tag{12}$$

Proof. From the definition (6), the connection matrix is pseudo-constant in t if and only if

$$B(x,t) \stackrel{\text{def}}{=} Y_\infty(x,qt)Y_\infty(x,t)^{-1} = Y_0(x,qt)Y_0(x,t)^{-1}.$$

Using (4),(5), we find that the only poles in $0 < |x| < \infty$ common to both sides are $x = qta_i$ ($i = 1,2$). Moreover (3) along with the normalization of $Y_\infty(x)$ imply that the left hand side behaves as $I + O(x^{-1})$ at $x = \infty$. Similarly (2) implies that the right hand side behaves like $O(x)$ at $x = 0$ (notice that D_0 is proportional to t). The proposition is an immediate consequence of these properties. □

4 Derivation of q-P_{VI}

The compatibility condition for the systems (1), (7) reads

$$A(x,qt)B(x,t) = B(qx,t)A(x,t) \tag{13}$$

where $A(x,t)$ and $B(x,t)$ are given respectively by (9) and (12). We will now work out the implications of (13) and find the q-P_{VI} equation.

Define $y = y(t)$, $z_i = z_i(t)$ $(i = 1, 2)$ by

$$A_{12}(y,t) = 0, \qquad A_{11}(y,t) = \kappa_1 z_1, \qquad A_{22}(y,t) = \kappa_2 z_2, \tag{14}$$

so that $z_1 z_2 = (y - ta_1)(y - ta_2)(y - a_3)(y - a_4)$. In terms of y, z_1, z_2 and (9)–(11), the matrix $A(x,t)$ can be parametrized as follows.

$$A(x,t) = \begin{pmatrix} \kappa_1((x-y)(x-\alpha) + z_1) & \kappa_2 w(x-y) \\ \kappa_1 w^{-1}(\gamma x + \delta) & \kappa_2((x-y)(x-\beta) + z_2) \end{pmatrix}.$$

Here

$$\begin{aligned}
\alpha &= \frac{1}{\kappa_1 - \kappa_2}[y^{-1}((\theta_1 + \theta_2)t - \kappa_1 z_1 - \kappa_2 z_2) - \kappa_2((a_1 + a_2)t + a_3 + a_4 - 2y)], \\
\beta &= \frac{1}{\kappa_1 - \kappa_2}[-y^{-1}((\theta_1 + \theta_2)t - \kappa_1 z_1 - \kappa_2 z_2) + \kappa_1((a_1 + a_2)t + a_3 + a_4 - 2y)], \\
\gamma &= z_1 + z_2 + (y + \alpha)(y + \beta) + (\alpha + \beta)y - a_1 a_2 t^2 - (a_1 + a_2)(a_3 + a_4)t - a_3 a_4, \\
\delta &= y^{-1}(a_1 a_2 a_3 a_4 t^2 - (\alpha y + z_1)(\beta y + z_2)).
\end{aligned}$$

The quantity $w = w(t)$ is related to the 'gauge' freedom, and does not enter the final result for the q-P_{VI} equation.

The compatibility (13) is equivalent to

$$A(qa_i t, qt)\,(qa_i t I + B_0(t)) = 0 \qquad (i = 1, 2) \tag{15}$$

$$(qa_i t I + B_0(t))\,A(a_i t, t) = 0 \qquad (i = 1, 2) \tag{16}$$

$$A_0(qt)B_0(t) = qB_0(t)A_0(t). \tag{17}$$

Substituting the parametrization above one obtains a set of q-difference equations among the quantities y, z_1,etc. We will not go into the details of the cumbersome calculation, but merely state the result.

Following [5] let us use the notations

$$\overline{y} = y(qt), \qquad \underline{y} = y(q^{-1}t)$$

and so forth. Introduce z by

$$z_1 = \frac{(y - ta_1)(y - ta_2)}{q\kappa_1 z}, \qquad z_2 = q\kappa_1(y - a_3)(y - a_4)z.$$

Then the matrix $B_0(t) = (B_{ij})$ is parametrized as follows:

$$B_{11} \;=\; \frac{-q\kappa_2\overline{z}}{1 - \kappa_2\overline{z}}\left(-\beta + \frac{t(a_1 + a_2) - y}{\kappa_2\overline{z}}\right),$$

$$
\begin{aligned}
B_{22} &= \frac{-q\kappa_1\overline{z}}{1-q\kappa_1\overline{z}}\left(-\overline{\alpha}+\frac{tq(a_1+a_2)-\overline{y}}{q\kappa_1\overline{z}}\right),\\
B_{12} &= \frac{q\kappa_2\overline{z}}{1-\kappa_2\overline{z}}w,\\
B_{21} &= \frac{q\kappa_1\overline{z}}{w(1-q\kappa_1\overline{z})}\left(tqa_1-\overline{\alpha}+\frac{tqa_2-\overline{y}}{q\kappa_1\overline{z}}\right)\left(ta_1-\beta+\frac{ta_2-y}{\kappa_2\overline{z}}\right)\\
&= \frac{q\kappa_1\overline{z}}{w(1-q\kappa_1\overline{z})}\left(tqa_2-\overline{\alpha}+\frac{tqa_1-\overline{y}}{q\kappa_1\overline{z}}\right)\left(ta_2-\beta+\frac{ta_1-y}{\kappa_2\overline{z}}\right).
\end{aligned}
$$

Set further

$$
b_1=\frac{a_1a_2}{\theta_1},\quad b_2=\frac{a_1a_2}{\theta_2},\quad b_3=\frac{1}{q\kappa_1},\quad b_4=\frac{1}{\kappa_2}. \tag{18}
$$

Theorem 3 *The equations (15)–(17) are equivalent to*

$$
\frac{y\overline{y}}{a_3a_4} = \frac{(\overline{z}-tb_1)(\overline{z}-tb_2)}{(\overline{z}-b_3)(\overline{z}-b_4)}, \tag{19}
$$

$$
\frac{z\overline{z}}{b_3b_4} = \frac{(y-ta_1)(y-ta_2)}{(y-a_3)(y-a_4)}, \tag{20}
$$

$$
\frac{\overline{w}}{w} = \frac{b_4}{b_3}.\frac{\overline{z}-b_3}{\overline{z}-b_4}. \tag{21}
$$

Here b_j's are given by (18). We have a single constraint

$$
\frac{b_1b_2}{b_3b_4}=q\frac{a_1a_2}{a_3a_4}.
$$

We call (19)–(20) q-P_{VI} system, or simply q-P_{VI} equation. Note that the number of parameters can be reduced to 4 by rescaling y, z, t; e.g. one can choose $a_1a_2=1, a_3a_4=1, b_1b_2=q, b_3b_4=1$. Written in the first order form, the map $(y,z)\mapsto(\overline{y},\overline{z})$ is birational. Upon elimination of z, however, $\overline{y}$ becomes double-valued as a function of y and $\underline{y}$.

One can verify without difficulty that the q-P_{VI} system (19)–(20) possesses the singularity confinement property in the sense of [1, 2]. At this moment we do not know how it is related to the other discrete P_V–P_I equations (see [2] and references therein).

Remark. It is worth mentioning that the q-P_{III} equation [2] has a very similar form

$$
\frac{\underline{w}\overline{w}}{a_3a_4}=\frac{(w-sa_1)(w-sa_2)}{(w-a_3)(w-a_4)}
$$

where $\overline{w}(s)=w(ps)$, $\underline{w}(s)=w(p^{-1}s)$, and $a_1,\cdots,a_4$ are arbitrary parameters. This can be obtained as a special case of the q-P_{VI} system by setting $t=s^2$, $q=p^2$, $b_i=pa_i$ $(i=1,2)$, $b_i=a_i$ $(i=3,4)$ and identifying $y(t)=\overline{w}(s)$, $z(t)=w(s)$. The linear problem given in [5, 7] also falls within the present framework. To see this, set $Y=D\Phi$ with $D=\mathrm{diag}(1,1,h^2,h^2)$

in the notation of [7], and rename the parameters q^2, x^2 and h^2 there by p, s and x. The linear system for q-P_{III} then takes the form (1), (7) with

$$A(x,s) = \frac{1}{x}A_0(s) + A_1(s), \qquad B(x,s) = \frac{1}{x}B_0(s) + B_1(s),$$

where A_j, B_j are 4×4 matrices given as follows.

$$A_0 = \begin{pmatrix} & & \alpha & 0 \\ & & p^{-1}\underline{\beta} + \tau/s & p^{-1}\underline{\beta} \\ \beta & 0 & & \\ p^{-1}\underline{\alpha} + \tau/s & p^{-1}\underline{\alpha} & & \end{pmatrix}, \qquad A_1 = \begin{pmatrix} \kappa & \kappa+\alpha & & \\ 0 & \tau/s & & \\ & & \xi & \xi+\beta \\ & & 0 & \tau/s \end{pmatrix},$$

$$B_0 = \begin{pmatrix} & & 0 & 0 \\ & & 1 & 0 \\ 0 & 0 & & \\ 1 & 0 & & \end{pmatrix}, \qquad B_1 = \begin{pmatrix} \dfrac{\kappa - \tau/s}{\alpha + \tau/s} & \dfrac{\kappa+\alpha}{\tau/s+\alpha} & & \\ 0 & 0 & & \\ & & \dfrac{\xi - \tau/s}{\beta + \tau/s} & \dfrac{\xi+\beta}{\tau/s+\beta} \\ & & 0 & 0 \end{pmatrix}.$$

Here $\kappa = -pa_1a_4/a_3$, $\xi = -pa_2$, $\tau = -pa_4$, $\alpha = pa_1a_4/w$, $\beta = pw/s$. The eigenvalues of A_i are

$$\begin{aligned} A_0 &: \pm cs^{-1/2}, \pm cp^{1/2}s^{-1/2} \qquad (c = p\sqrt{a_1a_4}) \\ A_1 &: \kappa, \xi, \tau s^{-1}, \tau s^{-1}, \end{aligned}$$

and $\det A(x,s)$ has zeroes at $x^2 = 1, -pa_2/a_1a_3$. Note that in this example the exponents at $x = 0, \infty$ are moving with respect to s while the zeroes of $\det A(x,s)$ are fixed. □

5 Continuous limit

From the construction one expects that in the continuous limit the q-P_{VI} equation reduces to the P_{VI} differential equation. Here by continuous limit we mean the limit $\epsilon = 1 - q \to 0$. In view of the relation (8) it is natural to set

$$\kappa_i = 1 + \epsilon K_i, \quad \theta_i = 1 - \epsilon\Theta_i, \quad a_i = 1 + \epsilon\alpha_i.$$

Note that

$$(1+\epsilon K_1)(1+\epsilon K_2)(1+\epsilon\alpha_1)(1+\epsilon\alpha_2)(1+\epsilon\alpha_3)(1+\epsilon\alpha_4) = (1-\epsilon\Theta_1)(1-\epsilon\Theta_2).$$

Redefinig $\mathbf{y} = y$ and $\mathbf{z}$ by

$$z_1 = \frac{1}{\kappa_1}(y-t)(y-1)(1-\epsilon y\mathbf{z})$$

we find

$$\frac{d\mathbf{y}}{dt} = \frac{\mathbf{y}(\mathbf{y}-1)(\mathbf{y}-t)}{t(t-1)}\left(2\mathbf{z} - \frac{\Theta_1+\Theta_2}{\mathbf{y}} - \frac{\alpha_3+\alpha_4}{\mathbf{y}-1} - \frac{\alpha_1+\alpha_2-1}{\mathbf{y}-t}\right) \tag{22}$$

$$\begin{aligned}\frac{d\mathbf{z}}{dt} = {} & \frac{-3\mathbf{y}^2+2(t+1)\mathbf{y}-t}{t(t-1)}\mathbf{z}^2 \\ & + \frac{(2\mathbf{y}-t-1)(\Theta_1+\Theta_2)+(2\mathbf{y}-1)(\alpha_1+\alpha_2-1)+(2\mathbf{y}-t)(\alpha_3+\alpha_4)}{t(t-1)}\mathbf{z} \\ & - \frac{K_1(K_2+1)}{t(t-1)} + \frac{\Theta_1\Theta_2}{(t-1)\mathbf{y}^2} + \frac{\alpha_1\alpha_2}{(\mathbf{y}-t)^2} - \frac{\alpha_3\alpha_4}{t(\mathbf{y}-1)^2}.\end{aligned} \tag{23}$$

This is a first order system equivalent to the P_{VI} differential equation.

6 Special solutions

At particular values of parameters, the system (19)–(20) decouples into a pair of q-Riccati equations, in exactly the same way as do the other discrete Painlevé equations [5, 6, 7]. Namely, assume that

$$\frac{b_1}{b_3} = q\frac{a_1}{a_3}, \qquad \frac{b_2}{b_4} = \frac{a_2}{a_4}.$$

Then (19), (20) are satisfied if

$$\overline{y} = a_3\frac{\overline{z}-tb_1}{\overline{z}-b_3}, \qquad \overline{z} = b_4\frac{y-ta_2}{y-a_4}.$$

The latter can be linearized in the standard way. Let

$$t = \frac{b_3}{b_1}s, \quad a = \frac{a_3}{a_4}, \quad b = \frac{a_2}{a_4}\frac{b_4}{b_1}, \quad c = \frac{a_3}{a_4}\frac{b_4}{b_3}.$$

Then

$$y = a_3\frac{u}{v}, \qquad \overline{z} = b_4\frac{u-(bs/c)v}{u-v/a}$$

where $u = u(s), v = v(s)$ are solutions of

$$\overline{u} = \frac{1}{1-(ab/c)s}\left((1-\frac{a}{c}s)u + \frac{1-b}{c}sv\right), \tag{24}$$

$$\overline{v} = \frac{1}{1-(ab/c)s}\left((1-\frac{a}{c})u + \frac{(1-bs)}{c}v\right). \tag{25}$$

In particular the q-hypergeometric functions

$$u = {}_2\phi_1\left(\begin{matrix}a & b\\ & c\end{matrix}; q, s\right), \qquad v = \frac{c-a}{c-1}\,{}_2\phi_1\left(\begin{matrix}a & qb\\ & qc\end{matrix}; q, s\right)$$

solve (24)–(25).

7 Discussions

In this note we studied a deformation of a linear q-difference system, which led to the q-P_{VI} equation. The argument presented here has a very general character. We feel the subject warrants further investigation to develop a general theory of deformation in the difference/q-difference setting, including τ-functions, symplectic structure, Schlesinger transforms and symmetries. Another interesting problem is to explore an analog of Okamoto's space of initial conditions [8], in connection with the affine Weyl group symmetry of the Painlevé equaitons [9]. This might shed light to the geometric meaning of the singularity confinement.

Acknowledgements. We wish to thank K. Aomoto, B. Grammaticos, K. Kajiwara, J. Matsuzawa, A. Ramani, and N. Yu. Reshetikhin for discussions and interest. This work is partly supported by Grant-in-Aid for Scientific Research on Priority Areas 231, the Ministry of Education, Science and Culture.

References

[1] B.Grammaticos, A.Ramani and V.G.Papageorgiou, Do integrable mappings have the Painlevé property?, *Phys. Rev. Lett.*,**67** (1991), 1825–1828.

[2] A.Ramani, B.Grammaticos and J.Hietarinta, Discrete versions of the Painlevé equations, *Phys. Rev. Lett.*,**67** (1991), 1829–1832.

[3] G.D.Birkhoff, The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations, *Proc.Am.Acad.Arts and Sciences*, **49** (1914) 521–568.

[4] M.Jimbo and T.Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II, *Physica* **2D** (1981) 407–448.

[5] B.Grammaticos, F.W.Nijhoff, V.G.Papageorgiou, A.Ramani and J.Satsuma, Linearization and solutions of the discrete Painlevé III equation, *Phys. Lett. A*,**185** (1994), 446–452.

[6] V.G.Papageorgiou, F.W.Nijhoff, B.Grammaticos and A.Ramani, Isomonodromic deformation problems for discrete analogues of Painlevé equations, *Phys. Lett. A*,**164** (1992), 57–64.

[7] K. Kajiwara, Y. Ohta and J. Satsuma, Casorati determinant solutions for the discrete Painlevé III equation *preprint* , 1994.

[8] K. Okamoto, Sur les feuilletages associés aux équation du seconde ordre à points crtiques fixes de P. Painlevé *Japan. J. Math.***5** (1979), 1–79.

[9] K. Okamoto, Studies on the Painlevé equations I. Sixth Painlevé equation P_{VI}, *Annali di Matematica pura ed applicata***CXLVI** (1987), 337–381.