Symplectic S-Cobodism Conjecture

- summary -

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§0 h and S Cobordism Theorems - review -

In this article the author would like to propose a natural generalization of celebrated h and S coboridism theorems to loop space. We will explain some part of the idea of the proof of an (easier) half of it, and also a finite dimensional analogue of the conjecture which can be proved under some additional assumptions. The detailed proof of them will appear elesewhere.

Let us start by reviewing famous results due to Smale etc. in differential topology. Let X be an oriented manifold with boundary such that $\partial X = M_- \cup -M_+$.

Definition 0.1. X is said to be an *h-coborsidm* if $\pi_1 M_- \simeq \pi_1 M_+ \simeq \pi_1 X$ and $H_*(X, M_-; \mathbf{Z}) = 0$.

Theorem 0.2. (h-cobordism, Smale, see [13]) *If X is an h-cobodism and* $\dim X \geq 6$, $\pi_1 X = \pi_1 M_\pm = 1$, *then X is diffeomorphic to $M_- \times [0,1]$.*

In the case when there is a fundamental group, there is one more obstruction that is the Whitehead torsion. Let us recall the definition of Whitehead group $wh^1(\pi_1 M)$. (See [14],[15] for detail.)

Let $GL(n, \Lambda)$ be the group of invertible $n \times n$ matrix with coefficient in the ring $\Lambda = \mathbf{Z}[\pi_1 M]$. We consider its subgroup $E^+(n)$ generated by matrices such as

(a)
$$\begin{pmatrix} 1 & & & & \\ & \ddots & & a_{i,j} & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}$$

(b)
$$\begin{pmatrix} 1 & & & & \\ & \ddots & & 0 & \\ & & a_i & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}$$

where a_i denotes an elements of $\pi_1 M$. Then the Whitehead group is by definition the inductive limit

$$\lim_{n \to \infty} \frac{GL(n, \Lambda)}{E^+(n)}.$$

Now we recall :

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Theorem 0.3. (S-cobordism, Barden-Mazur-Stalling, see [11]) *Let X be an h-cobordism. Then an invariant $\tau(X) \in wh^1(\pi_1 M)$, the Whitehead torsion, is defined and satisfies the following.*

(1) *If X is diffeomorphic to $M_- \times [0,1]$ then $\tau(X) = 0$.*
(2) *If $\tau(X) = 0$ and if $\dim X \geq 5$ then X is diffeomorphic to $M_- \times [0,1]$.*

Remark 0.4. It is also proved that for each $\tau \in wh^1(\pi_1 M)$ there exists an h-cobordism whose Whitehead torsion is equal to τ.

We recall that the main point of the proof of these theorems is to construct a function $f : X \to [0,1]$ such that f has no critical point and that $f^{-1}(0) = M_-$, $f^{-1}(1) = M_+$. These ideas were used furthermore to study pseudo isotopy of smooth manifolds. Let us put

$$C^\infty(X;0,1) = \{f : X \to [0,1] | f^{-1}(0) = M_-, f^{-1}(1) = M_+\}$$

$$\mathfrak{E} = \{f \in C^\infty(X;0,1) |\ f \text{ has no critical point}\}.$$

Theorem 0.5. *Let $X = M \times [0,1]$.*

(1) (Cerf [3]) *If $\pi_1 X = 1$ $\dim X \leq 5$ then $H_0(\mathfrak{E}) = 0$.*
(2) (Hatcher-Wagoner [9]) *If $\dim X \leq 7$, then*

$$H_0(\mathfrak{E}) \simeq wh^2(\pi_1 X) \oplus wh^1(\pi_1 X \times \mathbf{Z}_2)$$

See [9] for the definition of second Whitehead group. Eliashberg and Gromov developed symplectic analogue of pseudo isotopy theory for Lagrangian intersection in cotangent bundle by using generating functions. Here we are going to discuss symplectic analogue of h and S cobordism theorems by using Floer homology.

§1 The problem, earlier works and an example

We consider a symplectic manifold (X, ω) and one of the following situations.

(Case 1) We have a symplectic diffeomophism $\varphi : X \to X$. We assume that φ induces a trivial map on fundamental groups.

(Case 2) We suppose that we have two Lagrangian submanifolds L_1 and L_2 of X such that $\pi_1 L_i$ injects to $\pi_1 X$ and $\pi_1 L_1 = \pi_1 L_2$ in $\pi_1 X$.

We assume furthermore $\pi_2 X = 0$ in both cases. (One can replace this assumption by semi positivity of symplectic structure by using an argument in [10],[18].) We recall the following:

Definition 1.1. Let $H : X \times [0,1] \to \mathbf{R}$ be a function. The family of hamiltonian diffeomorphisms ψ_t associated to it is defined by the differential equation :

$$\begin{cases} \dfrac{d\psi_t}{dt} = X_{H(\cdot,t)} \\ \psi_0(x) = x \end{cases}$$

Here $X_{H(\cdot,t)}$ is the hamiltonian vector field which is defined by

$$\omega(X_{H(\cdot,t)}, V) = (dH(\cdot,t))(V).$$

We say that ψ is an exact symplectic diffeomorphism if there exists ψ_t as above such that $\psi_1 = \psi$.

Now in this article we concern with the following problem.

Problem 1.2.
(Case 1) When does there exist an exact symplectic diffemorphism ψ such that $\psi\varphi$ has no fixed point ?
(Case 2) When does there exist an exact symplectic diffemorphism ψ such that $\psi(L_1) \cap L_2$ is empty?

Both problems were studied by A. Floer by using Floer homology, in the case when X or L_i is simply connected.

Theorem 1.3. (Floer [5],[6])
(Case 1) Assume $\pi_1 X = 1$. Then one can define a Floer homology $HF(X, \varphi)$ such that $HF(X, \psi\varphi) \simeq HF(X, \varphi)$ for an exact symplectic diffeomorphism ψ, and that $HF(X, \varphi) = 0$ if φ has no fixed point.
(Case 2) Assume $\pi_1 L_i = 1$. Then one can define a Floer homology $HF(L_1, L_2)$, such that $HF(\psi L_1, L_2) \simeq HF(L_1, L_2)$ for an exact symplectic diffeomorphism ψ and $HF(\psi L_1, L_2) = 0$ if $L_1 \cap L_2 = \emptyset$.

Floer also gave a calculation of Floer homology in the case when φ is identity and $\Lambda_2 = \psi\Lambda_1$ for an exact symplectic diffeomorphism ψ. Namely

Theorem 1.4. (Floer [5],[6]) *Under the assumption of Theorem 1.3 we have :*

(Case 1) $HF(X, id) = H(X)$.
(Case 2) $HF(L_1, \psi(L_1)) = H(L_1)$, *if ψ is an exact symplectic diffeomorphism.*

Combining them, he found that there is no exact symplectic diffeomorphism ψ such that ψ has no fixed point (case 1), $\psi(L_1) \cap L_2 =$ (case 2), under the assumption of Theorem 1.3. More precise result (the estimate of the number of fixed points) is the celebrated conjecture by Arnold [1], which are studied by many mathematicians.

Theorem 1.4 (and more precise result on Arnold conjecture) was generalized by Hofer-Salamon[10],Ono[18],Lê-Ono[12] in Case 1 and by Oh[17] in Case 2. (Here we mention only the works using Floer homology.)

We now discuss the nonsimply connected case and shows by an example that the problem is more delicate in that case. Let us consider a two dimensional torus T^2 and two circles L_1,L_2 in it as in Figure 1.5.

*Figure*1.5.

They intersect to each other at two points p, q. Let A,B be the areas in Figure 1.5. So one can prove the following proposition by an elementary argument.

Proposition 1.6. *There exists an exact symplectic diffeomorphism $\psi : T^2 \to T^2$ such that $\psi(L_1) \cap L_2 = \emptyset$ if and only if $A \neq B$.*

This example shows that the answer to Problem 1.2 in not invariant under deformation of Lagrangian submanifold. We remark here that any deformations of Lagrangian submanifolds are realized by an exact symplectic diffeomorhpism, if and only if the Lagrangian submanifold is simply connected.

§2 Floer homology for nonsimply connected manifolds

To deal with Floer homology for nonsimply connected manifolds, we need to use Novikov's idea on Morse theory of multi-valued functions [16]. Novikov's idea was used in symplectic Floer homology by [10],[18],[19] to deal with π_2 of symplectic manifold. Here we use a similar idea but to deal with π_1. Let us construct a homomorphisms

$$a : \pi_1 X \to \mathbf{R}$$

in case 1 and

$$a : \pi_1 L_1 \to \mathbf{R}$$

in case 2. Their definitions are as follows. Let $\ell \in \pi_1 X$. Then, by assumption that $\varphi(\ell)$ is homotopic to ℓ, there exists an annulus A which bounds $\ell \cup -\varphi(\ell)$. We put

$$a(\ell) = \int_A \omega \tag{2.1}$$

Using the fact that $\pi_2 X = 1$, we can prove that (2.1) is invariant of the choice of the annulus and depends only on the homotopy class of ℓ.

In case 2, let $\ell \in \pi_1(L_1)$. By assumption, we find ℓ' in L_2, which is homototic to ℓ in X. We then find an annulus A which bounds $\ell \cup -\ell'$. Then $a(\ell)$ is defined by Formula (2.1).

Using this homomorphism a, we define Novikov ring Λ as follows. Λ is the set of all (infinite) sum

$$\sum_{\ell \in \pi_1 X} c_\ell [\ell]$$

such that, for each C, we have

$$\sharp \{\ell | c_\ell \neq 0, \ \ a(\ell) < C\} < \infty.$$

Let us define the product between two elements of Λ. Let $\sum_{\ell \in \pi_1 X} c_\ell [\ell]$ and $\sum_{\ell \in \pi_1 X} c'_\ell [\ell]$ be two elements of Λ. By assumption, there exists K such that

$$c_\ell = c'_\ell = 0$$

if $a(\ell) < K$. Now we put

$$b_\ell = \sum_{\ell_1 \cdot \ell_2 = \ell} c_{\ell_1} c_{\ell_2}$$

By the above remark, we find that $c_{\ell_1} c_{\ell_2} = 0$ unless $a(c_{\ell_1}) < a(c_\ell) - K$ and $a(c_{\ell_2}) < a(c_\ell) - K$. Thus the sum is, in fact, a finite sum and is well defined.

Next let C be an arbitrary number. Then if $\ell_1 \cdot \ell_2 = \ell$, $b_\ell \neq 0$ and $a(\ell) < C$, we have $a(\ell_1) < C - K$, $a(\ell_2) < C - K$. Among those ℓ_1, ℓ_2, the coefficients c_{ℓ_1} and c'_{ℓ_2} are nonzero only for finitely many choices. Therefore b_ℓ is nonzero for only finitely many ℓ with $a(\ell) < C$. Namely

$$\sharp\{\ell | b_\ell \neq 0, \ \ a(\ell) < C\} < \infty.$$

Thus $\sum_{\ell \in \pi_1 X} b_\ell[\ell]$ is an element of Λ. We let this element the product of $\sum_{\ell \in \pi_1 X} c_\ell[\ell]$ and $\sum_{\ell \in \pi_1 X} c'_\ell[\ell]$.

In case 2, we use $\pi_1(L_1)$ in place of $\pi_1 X$ and define Λ in the same way.

Example 2.2. Let us consider the case of two Lanrangians in the torus in Figure 1.5. Then $\pi_1 L_1 = \mathbf{Z}$ and the homomorphism a is $a(k) = k(B - A)$. Therefore

$$\Lambda = \begin{cases} \mathbf{Z}[t, t^{-1}] & \text{if } A = B \\ \mathbf{Z}[t][[t^{-1}]] & \text{if } A < B \\ \mathbf{Z}[t^{-1}][[t]] & \text{if } A > B \end{cases}$$

Thus the change of the Novikov ring reflects the change of answer to Problem 1.2 by the deformation of Lagrangian submanifold.

Now we are going to define Floer homology for nonsimply connected manifolds. We choose an almost complex structure $J : TX \to TX$ which is compatible to ω. (Namely we assume that $g(V, W) = \omega(V, JW)$ gives a Riemannian metric on X.) We also assume transversality. Namely, in case 1, we assume that the Jacobian of φ, at each fixed point, do not have 1 as eigenvalue and in case 2 we assume that L_1 and L_2 are transversal. Now we define a moduli space of pseudo holomorhic disks as follows. We discuss here only case 2. (The definition in case 1 is similar to case 2 and is also similar to simply connected case due to Floer and others.)

Let $p, q \in L_1 \cap L_2$. We put

$$\widetilde{\mathfrak{M}}(p, q) = \left\{ u : D^2 \to X \;\middle|\; \begin{array}{l} u_* J = J u_*, \\ u(-1) = p, \ \ u(1) = q, \\ u(\partial_1 D^2) \subset L_1 \\ u(\partial_2 D^2) \subset L_2 \end{array} \right\}$$

Here D^2 is the set of complex number z with $|z| \leq 1$. $\partial_1 D^2$ is set of complex number z with $|z| = 1$, $\operatorname{Im} z \geq 0$, and $\partial_2 D^2$ is set of complex number z with $|z| = 1$, $\operatorname{Im} z \leq 0$. There is an action of the group $\mathbf{R} = \operatorname{Aut}(D^2, -1, +1)$ on $\widetilde{\mathfrak{M}}(p, q)$. Let $\mathfrak{M}(p, q)$ be the quotient space. One can prove the following lemma in the same way as [5].

Lemma 2.3. *After generic perturbation, the set $\widetilde{\mathfrak{M}}(p, q)$ is a smooth manifold. There is a map $\mu : L_1 \cap L_2 \to \mathbf{Z}$ such that*

$$\dim \widetilde{\mathfrak{M}}(p, q) = \mu(p) - \mu(q)$$

So far our discussion is the same as the simply connected case. However we need to modify the argument when we study the compactness of the moduli space $\widetilde{\mathfrak{M}}(p,q)$. The point is that, in contrast to the simply connected case, the symplectic volume of the element u of $\widetilde{\mathfrak{M}}(p,q)$ depends on u. So we divide $\widetilde{\mathfrak{M}}(p,q)$, as follows. We fix $p_0 \in L_1$ and for each $p \in L_1 \cap L_2$ we fix a path ℓ_p joining p_0 to p. For each element u of $\widetilde{\mathfrak{M}}(p,q)$ we obtain a path $u(\partial_1 D^2)$ joining p to q. Then for each $\ell \in \pi_1(L_1)$ we define

$$\widetilde{\mathfrak{M}}_\ell(p,q) = \{u \in \widetilde{\mathfrak{M}}(p,q) | [\ell_q^{-1} \circ u(\partial_1 D^2) \circ \ell_p] = [\ell]\}$$

We define $\mathfrak{M}_\ell(p,q)$ in a similar way.

Lemma 2.4. *If $\mu(p) = \mu(q)+1$ then $\mathfrak{M}_\ell(p,q)$ is a finite set. Furthermore, for each C, the number of ℓ for which $\mathfrak{M}_\ell(p,q)$ is nonempty and $a(\ell) < C$ is finite.*

The main point of the proof of Lemma 2.4 is that if $u \in \mathfrak{M}_\ell(p,q)$ then the symplectic area of u is smaller than some constant plus $a(\ell)$. We then can use Gromov's compactness theorem [8] of the moduli space of pseudo holomorphic curves with bounded symplectic area and its variants to prove the finiteness in the lemma. Now we are ready to define Floer homology in the nonsimply connected case.

Definition 2.5.

$$CF_k(L_1,L_2) = \bigoplus_{p \in L_1 \cap L_2, \mu(p)=k} \Lambda[p]$$

$$\partial_k : CF_k(L_1,L_2) \to CF_{k-1}(L_1,L_2)$$

$$\partial([p]) = \sum_{\ell \in \pi_1 L_1} \sum_q \sharp \mathfrak{M}_\ell(p,q)[\ell][q]$$

Here $\sharp$ is the number counted with sign. So we need to define orientation on $\mathfrak{M}_\ell(p,q)$, which are omitted here. Lemma 2.4 implies that ∂ is well defined.

In a way similar to the usual Floer homology, we have :

Theorem 2.6. $\partial\partial = 0$.

Definition 2.7.

$$HF_k(L_1,L_2) = \frac{\operatorname{Ker} \partial_k}{\operatorname{Im} \partial_{k+1}}$$

Again, following the argument in the simply connected case, we can prove the following theorem.

Theorem 2.8. *If ψ is an exact symplectic diffeomorphism then*

$$HF(L_1,L_2) = HF(\psi L_1, L_2)$$

Corollary 2.9. *If $HF(L_1, L_2) \neq 0$ then there exists no exact symplectic diffeomorphism ψ such that $\psi L_1 \cap L_2 = \emptyset$.*

Let us consider again an example in Figure 1.5. It is easy to see that boundary operator ∂ is given by $\partial[p] = (1+t)[q]$. On the other hand, we find that $1+t$ is invertible in Novikov ring if and only if $A \neq B$. Therefore, in the case of the example in Figure 1.5, we find that $HF(L_1, L_2) = 0$ *if and only if* there exists an exact symplectic diffeomorphism ψ such that $\psi L_1 \cap L_2 = \emptyset$.

Now it is natural to ask whether this is true or not in general. As we will see in next section, the answer is negative.

At the end of this section, we remark that the same construction works in case 1.

§3 Whitehead Torsion and Floer homology

We again consider the situation of case 2. We also assume that $HF(L_1, L_2) = 0$. We modify the definition of Whitehead group as follows. We consider the group $E^{++}(n)(\Lambda)$ which is generated by the matrix (a),(b) in the definition of the Whitehead group. But we assume that a_i in (b) is written as

$$a_i = \sum_{\ell} c_\ell [\ell]$$

and there exists ℓ_0 such that $c_{\ell_0} = 1$ and $c_\ell = 0$ for all ℓ with $a(\ell) \leq a(\ell_0)$. (Namely the elements whose reading term is $[\ell]$.) Then we put

$$wh^1_+(\Lambda) = \lim_{n \to \infty} \frac{GL(n, \Lambda)}{E^{++}(n)}.$$

Now our main result is as follows :

Theorem 3.1. *There exists a Whitehead Torsion $\tau(L_1, L_2) \in wh^1_+(\Lambda)$ such that*

(1) $\tau(\psi L_1, L_2) = \tau(L_1, L_2)$ *for an exact symplectic diffeomorphism ψ.*
(2) $\tau(L_1, L_2) = 0$ *if $L_1 \cap L_2 = \emptyset$.*

Corollary 3.2. *If $\tau(L_1, L_2) \neq 0$ then there exist no exact symplectic diffeomorphism ψ such that $\psi L_1 \cap L_2 = \emptyset$.*

Symplectic S-Cobordism Conjecture 3.3. If $HF(L_1, L_2) = 0$ and $\tau(L_1, L_2) = 0$, then there exists an exact symplectic diffeomorphism ψ such that $\psi L_1 \cap L_2 = \emptyset$.

Let us give an example of Theorem 3.1. We take an h-cobordism X such that $\partial X = M \cup -M$ and $\tau(X) \neq 0$. (There are many examples for such X by the existence result of h-cobordism.) We patch M and $-M$ in the boundary of X to get Y. There is a smooth map $f : Y \to S^1$ induced by an element of $C^\infty(X; 0, 1)$.

In the cotangent bundle T^*Y, we consider two Lagrangian submanifolds : L_1 is the 0 section and L_2 is the graph of du. Then we are in the situation of Case 2. In fact the assumption $\pi_2 = 0$ may not be satisfied. However, in the case of cotangent bundle, we do not need this assumption, since there is no pseudo holomorhpic sphere in the cotangent bundle.

In this case the map $a : \pi_1 L_1 \to \mathbf{R}$ is given by $a(k\ell) = k$. Here $\pi_1 L_1 \simeq \mathbf{Z} \times \pi_1 M$. Therefore our Novikov ring is $\Lambda = \mathbf{Z}[\pi_1 M][t][[t^{-1}]]$. There is a natural injective homomorphism $wh^1(\pi_1 M) \to wh^1_+(\Lambda)$. We then have :

Lemma 3.4. $HF(L_1, L_2) = 0$. $\tau(L_1, L_2) = \tau(X)$

Thus we have a pair of Lagrangian submanifolds for which Floer homology vanishes but Whitehead torsion is nonzero.

In order to construct such an example in case 1, one may use the following :

Problem 3.5. Is there any *symplectic* manifold X which is homotopy equivalent to $M \times S^1$ but its (usual) Whitehead torsion is nonzero ?

Probably one can construct such an example using the technique of symplectic surgery. (Gompf [7].) If there exists X as in Problem 3.5, we take $f : X \to S^1$ and consider a hamilton vector field generated by it to obtain symplectic diffeomorphism φ. We find that $HF(X, \varphi) = 0$ while $\tau(\varphi) \neq 0$. Thus there is no exact symplectic diffeomorphism ψ such that $\psi\varphi$ has no fixed point. We remark that in this case there is a symplectic diffeomorphism φ' from X to itself without critical point, since there is a closed one form on X without critical point. (This is a consequence of Thurston's result on the existence of codimension one foliation on manifold with Euler number 0.) Hence $\varphi^{-1}\varphi'$ is not an exact symplectic diffeomorphism though the Floer homology of φ and φ' are equal.

Let us sketch very briefly the proof of Theorem 3.1. We remark that we have an acyclic complex, $CF(L_1, L_2)$, with a prefered basis. (Namely the basis given by the generator corresponding to the critical points.) Hence we can define its Whitehead Torsion $\tau(L_1, L_2)$ by [14] §4. Let us recall its definition in the case when the image of the boundary operator is a free Λ module. We choose a basis to each of $\operatorname{Im} \partial_{k+1} = \operatorname{Ker} \partial_k$. It gives a basis $\operatorname{Coker} \partial_k$ since $\operatorname{Coker} \partial_k \simeq \operatorname{Im} \partial_{k-1}$. Thus we have a basis of each $CF_k(L_1, L_2)$. The base change of this basis and prefered basis gives an element of $GL(n, \Lambda)$ and hence its class $\tau_k(L_1, L_2)$ in $Wh^1_+(\Lambda)$. We put

$$\sum_k (-1)^k \tau_k(L_1, L_2) = \tau(L_1, L_2)$$

One can prove that it is independent of the choice of the basis of $\operatorname{Im} \partial_k$.

To prove $\tau(\psi L_1, L_2) = \tau(L_1, L_2)$, we consider ψ_t as in the definition of exact diffeomorphism and prove that $\tau(\psi_t L_1, L_2)$ is independent of t. The point which we need to remark first is that the Novikov ring do not change.

We prove $\tau(\psi L_1, L_2) = \tau(L_1, L_2)$ in two steps. In the case when the number of intersection points $L_1 \cap L_2$ do not change, the Floer's chain complex does not change as an abelian group, but the boundary operator may change. To prove the invariance of $\tau(\psi_t L_1, L_2)$ of this case, we use the fact that there is a filtration in Floer's chain complex using action functional itself which is invariant of this change. This argument is used in the context of Gauge theory by Braam-Donaldson [2].

The other case is when the order of $L_1 \cap L_2$ changes. This case is very similar to the bifurcation $x^3 + tx$ as $t \in (-\epsilon, \epsilon)$. One can prove that the pseudo holomorphic disk behave in exactly the same way as gradient line behave for $x^3 + tx$. Thus the proof goes by imitating the proof in the case of Morse theory.

Finally we point out the following conjecture, which may follow in a similar way to the pseudo isotopy theory.

Conjecture 3.6. Let X be as in Problem 3.5. Assume that $wh^2(\pi_1 X) \neq 0$. Then there exists two symplectic diffeomorhpisms φ φ' such that

(1) φ and φ' do not have fixed points.
(2) $\varphi \circ \varphi'^{-1}$ is an exact symplectic diffeomorhpism.
(3) Let ψ_t be a family of exact symplectic diffeomorphism such that $\psi_0 = id$ and $\psi_t = \varphi \circ \varphi'^{-1}$. Then there exists t such that $\psi_t \circ \varphi'$ has a fixed point.

Probably Eliashberg and Gromov proved a similar result for Lagrangian intersection.

§4 Cancellation of critical points in Novikov cohomology

Let us consider a closed one form v on a closed manifold M (of finite dimension). Then the following problem is a finite dimensional analogue of the discussion of the argument of the preceding sections.

Problem 4.1. When there exists a function f such that $v + df$ has no critical point.

Here the critical point of v means the zero of v. We remark that integration of v along a curve in a homotopy class, gives a map $a : \pi_1 X \to \mathbf{R}$. Using it we define Novikov ring Λ in a similar way. We consider a vector field V which is the dual to u. Using it we construct a chain complex $CN(X, v)$ with Λ coefficient. This is the construction of Novikov (with small modification, that is to involve a bit more fundamental group.) We let $HN(X, v)$ be its homology. If it vanishes, we can again find a Whitehead torsion $\tau(X, v) \in wh^1_+(\Lambda)$. Then, following Novikov [16], we can prove

Theorem 4.2. *If $HN(X, v) \neq 0$ or if $HN(X, v) = 0$, $\tau(X, v) \neq 0$, then for every function f, $v + df$ has a critical point.*

Novikov discussed a converse of it in the case when the image of $a : \pi_1 X \to \mathbf{R}$ is isomorphic to $\mathbf{Z}$. The trouble in general case is that then the level surface (integral submanifold of u) can be dense in M and hence there is no analogy of handle decomposition. However we can prove the following :

Proposition 4.3. *If $HN(X, v) = 0$, $\tau(X, v) = 0$, and if there is no critical point of v of index $0, 1, 2, n, n-1, n-2$, then there exists a function f such that $v + df$ has no critical point. ($n = \dim M$.)*

We remark that locally v is an exterior derivative of a function and hence the index of critical point makes sense.

It seems very likely that the unpleasant assumption on the critical point of index $0, 1, 2, n, n-1, n-2$ can be removed.

We remark also that Proposition 4.3 implies symplectic S-cobordism conjecture in the case if two Lagrangian submanifolds are enough close to each other and the assumption on the critical point of index $0, 1, 2, n, n-1, n-2$ is satisfied.

We do not discuss the proof of Proposition 4.3. However let us mention a lemma and its analogue in loop space.

Lemma 4.4. *Let p and q are nondegenerate critical points of v. Let ℓ be an integral curve of V joining p and q. We assume that, for every other integral curve ℓ' of V joining p and q, we have*

$$\int_{\ell'} v > \int_{\ell} v.$$

Then, there there exists f such that the critical point set of $v + df$ is the critical point set of v minus $\{p, q\}$.

This is a generalization of Morse's lemma on cancellation of critical points. (See [13].) Its infinite dimensional analogue is the following conjecture, which the author believe to be the first step to the theory of infinite dimensional surgery.

Conjecture 4.5. Let X, L_1, L_2 be as in Case 1 and $p, q \in L_1 \cap L_2$. Assume that $u \in \mathfrak{M}(p, q)$. Assume also that, for each $u' \in \mathfrak{M}(p, q)$ different from u, we have

$$\int_{u'} \omega > \int_{u} \omega.$$

Then there exists an exact symplectic diffeomorhpism ψ such that

$$\psi L_1 \cap L_2 = L_1 \cap L_2 - \{p, q\}.$$

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