ANOTHER LOOK AT THE DIFFERENTIAL OPERATORS ON QUANTUM MATRIX SPACES AND ITS APPLICATIONS

Dedicated to Professor Kiyosato Okamoto
on the occasion of sixtieth birthday

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Abstract. The present paper gives a new view point for the differential operators with respect to the coordinates of the quantum matrix space. A special emphasis is put on an inductive construction of these differential operators from the q-difference operators defined on each columns (resp. rows). This idea for understanding our operators provides us with two important applications: (1) a construction of the q-oscillator representations of the quantized enveloping algebra $U_q(\mathfrak{sp}_{2m})$ of symplectic Lie algebra $\mathfrak{sp}_{2m}$ and an explicit description of their tensor powers; (2) a new definition of a quantum analogue of hypergeometric equations of many variables.

0. Introduction

This paper deals with a further investigation of the differential operators with respect to the *coordinates* of the quantum matrix space introduced in [HiW] and [NUW1]. Our study meet here other two works on the analysis on quantum homogeneous spaces in [NUW2, 3] and [N1].

The principal aim of this paper is to provide a new point of view to these operators. They were defined originally based on the following idea: focussing on the relations among three kinds of operators, i.e., polarization-, multiplication- and differntial-operators, in the classical situation, we regarded them as linear equations with differantial operators as unknowns, and solved the corresponding equations to get the quantum group counterparts. We show, contrary to this original definition, our differential operators can be described inductively by the operators of smaller sizes. In this way we see our operators are woven from much well-understood q-difference operators (cf. [NUW1], §5).

Applications include a construction of q-oscillator representations of the quantized enveloping algebra $U_q(\mathfrak{sp}_{2m})$ of Lie algebra of the symplectic group on the quantum matrix space, and a formulation of quantum version of the Gelfand hypergeometric equations associated with Grassmannians. The former is a necessary basic step for further study of a quantum dual pair $(\mathfrak{sp}_{2m}, \mathfrak{so}_n)$, the most important example among dual pairs. This is a direct continuation of our recent study in [NUW2, 3] on the quantum example of dual pair $(\mathfrak{sl}_2, \mathfrak{so}_n)$, spherical harmonics and the Capelli identity. To go further to define and to systematically investigate general quantum dual pairs,

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it is indispensable to understand the interrelation between our differential operators on the quantum matrix spaces of different sizes.

The latter application is, on the other hand, expected to be another basic foundation for harmonic analysis on quantum homogeneous spaces (cf. [N1]). Our operators enable us to give a natural definition for the q-hypergeometric equations in the sense of the quantum group version of the generalized hypergeometric equations treated in [G] and [GZK].

The content of each section is as follows: In §1, we review basic facts on q-difference operators and their connection with the quantum enveloping algebra $U_q(\mathfrak{gl}_n)$. In §2, after recalling the definition of differential operators in [NUW1] with constant coefficients on the quantum square matrix spaces, we generalize those kind of operators for rectangular matrix cases. We reconstruct these operators in §3 from the q-difference operators defined on each columns (resp. rows). §4 is devoted to the construction of the q-oscillator representation and its tensor power in a natural way. We formulate a quantum version of Gelfand's hypergeometric equations in the final section §5.

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1. From q-difference operators to the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$

Throughout the paper the ground field $\mathbb{K}$ will be fixed as the rational function field $\mathbb{Q}(q)$ of one variable q.

1.1. Partial q-difference operators:

Let $\mathcal{A} = \mathbb{K}[t_1, \cdots, t_n]$ denote the q-commutative ring with the relations $t_i t_j = q t_j t_i$ for $i < j$. We define the mutually commutative automorphisms $\gamma_i \, (i = 1, \ldots, n)$ as follows:

$$(\gamma_i \varphi)(t_1, \ldots, t_i, \ldots, t_n) = \varphi(t_1, \ldots, qt_i, \ldots, t_n) \quad \text{for} \quad \varphi \in \mathcal{A}.$$

Using γ_i we can define two types of (partial) q-difference operators as follows:

$$\partial_i^L \, \varphi = t_i^{-1} \frac{\gamma_i - \gamma_i^{-1}}{q - q^{-1}} \varphi, \qquad \partial_i^R \, \varphi = \left(\frac{\gamma_i - \gamma_i^{-1}}{q - q^{-1}} \varphi \right) t_i^{-1}.$$

In other words, these operators are taken as forms

$$t_i \, \partial_i^L = \{\gamma_i\}, \qquad t_i^{\circ} \, \partial_i^R = \{\gamma_i\}. \tag{1.1.1}$$

Here t_i and t_i° stand for the left and right multiplication operators respectively by the element t_i, and also $\{\gamma_i\}$ represents a difference operator (of symmetric form) given by

$$\{\gamma_i\} = \frac{\gamma_i - \gamma_i^{-1}}{q - q^{-1}}. \tag{1.1.2}$$

The following commutation relations are easily derived from their definition:

$$
\begin{aligned}
&t_it_j - qt_jt_i = 0, \quad \partial_i^L\partial_j^L - q\partial_j^L\partial_i^L = 0 \qquad (i<j)\\
&t_i^\circ t_j^\circ - q^{-1}t_j^\circ t_i^\circ = 0, \quad \partial_i^R\partial_j^R - q^{-1}\partial_j^R\partial_i^R = 0 \qquad (i<j),\\
&t_it_j^\circ - t_j^\circ t_i = 0, \quad \partial_i^L\partial_j^R - \partial_j^R\partial_i^L = 0 \qquad (\forall\, i,\ j),\\
&\partial_i^L t_i - q^{\mp 1}t_i\partial_i^L = \gamma_i^{\pm 1}, \quad \partial_i^R t_i^\circ - q^{\mp 1}t_i^\circ\partial_i^R = \gamma_i^{\pm 1} \qquad (\forall\, i),\\
&\partial_i^L t_j - q^{\mp 1}t_j\partial_i^L = 0, \qquad \partial_i^R t_j^\circ - q^{\pm 1}t_j^\circ\partial_i^R = 0 \qquad (i \lessgtr j).
\end{aligned}
$$

Also we see for any i $(1 \le i \le n)$

$$
\begin{aligned}
\partial_i^L t_i^\circ - q^{\mp 1}t_i^\circ\partial_i^L &= \gamma_i^{\pm 1}\,\mathrm{Int}(t_i),\\
\partial_i^R t_i - q^{\mp 1}t_i\partial_i^R &= \gamma_i^{\pm 1}\,\mathrm{Int}(t_i)^{-1},
\end{aligned}
$$

where

$$
\mathrm{Int}(t_i) = t_i^{-1}t_i^\circ = t_i^\circ t_i^{-1} = \gamma_1\cdots\gamma_{i-1}\gamma_{i+1}^{-1}\cdots\gamma_n^{-1}. \tag{1.1.3}
$$

Note that if $i \neq j$ then the corresponding operators commute with each other:

$$
\partial_i^L t_j^\circ - t_j^\circ\partial_i^L = 0, \quad \partial_i^R t_j - t_j\partial_i^R = 0.
$$

As well as (1.1.1) it is convenient for us to rewrite the relations into the following form:

$$
\begin{aligned}
&\partial_i^L\, t_i = \{q\gamma_i\}, \qquad \partial_i^R\, t_i^\circ = \{q\gamma_i\},\\
&t_i\,\partial_i^R = \mathrm{Int}(t_i)^{-1}\{\gamma_i\}, \qquad \partial_i^R\, t_i = \mathrm{Int}(t_i)^{-1}\{q\gamma_i\},\\
&t_i^\circ\,\partial_i^L = \mathrm{Int}(t_i)\{\gamma_i\}, \qquad \partial_i^L\, t_i^\circ = \mathrm{Int}(t_i)\{q\gamma_i\}.
\end{aligned}
$$

1.2. Realization of the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$:

Motivated by the classical situation that the general linear group $GL(n)$ acts on the polynomial ring with n variables, we introduce the following operators acting on $\mathcal{A}$:

$$
\begin{cases}
e_i = \gamma_i\, t_i^\circ\partial_{i+1}^R \quad (1 \le i < n),\\
f_i = \gamma_i^{-1}t_{i+1}^\circ\partial_i^R \quad (1 \le i < n),\\
\gamma_i^{\pm 1} = \gamma_i^{\pm 1} \quad (1 \le i \le n).
\end{cases} \tag{1.2}
$$

An easy calculation then shows that these operators give a representation of $U_q(\mathfrak{gl}_n)$ under the assignment

$$
\hat{e}_i \leftarrow e_i,\ \hat{f}_i \leftarrow f_i,\ q^{\pm\varepsilon_i} \leftarrow \gamma_i^{\pm 1}
$$

Here the q-deformed algebra $U_q(\mathfrak{gl}_n)$ is, as usual, defined an associative algebra generated by the symbols $q^{\pm\varepsilon_i}\,(1 \le i \le n)$, $\hat{e}_j, \hat{f}_j\,(1 \le j < n)$ with the following relations:

$$
\begin{aligned}
&q^0 = 1,\ q^\lambda q^\mu = q^{\lambda+\mu},\\
&q^\lambda \hat{e}_j q^{-\lambda} = q^{\langle \lambda, \varepsilon_j - \varepsilon_{j+1}\rangle}\hat{e}_j,\ q^\lambda \hat{f}_j q^{-\lambda} = q^{-\langle \lambda, \varepsilon_j - \varepsilon_{j+1}\rangle}\hat{f}_j,\\
&\hat{e}_i\hat{f}_j - \hat{f}_j\hat{e}_i = \delta_{ij}\frac{q^{\varepsilon_j-\varepsilon_{j+1}} - q^{-\varepsilon_j+\varepsilon_{j+1}}}{q-q^{-1}},\\
&\hat{e}_i\hat{e}_j = \hat{e}_j\hat{e}_i,\ \hat{f}_i\hat{f}_j = \hat{f}_j\hat{f}_i \quad (|i-j|>1),\\
&\hat{e}_i^2\hat{e}_j - (q+q^{-1})\hat{e}_i\hat{e}_j\hat{e}_i + \hat{e}_j\hat{e}_i^2 = 0 \quad (|i-j|=1),\\
&\hat{f}_i^2\hat{f}_j - (q+q^{-1})\hat{f}_i\hat{f}_j\hat{f}_i + \hat{f}_j\hat{f}_i^2 = 0 \quad (|i-j|=1),
\end{aligned}
$$

where $\langle\ ,\ \rangle$ stands for the canonical symmetric bilinear form such that $\langle \varepsilon_i, \varepsilon_j\rangle = \delta_{ij}$ and $q^\lambda = q^{a_1\varepsilon_1}\cdots q^{a_n\varepsilon_n}$ for an integral linear combination $\lambda = a_1\varepsilon_1 + \cdots + a_n\varepsilon_n$. The last two cubic equations in the above relations are sometimes referred to as the *Serre relations*.

The algebra $U_q(\mathfrak{gl}_n)$ has also a Hopf algebra structure. We take here the following convention of the comultiplication Δ:

$$
\begin{aligned}
&\Delta(q^\lambda) = q^\lambda \otimes q^\lambda,\\
&\Delta(\hat{e}_j) = \hat{e}_j \otimes 1 + q^{\varepsilon_j-\varepsilon_{j+1}} \otimes \hat{e}_j \quad (1 \le j < n),\\
&\Delta(\hat{f}_j) = \hat{f}_j \otimes q^{-\varepsilon_j+\varepsilon_{j+1}} + 1 \otimes \hat{f}_j \quad (1 \le j < n).
\end{aligned}
$$

The counit ε and antipode S are respectively given for the generators as:

$$
\begin{aligned}
&\varepsilon(q^\lambda) = 1, \quad \varepsilon(\hat{e}_j) = \varepsilon(\hat{f}_j) = 0,\\
&S(q^\lambda) = q^{-\lambda}, \quad S(\hat{e}_j) = -q^{-\varepsilon_j+\varepsilon_{j+1}}\hat{e}_j, \quad S(\hat{f}_j) = -\hat{f}_j q^{\varepsilon_j-\varepsilon_{j+1}},
\end{aligned}
$$

for $1 \le j < n$.

Remark: We note that the quantized enveloping algebra $U_q(\mathfrak{sl}_n)$ is, by definition, the Hopf algebra generated by the elements $k_i^{\pm1}$, $\hat{e}_j, \hat{f}_j\,(1 \le j \le n-1)$ where $k_i = q^{\varepsilon_i-\varepsilon_{i+1}}$ instead of q^{ε_i}'s in the fundamental relations of $U_q(\mathfrak{gl}_n)$.

Although the above action on $\mathcal{A}$ is *not* a faithfull representation of $U_q(\mathfrak{gl}_n)$, it gives us some insight for the natural coalgebra structure of $U_q(\mathfrak{gl}_n)$. We will explain this below.

The first thing to note is that with the comultiplication rule above the multiplication $\mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ is a $U_q(\mathfrak{gl}_n)$ homomorphism. Classically, this is called the Leibniz rule. Conversely if we require this situation to hold under representation of $U_q(\mathfrak{gl}_n)$ by the difference operators above, the natural "Leibniz rule" should be compatible with the comultiplication rule. Acutually the Leibniz rule suggests the latter.

Let us first see how comes the action $\{\gamma_i\}$ on a product of two functions in $\mathcal{A}$:

$$
\begin{aligned}
\Delta(\{\gamma_i\}) =& \gamma_i \otimes \{\gamma_i\} + \{\gamma_i\} \otimes \gamma_i^{-1}\\
=& \{\gamma_i\} \otimes \gamma_i + \gamma_i^{-1} \otimes \{\gamma_i\}.
\end{aligned}
$$

We notice here two possibilities to choose, and let us take, for example, the first one. On the other hand, for $\varphi, \psi \in \mathcal{A}$, we observe

$$\begin{aligned} t_i^\circ(\varphi\psi) &= \varphi\psi t_i = \varphi(t_i^\circ\psi) \\ &= \varphi t_i(t_i^{-1}t_i^\circ\psi) = (t_i^\circ\varphi)(\mathrm{Int}(t_i)\psi). \end{aligned}$$

This formula may be looked upon

$$\Delta(t_i^{\circ\,-1}) = 1 \otimes t_i^{\circ\,-1} = t_i^{\circ\,-1} \otimes \gamma_1^{-1}\cdots\gamma_{i-1}^{-1}\gamma_{i+1}\cdots\gamma_n.$$

These formulas then make the Leibniz rule $\Delta(\partial_i^R)$ for the difference operator of ∂_i^R as

$$\begin{aligned} \Delta(\partial_i^R) &= \Delta(t_i^{\circ\,-1})\Delta(\{\gamma_i\}) \\ &= \gamma_i \otimes \partial_i^R + \partial_i^R \otimes \gamma_1^{-1}\cdots\gamma_{i-1}^{-1}\gamma_i^{-1}\gamma_{i+1}\cdots\gamma_n. \end{aligned}$$

With these we are finally led to the following

$$\begin{aligned} &\Delta(t_{i+1}^\circ\partial_i^R) \\ =&(1\otimes t_{i+1}^\circ)\gamma_i \otimes \partial_i^R + (t_{i+1}^\circ \otimes \gamma_1\cdots\gamma_i\gamma_{i+2}^{-1}\cdots\gamma_n^{-1})\partial_i^R \otimes \gamma_1^{-1}\cdots\gamma_{i-1}^{-1}\gamma_i^{-1}\gamma_{i+1}\cdots\gamma_n \\ =&\gamma_i \otimes t_{i+1}^\circ\partial_i^R + t_{i+1}^\circ\partial_i^R \otimes \gamma_{i+1} \end{aligned}$$

or

$$\Delta(f_i) = \Delta(\gamma_i^{-1})\Delta(t_{i+1}^\circ\partial_i^R) = 1 \otimes f_i + f_i \otimes \gamma_i^{-1}\gamma_{i+1}.$$

This is really the comultiplication of $\hat{f}_j$ described above. Quite similarly, we can get to the comultiplication rules for other operators.

1.3. L-operators:

Throughout this paper, we will extensively make use of the so-called L-operators $L_{ij}^+, L_{ij}^- \in U_q(\mathfrak{gl}_n)$ as in [RTF],[NUW1]; these play the analogous role of root vectors of the Lie algebra $\mathfrak{gl}_n$. As is shown in [J], there exists a unique family of elements $\hat{E}_{ij}\,(1 \le i \ne j \le n)$ in $U_q(\mathfrak{gl}_n)$ such that

$$\begin{aligned} &\hat{E}_{j.j+1} = \hat{e}_j,\ \hat{E}_{j+1,j} = \hat{f}_j \\ &\hat{E}_{ij} = \hat{E}_{ik}\hat{E}_{kj} - q^{\pm 1}\hat{E}_{kj}\hat{E}_{ik} \quad (i \lesseqgtr k \lesseqgtr j). \end{aligned}$$

Using these elements, we define the elements $L_{ij}^\pm \in U_q(\mathfrak{gl}_n)$ as follows:

$$L_{ij}^+ = \begin{cases} q^{\varepsilon_i} & (i=j) \\ (q-q^{-1})q^{\varepsilon_i}\hat{E}_{ji} & (i<j) \\ 0 & (i>j), \end{cases}$$

and

$$L_{ij}^- = \begin{cases} q^{-\varepsilon_i} & (i=j) \\ -(q-q^{-1})q^{-\varepsilon_j}\hat{E}_{ji} & (i>j) \\ 0 & (i<j). \end{cases}$$

Concerning the Hopf algebra structure of $U_q(\mathfrak{gl}_n)$, we should remark that, in their matrix form, the L-operators $L^{\pm} = (L^{\pm}_{ij})_{1 \leq i,j \leq n}$ satisfy

$$\Delta(L^{\pm}) = L^{\pm} \otimes L^{\pm}, \tag{1.3.1}$$

$$S(L^{\pm})L^{\pm} = L^{\pm}S(L^{\pm}) = 1 \quad \text{and} \quad \varepsilon(L^{\pm}) = 1. \tag{1.3.2}$$

The square S^2 of the antipode S, which is an automorphism of the Hopf algebra $U_q(\mathfrak{gl}_n)$, is explicitly given by

$$S^2(a) = q^{-2\rho} a q^{2\rho} \quad \text{for any } a \in U_q(\mathfrak{gl}_n), \tag{1.3.3}$$

where $q^{2\rho}$ is the group-like element of $U_q(\mathfrak{gl}_n)$ corresponding to the sum of positive roots given by

$$2\rho = \sum_{j=1}^{n} 2(n-j)\varepsilon_j.$$

For the later use, we remark that the description of those action of $U_q(\mathfrak{gl}_n)$ on $\mathcal{A}$ above is given by these L-operators in the following matrix form:

$$\frac{1}{q-q^{-1}} L(1).\varphi = \left(\begin{bmatrix} \partial_1^R \\ \partial_2^R \\ \vdots \\ \partial_n^R \end{bmatrix} \varphi \right) \begin{bmatrix} t_1 & t_2 & \dots & t_n \end{bmatrix}. \tag{1.3.4}$$

Here the matrix form of L-operators with the spectral parameter λ is defined by

$$L(\lambda) = (L_{ij}(\lambda))_{1 \leq i,j \leq n}, \tag{1.3.5}$$

where

$$L_{ij}(\lambda) = \lambda L^{+}_{ij} - \lambda^{-1} L^{-}_{ij} \in U_q(\mathfrak{gl}_n) \otimes \mathbb{K}[\lambda, \lambda^{-1}]. \tag{1.3.6}$$

2. Differential operators on quantum matrix spaces

In this section, we review the quantum analogue of differential operators constructed in [NUW1].

We first recall briefly some facts on quantum R-matrices related to the L-operators defined in the preceding section. The description of the quantum matrix space will be given not only for the square matrices case but the rectangular case in general.

2.1. Quantum matrix spaces:

Let (ρ, V) be a vector representation of $U_q(\mathfrak{gl}_n)$. That is to say, V is the n-dimensional vector spaces over $\mathbb{K}$ with canonical basis $(v_1, \cdots, v_n)$ and its left $U_q(\mathfrak{gl}_n)$ module structure is given by

$$\begin{cases} \rho(L^{+}_{ii})v_k = q^{\delta_{ki}} v_k, & \rho(L^{+}_{ij})v_k = (q-q^{-1})\delta_{ki} v_j \quad (i<j) \\ \rho(L^{-}_{ii})v_k = q^{-\delta_{ki}} v_k, & \rho(L^{-}_{ij})v_k = -(q-q^{-1})\delta_{ki} v_j \quad (i>j). \end{cases}$$

We now define $n^2 \times n^2$-matrices $R^{\pm}$ as

$$R^{\pm} = \sum_{i,j=1}^{n} e_{ij} \otimes \rho(L_{ij}^{\pm}),$$

where $e_{ij} \in \mathrm{End}_{\mathbb{K}}(V)\,(1 \leq i, j \leq n)$ are the matrix units with respect to the basis $(v_j)_{1 \leq j \leq n}$ and $\otimes$ stands for the Kronecker product of matrices. More precisely they can be written as

$$\begin{aligned} R^{+} &= \sum_{i,j=1}^{n} q^{\delta_{ij}} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i<j} e_{ij} \otimes e_{ji}, \\ R^{-} &= \sum_{i,j=1}^{n} q^{-\delta_{ij}} e_{ii} \otimes e_{jj} - (q - q^{-1}) \sum_{i>j} e_{ij} \otimes e_{ji}. \end{aligned} \tag{2.1.1}$$

It is well known that these matrices satisfy the Yang-Baxter equation

$$R_{12}^{\varepsilon} R_{13}^{\varepsilon} R_{23}^{\varepsilon} = R_{23}^{\varepsilon} R_{13}^{\varepsilon} R_{12}^{\varepsilon} \quad (\varepsilon = \pm) \tag{2.1.2}$$

in $\mathrm{End}_{\mathbb{K}}(V_1 \otimes_{\mathbb{K}} V_2 \otimes_{\mathbb{K}} V_3)$ with $V_a = V$. Here the subscripts a, b for $R_{a,b}^{\varepsilon}$ indicates the pair of components it should act on non-trivially. We also recall that $R^{+} - R^{-} = (q - q^{-1})P$ with the matrix $P = \sum_{i,j} e_{ij} \otimes e_{ji}$ representing the flip operator: $v \otimes w \mapsto w \otimes v$. Further note that $(R_{12}^{+})^{-1} = R_{21}^{-}$ and ${}^{t}R_{12}^{+} = R_{21}^{\pm}$, where ^{t} denotes the transposition of matrix.

Using these R-matrices we can rewrite the fundamental relations of $U_q(\mathfrak{gl}_n)$ by the following way:

$$R^{+} L_1^{\varepsilon} L_2^{\varepsilon} = L_2^{\varepsilon} L_1^{\varepsilon} R^{+} \quad (\varepsilon = \pm) \quad \text{and} \quad R^{+} L_1^{+} L_2^{-} = L_2^{-} L_1^{+} R^{+}. \tag{2.1.3}$$

To specify the size of matrix $R^{\pm}$, $L^{\pm}$, we will write them, if necessary, as $R^{[n]\,\pm}$, $L^{[n]\,\pm}$ respectively.

Recall the definition of the coordinate ring $\mathcal{A}(\mathrm{Mat}_q(m \times n))$ of the quantum matrix space $\mathrm{Mat}_q(m \times n)$. It is the associative algebra generated by the "canonical coordinate" $t_{rs}\,(1 \leq r \leq m, 1 \leq s \leq n)$ with the following commutation relations:

$$\begin{aligned} t_{rj} t_{sj} &= q t_{sj} t_{rj} \qquad (1 \leq r < s \leq k,\ 1 \leq j \leq n), \\ t_{ri} t_{rj} &= q t_{rj} t_{ri} \qquad (1 \leq r \leq k,\ 1 \leq i < j \leq n), \\ t_{rj} t_{si} &= t_{si} t_{rj} \qquad (1 \leq r < s \leq k,\ 1 \leq i < j \leq n), \\ t_{ri} t_{sj} - t_{sj} t_{ri} &= (q - q^{-1}) t_{rj} t_{si} \qquad (1 \leq r < s \leq k,\ 1 \leq i < j \leq n). \end{aligned}$$

If we put $T = (t_{rs})_{1 \leq r \leq m, 1 \leq s \leq n}$ as in the matrix form, then the commutation relations above are equivalently written by the Yang-Baxter equation

$$R^{[m]\,+} T_2 T_1 = T_1 T_2 R^{[n]\,+}. \tag{2.1.4}$$

In the case of $m = n$, we will denote for abbreviation $\mathcal{A}(\mathrm{Mat}_q(n \times n))$ by $\mathcal{A}(\mathrm{Mat}_q(n))$. This algebra $\mathcal{A}(\mathrm{Mat}_q(n))$ has a distinguished central element called the *quantum determinant*:

$$\det_q = \det_q(T) \stackrel{\text{def}}{=} \sum_{\sigma \in \mathfrak{S}_n} (-q)^{\ell(\sigma)} t_{\sigma(1)1} \cdots t_{\sigma(n)n}.$$

Here $\mathfrak{S}_n$ is the permutation group of the index set $\{1, 2, \ldots, n\}$ and, for each $\sigma \in \mathfrak{S}_n$, $\ell(\sigma)$ stands for the number of inversions in σ. Further the coordinate ring of $\mathcal{A}(GL_q(n))$ of the quantum general linear group $GL_q(n)$ is then defined to be the localization $\mathcal{A}(\mathrm{Mat}_q(n))[\det_q(T)^{-1}]$ of $\mathcal{A}(\mathrm{Mat}_q(n))$ with respect to $\det_q(T)$. This algebra has a natural structure of Hopf algebra such that

$$\Delta(t_{ij}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj}, \quad \varepsilon(t_{ij}) = \delta_{ij}, \tag{2.1.5}$$

for $1 \leq i, j \leq n$. In terms of the matrix T, these formulas will be written as $\Delta(T) = T \otimes T$ and $\varepsilon(T) = I$. Note also that $\det_q(T)$ is a group like element, namely, $\Delta(\det_q(T)) = \det_q(T) \otimes \det_q(T)$, $\varepsilon(\det_q(T)) = 1$. The antipode of $\mathcal{A}(GL_q(n))$, we also denote it by S, is a $\mathbb{K}$ algebra anti-automorphism such that

$$S(T)T = TS(T) = I, \quad S(\det_q(T)) = \det_q(T)^{-1} \tag{2.1.6}$$

where $S(T) = (S(t_{ij}))_{1 \leq i,j \leq n}$.

Since $\mathcal{A}(GL_q(n))$ has a natural two-sided comodule structure over itself, the algebra $\mathcal{A}(GL_q(n))$ becomes a $U_q(\mathfrak{gl}_n)$-bimodule structure through a pairing $(\ ,\ )$ defined by

$$(L_1^{\pm}, T_2) = R^{\pm}, \quad (L^{\pm}, \det_q(T)) = q^{\pm} I.$$

To be more precise, the left and right actions are given by

$$a.\varphi = (id \otimes a) \circ \Delta(\varphi) \in \mathcal{A}(GL_q(n)),$$
$$\varphi.a = (a \otimes id) \circ \Delta(\varphi) \in \mathcal{A}(GL_q(n))$$

for $a \in U_q(\mathfrak{gl}_n)$, $\varphi \in \mathcal{A}(GL_q(n))$, where id denotes the identity operator on $\mathcal{A}(GL_q(n))$. It should be also noted that the algebra $\mathcal{A}(\mathrm{Mat}_q(m \times n))$ has a natural structure of a two-sided comodule over the pair of Hopf algebras $(\mathcal{A}(GL_q(n)), \mathcal{A}(GL_q(m)))$. Accordingly, it becames a bimodule over the pair $(U_q(\mathfrak{gl}_n), U_q(\mathfrak{gl}_m))$. One can write down these actions of L-operators on $T \in \mathcal{A}(\mathrm{Mat}_q(m \times n))$ in a concrete form:

$$L_1^{[n]\,\pm}.T_2 = T_2 R^{[n]\,\pm}, \quad T_2.L_1^{[m]\,\pm} = R^{[m]\,\pm} T_2. \tag{2.1.7}$$

We note also, for $n = m$

$$L^{\pm}.\det_q(T) = \det_q(T).L^{\pm} = q^{\pm}\det_q(T). \tag{2.1.8}$$

2.2. Differential operators on quantum matrix spaces:

We now recall briefly the definition of the differential operators on $\mathcal{A}(GL_q(n))$ which are similar objects of the partial differentiation (differential operators with constant coefficients) ∂/∂_{ij} ([NUW1], Theorem 2.1).

Theorem & Definition. *There exists a unique family of linear operators* $\partial_{ij}(\lambda) = \lambda\partial_{ij}^{+} - \lambda^{-1}\partial_{ij}^{-} \in \mathrm{End}_{\mathbb{K}}(\mathcal{A}(GL_q(n)))\otimes\mathbb{K}[\lambda,\lambda^{-1}](1 \le i,j \le n)$, *with the spectral parameter* λ, *satisfying both of the following equations:*

$$L_{ji}(\lambda) = (q-q^{-1})\sum_{k=1}^{n} t_{ki}^{\circ}\partial_{kj}(\lambda), \tag{2.2.1}$$

$$L_{ji}(\lambda)^{\circ} = (q-q^{-1})\sum_{k=1}^{n} t_{jk}\partial_{ik}(\lambda), \tag{2.2.2}$$

where t_{ki}° *and* $L_{ji}(\lambda)^{\circ}$ *stand for the actions from the right. Moreover we call the operator* $\partial_{ij} = \partial_{ij}(1)$ *for the differential operator with respect to the coordinates on* $GL_q(n)$. $\square$

It should be remarked that those operators ∂_{ij} defined above are in fact, acting not only on $\mathcal{A}(GL_q(n))$ but also on $\mathcal{A}(\mathrm{Mat}_q(n))$ (see Proposition 5.1 in [NUW1]). This property works essentially for the discussion below.

From now on, we consider the rectangular case. That is, we have the following proposition concerned with the existence for such kind of "differential operator" with constant coefficients on the space $\mathrm{Mat}_q(m\times n)$.

Proposition 2.2. *For the natural action of* $U_q(\mathfrak{gl}_n)$ *(resp.* $U_q(\mathfrak{gl}_m)$*) from the left (resp. right) on* $\mathcal{A}(\mathrm{Mat}_q(m\times n))$ *there exist a unique family of linear operators* $\overline{\partial}_{ij}\,(1 \le i \le m, 1 \le j \le n) \in \mathrm{End}_{\mathbb{K}}(\mathcal{A}(\mathrm{Mat}_q(m\times n)))$ *such that (in the matrix form)*

$$L^{[n]}.\varphi = (q-q^{-1})({}^{t}\overline{\partial}\varphi)\,T,$$
$$\varphi.L^{[m]} = (q-q^{-1})T\,({}^{t}\overline{\partial}\varphi).$$

respectively, for the matrix $\overline{\partial} = (\overline{\partial}_{ij})_{1\le i\le m, 1\le j\le n}$. *Here*

$$L^{[n]} = L^{[n]}(1) = L^{[n]+} - L^{[n]-}.$$

For the case $m = n$ *this implies the above theorem for* $\lambda = 1$.

Furthermore, we observe that these definitions are compatible with the embedding of $\mathcal{A}(\mathrm{Mat}_q(m\times n))$ *into* $\mathcal{A}(\mathrm{Mat}_q(N))$ *for any* $N \ge \max(m,n)$. *More precisely the restriction of* ∂_{ij} *defined on* $\mathcal{A}(GL_q(N))$ *coincides with* $\overline{\partial}_{ij}$ *on* $\mathcal{A}(\mathrm{Mat}_q(m\times n))$*:*

$$\partial_{ij}|_{\mathcal{A}(\mathrm{Mat}_q(m\times n))} = \overline{\partial}_{ij}. \quad \square$$

Remark: We remark that as well as the formula (2.4), the following one which comes from the very definition of Euler's degree operator, holds:

$$\{\gamma\} = \sum_{i=1}^{n} t_i\partial_i^R.$$

For the proof of this proposition, we need some more knowledge of basic properties of ∂_{ij}. We will give the proof in the next section.

2.3. Fundamental properties of differential operators on $\mathrm{Mat}_q(m\times n)$:

We now write down several commutation relations among the multiplication operators and the differential operators which are disccussed in [NUW1] in the case of $m = n$.

Proposition 2.3. *As linear operators on $\mathcal{A}(GL_q(n))$ we have*

(1) *The operators $\partial_{ij}(\lambda)$ satisfy the following commutation relations:*

$$R_{21}^{\pm}\partial_2(\lambda)\partial_1(\lambda) = \partial_1(\lambda)\partial_2(\lambda)R_{12}^{\pm},$$
$$R_{12}^{+}\partial_2^{+}\partial_1^{-} = \partial_1^{-}\partial_2^{+}R_{21}^{-}.$$

(2) *The commutation relations for $\partial_{ij}^{\pm}$ with the left and right multiplication operators are*

$$\partial_1^{\pm}T_2 = T_2\partial_1^{\pm\,{}^{t_1}}R_{12}^{\pm},$$
$$\partial_1^{\pm}T_2^{\circ} = {}^{t_1}R_{12}^{\pm}\,T_2^{\circ}\partial_1^{\pm},$$

where ${}^{t_1}R_{12}^{\pm}$ stands for the matrices obtained from $R_{12}^{\pm}$ by the transposition in the first factor. □

Notice that the proof of the above commutation relations follows directly from the Leibniz rule of $\partial_{ij}(\lambda)$:

Leibniz rule ([NUW1] Proposition 4.1): For $\varphi_1, \varphi_2 \in \mathcal{A}(GL_q(n))$,

$${}^t\partial(\lambda\mu)(\varphi_1\varphi_2) = \mu^{\pm 1}({}^t\partial(\lambda)\varphi_1)(L^{\pm\circ}.\varphi_2) + \lambda^{\mp 1}(L^{\mp}.\varphi_1)({}^t\partial(\mu)\varphi_2).$$

Moreover if we write down the commutation relations for ∂_{ij} not only themselves (which act on $\mathcal{A}(\mathrm{Mat}_q(n))$) but also with the multiplication operators in componentwise, then we see that

$$\begin{aligned}
\partial_{rj}\partial_{sj} &= q\partial_{sj}\partial_{rj} \qquad (1 \le r < s \le n,\ 1 \le j \le n),\\
\partial_{ri}\partial_{rj} &= q^{-1}\partial_{rj}\partial_{ri} \qquad (1 \le r \le n,\ 1 \le i < j \le n),\\
\partial_{ri}\partial_{sj} &= \partial_{sj}\partial_{ri} \qquad (1 \le r < s \le n,\ 1 \le i < j \le n),\\
\partial_{rj}\partial_{si} - \partial_{si}\partial_{rj} &= (q - q^{-1})\partial_{ri}\partial_{sj} \qquad (1 \le r < s \le n,\ 1 \le i < j \le n),
\end{aligned}$$

and

$$\partial_{ij}t_{k\ell} - q^{-\delta_{\ell j}}t_{k\ell}\partial_{ij} + (q - q^{-1})\delta_{\ell j}\sum_{\alpha<j} t_{k\alpha}\partial_{i\alpha} = \delta_{\ell j}L_{ki}^{+\circ}, \tag{2.3.1}$$

$$\partial_{ij}t_{k\ell} - q^{\delta_{\ell j}}t_{k\ell}\partial_{ij} - (q - q^{-1})\delta_{\ell j}\sum_{\alpha>j} t_{k\alpha}\partial_{i\alpha} = \delta_{\ell j}L_{ki}^{-\circ}, \tag{2.3.2}$$

$$\partial_{ij}t_{k\ell}^{\circ} - q^{-\delta_{ik}}t_{k\ell}^{\circ}\partial_{ij} + (q - q^{-1})\delta_{ik}\sum_{i<\alpha} t_{\alpha\ell}^{\circ}\partial_{\alpha j} = \delta_{ik}L_{j\ell}^{+}, \tag{2.3.3}$$

$$\partial_{ij}t_{k\ell}^{\circ} - q^{\delta_{ik}}t_{k\ell}^{\circ}\partial_{ij} - (q - q^{-1})\delta_{ik}\sum_{i>\alpha} t_{\alpha\ell}^{\circ}\partial_{\alpha j} = \delta_{ik}L_{j\ell}^{-}, \tag{2.3.4}$$

respectively.

Remark: The above commutation relation ∂_{ij}'s among themselves is just the same as those of t_{ij}'s under the arrangement in the reverse order with respect to their columns.

Proof of Proposition 2.2: It is clear that the embedding of $\mathcal{A}(\mathrm{Mat}_q(m \times n))$ into large $\mathcal{A}(\mathrm{Mat}_q(N))$ can be described by a pair of m-tuples I and n-tuples J given by

$$I = \{i_1, \ldots, i_m\}\,(1 \le i_1 < \cdots < i_m \le N), \quad J = \{j_1, \ldots, j_n\}\,(1 \le j_1 < \cdots < j_n \le N)$$

for any fixed N satisfying $N \ge \max(m, n)$. We denote such embedding by ρ_{IJ}. Then it is also clear that the restriction of L-operators for this embedding are compatible with the actions on $\mathcal{A}(\mathrm{Mat}_q(m \times n))$. That is,

$$L_{ji}^{[N]\pm}|_{\rho_{IJ}(\mathcal{A}(\mathrm{Mat}_q(m\times n)))} = \begin{cases} L_{ji}^{[m]\pm} & (i \in I,\ j \in J) \\ 0 & (\text{otherwise}) \end{cases}$$

for the left actions. A similar formula holds for the right actions. On the other hand, using the commutation relations for ∂_{ij} with the multiplication operators, we can show that ∂_{ji} defines the zero operator on the space $\rho_{IJ}(\mathcal{A}(\mathrm{Mat}_q(m \times n)))$ except for the cases $i \in I$ and $j \in J$ by a simple inductive argument. Thus, recalling the defining relations of ∂_{ij} on $\mathcal{A}(GL_q(N))$, we find for $i \in I$, $j \in J$ that

$$L_{ji}^{[m]} = (q - q^{-1}) \sum_{k \in I} t_{ki}^{\circ} \partial_{ji}|_{\rho_{IJ}(\mathcal{A}(\mathrm{Mat}_q(m\times n)))}, \tag{2.3.5}$$

$$L_{ji}^{[n]\circ} = (q - q^{-1}) \sum_{k \in J} t_{jk} \partial_{ik}|_{\rho_{IJ}(\mathcal{A}(\mathrm{Mat}_q(m\times n)))}. \tag{2.3.6}$$

These relations guarantees the operator $\overline{\partial}_{ij}$ to be well-defined.

The uniqueness of the operators can also be shown by induction. In fact, by Proposition 5.2 in [NUW1], we know that the restriction of our operator ∂_{ij}, hence $\overline{\partial}_{ij}$ defines the q-difference operator on each column (resp. row). The uniqueness of operator on each column (resp. row) follows immediately from this fact. The Leibniz rule

$$\partial_{ji}(\varphi_1 \varphi_2) = \sum_k (\partial_{ki}\varphi_1)(L_{kj}^{\pm\circ}.\varphi_2) + \sum_k (L_{ik}^{\mp}.\varphi_1)(\partial_{jk}\varphi_2) \tag{2.3.7}$$

guarantees the uniqueness of operator $\overline{\partial}_{ij}$ for the general case. This proves the assertion of Proposition 2.2. $\square$

Remark: From the proof above, we note that the uniqueness of operators $\overline{\partial}_{ij}$ follows simply from the two facts: (1) the restriction of $\overline{\partial}_{ij}$ on either each columns or rows gives the q-difference operators, (2) either one of the two relations (2.3.5) and (2.3.6) holds.

3. Weaving q-difference operators into differential operators on quantum matrix spaces

In [NUW1] we showed that the restriction of our differntial operators to each column (resp. row) gave q-difference operators. In this section we will conversely show that those differential operators are obtained from q-difference operators as a kind of "normal product" of operators.

We consider the following situation. We regard $\mathcal{A}(\mathrm{Mat}_q(m \times n))$ as the m-fold tensor product of $\mathcal{A} = \mathbb{K}[t_1, \cdots, t_n]$ through the multiplication of $\mathcal{A}(\mathrm{Mat}_q(m \times n))$:

$$\mathcal{A}^{\otimes m} = \mathcal{A} \otimes \cdots \otimes \mathcal{A} \xrightarrow{\sim} \mathcal{A}(\mathrm{Mat}_q(m \times n)).$$

In this paragraph, for simplicity, we denote by ∂^L_{rs} and ∂^R_{rs} for the q-difference operators with respect to the variable t_{rs}:

$$\partial^L_{rs}\,\varphi = t_{rs}^{-1}\,\frac{\gamma_{rs} - \gamma_{rs}^{-1}}{q - q^{-1}}\varphi, \qquad \partial^R_{rs}\,\varphi = \left(\frac{\gamma_{rs} - \gamma_{rs}^{-1}}{q - q^{-1}}\varphi\right) t_{rs}^{-1}.$$

Here γ_{rs} stands for the automorphism defined by the q-shift operator $t_{ij} \to q^{\delta_{ir}\delta_{js}} t_{ij}$.

Theorem 3.1. *Via the natural embedding $\mathcal{A}^{\otimes m} = {}^t(\mathcal{A} \otimes \cdots \otimes \mathcal{A}) \hookrightarrow \mathcal{A}(\mathrm{Mat}_q(m \times n))$, the differential operators $\overline{\partial}_{i\,j}$ and the right multiplication operators $\overline{t}^{\circ}_{ij}$ on $\mathcal{A}^{\otimes m}$ are respectively given by the following formulas:*

$$\begin{aligned}
\overline{\partial}_{i\,j} = \sum_{\ell \geq 0} (q - q^{-1})^{\ell} \sum_{\substack{1 \leq i_1 < i_2 < \cdots < i_\ell < i \\ j < j_1 < j_2 < \cdots < j_\ell \leq n}} & \gamma_{1\,j} \otimes \cdots \otimes \gamma_{i_1 - 1\,j} \otimes t^{\circ}_{i_1\,j_1} \partial^R_{i_1\,j} \\
& \otimes \gamma_{i_1+1\,j_1} \otimes \cdots \otimes \gamma_{i_2-1\,j_1} \otimes t^{\circ}_{i_2\,j_2} \partial^R_{i_2\,j_1} \otimes \gamma_{i_2+1\,j_2} \otimes \cdots \otimes \gamma_{i_3-1\,j_2} \\
& \otimes t^{\circ}_{i_3\,j_3} \partial^R_{i_3\,j_2} \otimes \gamma_{i_3+1\,j_3} \otimes \cdots \otimes \gamma_{i_{\ell-1}\,j_{\ell-1}} \otimes t^{\circ}_{i_\ell\,j_\ell} \partial^R_{i_\ell\,j_{\ell-1}} \otimes \gamma_{i_\ell+1\,j_\ell} \otimes \cdots \\
& \otimes \gamma_{i-1\,j_\ell} \otimes \partial^R_{i\,j_\ell} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(m-i)\text{-times}},
\end{aligned}$$

and

$$\begin{aligned}
\overline{t}^{\circ}_{ij} = \overbrace{1 \otimes \cdots \otimes 1}^{(i-1)\text{-times}} \otimes \sum_{\ell \geq 0} \{-(q - q^{-1})\}^{\ell} \sum_{\substack{i < i_1 < i_2 < \cdots < i_\ell \leq m \\ j < j_\ell < j_{\ell-1} < \cdots < j_1 \leq n}} & t^{\circ}_{i\,j_1} \otimes \gamma^{-1}_{i+1\,j_1} \\
& \otimes \cdots \otimes \gamma^{-1}_{i_1-1\,j_1} \otimes t^{\circ}_{i_1\,j_2} \partial^R_{i_1\,j_1} \otimes \gamma^{-1}_{i_1+1\,j_2} \otimes \cdots \otimes \gamma^{-1}_{i_2-1\,j_2} \otimes t^{\circ}_{i_2\,j_3} \partial^R_{i_2\,j_2} \\
& \otimes \gamma^{-1}_{i_2+1\,j_3} \otimes \cdots \otimes \gamma^{-1}_{i_3-1\,j_3} \otimes t^{\circ}_{i_3\,j_4} \partial^R_{i_3\,j_3} \otimes \gamma^{-1}_{i_3+1\,j_4} \otimes \cdots \otimes \gamma^{-1}_{i_{\ell-1}\,j_\ell} \\
& \otimes t^{\circ}_{i_\ell\,j} \partial^R_{i_\ell\,j_\ell} \otimes \gamma^{-1}_{i_\ell+1\,j} \otimes \cdots \otimes \gamma^{-1}_{m\,j}. \quad \square
\end{aligned}$$

The proof of this theorem can be immediately done by an induction using the following key proposition under a step-by-step discussions. For this purpose, we introduce

an ad-hoc notation on the operators as folows. According to the division of $m \times n$ matrix into two parts horizontally with respect to their rows, say (Top) and $(Bottom)$ for a while, we have in $\mathcal{A} = \mathcal{A}(\mathrm{Mat}_q(m \times n))$ two subalgebras $\overline{\mathcal{A}}$ and $\underline{\mathcal{A}}$ which are respectively generated by the coodinates for (Top)- and $(Bottom)$-indices. If an operator $a \in \mathrm{End}_{\mathbb{K}}(\mathcal{A}(\mathrm{Mat}_q(m \times n)))$ also stabilizes $\overline{\mathcal{A}}$ or $\underline{\mathcal{A}}$, then we write the operator a restricted to $\overline{\mathcal{A}}$ or $\underline{\mathcal{A}}$ respectively as $\overline{a}$ or $\underline{a}$ to distinguish where to act. To put an emphasis on a that we let it act on the whole $\mathcal{A}$, we denote it by $a = \underline{\overline{a}}$. Our concern here is a description of operators on $\mathcal{A}$ which can be reduced in a form $\underline{\overline{a}} = \sum_i \overline{b}_i \otimes \underline{c}_i \in \mathrm{End}_{\mathbb{K}}(\overline{\mathcal{A}}) \otimes \mathrm{End}_{\mathbb{K}}(\underline{\mathcal{A}})$.

Proposition 3.2. *With the convention on notation above, we have an expression for the right multiplication operator $t^{\circ}_{\ell\, j}$ as*

$$\begin{cases} \underline{\overline{t}}^{\circ}_{\ell\, j} = \overline{t}^{\circ}_{\ell\, j} \otimes \underline{\gamma}_j^{-1} - (q - q^{-1}) \sum_{j<\alpha,\, \lambda \in (Bottom)} \overline{t}^{\circ}_{\ell\, \alpha} \otimes \underline{t}^{\circ}_{\lambda\, j} \underline{\partial}_{\lambda\, \alpha} & \text{for } \ell \in (Top) \\ \underline{\overline{t}}^{\circ}_{\ell\, j} = 1 \otimes \underline{t}^{\circ}_{\ell\, j} & \text{for } \ell \in (Bottom), \end{cases}$$

and for the differential operator $\partial_{\ell\, i}$ as

$$\begin{cases} \underline{\overline{\partial}}_{\ell\, i} = \overline{\partial}_{\ell\, i} \otimes 1 & \text{for } \ell \in (Top) \\ \underline{\overline{\partial}}_{\ell\, i} = \overline{\gamma}_i \otimes \underline{\partial}_{\ell\, i} + (q - q^{-1}) \sum_{i<\beta,\, \kappa \in (Top)} \overline{t}^{\circ}_{\kappa\beta} \overline{\partial}_{\kappa\, i} \otimes \underline{\partial}_{\ell\, \beta} & \text{for } \ell \in (Bottom). \end{cases}$$

Proof: For the right multiplication operators, it is clear that as far as the case $\ell \in (Bottom)$ and, the above formula is equivalent to the following:

$$\begin{aligned} \underline{\overline{t}}^{\circ}_{\ell\, j} &= \overline{t}^{\circ}_{\ell\, j} \otimes \underline{\gamma}_j^{-1} - (q - q^{-1}) \sum_{j<\alpha} \overline{t}^{\circ}_{\ell\, \alpha} \otimes \underline{L}_{\alpha\, j} \\ &= \sum_{j \le \alpha} \overline{t}^{\circ}_{\ell\, \alpha} \otimes \underline{L}^{-}_{\alpha\, j}. \end{aligned}$$

We will first show this. Since $\mathcal{A}(\mathrm{Mat}_q(m \times n))$ ia a graded algebra we take an inductive way on the degree of monomials to prove it as follows.

Let $\psi \in \mathcal{A}(\mathrm{Mat}_q(m \times n))$, $\varphi \in \mathcal{A}(Bottom)$. Then for $\ell \in (Top)$, it suffices to prove that

$$t^{\circ}_{\ell\, j}(\psi\varphi) = (\psi\varphi) t_{\ell\, j} = \sum_{j \le \alpha} (\psi t_{\ell\, \alpha}) L^{-}_{\alpha j} . \varphi. \tag{3.1}$$

Suppose that $\varphi = t_{pk}$ for $k \in (Bottom)$. Then, if $p \le j$, the simple commutation relation between $t_{\ell\, j}$ and t_{pk} says

$$t^{\circ}_{\ell\, j}(\psi t_{pk}) = q^{-\delta_{jk}} (\psi t_{\ell\, j}) t_{pk} = (\psi t_{\ell\, j}) \gamma_j^{-1} . t_{pk}. \tag{3.2}$$

For $p > j$, the relation

$$t_{pk} t_{\ell\, j} = t_{\ell\, j} t_{pk} - (q - q^{-1}) t_{\ell\, k} t_{pj}$$

implies

$$t^{\circ}_{\ell\, j}(\psi t_{pk}) = (\psi t_{\ell\, j}) t_{pk} - (q - q^{-1}) \psi t_{\ell\, k} t_{pj}.$$

Since $L^-_{jj} = \gamma_j^{-1}$ and $L^-_{ji}.t_{pq} = -(q-q^{-1})t_{pi}\delta_{jq}\,(j > i)$, those formulas respectively amount to the desired relation for the polynomials of degree 1. For a general $\varphi \in \mathcal{A}(Bottom)$, it then suffices to show that if, the formula (3.2) holds for two elements $\varphi_1, \varphi_2 \in \mathcal{A}(Bottom)$, so does for the product $\varphi_1\varphi_2$.

$$\begin{aligned} t^\circ_{\ell\, j}(\psi\varphi_1\varphi_2) &= \sum_{j\le\alpha}((\psi\varphi_1)t_{\ell\,\alpha})(L^-_{\alpha j}.\varphi_2) \\ &= \sum_{j\le\alpha}(\sum_{\alpha\le\beta}(\psi t_{\ell\,\beta})(L^-_{\beta\alpha}.\varphi_1))(L^-_{\alpha j}.\varphi_2) \\ &= \sum_{j\le\beta}(\psi t_{\ell\,\beta})\sum_{j\le\alpha\le\beta}(L^-_{\beta\alpha}.\varphi_1))(L^-_{\alpha j}.\varphi_2) \\ &= \sum_{j\le\beta}(\psi t_{\ell\,\beta})L^-_{\beta j}.(\varphi_1\varphi_2). \end{aligned}$$

The last equality follows immediately from the comultiplication formula of L-operators $\{L^-_{\beta j}\}$: $\Delta(L^-_{\beta j}) = \sum_{j\le\alpha\le\beta} L^-_{\beta\alpha}\otimes L^-_{\alpha j}$. This completes the assertion on the right multiplication operators.

Next we will prove the assertion on differential operators. It of course suffices to show that for the case $\ell \in (Bottom)$

$$\underline{\overline{\partial}}_{\ell\, i} = \sum_{i\le\beta}\overline{L}^+_{i\beta}\otimes\underline{\partial}_{\ell\,\beta}. \tag{3.3}$$

For a moment, we denote by $p_{\ell\, i}$ the right hand side of (3.3) for $\ell \in (Bottom)$ and put $p_{\ell\, i} = \underline{\overline{\partial}}_{\ell\, i}$ for $\ell \in (Top)$. Then we see

$$\begin{aligned} &(q-q^{-1})\sum_\ell \underline{\overline{t}}^\circ_{\ell\, j}p_{\ell\, i} \\ =&(q-q^{-1})\sum_{\ell\in(Top)}\underline{\overline{t}}^\circ_{\ell\, j}p_{\ell\, i} + (q-q^{-1})\sum_{\ell\in(Bottom)}\underline{\overline{t}}^\circ_{\ell\, j}p_{\ell\, i} \\ =&(q-q^{-1})\{\sum_{\ell\in(Top)}(\sum_{j\le\alpha}\overline{t}^\circ_{\ell\,\alpha}\otimes\underline{L}^-_{\alpha j})(\overline{\partial}_{\ell\, i}\otimes 1) + \sum_{\ell\in(Bottom)}(1\otimes\underline{t}^\circ_{\ell\, j})(\sum_{i\le\beta}\overline{L}^+_{i\beta}\otimes\underline{\partial}_{\ell\,\beta})\} \\ =&(q-q^{-1})\{\sum_{\ell\in(Top)}\sum_{j\le\alpha}\overline{t}^\circ_{\ell\,\alpha}\overline{\partial}_{\ell\, i}\otimes\underline{L}^-_{\alpha j} + \sum_{\ell\in(Bottom)}\sum_{i\le\beta}\overline{L}^+_{i\beta}\otimes\underline{t}^\circ_{\ell\, j}\underline{\partial}_{\ell\,\beta}\} \\ =&\sum_{j\le\alpha}\overline{L}_{i\alpha}\otimes\underline{L}^-_{\alpha j} + \sum_{i\le\beta}\overline{L}^+_{i\beta}\otimes\underline{L}_{\beta\, j}. \end{aligned}$$

This implies that the right hand side coincides with a comultiplication of $\underline{\overline{L}}_{ij}$, that is, the action of L_{ij} on $\mathrm{End}_{\mathbb{K}}(\mathcal{A}(Top))\otimes\mathrm{End}_{\mathbb{K}}(\mathcal{A}(Bottom))$. Hence by a simple induction we obtain

$$(q-q^{-1})\sum_\ell \underline{\overline{t}}^\circ_{\ell\, j}p_{\ell\, i} = \underline{\overline{L}}_{ij}.$$

By the uniqueness of operators, which is guaranteed by Proposition 2.2 (see also Remark at the end of §2.2), we have $p_{ij} = \underline{\partial}_{ij}$. Hence completes the proof of the proposition. □

Remark: We can discuss similar weaving formulas from other directions. We have many combinations for them. For exmaple, those for left multiplication and for right difference operators, and also weaving formulas from left and right directions instead of up-down ones, etc.

4. $U_q(\mathfrak{sp}_{2N})$ AND ITS OSCILLATOR REPRESENTATION

From now on, we use the better symbol ∂_{ij} instead of $\overline{\partial}_{ij}$ for the differential operators on $\mathrm{Mat}_q(m \times n)$.

4.1. Hopf algebra $U_q(\mathfrak{sp}_{2N})$:

First of all, let us recall that the definition of a quantized enveloping algebra $U_q(\mathfrak{sp}_{2N})$ of the symplectic Lie algebra $\mathfrak{sp}_{2N}$. The algebra $U_q(\mathfrak{sp}_{2N})$ is a Hopf algebra with generators $\hat{e}_i$, $\hat{f}_i$, $k_i^{\pm 1}$ $(1 \le i \le N)$ where $k_i = q^{\varepsilon_i - \varepsilon_{i+1}}$ $(1 \le i \le N-1)$, $k_N = q^{2\varepsilon_N}$ subject to the following three types of relations:

(1) $$\hat{e}_i,\ \hat{f}_i,\ k_i^{\pm 1}\ (1 \le i \le N-1) \quad \text{generate } U_q(\mathfrak{sl}_N),$$

(2)
$$\begin{cases}
k_N \hat{e}_N k_n^{-1} = q^4 \hat{e}_N,\ k_N \hat{e}_{N-1} k_N^{-1} = q^{-2} \hat{e}_{N-1},\ k_{N-1} \hat{e}_N k_{N-1}^{-1} = q^{-2} \hat{e}_N, \\
k_N \hat{f}_N k_n^{-1} = q^{-4} \hat{f}_N,\ k_N \hat{f}_{N-1} k_N^{-1} = q^2 \hat{f}_{N-1},\ k_{N-1} \hat{f}_N k_{N-1}^{-1} = q^2 \hat{f}_N, \\
\hat{e}_N \hat{f}_N - \hat{f}_N \hat{e}_N = \frac{k_N - k_N^{-1}}{q^2 - q^{-2}}, \\
\hat{e}_N^2 \hat{e}_{N-1} - (q + q^{-1}) \hat{e}_N \hat{e}_{N-1} \hat{e}_N + \hat{e}_{N-1} \hat{e}_N^2 = 0, \\
\hat{e}_{N-1}^3 \hat{e}_N - (q^2 + 1 + q^{-2})(\hat{e}_{N-1}^2 \hat{e}_N \hat{e}_{N-1} - \hat{e}_{N-1} \hat{e}_N \hat{e}_{N-1}^2) - \hat{e}_N \hat{e}_{N-1}^3 = 0, \\
\hat{f}_N^2 \hat{f}_{N-1} - (q + q^{-1}) \hat{f}_N \hat{f}_{N-1} \hat{f}_N + \hat{f}_{N-1} \hat{f}_N^2 = 0, \\
\hat{f}_{N-1}^3 \hat{f}_N - (q^2 + 1 + q^{-2})(\hat{f}_{N-1}^2 \hat{f}_N \hat{f}_{N-1} - \hat{f}_{N-1} \hat{f}_N \hat{f}_{N-1}^2) - \hat{f}_N \hat{f}_{N-1}^3 = 0,
\end{cases}$$

and for $1 \le j \le N-2$

(3)
$$\begin{cases}
k_j \hat{e}_N = \hat{e}_N k_j,\ k_j \hat{f}_N = \hat{f}_N k_j, \\
\hat{e}_N \hat{e}_j = \hat{e}_j \hat{e}_N,\ \hat{e}_j \hat{e}_N = \hat{e}_N \hat{e}_j, \\
\hat{f}_N \hat{f}_j = \hat{f}_j \hat{f}_N,\ \hat{f}_j \hat{f}_N = \hat{f}_N \hat{f}_j.
\end{cases}$$

The comultiplication, which we also denote by Δ, is given by

$$\Delta(\hat{e}_j) = \hat{e}_j \otimes 1 + k_j \otimes \hat{e}_j, \quad \Delta(\hat{f}_j) = \hat{f}_j \otimes k_j^{-1} + 1 \otimes \hat{f}_j,$$
$$\Delta(k_j^{\pm 1}) = k_j^{\pm 1} \otimes k_j^{\pm 1}, \quad \text{for } 1 \le j \le N,$$

and denote also its counit and antipode by ε and S (which is uniquely determined by Δ and ε, see e.g. [Ab]), respectively:

$$\varepsilon(\hat{e}_j) = \varepsilon(\hat{f}_j) = 0, \quad \varepsilon(k_j^{\pm 1}) = 1, \quad \text{for } 1 \le j \le N.$$

4.2. The q-oscillator representation of $U_q(\mathfrak{sp}_{2N})$:

Let us consider the q-commutative polynomial ring $\mathcal{A} = \mathbb{K}[t_1, \dots, t_N]$, i.e, the relations $t_i t_j = q t_j t_i \, (i < j)$ hold among variables t_j. Then the (q-)oscillator representation of $U_q(\mathfrak{sp}_{2N})$ is realized on $\mathcal{A}$.

Theorem 4.1. *The action ω on the generators of $U_q(\mathfrak{sp}_{2N})$ defined below gives a left module structure on $\mathcal{A} = \mathbb{K}[t_1, \dots, t_N]$:*

$$\begin{cases} \omega(\hat{e}_i) = \gamma_i \, t_i^{\circ} \partial_{i+1}^R, \quad \omega(\hat{f}_i) = \gamma_i^{-1} t_{i+1}^{\circ} \partial_i^R, \\ \omega(k_i) = \gamma_i \gamma_{i+1}^{-1} \quad (1 \le i \le N-1) \\ \omega(\hat{e}_N) = \frac{1}{[2]} \partial_N^{R\,2}, \quad \omega(\hat{f}_N) = \frac{1}{[2]} t_N^{\circ\,2}, \quad \omega(k_N) = q\gamma_N^2, \end{cases}$$

where $[\ell]$ stands for the q-integer: $[\ell] = \frac{q^\ell - q^{-\ell}}{q - q^{-1}}$. $\square$

Proof: The first three actions $\omega(\hat{e}_i), \omega(\hat{f}_i), \omega(k_i) \, (1 \le i \le N-1)$ just correspond to the part of actions of $U_q(\mathfrak{sl}_N)$ in $U_q(\mathfrak{sp}_{2N})$. Moreover, as well as in the case of $U_{q^2}(\mathfrak{sl}_2)$ (cf. [NUW3]) we have easily find that

$$[\partial_N^{R\,2}, \, t_N^{\circ\,2}] = [2] \frac{q\gamma_N^2 - q^{-1}\gamma_N^{-2}}{q - q^{-1}}.$$

It is easy to see that among the commuting family of those operators keep up their relations. We need therefore only to show that the Serre relations

$$\hat{e}_N^2 \hat{e}_{N-1} - (q + q^{-1}) \hat{e}_N \hat{e}_{N-1} \hat{e}_N + \hat{e}_{N-1} \hat{e}_N^2 = 0,$$
$$\hat{e}_{N-1}^3 \hat{e}_N - (q^2 + 1 + q^{-2})(\hat{e}_{N-1}^2 \hat{e}_N \hat{e}_{N-1} - \hat{e}_{N-1} \hat{e}_N \hat{e}_{N-1}^2) - \hat{e}_N \hat{e}_{N-1}^3 = 0$$

between the operators $\hat{e}_{N-1}$ and $\hat{e}_N$, and those of $\hat{f}_{N-1}$ and $\hat{f}_N$, respectively. Since their proof is just a direct calculation as in [Ha], we omit it. $\square$

We call this representation ω on $\mathcal{A} = \mathbb{K}[t_1, \dots, t_N]$ the *(q-)oscillator representation* of $U_q(\mathfrak{sp}_{2N})$. This is not irreducible but breaks into two irreducible components which consist of the polynomials respectively of even and odd degrees. Since the quantized enveloping algebra $U_{q^2}(\mathfrak{sl}_2)$ is isomorphic to $U_q(\mathfrak{sp}_2)$ as a Hopf algebra, the oscillator representation of $U_{q^2}(\mathfrak{sl}_2)$ constructed in [NUW2],[NUW3] is a special (rank 1) case of the above.

Remark: In the paper [Ha], Hayashi constructed explicitly the oscillator representations of $U_q(\mathfrak{sp}_{2N})$. Our description of the operators utilizes the natural ring structure of representation spaces. This has an advantage when considering their tensor power representations (see Theorem 4.2 below).

Recall the identification under natural multiplication of $\mathcal{A}(\mathrm{Mat}_q(m \times N))$:

$$\mathcal{A}^{\otimes m} = \mathcal{A} \otimes \cdots \otimes \mathcal{A} \xrightarrow{\sim} \mathcal{A}(\mathrm{Mat}_q(m \times N)).$$

Through this identification we obtain naturally the m-fold tensor product representation $\omega^{\otimes m}$ of the oscillator representation ω of $U_q(\mathfrak{sp}_{2N})$ by the comultiplication Δ:

Theorem 4.2. *The actions $\omega^{\otimes m}$ of $U_q(\mathfrak{sp}_{2N})$ on $\mathcal{A}(\mathrm{Mat}_q(m \times N))$ are given by the following equations:*

$$\omega^{\otimes m}(L_{ji}) = (q - q^{-1}) \sum_{k=1}^{m} t^{\circ}_{ki} \partial_{kj} \quad (1 \leq i, j \leq N),$$

$$\omega^{\otimes m}(\hat{e}_N) = \frac{1}{[2]} \sum_{k=1}^{m} q^{k-1} \partial^2_{kN},$$

$$\omega^{\otimes m}(\hat{f}_N) = \frac{1}{[2]} \sum_{k=1}^{m} q^{k-m} t^{\circ 2}_{kN},$$

$$\omega^{\otimes m}(k_N) = q^{m+2\varepsilon_N}.$$

Here the first equations indicates to the $\mathfrak{sl}_N$-part of $\mathfrak{sp}_{2N}$.

Proof: The $\mathfrak{sl}_N$-part is easily seen from the comultiplication rule (1.3.1). Also the last formula for $\omega^{\otimes m}(k_N)$ is clear.

We prove the rest of the assertions by induction on m. For $\hat{e}_N$, we utilize the identification

$$\mathcal{A}^{\otimes m} \simeq \mathcal{A}^{\otimes m-1} \otimes \mathcal{A} \xrightarrow{\sim} \mathcal{A}(\mathrm{Mat}_q((m-1) \times N)) \otimes \mathcal{A}.$$

For $\hat{e}_N$, we first recall its comultiplication:

$$\Delta(\hat{e}_N) = \hat{e}_N \otimes 1 + k_N \otimes \hat{e}_N.$$

Then we have from this and our induction assumption that

$$\begin{aligned} \omega^{\otimes m}(\hat{e}_N) &= \omega^{\otimes m-1}(\hat{e}_N) \otimes 1 + \omega^{\otimes m-1}(k_N) \otimes \omega(\hat{e}_N) \\ &= \frac{1}{[2]} \sum_{k=1}^{m-1} q^{k-1} \overline{\partial^2_{kN}} \otimes 1 + q^{m-1} \gamma^2_N \otimes (\frac{1}{[2]} \partial^2_{mN}). \end{aligned}$$

On the other hand, we have from Proposition 3.2

$$\underline{\partial}_{\ell\, i} = \overline{\partial}_{\ell\, i} \otimes 1 \quad \text{for } \ell < m, \text{ and } \quad \underline{\partial}_{m\, i} = \overline{\gamma}_i \otimes \underline{\partial}_{m\, i},$$

so that

$$\sum_{k=1}^{m} q^{k-1} \partial^2_{kN} = \sum_{k=1}^{m-1} q^{k-1} \overline{\partial}^2_{kN} \otimes 1 + q^{m-1} \overline{\gamma}^2_N \otimes \underline{\partial}^2_{mN}.$$

Combining this with the above expression of $\omega^{\otimes m}(\hat{e}_N)$, we come to the conclusion.

For $\hat{f}_N$, we utilize the identification

$$\mathcal{A}^{\otimes m} \simeq \mathcal{A} \otimes \mathcal{A}^{\otimes m-1} \xrightarrow{\sim} \mathcal{A} \otimes \mathcal{A}(\mathrm{Mat}_q((m-1) \times N)),$$

instead. Then the proof is quite paralell to the case of $\hat{e}_N$. $\square$

5. Quantum Grassmannians and q-hypergeometric series

As another application of our construction of the q-differential operators, we will briefly give a formulation of q-hypergeometric equations. By virtue of our differential operators, we can give a more direct analogue of Gelfand's hypergeometric equation than that given in [N1]. The difference between the formulation in [N1] and ours is implicit as long as the $2 \times n$ case is treated.

We review the main feature of Gelfand's generalized hypergeometric equations introduced in [G] (see also [GZ] and [GZK]) associated with the Grassmannian $G_{m,n}$. Let us consider the matrix space $\mathrm{Mat}(m \times n)$, on which GL_m and GL_n act from the left and right respectively. The system of equations (Gelfand Hypergeometric Equation) consists of three parts: (1) the relative invariance under GL_m from the left; (2) homogeneity under the diagonal subgroup of GL_n from the right; (3) the second order differential equations defined by a family of (pseudo-) Laplacians which bears a finite dimensional representation of GL_n.

With the same spirit, a quantum version of hypergeometric equations was introduced in [N1]. The most difficult point there was the part (3). In that paper, instead of considering the operators $\Delta_{ij}^{sr} = \partial^2/\partial t_{ri}\partial t_{sj} - \partial^2/\partial t_{si}\partial t_{rj}$, Noumi used the operators $\sum_{1\leq r<s\leq m}(t_{ri}t_{sj} - t_{si}t_{rj})\Delta_{ij}^{sr}$ as the classical counterparts. For $m = 2$, the difference between the two is just the multiplication of $t_{1i}t_{2j} - t_{2i}t_{1j}$. The reason why Noumi adopted these operators was that the latter operators are written in terms of the infinitesimal generators of the right action of GL_n, so that these operators can have the quantum counterpart in the action of $U_q(\mathfrak{gl}_n)$. Along this direction, Nakatani studied the case for $m = 3, n = 6$ in [Na]. More general cases are discussed in [NaN] with a little bit different formulation by using affinized coordinates. But these transition are seemingly different from the Gelfand's definition and the relation between the two is not clear even in the classical case.

In the rest of this paper, we propose a direct analogue of the equations in terms of our differential operarors.

5.1. The operators $\Box_{ij}^{sr}$:

As explained above, our concern is to consider the direct analogue of the operators $\partial^2/\partial t_{ri}\partial t_{sj} - \partial^2/\partial t_{si}\partial t_{rj}$. Two points are to be checked here: (1) the compatibility of this family of operators and the other actions of GL_m and the diagonal subgroup of GL_n; (2) how the infinitesimal action of GL_n gives the shift of the parameters. The second one gives so-called contiguity relations. To see these points for our quantum case, we first calculate the commutation relations between our operators and left and right actions of $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$, respectively. Representation-theoretically, this should be given by the behavior of the differential operators ∂_{ij}'s under suitable adjoint actions of $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$. Let us define two adjoint actions ad and $\tilde{ad}$ as follows: for $a \in U_q(\mathfrak{gl}_n)$ (resp. $a \in U_q(\mathfrak{gl}_m)$) and $\varphi \in \mathrm{End}_{\mathbb{K}}(\mathcal{A}(\mathrm{Mat}_q(m \times n)))$,

$$ad(a)\varphi = \sum_i a_i^{(1)}\varphi S(a_i^{(2)}), \qquad \tilde{ad}(a)\varphi = \sum_i a_i^{(2)\circ}\varphi S(a_i^{(1)})^{\circ},$$

where the comultiplication is given as $\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$. Here we distinguish the left and right actions of $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$ by the superscript $^\circ$ for elements

$a \in U_q(\mathfrak{gl}_m)$. It is immediately to see that the former adjoint action ad gives a left action, but the latter $\tilde{ad}$ defines a right action. Note that under these two adjoint actions, the algebra $\mathrm{End}_{\mathbb{K}}(\mathcal{A}(\mathrm{Mat}_q(m \times n)))$ becomes an algebra with both $U_q(\mathfrak{gl}_n)$- and $U_q(\mathfrak{gl}_m)$-symmetries. This means that the unit $\mathbb{K} \to \mathrm{End}_{\mathbb{K}}(\mathcal{A}(\mathrm{Mat}_q(m \times n)))$ and the composition of endmorphisms $\mathrm{End}_{\mathbb{K}}(\mathcal{A}(\mathrm{Mat}_q(m \times n))) \otimes \mathrm{End}_{\mathbb{K}}(\mathcal{A}(\mathrm{Mat}_q(m \times n))) \to \mathrm{End}_{\mathbb{K}}(\mathcal{A}(\mathrm{Mat}_q(m \times n)))$ are $U_q(\mathfrak{gl}_n)$- and $U_q(\mathfrak{gl}_m)$-homomorphisms. Explicitly, these adjoint actions of L-operators are respectively given by

$$ad(L^{\pm}_{ij})\varphi = \sum_k L^{\pm}_{ik}\varphi S(L^{\pm}_{kj}), \qquad \tilde{ad}(L^{\pm}_{ij})\varphi = \sum_k L^{\pm\circ}_{kj}\varphi S(L^{\pm}_{ik})^{\circ}, \tag{5.1.1}$$

and it is often convenient to write these actions in the matrix form as follows:

$$ad(L^{\pm})\varphi = L^{\pm}\varphi S(L^{\pm}), \qquad \tilde{ad}({}^tL^{\pm})\varphi = {}^tL^{\pm}\varphi {}^tS(L^{\pm})^{\circ}.$$

Proposition 5.1. *The differential operators ∂_{ij} bear the contragredient of vector representation under the adjoint actions of $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$ both from the right and the left. The explicit form of the actions are:*

$$\begin{cases} ad(L^{\pm}_1)^t\partial_2 &= (R^{\pm}_{12})^{-1\,t}\partial_2, \\ \tilde{ad}({}^tL^{\pm}_1)^t\partial_2 &= {}^t\partial_2(R^{\pm}_{12})^{-1}. \end{cases}$$

Proof: Since the definition of the differential operators is compatibile with the restriction to smaller matrix spaces, we may assume $m = n$, by embedding to square matrix of bigger size if necessary. Then the proof is seen from the Yang-Baxter equation for L-operators: for $\varphi \in \mathcal{A}(\mathrm{Mat}_q(m \times n))$ and $\epsilon = \pm$,

$$\begin{aligned} ad(L^{\pm}_1)^t\partial^{\epsilon}_2\varphi &= L^{\pm\,t}_1\partial^{\epsilon}_2 S(L^{\pm}_1)\varphi \\ &= (q-q^{-1})^{-1}L^{\pm}_1.(S(T_2)(L^{\epsilon\circ}_2.(S(L^{\pm}_1)\varphi))) \\ &= (q-q^{-1})^{-1}L^{\pm}_1.(S(T_2))(L^{\pm}_1L^{\epsilon\circ}_2.(S(L^{\pm}_1)\varphi)) \\ &= (q-q^{-1})^{-1}(R^{\pm}_{12})^{-1}S(T_2)L^{\epsilon\circ}_2\varphi) \\ &= (R^{\pm}_{12})^{-1\,t}\partial^{\epsilon}_2\varphi. \end{aligned}$$

Since $\partial = \partial^+ - \partial^-$, this calculation clearly gives our first assertion. The second one is similar. $\square$

For $1 \le r < s \le m$, $1 \le i < j \le n$, we put

$$\square^{rs}_{ij} = \partial_{ri}\partial_{sj} - q^{-1}\partial_{rj}\partial_{si}. \tag{5.1.2}$$

Then under the adjoint actions of $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$, the any operator of the form $\square^{rs}_{ij}$ generates a finite dimensional representation. This is clearly seen from Proposition 5.1 and that $\square^{rs}_{ij}$ is in the square tensor of ∂_{ij}'s. We will give more explicit formula for this representation below.

Proposition 5.2. *Let W be the linear space spanned by $\Box_{ij}^{rs}$'s ($1 \leq r < s \leq m$, $1 \leq i < j \leq n$). Then W is closed under the joint adjoint actions of $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$. More precisely, W is equivalent to $\lambda_n^{*(2)} \otimes {}^t\lambda_m^{*(2)}$ as a $U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m)$-module. Here $\lambda_n^{*(2)}$ or ${}^t\lambda_m^{*(2)}$ respectively stands for the exterior square of the contragredient of vetctor representation of $U_q(\mathfrak{gl}_n)$ or $U_q(\mathfrak{gl}_m)$.*

Proof: Considering the q-exterior algebra and Proposition 5.1 above, we can prove the assertion with the same technique as in §3 of [NUW1]. In fact, since ad (resp. $\tilde{ad}$) gives a structure of $U_q(\mathfrak{gl}_n)$-symmmetry (resp. $U_q(\mathfrak{gl}_m)$-symmetry), from Proposition 5.1 we see

$$ad(L_0^{\pm})^t\partial_1\,{}^t\partial_2 = (R_{01}^{\pm})^{-1}(R_{02}^{\pm})^{-1\,t}\partial_1\,{}^t\partial_2, \quad \tilde{ad}({}^tL_0^{\pm})^t\partial_1\,{}^t\partial_2 = {}^t\partial_1\,{}^t\partial_2(R_{01}^{\pm})^{-1}(R_{02}^{\pm})^{-1}.$$

Thus, instead of showing the detailed calculations, we just present the explicit results:

$$\begin{aligned} ad(L_{\alpha\beta}^{+})\Box_{ij}^{rs} &= q^{-\delta_{\alpha i}-\delta_{\alpha j}}\delta_{\alpha\beta}\Box_{ij}^{rs} \\ &\quad -(q-q^{-1})\{\delta_{\alpha<i}\delta_{\beta i}\Box_{\alpha j}^{rs} - q\delta_{\alpha<i}\delta_{\beta j}\Box_{\alpha i}^{rs} + \delta_{i<\alpha<j}\delta_{\beta j}\Box_{i\alpha}^{rs}\}, \qquad (5.1.3) \\ ad(L_{\alpha\beta}^{-})\Box_{ij}^{rs} &= q^{\delta_{\alpha i}+\delta_{\alpha j}}\delta_{\alpha\beta}\Box_{ij}^{rs} \\ &\quad +(q-q^{-1})\{\delta_{\alpha>j}\delta_{\beta j}\Box_{i\alpha}^{rs} - q^{-1}\delta_{\alpha>j}\delta_{\beta i}\Box_{j\alpha}^{rs} + \delta_{j>\alpha>i}\delta_{\beta i}\Box_{\alpha j}^{rs}\}, \qquad (5.1.4) \\ \tilde{ad}(L_{\alpha\beta}^{+})\Box_{ij}^{rs} &= q^{-\delta_{\beta r}-\delta_{\beta s}}\delta_{\alpha\beta}\Box_{ij}^{rs} \\ &\quad -(q-q^{-1})\{\delta_{\beta>s}\delta_{\alpha s}\Box_{ij}^{r\beta} - q\delta_{\beta>s}\delta_{\alpha r}\Box_{ij}^{s\beta} + \delta_{s>\beta>r}\delta_{\alpha r}\Box_{ij}^{\beta s}\}, \qquad (5.1.5) \\ \tilde{ad}(L_{\alpha\beta}^{-})\Box_{ij}^{rs} &= q^{\delta_{\beta r}+\delta_{\beta s}}\delta_{\alpha\beta}\Box_{ij}^{rs} \\ &\quad +(q-q^{-1})\{\delta_{\beta<r}\delta_{\alpha r}\Box_{ij}^{\beta s} - q^{-1}\delta_{\beta<r}\delta_{\alpha s}\Box_{ij}^{\beta r} + \delta_{s>\beta>r}\delta_{\alpha s}\Box_{ij}^{r\beta}\}. \qquad (5.1.6) \end{aligned}$$

Remark: Since $\Box_{ij}^{rs} = -q^{-1}(\partial_{rj}\partial_{si} - q\partial_{sj}\partial_{ri})$, the correspondence between left and right actions are given by the relpacement

$$\alpha \longleftrightarrow \beta; \quad q \longleftrightarrow q^{-1}, \quad + \longleftrightarrow -; \quad i \longleftrightarrow r, \quad j \longleftrightarrow s,$$

and of roles of super- and sub-scripts of $\Box_{ij}^{rs}$ coming from indices of colums and rows, respectively.

Theorem 5.3. *Under the adjoint actions of $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$, the linear space W spanned by the operators $\Box_{ij}^{rs}$'s ($1 \leq r < s \leq m$, $1 \leq i < j \leq n$) bears a finite dimensional representation respectively.* □

This theorem is in fact clear from the following commutation relations between L-operators $L_{ij}^{\pm}$'s and pseudo Laplacians $\Box_{ij}^{rs}$'s.

$$\begin{aligned} &\Box_{ij}^{rs}L_{\alpha\beta}^{+} - q^{\delta_{\alpha i}+\delta_{\alpha j}}L_{\alpha\beta}^{+}\Box_{ij}^{rs} \\ &=(q-q^{-1})\{\delta_{i>\alpha}L_{i\beta}^{+}\Box_{\alpha j}^{rs} + \delta_{j>\alpha>i}L_{j\beta}^{+}\Box_{i\alpha}^{rs} - q^{-1}\delta_{i>\alpha}L_{j\beta}^{+}\Box_{\alpha i}^{rs}\} \end{aligned} \qquad (L1)$$

$$\begin{aligned} &\Box_{ij}^{rs}L_{\alpha\beta}^{-} - q^{-\delta_{\alpha i}-\delta_{\alpha j}}L_{\alpha\beta}^{-}\Box_{ij}^{rs} \\ &=(q-q^{-1})\{-\delta_{\alpha>j}L_{j\beta}^{-}\Box_{i\alpha}^{rs} - \delta_{j>\alpha>i}L_{i\beta}^{-}\Box_{\alpha j}^{rs} + q\delta_{\alpha>j}L_{i\beta}^{-}\Box_{j\alpha}^{rs}\}, \end{aligned} \qquad (L2)$$

for the left actions of $L^{\pm} \in U_q(\mathfrak{gl}_n)$,

(R1)
$$\begin{aligned}&\Box^{rs}_{ij}L^{+\circ}_{\alpha\beta} - q^{\delta_{r\beta}+\delta_{s\beta}}L^{+\circ}_{\alpha\beta}\Box^{rs}_{ij}\\ =&(q-q^{-1})\{\delta_{s<\beta}L^{+\circ}_{\alpha s}\Box^{r\beta}_{ij} + \delta_{r<\beta<s}L^{+\circ}_{\alpha r}\Box^{\beta s}_{ij} - q^{-1}\delta_{s<\beta}L^{+\circ}_{\alpha r}\Box^{s\beta}_{ij}\}\end{aligned}$$

(R2)
$$\begin{aligned}&\Box^{rs}_{ij}L^{-\circ}_{\alpha\beta} - q^{-\delta_{r\beta}-\delta_{s\beta}}L^{-\circ}_{\alpha\beta}\Box^{rs}_{ij}\\ =&(q-q^{-1})\{-\delta_{r>\beta}L^{-\circ}_{\alpha r}\Box^{rs}_{ij} - \delta_{s>\beta>r}L^{-\circ}_{\alpha s}\Box^{r\beta}_{ij} + q\delta_{r>\beta}L^{-\circ}_{\alpha s}\Box^{\beta r}_{ij}\},\end{aligned}$$

for the right actions of $L^{\pm} \in U_q(\mathfrak{gl}_m)$,

where we put

$$\delta_{\alpha>\beta} = \begin{cases} 1 & \text{for } \alpha > \beta, \\ 0 & \text{otherwise} \end{cases}$$

etc.

The proof of the above commutation relations are done by a small amount of calculations by using the following commutation relations:

Lemma 5.4. *The commutation relations between the L-operators and the differential operators are given as follows:*

$$\begin{aligned}R^{\pm}L^{\pm t}_1\partial_2 &= {}^t\partial_2 L^{\pm}_1,\\ L^{\pm\circ t}_1\partial_2 R^{\pm} &= {}^t\partial_2 L^{\pm\circ}_1.\end{aligned}$$

Proof: Direct calculation from the definition. Since their proofs are similar to Proposition 4.3 in [NUW1], we leave them to the readers. □

Remarks: (1) When we apply this lemma to the proof of the formulas (L1), (L2) and (R1), (R2), the following paraphrased formulas are used:

$$\begin{aligned}\partial_{\beta i}L^{\pm}_{j\alpha} &= q^{\pm\delta_{ij}}L^{\pm}_{j\alpha}\partial_{\beta i} \pm (q-q^{-1})\delta_{j\lesseqgtr i}L^{\pm}_{i\alpha}\partial_{\beta j},\\ \partial_{\beta i}L^{\pm\circ}_{j\alpha} &= q^{\pm\delta_{\alpha\beta}}L^{\pm\circ}_{j\alpha}\partial_{\beta i} \pm (q-q^{-1})\delta_{\beta\lesseqgtr\alpha}L^{\pm\circ}_{j\beta}\partial_{\alpha i}.\end{aligned}$$

(2) For the proof of Theorem 5.3, the first formula is for the size n (left action), and the latter for m (right action), respectively.

(3) Let $\Delta(a) = \sum_i a^{(1)}_i \otimes a^{(2)}_i$ be the comultiplication formula for $a \in U_q(\mathfrak{gl}_n)$. Then for an operator $w \in \mathrm{End}_{\mathbb{K}}(\mathcal{A}(\mathrm{Mat}_q(m\times n)))$, we have

$$wa = \sum_i a^{(2)}_i ad(S^{-1}(a^{(1)}_i))w.$$

From this formula, we can prove Theorem 5.3 by using formulas of two adjoint actions of L-operators given in Proposition 5.2. Actually, the commutation relations (L1), (L2) and (R1), (R2) can be seen from the explicit descriptions of the adjoint actions of another L-operator $\tilde{L}^{\pm}$ defined by $\tilde{L}^{\pm}_{ij} = S(L^{\mp}_{ji})$ (cf. [NUW1], Appendix B). That is,

(L1), (L2) (resp.(R1), (R2)) are directly reduced from the explicit formulas of $\Box_{ij}^{rs}$'s under two actions $ad(\tilde{L}^{\pm})$ (resp. $\tilde{ad}({}^t\tilde{L}^{\pm})$). However if you compare the explicit forms, you will soon find the latter formulas (L1), (L2) and (R1), (R2) are more convenient than the formulas of the adjoint actions of $L_{ij}^{\pm}$'s, at least for proving the theorem.

5.2. Quantum hypergeometric equation:

As Theorem 5.3 means both the compatibility and the contiguity relations, we propose the following equations as a quantum version of the (generalized) hypergeometric equations.

Definition. *(QHGE)*

$$L_{rs}^{\pm\circ}.\Phi = \delta_{rs} q^{\mp\delta_{rs}}\Phi \quad (1 \leq r, s \leq m), \tag{1}$$

$$L_{ii}^{\pm}.\Phi = q^{\pm\lambda_i}\Phi \quad (1 \leq i \leq n), \tag{2}$$

$$\Box_{ij}^{rs}\Phi = 0 \quad (1 \leq r, s \leq m,\ 1 \leq i, j \leq n), \tag{3}$$

where the parameters λ_i satisfy $\sum_{i=1}^{n} \lambda_i = m$. $\square$

Remark: The relations (R1) and (R2) in Theorem 5.1 guarantee the compatibility of the equations (1) and (3). The relations (L1) and (L2) in Theorem 5.1 shows that the algebra $U_q(\mathfrak{gl}_n)$ stabilizes the set of the solutions to the equations (1) and (3). The commutation relations of $L_{ii}^{\pm}$ and $L_{ab}^{\pm}$ in $U_q(\mathfrak{gl}_n)$ implies the contiguity relations, i.e, how the parameters λ_i's get shifted under the action of $U_q(\mathfrak{gl}_n)$. The last remark is just the same as clarified in [N1]. In the similar way, we may also formulate the contiguity relations of a quantum version of the generalized hypergeometric system of "confluent type" using the other maxmal abelian subalgebra of $U_q(\mathfrak{gl}_n)$ instead of the equation (2), as classicaly discussed in [KHT].

Remark: The relations among $\Box_{ij}^{rs}$'s and L-operators reduce the numbers of equations in (3). In fact, we can verify the following assertions:

Lemma 5.5. *Suppose $m \leq n$. For fixed i, j, the following implications hold:*

$$\Box_{ij}^{r\,r+1}\Phi = 0 \quad \textit{for } r = 1, 2, \cdots, m-1 \Longrightarrow \Box_{ij}^{rs}\Phi = 0 \quad \textit{for any } 1 \leq r, s \leq m.$$

Also for fixed r, we have

$$\Box_{j\,j+1}^{rm}\Phi = 0 \quad \textit{for } j = m+1, \cdots, n-1 \Longrightarrow \Box_{ij}^{rm}\Phi = 0 \quad \textit{for any } 1 \leq i, j \leq n. \quad \square$$

Moreover, if we consider the actions from the right and left, we see a version of second order Capelli identities:

Proposition 5.6. *Suppose that* $1 \leq r < s \leq m, 1 \leq i < j \leq n$. *Let* $L(\lambda) = \lambda L^+ - (\lambda)^{-1} L^-$ *denotes the* L*-operator with a spectral parameter* λ. *Then we have*

$$\begin{aligned} &L_{ir}(q)L_{js}(1) - q^{-1}L_{is}(q)L_{jr}(1) \\ &\qquad = q(q-q^{-1})^2 \sum_{1\leq\alpha<\beta\leq m} \det{}_q \begin{bmatrix} t_{\alpha r} & t_{\alpha s} \\ t_{\beta r} & t_{\beta s} \end{bmatrix}^{\circ} \Box_{ij}^{\alpha\beta}, \\ &L_{ir}(q)^{\circ}L_{js}(1)^{\circ} - qL_{is}(q)^{\circ}L_{jr}(1)^{\circ} \\ &\qquad = q(q-q^{-1})^2 \sum_{1\leq\alpha<\beta\leq n} \det{}_q \begin{bmatrix} t_{i\alpha} & t_{i\beta} \\ t_{j\alpha} & t_{j\beta} \end{bmatrix} \Box_{\alpha\beta}^{rs}, \end{aligned}$$

where $^{\circ}$ *indicates the action from the right.*

Proof: The second identity is easily obtained by the same method using the q-exterior algebra as in the proof of the Capelli identity (see [NUW1], §3). In contrast with this, the first identity can be still proven in the quite similar way but we use the q^{-1}-exterior algebra instead. The detailed calculations are left to the readers. □

Remarks: (1) Put

$$\begin{aligned} C^{rs}(\lambda) &= L_{rr}(q\lambda)L_{ss}(\lambda) - q^{-1}L_{rs}(q\lambda)L_{sr}(\lambda), \\ C_{ij}(\lambda) &= L_{ii}(q\lambda)^{\circ}L_{jj}(\lambda)^{\circ} - qL_{ji}(q\lambda)^{\circ}L_{ij}(\lambda)^{\circ}, \end{aligned}$$

where λ is a spectral parameter. Then, in particular, the above proposition asserts:

$$\begin{aligned} C^{rs}(1) &= q(q-q^{-1})^2 \sum_{1\leq\alpha<\beta\leq m} \det{}_q \begin{bmatrix} t_{\alpha r} & t_{\alpha s} \\ t_{\beta r} & t_{\beta s} \end{bmatrix}^{\circ} \Box_{rs}^{\alpha\beta}, \\ C_{ij}(1) &= q(q-q^{-1})^2 \sum_{1\leq\alpha<\beta\leq n} \det{}_q \begin{bmatrix} t_{i\alpha} & t_{i\beta} \\ t_{j\alpha} & t_{j\beta} \end{bmatrix} \Box_{\alpha\beta}^{ij}, \end{aligned}$$

respectively. From this formula, we see that solutions to our equations also satisfy the equations defined in [N1] and [Na], because there the equations $C_{ij}(1)\Phi = 0\,(1 \leq i < j \leq n)$ are employed instead of (3) in QHGE.

(2) Higher order variants of Capelli-like identities, similarly to Proposition 5.6, for the shifted quantum determinant of L-operators can be obtained by the same way as above.

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