A topological characterization of **SU**

Ken-ichi TOKUNAGA
Department of Mathematics,
Faculty of Science,
Kyoto University.

May 27, 1994

1 Introduction

Let **SU** be the special unitary group of infinite deminsion. The cohomology ring of **SU**

$$H^*(\mathbf{SU};\mathbf{Z}) = \bigwedge(x_3, x_5, \cdots, x_{2i+1}, \cdots)$$

and Bott periodicity theorem [4] of complex case:

$$\mathbf{BU} \simeq \Omega\mathbf{SU}, \mathbf{U} \simeq \Omega\mathbf{BU}$$

are well known. In this paper, we shall show a topological characterization of **SU** using above two results.

Theorem 1.1 *Let X be a finite type CW complex which is a homotopy associative H-space. If X has following properties:*

(1) as an algebra $H^*(X;\mathbf{Z}) = \bigwedge(x_3, x_5, \cdots, x_{2i+1}, \cdots)$
where $|x_{2i+1}| = 2i+1$ and each x_{2i+1} is a primitive element ;

(2) there exist two maps

$$\begin{aligned} j' : \Sigma\mathbf{CP}^\infty &\longrightarrow X \\ \lambda' : \Sigma^2 X &\longrightarrow X \end{aligned}$$

such that

(a) $j'^* : H^*(X;\mathbf{Z}) \longrightarrow H^*(\Sigma\mathbf{CP}^\infty;\mathbf{Z})$ *is epic and*
(b) $(\lambda' \circ (\Sigma^2 j'))^* : x_{2i+1} \longmapsto \pm i s_3 t^{i-1}$,

where s_3 *is a generator of* $H^*(S^3;\mathbf{Z})$ *and* t *is a generator of* $H^*(\mathbf{CP}^\infty;\mathbf{Z})$.

Then the map

$$\widetilde{Ad^2\lambda'} : X \to \Omega^2(X\langle 3\rangle)$$

is a homotopy equivalence.

We denote the 3-connected fiber space of X by $X\langle 3\rangle$. Ad is a natural equivalence

$$[\Sigma A, B] \xrightarrow{\sim} [A, \Omega B].$$

Then the map $Ad^2\lambda'$ induces $\widetilde{Ad^2\lambda'} : X \to \Omega^2(X\langle 3\rangle)$.

Theorem 1.2 *If* X *satisfies above conditions, then* X *is homotopy equivalent to* $\mathbf{SU}$.

In Section 2, we will prove Theorem 1.1. The two map j' and λ' are naturally constructed in case of $\mathbf{SU}$. This is discussed in Section 3. Miller [7] showed that $\mathbf{U}$ is stably homotopy equivalent to $\bigvee_{k\geq 1}\mathbf{BU}_k^{ad_k}$. Moreover when $k=1$, the inclusion map $\mathbf{BU}_1^{ad_1} \to \mathbf{U}$ is induced by an unstable map. Making use of these facts, we will build a homotopy equivalent map $\mathbf{SU} \to X$ in Section 4.

Through this paper, we denote $s_n \in H^*(S^n;\mathbf{Z})$ and $t \in H^*(\mathbf{CP}^\infty;\mathbf{Z})$ to be generators of cohomology rings. And let a canonical line bundle over $\mathbf{CP}^1$ (resp. $\mathbf{CP}^\infty$) be η (resp. η_∞) whose first Chern class is s_2 (resp. t). Let q be the natural inclusion map $\mathbf{SU} \to \mathbf{U}$ which identifies $x_{2i+1} \in H^*(\mathbf{SU};\mathbf{Z})$ and $H^*(\mathbf{U};\mathbf{Z})$.

I would like to thank Professor Akira Kono for a lot of advice and encouragement.

2 The periodicity

In this section, we shall prove Theorem 1.1.

First we describe elementary results from the general theory of spectral sequences and Hopf-algebras. Let X be a finite type CW complex. We assume that X is an associative H-space,

$$H^*(X;\mathbf{Z}) = \bigwedge(x_3, x_5, \cdots, x_{2i+1}, \cdots),$$

and each x_{2i+1} is a primitive element of degree $2i+1$.

Lemma 2.1 *We get the following properties.*

(1) $H_*(X;\mathbf{Z}) = \bigwedge(\beta_3, \beta_5, \cdots, \beta_{2i+1}, \cdots)$
where β_{2i+1} *is the dual of* x_{2i+1} *and a primitive element for each* i.

(2) $H_*(\Omega X;\mathbf{Z}) = \mathbf{Z}[b_2, b_4, \cdots, b_{2i}, \cdots]$
where $b_{2i} = \tau(\beta_{2i+1})$ *is the image of the homology transgression map of* β_{2i+1}.

Moreover we asuume that there exists a map

$$j' : \Sigma\mathbf{CP}^\infty \longrightarrow X$$

such that $j'^* : H^*(X;\mathbf{Z}) \to H^*(\Sigma\mathbf{CP}^\infty;\mathbf{Z})$ *is epic. Then we get*

(3) the coproduct of b_{2i} *is*

$$\triangle_*(b_{2i}) = \sum_{j+k=i} b_{2j} \otimes b_{2k}.$$

See [8] for the proofs of (1) and (2).

Proof of (3). We define $\tau_i \in H_{2i}(\mathbf{CP}^\infty;\mathbf{Z})$ to be the dual of $t^i \in H^{2i}(\mathbf{CP}^\infty;\mathbf{Z})$. From assumption we get

$$Adj'_*(\tau_i) = b_{2i}$$

by changing the sign of b_{2i} if necessary. By definition, the coproduct of τ_i is

$$\triangle_*(\tau_i) = \sum_{j+k=i} \tau_j \otimes \tau_k.$$

Adj'_* is a homomorphism of coalgebra. Hence we come to a conclusion. ∎

Lemma 2.2 *On condition that Lemma 2.1 (3) holds, we obtain the following properties.*

(1) $H^*(\Omega X;\mathbf{Z}) = \mathbf{Z}[y_2, y_4, \cdots, y_{2i}, \cdots]$
where y_{2i} *is the dual of* ${b_2}^i$ *and the coproduct of* y_{2i} *is*

$$\mu(y_{2i}) = \sum_{j+k=i} y_{2j} \otimes y_{2k}.$$

(2) $H^*(\Omega^2 X;\mathbf{Z}) = \bigwedge(z_1, z_3, \cdots, z_{2i-1}, \cdots)$
where $z_{2i-1} = \sigma(y_{2i})$ *is the image of the cohomology suspension map of* y_{2i} *and* z_{2i-1} *is a primitive element for each* i.

Next we compute the image of the cohomology suspension of x_{2i+1}.

Lemma 2.3 *Let* $t_1, t_2, \cdots, t_n$ *be indeterminate elements and* σ_i *its* i*-th fundamental symmetric function. We define an integral polynomial* S_n *of* n *variables by*

$$t_1^n + t_2^n + \cdots + t_n^n = S_n(\sigma_1, \sigma_2, \cdots, \sigma_n).$$

Then the image of the cohomology suspension map

$$\sigma : H^{2i+1}(X;\mathbf{Z}) \longrightarrow H^{2i}(\Omega X;\mathbf{Z})$$

of x_{2i+1} *is*

$$\sigma(x_{2i+1}) = S_i(y_2, y_4, \cdots, y_{2i}).$$

Proof. By [8] Chap.4 Lemma 2.16, $\sigma(x_{2i+1})$ is a primitive element. By [8] Chap.4 Theorem 2.18, the submodule of all primitive elements of $H^{2i}(\Omega X;\mathbf{Z})$ is generated by $S_i(y_2, y_4, \cdots, y_{2i})$. Then there exists an integer a such that

$$\sigma(x_{2i+1}) = aS_i(y_2, y_4, \cdots, y_{2i}).$$

We compute by Newton formula,

$$S_i(y_2, y_4, \cdots, y_{2i}) = y_2^i + \text{ other terms orthogonal to } b_{2i}.$$

Then

$$\begin{aligned} a &= a\langle S_i(y_2, y_4, \cdots, y_{2i}), b_{2i}\rangle \\ &= \langle \sigma(x_{2i+1}), b_{2i}\rangle \\ &= \langle x_{2i+1}, \beta_{2i+1}\rangle \\ &= 1. \end{aligned}$$

Proof of Theorem 1.1. First we denote

$$\begin{array}{rcccl} & k & = & \lambda' \circ \Sigma^2 j' & : \Sigma^3 \mathbf{CP}^\infty \to \Sigma^2 X \to X \\ k_1 = & Adk & = & Ad\lambda' \circ \Sigma j' & : \Sigma^2 \mathbf{CP}^\infty \to \Sigma X \to \Omega X \\ k_2 = & Adk_1 & = & Ad^2\lambda' \circ j' & : \Sigma^1 \mathbf{CP}^\infty \to X \to \Omega^2 X. \end{array}$$

We shall show that the image of k_2^* includes all generators of $H^*(\Sigma \mathbf{CP}^\infty; \mathbf{Z})$. Consider the following commutative diagram:

$$\begin{array}{ccc} H^{2i+1}(X; \mathbf{Z}) & \xrightarrow{k^*} & H^{2i+1}(\Sigma^3 \mathbf{CP}^\infty; \mathbf{Z}) \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \text{suspension iso.}} \\ H^{2i}(\Omega X; \mathbf{Z}) & \xrightarrow{k_1^*} & H^{2i}(\Sigma \mathbf{CP}^\infty; \mathbf{Z}) \end{array}$$

By condition (2b) and Lemma 2.3, we obtain

$$k_1^*(S_i(y_2, y_4, \cdots, y_{2i})) = \pm i s_2 t^i.$$

Since in $\tilde{H}^*(\Sigma^2 \mathbf{CP}^\infty; \mathbf{Z})$ cup products vanish, k_1^* maps decomposable elements to 0. By Newton formula, we have

$$S_i(y_2, y_4 \cdots, y_{2i}) = \pm i y_{2i} + \text{ (decomposable elements)}.$$

Then

$$\begin{array}{rcl} \pm k_1^*(i y_{2i}) & = & \pm i s_2 t^i \\ k_1^*(y_{2i}) & = & \pm s_2 t^i. \end{array}$$

Consider the following commutative diagram:

$$\begin{array}{ccc} H^{2i}(\Omega X; \mathbf{Z}) & \xrightarrow{k_1^*} & H^{2i}(\Sigma^2 \mathbf{CP}^\infty; \mathbf{Z}) \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \text{suspension iso.}} \\ H^{2i-1}(\Omega^2 X; \mathbf{Z}) & \xrightarrow{k_2^*} & H^{2i-1}(\Sigma \mathbf{CP}^\infty; \mathbf{Z}) \end{array}$$

Then we obtain

$$k_2^*(z_{2i-1}) = \pm s_1 t^i.$$

Consequently k_2^* is an epimorphism. Condition (2a) implies that j'^* is a bijection between generators of algebras. Then Image $(Ad^2\lambda')^*$ includes all generators $\{x_{2i+1}\}$ of algebra $H^*(X;\mathbf{Z})$. So does $\widetilde{Ad^2\lambda'}^* : H^*(\Omega^2(X\langle 3\rangle);\mathbf{Z}) \to H^*(X;\mathbf{Z})$. By [6] Lemma 2.2, $\widetilde{Ad^2\lambda'}^*$ is an isomorphism at each degree. Since both sides are free $\mathbf{Z}$ modules, $\widetilde{Ad^2\lambda'}_*$ is an isomorphism also. Since X and $\Omega^2X\langle 3\rangle$ are simply connected, $\widetilde{Ad^2\lambda'}$ is a homotopy equivalence by J.H.C.Whitehead's theorem. ∎

3 Periodicity theorem for SU

We shall prove periodicity theorem for **SU** by using Theorem 1.1. In other words, we shall construct two maps j'_{SU} and λ'_{SU} which satisfy conditions of Theorem 1.1.

Let ξ_n be the universal vector bundle over $\mathbf{BU}_n$. We define

$$\xi := \varinjlim(\xi_n - n),$$

where n is a trivial vector bundle of n dimension. Let λ be the classifying map of $(\eta - 1)\hat{\otimes}\xi \in \tilde{K}(\Sigma^2\mathbf{BU})$. This map λ has a lift $\tilde{\lambda} : \Sigma^2\mathbf{BU} \to \mathbf{BSU}$.

We now come to construct two maps j'_{SU} and λ'_{SU}. We define

$$j'_{\mathrm{SU}} := Ad\tilde{\lambda} \circ (\Sigma j) \tag{1}$$

$$\lambda'_{\mathrm{SU}} := Ad\tilde{\lambda} \circ (\Sigma q) \circ (\Sigma\pi) \tag{2}$$

where $q : \mathbf{BSU} \to \mathbf{BU}$ is a natural inclusion and denote $\pi : \Sigma\Omega\mathbf{BSU} \to \mathbf{BSU}$ to be $Ad^{-1}(1_{\Omega\mathrm{BSU}})$.

Lemma 3.1 *The map*

$$j'_{\mathrm{SU}}{}^* : H^*(\mathbf{SU};\mathbf{Z}) \longrightarrow H^*(\Sigma\mathbf{CP}^\infty;\mathbf{Z})$$

is an epimorphism.

Proof . First we consider the following commutative diagram:

$$\begin{array}{ccc} H^*(\mathbf{BSU};\mathbf{Z}) & \xrightarrow{Ad^{-1}j'_{\mathrm{SU}}{}^*} & H^*(\Sigma^2\mathbf{CP}^\infty;\mathbf{Z}) \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \text{suspension iso.}} \\ H^{*-1}(\mathbf{SU};\mathbf{Z}) & \xrightarrow{j'_{\mathrm{SU}}{}^*} & H^{*-1}(\Sigma\mathbf{CP}^\infty;\mathbf{Z}). \end{array}$$

By (1), $Ad^{-1}j'_{\mathrm{SU}} = \tilde{\lambda} \circ (\Sigma^2 j)$ and $q \circ \tilde{\lambda} = \lambda$, the proof completes if the map

$$H^*(\mathbf{BU}; \mathbf{Z}) \xrightarrow{\lambda} H^*(\Sigma^2\mathbf{BU}; \mathbf{Z}) \xrightarrow{\Sigma^2 j} H^*(\Sigma^2\mathbf{CP}^\infty; \mathbf{Z}) \tag{3}$$

is epic. We will calculate the image of total Chern class $c = 1 + c_1 + c_2 + \cdots \in H^*(\mathbf{BU}; \mathbf{Z})$. This is just the total Chern class of the virtual bundle $(\eta - 1)\hat{\otimes}(\eta_\infty - 1)$ which the map $\lambda \circ \Sigma^2 j$ classifies. Since

$$c((\eta - 1)\hat{\otimes}(\eta_\infty - 1)) = 1 + \sum_{i=1}^{\infty} (-1)^i s_2 t^i$$

(See [6]),we obtain

$$(\lambda \circ \Sigma^2 j)^*(c_{i+1}) = (-1)^i s_2 t^i.$$

∎

Lemma 3.2 $(\lambda'_{\mathrm{SU}} \circ \Sigma^2 j'_{\mathrm{SU}})^*(x_{2i+1}) = \pm i s_3 t^{i-1}$

Proof . Since $\sigma(c_{i+1}) = x_{2i+1}$, suffice it to say that

$$(Ad^{-1}\lambda'_{\mathrm{SU}} \circ \Sigma^3 j'_{\mathrm{SU}})^*(c_{i+1}) = \pm i s_4 t^{i-1}.$$

By (1) and (2) we can easily get the next factorizations,

$$\Sigma^3 j'_{\mathrm{SU}} = \Sigma^3 Ad\tilde{\lambda} \circ \Sigma^4 j$$

$$Ad^{-1}\lambda'_{\mathrm{SU}} = \tilde{\lambda} \circ \Sigma^2 q \circ \Sigma^2 \pi.$$

Then we get

$$\begin{aligned} Ad^{-1}\lambda'_{\mathrm{SU}} \circ \Sigma^3 j'_{\mathrm{SU}} &= \tilde{\lambda} \circ \Sigma^2 q \circ \Sigma^2 \pi \circ \Sigma^3 Ad\tilde{\lambda} \circ \Sigma^4 j \\ &= \tilde{\lambda} \circ \Sigma^2 q \circ \Sigma^2 \tilde{\lambda} \circ \Sigma^4 j \\ &= \tilde{\lambda} \circ \Sigma^2 \lambda \circ \Sigma^4 j \end{aligned}$$

We will calculate the total Chern class of the virtual bundle $(\eta - 1)\hat{\otimes}(\eta - 1)\hat{\otimes}(\eta_\infty - 1)$ over $\Sigma^4\mathbf{CP}^\infty \cong \Sigma^2\Sigma^2\mathbf{CP}^\infty$ classified by $\lambda \circ \Sigma^2\lambda \circ \Sigma^4 j$. We write

$s \in H^2(S^2;\mathbf{Z})$ to be the first component of cohomology of $S^2 \wedge S^2$ and s' to be the second. Note that $s = c_1(\eta\hat{\otimes}1)$, $s' = c_1(1\hat{\otimes}\eta)$, and recall that $t = c_1(\eta_\infty)$.

$$
\begin{aligned}
& c((\eta-1)\hat{\otimes}(\eta-1)\hat{\otimes}(\eta_\infty-1)) \\
= \;& c(\eta\hat{\otimes}\eta\hat{\otimes}\eta_\infty)c(1\hat{\otimes}\eta\hat{\otimes}\eta_\infty)^{-1}c(\eta\hat{\otimes}1\hat{\otimes}\eta_\infty)^{-1}c(\eta\hat{\otimes}\eta\hat{\otimes}1)^{-1} \\
& c(\eta\hat{\otimes}1\hat{\otimes}1)c(1\hat{\otimes}\eta\hat{\otimes}1)c(1\hat{\otimes}1\hat{\otimes}\eta_\infty) \\
= \;& (1+s+s'+t)(1+s'+t)^{-1}(1+s+t)^{-1}(1+s+s')^{-1} \\
& (1+s)(1+s')(1+t) \\
= \;& (1+s)(1+s(1+s'+t)^{-1})(1-s(1+s+t)^{-1})(1-s(1+s+s')^{-1}) \\
= \;& 1+s+s(1+s'+t)^{-1}-s(1+s+t)^{-1}-s(1+s+s')^{-1} \\
= \;& 1+s \\
& +s(1+\sum_{i=1}^{\infty}(-1)^i(s'+t)^i) \\
& -s(1+\sum_{i=1}^{\infty}(-1)^i(s+t)^i) \\
& -s(1+\sum_{i=1}^{\infty}(-1)^i(s+s')^i) \\
= \;& 1+s \\
& +s(1+\sum_{i=1}^{\infty}(-1)^i(t^i+is't^{i-1})) \\
& -s(1+\sum_{i=1}^{\infty}(-1)^i(t^i+ist^{i-1})) \\
& -s(1-s-s'+2ss') \\
= \;& 1+\sum_{i=2}^{\infty}(-1)^i iss't^{i-1},
\end{aligned}
$$

because $s^2 = 0$ and $s'^2 = 0$. Thus we obtain

$$(Ad^{-1}\lambda'_{\mathrm{SU}} \circ \Sigma^3 j'_{\mathrm{SU}})^*(c_{i+1}) = (-1)^i i s_4 t^{i-1}$$

where $s_4 \in H^4(S^4;\mathbf{Z})$ is $ss' \in H^4(S^2 \wedge S^2;\mathbf{Z})$. ∎

4 Characterization theorem

As a matter of fact, the condition of Theorem 1.1 gives characterization of **SU**. In this section we shall prove it.

In [7] and [5], there exists a stable homotopy equivalence

$$\mathbf{U} \cong \bigvee_{k \geq 1} \mathbf{BU}_k^{ad_k}.$$

When $k = 1$, the inclusion map is induced by an unstable map

$$\mathbf{BU}_1^{ad_1} \cong \Sigma\mathbf{CP}^\infty \xrightarrow{j} \mathbf{U}.$$

Remark that $\mathbf{BU}_1 \simeq \mathbf{CP}^\infty$ and its adjoint representaion of $\mathbf{U}_1$ is trivial. Thus its Thom space is $\Sigma\mathbf{CP}^\infty$. There exists a stable map $p : \mathbf{U} \to \Sigma\mathbf{CP}^\infty$ such that $p \circ (\Sigma^\infty j) = id$. Note that j^* is epic in cohomology rings. We will use the same symbol for expressing the corresponding adjoint map of a stable map.

Lemma 4.1 *The map*

$$\Sigma\mathbf{CP}^\infty \xrightarrow{j'_{\mathrm{SU}}} \mathbf{SU} \xrightarrow{q} \mathbf{U} \xrightarrow{p} \Sigma\mathbf{CP}^\infty$$

is a stable homotopy equivalence.

Proof . First we show

$$p^*(s_1 t^i) = \pm x_{2i+1} \pm (\text{decomposable elements}) \quad \text{for } i \geq 1.$$

This follows immediatly the fact that $j^*(x_{2i+1}) = \pm s_1 t^i$ and cup products vanish in $\tilde{H}^*(\Sigma\mathbf{CP}^\infty; \mathbf{Z})$.

Since ${j'_{\mathrm{SU}}}^*$ maps decomposable elements to zero,

$$\begin{aligned} & {j'_{\mathrm{SU}}}^* \circ q^* \circ p^*(s_1 t^i) \\ = \; & {j'_{\mathrm{SU}}}^* \circ q^*(\pm x_{2i+1} \pm (\text{decomposable elements})) \\ = \; & {j'_{\mathrm{SU}}}^*(\pm x_{2i+1} \pm (\text{decomposable elements})) \\ = \; & \pm s_1 t^i \end{aligned}$$

Thus this map induces an isomorphism between cohomology rings. $\Sigma\mathbf{CP}^\infty$ is simply connected and its homology group is free. Then this map is stably homotopy equivalent by J.H.C.Whitehead's theorem. ∎

Now we label this stable homotopy equivalent map by h. We assume that X satisfies the condition of Theorem 1.1 and we write

$$H^*(X;\mathbf{Z}) = \bigwedge(x'_3, x'_5, \cdots, x'_{2i+1}, \cdots).$$

By Theorem 1.1, X is an infinite loop space. Thus we can define

$$\xi : Q(X) \to X$$

such that

$$\xi \circ \iota \simeq 1_X,$$

where $Q(X) := \varinjlim \Omega^n \Sigma^n X$ and $\iota := \varinjlim Ad^n(1_{\Sigma^n X})$. Now we prove Theorem 1.2. All we have to do is to prove the next lemma.

Lemma 4.2 *The map*

$$\phi : \mathbf{SU} \xrightarrow{q} \mathbf{U} \xrightarrow{p} Q(\Sigma\mathbf{CP}^\infty) \xrightarrow{h^{-1}} Q(\Sigma\mathbf{CP}^\infty) \xrightarrow{Q(j')} Q(X) \xrightarrow{\xi} X$$

is homotopy equivalent.

Proof . We consider the following commutative diagram:

$$\begin{array}{ccc}
\Sigma\mathbf{CP}^\infty & \xrightarrow{j'_{\mathrm{SU}}} & \mathbf{SU} \\
 & & \downarrow{\scriptstyle q} \\
\| & & \mathbf{U} \\
 & & \downarrow{\scriptstyle p} \\
\Sigma\mathbf{CP}^\infty & \xrightarrow{h} & Q(\Sigma\mathbf{CP}^\infty) \\
 & & \downarrow{\scriptstyle h^{-1}} \\
\| & & \\
\Sigma\mathbf{CP}^\infty & \xrightarrow{\iota} & Q(\Sigma\mathbf{CP}^\infty) \\
\downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle Q(j')} \\
X & \xrightarrow{\iota} & Q(X) \\
 & \searrow\!\!\searrow & \downarrow{\scriptstyle \xi} \\
 & & X
\end{array}$$

where ι is a natural inclusion map. Commutativity is trivial by definition. We now compute cohomology rings of above spaces. Both j' and j'_{SU} induce bijections between generators of cohomology rings. Then the map ϕ of right column induces a bijection between generators, too. Thus ϕ^* is an isomorphism of cohomology at each degree. Since both X and $\mathbf{SU}$ are simply connected, ϕ is a homotopy equivalence by J.H.C.Whitehead's theorem. ∎

References

[1] M. F. Atiyah. *K-Theory*. Benjamin Press, New York, 1967.

[2] M. F. Atiyah and R. Bott. On the periodicity theorem for complex vector bundles. *Acta Math.*, Vol. 112, pp. 229–247, 1964.

[3] R. Bott. The space of loops on Lie groups. *Mich. Math. J.*, Vol. 5, pp. 35–61, 1958.

[4] R. Bott. The stable homotopy of the classical groups. *Ann. of Math.*, Vol. 70, pp. 313–337, 1959.

[5] M. C. Crabb. On the stable splitting of $U(n)$ and $\Omega U(n)$. In *Algebraic Topology Barcelona 1986*, Lect. Notes in Math. 1298. Springer-Verlag, 1987.

[6] Akira Kono and Ken-ichi Tokunaga. A topological proof of Bott periodicity theorem and a characterization of BU. *Kyoto Math. preprint series 94-08*, 1994.

[7] H. Miller. Stable splittings of Stiefel manifolds. *Topology*, Vol. 24, pp. 411–419, 1985.

[8] Mamoru Mimura and Hirosi Toda. *Topology of Lie Groups I and II.* Trans. of AMS, 1991.

[9] Hirosi Toda. A topological proof of theorems of Bott andfor homotopy groups of unitary groups. *Memoirs of the College of Science, University of Kyoto, Series A*, Vol. XXXII,, 1959.