A topological proof of Bott periodicity theorem and a characterization of **BU**

Akira KONO Ken-ichi TOKUNAGA

Department of Mathematics,
Faculty of Science,
Kyoto University.

June 1, 1994

1 Introduction

The purpose of this paper is to give a simple and topological proof of the Bott periodicity theorem $\mathbf{BU} \simeq \Omega^2\mathbf{BSU}$ and a characterization of $\mathbf{BU}$.

Let X be a finite type CW complex which is an H-space satisfying the following properties:

(1) as an algebra, $H^*(X;\mathbf{Z}) = \mathbf{Z}[c_1, c_2, \cdots, c_n, \cdots]$ where $|c_i| = 2i$;

(2) there exist two maps

$$j : \mathbf{CP}^\infty \longrightarrow X \quad \text{and}$$
$$\lambda : S^2 \wedge X \longrightarrow X$$

such that

$$(\lambda \circ (S^2 \wedge j))^* : H^*(X;\mathbf{Z}) \to H^*(S^2 \wedge \mathbf{CP}^\infty;\mathbf{Z}) \quad \text{is epic and}$$
$$Ad^2\lambda : X \to \Omega^2 X \quad \text{is an H-map.}$$

Denote the homotopy fiber of $c_1 : X \to \mathbf{CP}^\infty = K(\mathbf{Z}, 2)$ by $X\langle 2\rangle$. Then $Ad^2\lambda$ induces a map $\widetilde{Ad^2}\lambda : X \to \Omega^2(X\langle 2\rangle)$ since $\Omega^2(X\langle 2\rangle)$ is the connected component of $\Omega^2 X$ containing the constant loop and X is connected. The map $\widetilde{Ad^2}\lambda$ is a homotopy equivalence (see Theorem 2.1).

Note that $\mathbf{BU}$ is an H-space and satisfies (1). Let λ be the classifying map of

$$\tilde{K}(\cdot) \to \tilde{K}(S^2 \wedge \cdot)$$

defined by

$$x \mapsto (\eta - 1)\hat{\otimes}x$$

and j be the classifying map of $\eta_\infty - 1$ where η (resp. η_∞) is the cannonical line bundle over $S^2 = \mathbf{CP}^1$ (resp. $\mathbf{CP}^\infty$). In section 3 we show j and λ satisfy (2). Therefore Theorem 2.1 is a topological proof of the Bott periodicity thorem.

Now consider the Segal splitting

$$\epsilon_{\mathrm{C}} : \mathbf{BU} \longrightarrow Q(\mathbf{CP}^\infty),$$

where $Q(\cdot) = \varinjlim \Omega^n \Sigma^n(\cdot)$. Since X is an infinite loop space by Theorem 2.1, there is an infinite loop map

$$\xi_X : Q(X) \longrightarrow X.$$

In section 4 we show

$$\Phi = \xi_X \circ Q(j) \circ \epsilon_{\mathrm{C}} : \mathbf{BU} \longrightarrow X$$

is a homotopy equivalence. Thus the conditions (1) and (2) are a characterization of $\mathbf{BU}$.

2 A periodic H-space

In this section, we shall give the topological condition that an H-space have periodicity.

Theorem 2.1 *Let X be a finite type CW-complex which is an H-space. Suppose*

(1) as an algebra, $H^*(X;\mathbf{Z}) = \mathbf{Z}[c_1, c_2, \cdots, c_n, \cdots]$ *where* $|c_i| = 2i$ *;*

(2) there exist two maps

$$j : \mathbf{CP}^\infty \longrightarrow X$$
$$\lambda : S^2 \wedge X \longrightarrow X$$

such that

(a) $(\lambda \circ (S^2 \wedge j))^* : H^*(X;\mathbf{Z}) \to H^*(S^2 \wedge \mathbf{CP}^\infty;\mathbf{Z})$ *is an epimorphism;*

(b) $Ad^2\lambda : X \to \Omega^2 X$ *is an H-map.*

Then the map

$$\widetilde{Ad^2\lambda} : X \longrightarrow \Omega^2(X\langle 2\rangle)$$

is a homotopy equivalence.

Before we prove this theorem, we will state some remarks. If X satisfies the conditions of Theorem 2.1 , X is simply connected. Given a simply connected topological space X, we shall denote $X\langle 2\rangle$ to be the 2-connected fiber space of X, and Ad to be a natural equivalence

$$[\Sigma A, B] \xrightarrow{\sim} [A, \Omega B].$$

Note that $\Omega^2(X\langle 2\rangle)$ is a connected component of $\Omega^2 X$.

First we show an algebraic lemma. Let R be a PID with unit, and $A = \oplus A_n, B = \oplus B_n$ graded R-algebras.

Lemma 2.2 *Let* A_n *and* B_n *be free* R*-modeles with finite rank and* $rank A_n = rank B_n$ *for all* n*. If there exist a homomorphism of* R*-algebra* $f : A \to B$ *such that all generators of* B *is included in the image of* f*, then* $f_n : A_n \to B_n$ *is an isomorphism as* R*-modules for all* n*.*

Proof. This follows immediately from the fact that f_n is an epimorphism for all n. □

Proof of Theorem 2.1. First we write

$$k = \lambda \circ (S^2 \wedge j) : S^2 \wedge \mathbf{CP}^\infty \to X,$$
$$k_1 = Ad\lambda \circ (S^1 \wedge j) : S^1 \wedge \mathbf{CP}^\infty \to \Omega X \text{ and}$$
$$k_2 = Ad^2\lambda \circ j : \mathbf{CP}^\infty \to \Omega^2 X.$$

Note that $k_1 = Adk$,and $k_2 = Adk_1$. Let $\alpha_1 \in H^*(S^1; \mathbf{Z})$, $\alpha_2 \in H^*(S^2; \mathbf{Z})$ and $\beta \in H^*(\mathbf{CP}^\infty; \mathbf{Z})$ be generators as cohomology rings. Then we can write

$$H^{2i+1}(S^1 \wedge \mathbf{CP}^\infty; \mathbf{Z}) =< \alpha_1 \times \beta > \hookrightarrow H^{2i+1}(S^1 \times \mathbf{CP}^\infty; \mathbf{Z})\ i = 0, 1, 2, \cdots$$

$$H^{2i+2}(S^2 \wedge \mathbf{CP}^\infty; \mathbf{Z}) =< \alpha_2 \times \beta > \hookrightarrow H^{2i+2}(S^2 \times \mathbf{CP}^\infty; \mathbf{Z})\ i = 1, 2, \cdots .$$

Let σ be the cohomology suspension map : $H^*(X; \mathbf{Z}) \to H^{*-1}(\Omega X; \mathbf{Z})$. Then

$$H^*(\Omega X; \mathbf{Z}) = \bigwedge(\sigma(c_1), \sigma(c_2), \cdots, \sigma(c_n), \cdots).$$

Since $k^* : H^*(X; \mathbf{Z}) \to H^*(S^2 \wedge X; \mathbf{Z})$ is an epimorphism, $k^*(c_i) = \pm\alpha_2\beta^{i-1}$ $(i = 1, 2, \cdots)$, where $\alpha_2\beta^{i-1}$ is $\alpha_2 \times \beta^{i-1}$ precisely. From now on we shall often write in this way if there is no risk of misunderstanding. The adjoint map and the suspension map are commutative. Thus $k_1^*(\sigma(c_i)) = \pm\alpha_1\beta^{i-1}$ $(i = 2, 3, \cdots)$. It follows from this equation that the sequence

$$0 \to \mathbf{Ker}(k_1^*) \to H^i(\Omega X; \mathbf{Z}) \xrightarrow{k_1^*} H^i(S^1 \wedge \mathbf{CP}^\infty; \mathbf{Z}) \to 0$$

is splitting exact for $i \geq 2$. Therefore its dual sequence

$$0 \to H_i(S^1 \wedge \mathbf{CP}^\infty; \mathbf{Z}) \xrightarrow{k_{1*}} H_i(\Omega X; \mathbf{Z}) \to \mathbf{Coker}(k_{1*}) \to 0$$

is also splitting exact for $i \geq 2$.

Recall that since ΩX is an H-space $H_*(\Omega X; \mathbf{Z})$ is a Hopf algebra over $\mathbf{Z}$. Then we can choose generators ξ_i which are primitive elements of degree $2i - 1$ such that

$$H_*(\Omega X; \mathbf{Z}) = \bigwedge(\xi_1, \xi_2, \cdots, \xi_n, \cdots).$$

See [9]. $H_*(S^1 \wedge \mathbf{CP}^\infty; \mathbf{Z})$ is a coalgebra over $\mathbf{Z}$, and $\theta_i = [\alpha_1\beta^i \mapsto 1] \in H_{2i+1}(S^1 \wedge \mathbf{CP}^\infty; \mathbf{Z})$ is a primitive element. Since k_{1*} is a coalgebra homomorphism and the above short exact sequence splits, $k_{1*}(\theta_i) = \pm\xi_{i+1}$ $(i = 1, 2, \cdots)$.

The map k_1 has a lift

$$\widetilde{k_1} : S^1 \wedge \mathbf{CP}^\infty \to \Omega(X\langle 2\rangle)$$

and

$$\widetilde{k_{1*}}(\theta_i) = \xi_{i+1} \in H_*(\Omega(X\langle 2\rangle); \mathbf{Z}) = \bigwedge(\xi_2, \xi_3, \cdots).$$

Let

$$\widetilde{k_2} = \widetilde{Ad^2\lambda} \circ j : \mathbf{CP}^\infty \to \Omega^2(X\langle 2\rangle)$$

be a lift of k_2. Note that $\widetilde{k_2}$ is the adjoint map of $\widetilde{k_1}$. Let τ be the homology transgression map : $H_*(\Omega(X\langle 2\rangle); \mathbf{Z}) \to H_{*-1}(\Omega^2(X\langle 2\rangle); \mathbf{Z})$. Since the adjoint map and the transgression map are commutative, the image of the map

$$\widetilde{k_2}_* : H_*(\mathbf{CP}^\infty; \mathbf{Z}) \longrightarrow H_*(\Omega^2(X\langle 2\rangle); \mathbf{Z}) = \mathbf{Z}[\tau(\xi_2), \tau(\xi_3), \cdots, \tau(\xi_n), \cdots]$$

includes generators $\tau(\xi_n)$. So does the image of the map $\widetilde{Ad^2\lambda}_*$. By Lemma 2.2, $\widetilde{Ad^2\lambda}_*$ is an isomorphism at each degree. Since X and $\Omega^2(X\langle 2\rangle)$ are simply connected, $\widetilde{Ad^2\lambda}$ is a homotopy equivalence by J.H.C.Whitehead's theorem. $\square$

3 A new proof of Bott periodicity

Bott periodicity (the complex case) implys that $\mathbf{BU}$ is homotopy equivalent to $\Omega^2\mathbf{BSU}$. Note that $\mathbf{BSU} \simeq \mathbf{BU}\langle 2\rangle$. The aim of this section is to give a new proof of Bott periodicity of complex case. By Theorem 2.1, it will be sufficient to construct two maps $j : \mathbf{CP}^\infty \to \mathbf{BU}$ and $\lambda : S^2 \wedge \mathbf{BU} \to \mathbf{BU}$ which have the properties given in Theorem 2.1.

Since $\mathbf{BU}$ is a classifying space of K-thory, it has an H-space structure corresponding to Whitney sum. Let $\mu : \mathbf{BU} \times \mathbf{BU} \to \mathbf{BU}$ be this structure. Denote an algebra generator of $H^*(S^2; \mathbf{Z})$ (resp. $H^*(\mathbf{CP}^\infty; \mathbf{Z})$) by α (resp. β). The Hopf line bundle over $S^2 = \mathbf{CP}^1$ is denoted by η whose first Chern class $c_1(\eta) = \alpha$, and the canonical line bundle over $\mathbf{CP}^\infty$denotes η_∞ whose first Chern class $c_1(\eta_\infty) = \beta$. Now we shall construct maps j and λ as elements of $\tilde{K}(\mathbf{CP}^\infty)$ and $\tilde{K}(S^2 \wedge \mathbf{BU})$ respectively. We define $j = \eta_\infty - 1$, where 1 is a trivial 1-dim vector bundle.

Define

$$\xi := \varinjlim(\xi_n - n),$$

where $\xi_n \to \mathbf{BU}_n$ is the universal vector bundle. Note that the restriction of $(\eta - 1)\hat{\otimes}\xi$ to $S^2 \vee \mathbf{BU}$ is trivial. This means that this element is in $\tilde{K}(S^2 \wedge \mathbf{BU})$. Now we define $\lambda = (\eta - 1)\hat{\otimes}\xi$.

Proposition 3.1 *The map*

$$((\lambda \circ (S^2 \wedge j))^* : H^*(\mathbf{BU}; \mathbf{Z}) \to H^*(S^2 \wedge \mathbf{CP}^\infty; \mathbf{Z})$$

is an epimorphism.

Proof. We will compute $((\lambda \circ (S^2 \wedge j))^*(c)$, where c is the total Chern class $c = 1 + c_1 + c_2 + \cdots \in H^*(\mathbf{BU}; \mathbf{Z})$. From now on, we simply write $S^2 \wedge j$ to be j. Then

$$j^* \circ \lambda^*(c) = c(j^*(\lambda)).$$

The right side of the above equation is the total Chern class of the element $j^*(\lambda) \in \tilde{K}(S^2 \wedge \mathbf{CP}^\infty)$. Thus

$$\begin{aligned} c(j^*(\lambda)) &= c((\eta - 1)\hat{\otimes}(\xi_1 - 1)) \\ &= c(\eta\hat{\otimes}\xi_1)c(\eta\hat{\otimes}1)^{-1}c(1\hat{\otimes}\xi_1)^{-1} \\ &= (1 + \alpha + \beta)(1 + \alpha)^{-1}(1 + \beta)^{-1} \\ &= (1 + \alpha)^{-1}(1 + \alpha(1 + \beta)^{-1}) \\ &= (1 - \alpha)(1 + \alpha(\sum_{i=0}^{\infty}(-1)^i\beta^i)) \\ &= 1 + \sum_{i=1}^{\infty}(-1)^i\alpha\beta^i \end{aligned}$$

since $\alpha^2 = 0$ and $j^*(\xi) = \xi_1 - 1, \xi_1 = \eta_\infty$. We obtain

$$(\lambda \circ j)^*(c_{i+1}) = (-1)^i\alpha\beta^i.$$

Therefore it is an epimorphism. □

Proposition 3.2 *The map*

$$Ad^2\lambda : \mathbf{BU} \longrightarrow \Omega^2\mathbf{BU}$$

is an H-map.

Proof. We must show that the following diagram

$$\begin{array}{ccc} \mathbf{BU} \times \mathbf{BU} & \xrightarrow{\mu} & \mathbf{BU} \\ {\scriptstyle Ad^2\lambda \times Ad^2\lambda}\downarrow & & {\scriptstyle Ad^2\lambda}\downarrow \\ \Omega^2\mathbf{BU} \times \Omega^2\mathbf{BU} & \xrightarrow{\Omega^2\mu} & \Omega^2\mathbf{BU} \end{array}$$

is homotopy commutative. Its adjoint diagram is the following:

$$\begin{array}{ccc} S^2 \wedge (\mathbf{BU} \times \mathbf{BU}) & \xrightarrow{S^2 \wedge \mu} & S^2 \wedge \mathbf{BU} \\ \downarrow & & \lambda \downarrow \\ \mathbf{BU} \times \mathbf{BU} & \xrightarrow{\mu} & \mathbf{BU}. \end{array}$$

Since these maps are classifying the same element

$$(\eta - 1)\hat{\otimes}(\xi \times \xi) \in \tilde{K}(S^2 \wedge (\mathbf{BU} \times \mathbf{BU})) = [S^2 \wedge (\mathbf{BU} \times \mathbf{BU}), \mathbf{BU}],$$

this diagram is obviously homotopy commutative. Then the first diagram is also homotopy commutative. □
By above propositions and Theorem 2.1, we obtain Bott periodicity theorem.

Corollary 3.3 (Bott periodicity theorem)

$$\mathbf{BU} \simeq \Omega^2 \mathbf{BSU}.$$

4 A topological characterization of BU

In fact, X is homotopy equivalent to BU if X equips the conditons of Theorem 2.1. We shall prove it in this section.

We define

$$Q(X) := \lim_{\rightarrow} \Omega^n \Sigma^n X.$$

Note that $Q(X)$ is an infinite loop space. Furthermore, if $f : A \to B$ is a map,

$$Q(f) = \lim_{\rightarrow} \Omega^n \Sigma^n f : Q(A) \to Q(B)$$

is an infinite loop map. If X is an infinite loop space, we define an infinite loop map

$$\xi_X = \lim_{\rightarrow} \epsilon_n^{-1} \circ \Omega^n Ad^{-n} \epsilon_n : Q(X) \to X,$$

where $\epsilon_n : X \to \Omega^n B^n$ is a homotopy equivalence.

By Segal [8] and Becker [3], there exists the Segal map $\epsilon_{\mathbf{C}} : \mathbf{BU} \to Q(\mathbf{CP}^\infty)$ such that $\xi_{\mathbf{BU}} \circ Q(j) \circ \epsilon_{\mathbf{C}} \simeq 1_{\mathbf{BU}}$. The Segal map $\epsilon_{\mathbf{C}}$ is a "section" of a principal fiber space. Thus we obtain $Q(\mathbf{CP}^\infty) \simeq \mathbf{BU} \times F$, where F is the homotopy fiber and the homotopy group of F, $\pi_i(F)$ is finite for all i. Thus its reduced homology group $\tilde{H}_i(F; \mathbf{Z})$ is finite for all i.

We note that if X satisfies the conditions of Theorem 2.1, $j = \xi_X \circ Q(j) \circ i$ where i is the inclusion map $\mathbf{CP}^\infty \to Q(\mathbf{CP}^\infty)$. Consider the following commutative diagram:

$$\begin{array}{ccccccc} & & \mathbf{CP}^\infty & & \xrightarrow{j} & & X \\ & & {\scriptstyle i}\downarrow & & & & \| \\ \mathbf{BU} & \xrightarrow{\epsilon_{\mathrm{C}}} & Q(\mathbf{CP}^\infty) & \xrightarrow{Q(j)} & Q(X) & \xrightarrow{\xi_X} & X. \end{array}$$

Since $H_*(X;\mathbf{Z})$ does not contain any torsion elements, there exists the algebra homomorphism φ such that the following diagram is commutative:

$$\begin{array}{ccccc} & & H_*(\mathbf{CP}^\infty;\mathbf{Z}) & \xrightarrow{j_*} & H_*(X;\mathbf{Z}) \\ & & {\scriptstyle i_*}\downarrow & & \| \\ H_*(\mathbf{BU};\mathbf{Z}) & \xrightarrow{\epsilon_{\mathrm{C}_*}} & H_*(Q(\mathbf{CP}^\infty);\mathbf{Z}) & \xrightarrow{(\xi_X \circ Q(j))_*} & H_*(X;\mathbf{Z}) \\ \| & & {\scriptstyle proj.}\downarrow & & \| \\ H_*(\mathbf{BU};\mathbf{Z}) & \xrightarrow{\cong} & H_*(Q(\mathbf{CP}^\infty);\mathbf{Z})/Tor & \xrightarrow{\varphi} & H_*(X;\mathbf{Z}). \end{array}$$

Then

$$\text{Image } \varphi \supset \text{Image } j_*.$$

Therefore the image of φ includes all algebra generators of $H_*(X;\mathbf{Z})$. By Lemma 2.2, φ is an isomorphism at each degree. Finally we obtain the following theorem.

Theorem 4.1 *If X satifies the conditions of Theorem 2.1, the map*

$$\xi_X \circ Q(j) \circ \epsilon_{\mathrm{C}} : \mathbf{BU} \to X$$

is homotopy equivalent.

References

[1] M. F. Atiyah. *K-Theory.* Benjamin Press, New York, 1967.

[2] M. F. Atiyah and R. Bott. On the periodicity theorem for complex vector bundles. *Acta Math.*, Vol. 112, pp. 229–247, 1964.

[3] J. C. Becker. Characteristic classes and K-thory. In *Lecture Note in Math. 428*, pp. 132–143. Springer-Verlag, 1974.

[4] R. Bott. The space of loops on Lie groups. *Mich. Math. J.*, Vol. 5, pp. 35–61, 1958.

[5] R. Bott. The stable homotopy of the classical groups. *Ann. of Math.*, Vol. 70, pp. 313–337, 1959.

[6] J. W. Milnor and J. C. Moore. On the structure of Hopf algebras. *Ann. of Math.*, Vol. 81, pp. 211–264, 1965.

[7] Mamoru Mimura and Hirosi Toda. *Topology of Lie Groups I and II.* Trans. of AMS, 1991.

[8] G. B. Segal. The stable homotopy of complex projective space. *Quart. J. Math.*, Vol. 2, No. 24, pp. 1–5, 1973.

[9] Hirosi Toda. A topological proof of theorems of Bott and Borel-Hirzebrurch for homotopy groups of unitary groups. *Memoirs of the College of Science, University of Kyoto, Series A*, Vol. XXXII,, 1959.