Log Smooth Deformation Theory

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Abstract

This paper gives a foundation of log smooth deformation theory. We study the infinitesimal liftings of log smooth morphisms and show that the log smooth deformation functor has a representable hull. This deformation theory gives, for example, the following two types of deformations: (1) relative deformations of a certain kind of a pair of an algebraic variety and a divisor of it, and (2) global smoothings of normal crossing varieties. The former is a generalization of the relative deformation theory introduced by Makio, and the latter coincides with the logarithmic deformation theory introduced by Kawamata and Namikawa.

1 Introduction

In this article, we formulate and develop the theory of *log smooth deformation*. Here, log smoothness (more precisely, logarithmic smoothness) is a concept in *log geometry* which is a generalization of "usual" smoothness of morphisms of algebraic varieties. Log geometry is a beautiful geometric theory which succesfully generalizes and unifies the scheme theory and the theory of torus embeddings. This theory was first planned and founded by Fontaine and Illusie, based on their idea of *log structures* on schemes, and further developed by Kato [5]. Recently, the importance of log geometry comes to be recognized by many geometers and applied to various fields of algebraic and arithmetic geometry. One of such applications can be seen in the recent work of Steenbrink [12]. In the present paper, we attempt to apply log geometry to extend the usual smooth deformation theory by using the concept of log smoothness.

Log smoothness is one of the most important concepts in log geometry, and is a log geometric generalization of usual smoothness. For example, varieties with toric singularities or normal crossing varieties may become log smooth over certain logarithmic points. Kato [5] showed that any log smooth morphism is written étale locally by the composition of a usual smooth morphism and a morphism induced by a homomorphism of monoids which essentially determines the log structures (Theorem 4.1). On the other hand, log smoothness is described by means of *log differentials* and *log derivations* similarly to usual smoothness by means of differentials and derivations. Hence if we consider

1991 *Mathematics Subject Classification*. Primary 13D10; Secondary 14D15, 14M25, 16S80.

the log smooth deformation by analogy with the usual smooth deformation, it is expected that the first order deformation is controled by the sheaf of log derivations. This intuition motivated this work and we shall see later that this is, in fact, the case.

In the present paper, we construct log smooth deformation functor by the concept of infinitesimal log smooth lifting. The goal of this paper is to show that this functor has a representable hull in the sense of Schlessinger [11], under certain conditions on the underlying schemes (Theorem 8.5). At the end of this paper, we give two examples of our log smooth deformation theory, which are summarized as follows:

1. Deformations with divisors (§10): Let X be a variety over a field k. Assume that the variety X is covered by étale open sets which are smooth over affine torus embeddings, and there exists a divisor D of X which is the union of the closures of codimension 1 torus orbits. Then, there exists a log structure $\mathcal{M}$ on X such that the log scheme $(X, \mathcal{M})$ is log smooth over k with trivial log structure. (The converse is also true in a certain excellent category of log schemes.) In this case, our log smooth deformation is a deformation of the pair (X, D). If X itself is smooth and D is a smooth divisor of X, our deformation coincides with the relative deformation studied by Makio [9].

2. Smoothings of normal crossing varieties (§11): If a connected scheme of finite type X over a field k is, étale locally, isomorphic to an affine normal crossing variety $\operatorname{Spec} k[z_1, \ldots, z_n]/(z_1 \cdots z_d)$, then we call X a normal crossing variety over k. If X is d–semistable (cf. [1]), there exists a log structure $\mathcal{M}$ on X such that $(X, \mathcal{M})$ is log smooth over a standard log point $(\operatorname{Spec} k, \mathbf{N})$ (Theorem 11.2). Then, our log smooth deformation is nothing but a smoothing of X. If the singular locus of X is connected, our deformation theory coincides with the one introduced by Kawamata and Namikawa [6].

The composition of this paper is as follows. We recall some basic notions of log geometry in the next section, and review the definition and basic properties of log smoothness in section 3. In section 4, we study the characterization of log smoothness by means of the theory of torus embeddings according to Illusie [3] and Kato [5]. In section 5, we recall the definitions and basic properties of log derivations and log differentials. In section 6 and section 7, we give proofs of theorems stated in section 4. Section 8 is devoted to the formulation of log smooth deformation theory. This section is the main section of this present paper. We prove the existance of a representable hull of the log smooth deformation functor in section 9. In section 10 and section 11, we give two examples of log smooth deformation. For the reader's convenience, in section 12, we give a proof of the result of Kawamata and Namikawa [6] which is relevant to our log smooth deformation.

The author would like to express his thanks to Professors Kazuya Kato and Yoshinori Namikawa for valuable suggestions and advice. The author is also very grateful to Professor Luc Illusie for valuable advice on this paper.

Convention. We assume that all monoids are commutative and have neutral elements. A homomorphism of monoids is assumed to preserve neutral elements. We write the binary operations of all monoids multiplicatively except in the case of $\mathbf{N}$ (the monoid of non–negative integers), $\mathbf{Z}$, etc., which we write additively. All sheaves on schemes are considered with respect to the étale topology.

2 Fine saturated log schemes

In this and subsequent sections, we use the terminology of log geometry basically as in [5]. Let X be a scheme. We view the structure sheaf $\mathcal{O}_X$ of X as a sheaf of monoids under multiplication.

Definition 2.1 (cf. [5, §1]) A *pre–log structure* on X is a homomorphism $\mathcal{M} \to \mathcal{O}_X$ of sheaves of monoids where $\mathcal{M}$ is a sheaf of monoids on X. A pre–log structure $\alpha : \mathcal{M} \to \mathcal{O}_X$ is said to be a *log structure* on X if α induces an isomorphism

$$\alpha : \alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\sim} \mathcal{O}_X^\times.$$

Given a pre–log structure $\alpha : \mathcal{M} \to \mathcal{O}_X$, we can construct the *associated log structure* $\alpha^{\mathrm{a}} : \mathcal{M}^{\mathrm{a}} \to \mathcal{O}_X$ functorially by

$$\mathcal{M}^{\mathrm{a}} = (\mathcal{M} \oplus \mathcal{O}_X^\times)/\mathcal{P} \tag{1}$$

and

$$\alpha^{\mathrm{a}}(x, u) = u \cdot \alpha(x)$$

for $(x, u) \in \mathcal{M}^{\mathrm{a}}$, where $\mathcal{P}$ is the submonoid defined by

$$\mathcal{P} = \{(x, \alpha(x)^{-1}) \mid x \in \alpha^{-1}(\mathcal{O}_X^\times)\}.$$

Here, in general, the quotient M/P of a monoid M with respect to a submonoid P is the coset space $M/\sim$ with induced monoid structure, where the equivalence relation $\sim$ is defined by

$$x \sim y \Leftrightarrow xp = yq \text{ for some } p, q \in P.$$

$\mathcal{M}^{\mathrm{a}}$ has a universal mapping property: if $\beta : \mathcal{N} \to \mathcal{O}_X$ is a log structure on X and $\varphi : \mathcal{M} \to \mathcal{N}$ is a homomorphism of sheaves of monoids such that $\alpha = \beta \circ \varphi$, then there exists a unique lifting $\varphi^{\mathrm{a}} : \mathcal{M}^{\mathrm{a}} \to \mathcal{N}$. Note that the monoid $\mathcal{M}^{\mathrm{a}}$ defined by (1) is the push–out of the diagram

$$\mathcal{M} \supset \alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\alpha} \mathcal{O}_X^\times$$

in the category of monoids, and the homomorphism α^{a} is induced by α and the inclusion $\mathcal{O}_X^\times \hookrightarrow \mathcal{O}_X$. We sometimes denote the monoid $\mathcal{M}^{\mathrm{a}}$ by $\mathcal{M} \oplus_{\alpha^{-1}(\mathcal{O}_X^\times)} \mathcal{O}_X^\times$. Note that we have the natural isomorphism

$$\mathcal{M}/\alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\sim} \mathcal{M}^{\mathrm{a}}/\mathcal{O}_X^\times. \tag{2}$$

Definition 2.2 By a *log scheme*, we mean a pair $(X, \mathcal{M})$ with a scheme X and a log structure $\mathcal{M}$ on X. A *morphism* of log schemes $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is a pair $f = (f, \varphi)$ where $f : X \to Y$ is a morphism of schemes and $\varphi : f^{-1}\mathcal{N} \to \mathcal{M}$ is a homomorphism of sheaves of monoids such that the diagram

$$\begin{array}{ccc} f^{-1}\mathcal{N} & \xrightarrow{\varphi} & \mathcal{M} \\ \downarrow & & \downarrow \\ f^{-1}\mathcal{O}_Y & \longrightarrow & \mathcal{O}_X \end{array}$$

commutes.

Definition 2.3 Let $\alpha : \mathcal{M} \to \mathcal{O}_X$ and $\alpha' : \mathcal{M}' \to \mathcal{O}_X$ be log structures on a scheme X. These log structures are said to be *equivalent* if there exists an isomorphism $\varphi : \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ such that $\alpha = \alpha' \circ \varphi$, i.e., there exists an isomorphism of log schemes $(X, \mathcal{M}) \xrightarrow{\sim} (X, \mathcal{M}')$ whose underlying morphism of schemes is the identity id_X. Let $\beta : \mathcal{N} \to \mathcal{O}_Y$ and $\beta' : \mathcal{N}' \to \mathcal{O}_Y$ be log structures on a scheme Y. Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ and $f' : (X, \mathcal{M}') \to (Y, \mathcal{N}')$ be morphisms of log schemes. Then f and f' are said to be *equivalent* if there exist isomorphisms $\varphi : \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ and $\psi : \mathcal{N} \xrightarrow{\sim} \mathcal{N}'$ such that $\alpha = \alpha' \circ \varphi$, $\beta = \beta' \circ \psi$ and the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\varphi} & \mathcal{M}' \\ \uparrow & & \uparrow \\ f^{-1}\mathcal{N} & \xrightarrow[f^{-1}\psi]{} & f^{-1}\mathcal{N}' \end{array}$$

commutes.

We denote the category of log schemes by **LSch**. For $(S, \mathcal{L}) \in \mathrm{Obj}(\mathbf{LSch})$, we denote the category of log schemes over $(S, \mathcal{L})$ by $\mathbf{LSch}_{(S,\mathcal{L})}$. The following examples play important roles in the sequel.

Example 2.4 On any scheme X, we can define a log structure by the inclusion $\mathcal{O}_X^{\times} \hookrightarrow \mathcal{O}_X$, called the *trivial* log structure. Thus, we have an inclusion functor from the category of schemes to that of log schemes. We often denote the log scheme $(X, \mathcal{O}_X^{\times} \hookrightarrow \mathcal{O}_X)$ simply by X.

Example 2.5 Let A be a commutative ring. For a monoid P , we can define a log structure canonically on the scheme $\mathrm{Spec}\, A[P]$, where $A[P]$ denotes the monoid ring of P over A, as the log structure associated to the natural homomorphism,

$$P \xrightarrow{\alpha} A[P].$$

This log structure is called the *canonical log structure* on $\mathrm{Spec}\, A[P]$. Thus we obtain a log scheme which we denote simply by $(\mathrm{Spec}\, A[P], P)$. A monoid homomorphism $P \to Q$ induces a morphism $(\mathrm{Spec}\, A[Q], Q) \to (\mathrm{Spec}\, A[P], P)$ of log schemes. Thus, we have a contravariant functor from the category of monoids to $\mathbf{LSch}_{\mathrm{Spec}\, A}$.

Example 2.6 Let Σ be a fan on $N_{\mathbf{R}} = \mathbf{R}^d$, $N = \mathbf{Z}^d$, and X_Σ the toric variety determined by the fan Σ over a commutative ring A. Then, we get an induced log structure on the scheme X_Σ by gluing the log structures associated to the homomorphisms

$$M \cap \sigma^\vee \longrightarrow A[M \cap \sigma^\vee],$$

for each cone σ in Σ, where $M = \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$. Thus, a toric variety X_Σ is naturally viewed as a log scheme over $\mathrm{Spec}\, A$, which we denote by (X_Σ, Σ).

Next, we define important subcategories of **LSch**. These subcategories are closely related with *charts* defined as follows.

Definition 2.7 Let $(X, \mathcal{M}) \in \mathrm{Obj}(\mathbf{LSch})$. A *chart* of $\mathcal{M}$ is a homomorphism $P \to \mathcal{M}$ from the constant sheaf of a monoid P which induces an isomorphism from the associated log structure P^{a} to $\mathcal{M}$.

Definition 2.8 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism in **LSch**. A *chart* of f is a triple $(P \to \mathcal{M}, Q \to \mathcal{N}, Q \to P)$, where $P \to \mathcal{M}$ and $Q \to \mathcal{N}$ are charts of $\mathcal{M}$ and $\mathcal{N}$, respectively, and $Q \to P$ is a homomorphism for which the diagram

$$\begin{array}{ccc} Q & \longrightarrow & P \\ \downarrow & & \downarrow \\ f^{-1}\mathcal{N} & \longrightarrow & \mathcal{M} \end{array}$$

is commutative.

Definition 2.9 (cf. [5, §2]) A log structure $\mathcal{M} \to \mathcal{O}_X$ on a scheme X is said to be *fine* if $\mathcal{M}$ has étale locally a chart $P \to \mathcal{M}$ with P a finitely generated integral monoid. Here, in general, a monoid M is said to be *finitely generated* if there exists a surjective homomorphism $\mathbf{N}^n \to M$ for some n, and a monoid M is said to be *integral* if $M \to M^{\mathrm{gp}}$ is injective, where M^{gp} denotes the Grothendieck group associated with M. A log scheme $(X, \mathcal{M})$ with a fine log structure $\mathcal{M} \to \mathcal{O}_X$ is called a *fine* log scheme.

We denote the category of fine log schemes by $\mathbf{LSch}^{\mathrm{f}}$. Similarly, we denote the category of fine log schemes over $(S, \mathcal{L}) \in \mathrm{Obj}(\mathbf{LSch}^{\mathrm{f}})$ by $\mathbf{LSch}^{\mathrm{f}}_{(S,\mathcal{L})}$, The category $\mathbf{LSch}^{\mathrm{f}}$ (resp. $\mathbf{LSch}^{\mathrm{f}}_{(S,\mathcal{L})}$) is a full subcategory of **LSch** (resp. $\mathbf{LSch}_{(S,\mathcal{L})}$). Both **LSch** and $\mathbf{LSch}^{\mathrm{f}}$ have fiber products. But the inclusion functor $\mathbf{LSch}^{\mathrm{f}} \hookrightarrow \mathbf{LSch}$ does not preserve fiber products (cf. Lemma 3.4). The inclusion functor $\mathbf{LSch}^{\mathrm{f}} \hookrightarrow \mathbf{LSch}$ has a right adjoint $\mathbf{LSch} \to \mathbf{LSch}^{\mathrm{f}}$ [5, (2.7)]. Then, the fiber product of a diagram $(X, \mathcal{M}) \to (Z, \mathcal{P}) \leftarrow (Y, \mathcal{N})$ in $\mathbf{LSch}^{\mathrm{f}}$ is the image of that in **LSch** by this adjoint functor. Note that the underlying scheme of the fiber product of $(X, \mathcal{M}) \to (Z, \mathcal{P}) \leftarrow (Y, \mathcal{N})$ in **LSch** is $X \times_Z Y$, but this is not always the case in $\mathbf{LSch}^{\mathrm{f}}$.

Next, we introduce more excellent subcategory of **LSch**.

Definition 2.10 Let M be a monoid and P a submonoid of M. The monoid P is said to be *saturated* in M if $x \in M$ and $x^n \in P$ for some positive integer n imply $x \in P$. An integral monoid N is said to be *saturated* if N is saturated in N^{gp}.

Example 2.11 Put $M = \mathbf{N}$ and $P = l \cdot M$ for an integer $l > 1$. Then P is saturated but not saturated in M.

Definition 2.12 A fine log scheme $(X, \mathcal{M}) \in \mathrm{Obj}(\mathbf{LSch}^{\mathrm{f}})$ is said to be *saturated* if the log structure $\mathcal{M}$ is a sheaf of saturated monoids.

We denote the category of fine saturated log schemes by $\mathbf{LSch}^{\mathrm{fs}}$. Similarly, we denote the category of fine saturated log schemes over $(S, \mathcal{L}) \in \mathrm{Obj}(\mathbf{LSch}^{\mathrm{fs}})$ by $\mathbf{LSch}^{\mathrm{fs}}_{(S,\mathcal{L})}$. The category $\mathbf{LSch}^{\mathrm{fs}}$ (resp. $\mathbf{LSch}^{\mathrm{fs}}_{(S,\mathcal{L})}$) is a full subcategory of $\mathbf{LSch}^{\mathrm{f}}$ (resp. $\mathbf{LSch}^{\mathrm{f}}_{(S,\mathcal{L})}$). The following lemma is an easy consequence of [5, Lemma (2.10)].

Lemma 2.13 *Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism in $\mathbf{LSch}^{\mathrm{fs}}$, and $Q \to \mathcal{N}$ a chart of $\mathcal{N}$, where Q is a finitely generated integral saturated monoid. Then there exists étale locally a chart $(P \to \mathcal{M}, Q \to \mathcal{N}, Q \to P)$ of f extending $Q \to \mathcal{N}$ such that the monoid P is also finitely generated, integral and saturated.*

Lemma 2.14 *The inclusion functor* $\mathbf{LSch}^{\mathrm{fs}} \hookrightarrow \mathbf{LSch}^{\mathrm{f}}$ *has a right adjoint.*

Proof. Let M be an integral monoid. Define

$$M^{\mathrm{sat}} = \{x \in M^{\mathrm{gp}} \mid x^n \in M \text{ for some positive integer } n\}.$$

Then M^{sat} is an integral saturated monoid. For any integral saturated monoid N and homomorphism $M \to N$, there exists a unique lifting $M^{\mathrm{sat}} \to N$. In this sense, M^{sat} is the universal saturated monoid associated with M. Let $(X, \mathcal{M})$ be a fine log scheme. Then we have étale locally a chart $P \to \mathcal{M}$. This chart defines a morphism $X \to \operatorname{Spec} \mathbf{Z}[P]$ étale locally. Let $X' = X \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[P^{\mathrm{sat}}]$. Then $X' \to \operatorname{Spec} \mathbf{Z}[P^{\mathrm{sat}}]$ induces a log structure $\mathcal{M}'$ by the associated log structure of $P^{\mathrm{sat}} \to \mathbf{Z}[P^{\mathrm{sat}}] \to \mathcal{O}_{X'}$, and $(X', \mathcal{M}')$ is a fine saturated log scheme. This procedure defines a functor $\mathbf{LSch}^{\mathrm{f}} \to \mathbf{LSch}^{\mathrm{fs}}$. It is easy to see that this functor is the right adjoint of the inclusion functor $\mathbf{LSch}^{\mathrm{fs}} \hookrightarrow \mathbf{LSch}^{\mathrm{f}}$. □

Corollary 2.15 $\mathbf{LSch}^{\mathrm{fs}}$ *has fiber products. More precisely, the fiber product of morphisms* $(X, \mathcal{M}) \to (Z, \mathcal{P}) \leftarrow (Y, \mathcal{N})$ *in* $\mathbf{LSch}^{\mathrm{fs}}$ *is the image of that in* $\mathbf{LSch}^{\mathrm{f}}$ *by the right adjoint functor of* $\mathbf{LSch}^{\mathrm{fs}} \hookrightarrow \mathbf{LSch}^{\mathrm{f}}$.

3 Log smooth morphisms

In this section, we review the definition and basic properties of log smoothness (cf. [5]).

Definition 3.1 Let $f : X \to Y$ be a morphism of schemes, and $\mathcal{N}$ a log structure on Y. Then the *pull–back* of $\mathcal{N}$, denoted by $f^*\mathcal{N}$, is the log structure on X associated with the pre–log structure $f^{-1}\mathcal{N} \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. A morphism of log schemes $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is said to be *strict* if the induced homomorphism $f^*\mathcal{N} \to \mathcal{M}$ is an isomorphism. A morphism of log schemes $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is said to be an *exact closed immersion* if it is strict and $f : X \to Y$ is a closed immersion in the ususal sense.

Exact closed immersions are stable under base change in $\mathbf{LSch}^{\mathrm{f}}$ [5, (4.6)].

Lemma 3.2 *Let* $\alpha : \mathcal{M} \to \mathcal{O}_X$ *and* $\alpha' : \mathcal{M}' \to \mathcal{O}_X$ *be fine log strctures on a scheme* X *with a homomorphism* $\varphi : \mathcal{M} \to \mathcal{M}'$ *of monoids such that* $\alpha = \alpha' \circ \varphi$. *Then,* φ *is an isomorphism if and only if* $\varphi \bmod \mathcal{O}_X^{\times} : \mathcal{M}/\mathcal{O}_X^{\times} \to \mathcal{M}'/\mathcal{O}_X^{\times}$ *is an isomorphism.*

The proof is straightforward.

Lemma 3.3 *Let* $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ *be a morphism of log schemes. Then, we have the natural isomorphism*

$$f^{-1}(\mathcal{N}/\mathcal{O}_Y^{\times}) \xrightarrow{\sim} f^*\mathcal{N}/\mathcal{O}_X^{\times}.$$

In particular, f *is strict if and only if the induced morphism*

$$f^{-1}(\mathcal{N}/\mathcal{O}_Y^{\times}) \xrightarrow{\sim} \mathcal{M}/\mathcal{O}_X^{\times}.$$

Proof. The first part is easy to see. For the second part, apply (2) and Lemma 3.2. □

Lemma 3.4 (cf. [4, (1.7)]) *Let*

$$(X, \mathcal{M}) \to (Z, \mathcal{P}) \leftarrow (Y, \mathcal{N}) \tag{3}$$

be morphisms in $\mathbf{LSch}^{\mathrm{fs}}$. *If* $(Y, \mathcal{N}) \to (Z, \mathcal{P})$ *is strict, then the fiber product of* (3) *in* $\mathbf{LSch}^{\mathrm{fs}}$ *is isomorphic to that in* $\mathbf{LSch}$. *In particular, the underlying scheme of the fiber product of* (3) *in* $\mathbf{LSch}^{\mathrm{fs}}$ *is isomorphic to* $X \times_Z Y$.

Proof. Let $P \to \mathcal{P}$ be a chart of $\mathcal{P}$, where P is a finitely generated integral saturated monoid. Since $(Y, \mathcal{N}) \to (Z, \mathcal{P})$ is strict, $\to \mathcal{P} \to \mathcal{N}$ is a chart of $\mathcal{N}$ by Lemma 3.2, Lemma 3.3, and (2). Take a chart

$$\begin{array}{ccc} P & \longrightarrow & M \\ \downarrow & & \downarrow \\ \mathcal{P} & \longrightarrow & \mathcal{M} \end{array}$$

of $(X, \mathcal{M}) \to (Z, \mathcal{P})$ extending $P \to \mathcal{P}$. Set $W = X \times_Z Y$. There exists an induced homomorphism $M \to \mathcal{O}_W$. Define a log structure on W by this homomorphism. Then this log scheme $(W, M \to \mathcal{O}_W)$ is the fiber product of (3) in $\mathbf{LSch}$. Since the associated log structure of $M \to \mathcal{O}_W$ is fine and saturated, (W, M) is, indeed, the fiber product of (3) in $\mathbf{LSch}^{\mathrm{fs}}$. $\square$

Definition 3.5 The exact closed immersion $t : (T', \mathcal{L}') \to (T, \mathcal{L})$ is said to be a *thickening of order* $\leq n$, if $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_T \to \mathcal{O}_{T'})$ is a nilpotent ideal such that $\mathcal{I}^{n+1} = 0$.

Lemma 3.6 (cf. [3]). *Let* $(T, \mathcal{L})$ *and* $(T', \mathcal{L}')$ *be fine log schemes. If* $(t, \theta) : (T', \mathcal{L}') \to (T, \mathcal{L})$ *is a thickening of order 1, there exists a commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} 1 & \to & 1+\mathcal{I} & \hookrightarrow & t^{-1}\mathcal{L} & \xrightarrow{\theta} & \mathcal{L}' & \to & 1 \\ & & \| & & \cap & & \cap & & \\ 1 & \to & 1+\mathcal{I} & \to & t^{-1}\mathcal{L}^{\mathrm{gp}} & \underset{\theta^{\mathrm{gp}}}{\to} & \mathcal{L}'^{\mathrm{gp}} & \to & 1, \end{array}$$

where $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_T \to \mathcal{O}_{T'})$, *such that the right square of this commutative diagram is cartesian.*

The proof is straightforward. Note that the multiplicative monoid $1 + \mathcal{I}$ is identified with the additive monoid $\mathcal{I}$ by $1 + x \mapsto x$ since $\mathcal{I}^2 = 0$.

Definition 3.7 (cf. [5, (3.3)]) Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism in $\mathbf{LSch}^{\mathrm{f}}$. f is said to be *log smooth* if the following conditions are satisfied:

1. f is locally of finite presentation,
2. for any commutative diagram

$$\begin{array}{ccc} (T', \mathcal{L}') & \xrightarrow{s'} & (X, \mathcal{M}) \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle f} \\ (T, \mathcal{L}) & \underset{s}{\longrightarrow} & (Y, \mathcal{N}) \end{array}$$

in $\mathbf{LSch}^{\mathrm{f}}$, where t is a thickening of order 1, there exists étale locally a morphism $g : (T, \mathcal{L}) \to (X, \mathcal{M})$ such that $s' = g \circ t$ and $s = f \circ g$.

The proofs of the following two propositions are straightforward and are left to the reader.

Proposition 3.8 *Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism in $\mathbf{LSch}^{\mathrm{f}}$. If f is strict, then f is log smooth if and only if f is smooth in the usual sense.*

Proposition 3.9 *For $(S, \mathcal{L}) \in \mathrm{Obj}(\mathbf{LSch}^{\mathrm{f}})$ and $(X, \mathcal{M}), (Y, \mathcal{N}) \in \mathrm{Obj}(\mathbf{LSch}^{\mathrm{f}}_{(S,\mathcal{L})})$, let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism in $\mathbf{LSch}^{\mathrm{f}}_{(S,\mathcal{L})}$. Assume that f is log smooth. If $(S', \mathcal{L}')$ is a log scheme over $(S, \mathcal{L})$, then the induced morphism*

$$(X, \mathcal{M}) \times_{(S,\mathcal{L})} (S', \mathcal{L}') \to (Y, \mathcal{N}) \times_{(S,\mathcal{L})} (S', \mathcal{L}')$$

is also log smooth.

4 Toroidal characterization of log smoothness

The following theorem is due to Kato [5], and we prove it in §6 for the reader's convenience.

Theorem 4.1 ([5, (3.5)]) *Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism in $\mathbf{LSch}^{\mathrm{f}}$. and $Q \to \mathcal{N}$ a chart of $\mathcal{N}$, where Q is a finitely generated integral monoid. Then the following conditions are equivalent.*

1. *f is log smooth.*
2. *There exists étale locally a chart $(P \to \mathcal{M}, Q \to \mathcal{N}, Q \to P)$ of f extending $Q \to \mathcal{N}$, where P is a finitely generated integral monoid, such that*
   (a) *$\mathrm{Ker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ and the torsion part of $\mathrm{Coker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ are finite groups of orders invertible on X,*
   (b) *$X \to Y \times_{\mathrm{Spec}\, \mathbf{Z}[Q]} \mathrm{Spec}\, \mathbf{Z}[P]$ is smooth (in the usual sense).*

Moreover, if $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is a log smooth morphism in $\mathbf{LSch}^{\mathrm{fs}}$ and $Q \to \mathcal{N}$ is a chart of $\mathcal{N}$ such that Q is finitely generated, integral and saturated, then there exists a chart $(P \to \mathcal{M}, Q \to \mathcal{N}, Q \to P)$ of f as above with P also saturated.

Remark 4.2 The proof of Theorem 4.1 in §6 shows that we can require in the condition 2. (a) that $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is injective without changing the conclusion.

We give some important examples of log smooth morphisms in the following. Let k be a field.

Definition 4.3 A log structure on $\mathrm{Spec}\, k$ is called a log structure of a *logarithmic point* if it is equivalent (Definition 2.3) to the associated log structure of $\alpha : Q \to k$, where Q is a monoid having no invertible element other than 1 and α is a homomorphism defined by

$$\alpha(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this log structure is equivalent to $Q \oplus k^{\times} \to k$. We denote the log scheme obtained in this way by $(\mathrm{Spec}\, k, Q)$. The log scheme $(\mathrm{Spec}\, k, Q)$ is called a *logarithmic point*. Especially, if $Q = \mathbf{N}$, the logarithmic point $(\mathrm{Spec}\, k, \mathbf{N})$ is said to be the *standard log point*.

If k is algebraically closed, any log structure on $\operatorname{Spec} k$ is equivalent to a log structure of logarithmic point (cf. [3]). Note that if we set $Q = \{1\}$, then the log structure of the logarithmic point induced by Q is the trivial log structure (Example 2.4).

Example 4.4 Let P be a submonoid of a group $M = \mathbf{Z}^d$ such that $P^{\mathrm{gp}} = M$ and that P is saturated. Let Q be a submonoid of P, which is saturated but not necessarily saturated in P. We assume the following:

1. the monoid Q has no invertible element other than 1,
2. the order of the torsion part of M/Q^{gp} is invertible in k.

Let $R = \mathbf{Z}[1/N]$ where N is the order of the torsion part of M/Q^{gp}. The latter assumption implies, by Theorem 4.1, that $(\operatorname{Spec} R[P], P) \to (\operatorname{Spec} R[Q], Q)$ (see Example 2.5) is log smooth. Define $\operatorname{Spec} k \to \operatorname{Spec} R[Q]$ by $\alpha : Q \to k$ as in Definition 4.3. Let X be a scheme over k which is smooth over $\operatorname{Spec} k \times_{\operatorname{Spec} R[Q]} \operatorname{Spec} R[P]$. Then we have a diagram

$$\begin{array}{ccc} X & & \\ \downarrow & & \\ \operatorname{Spec} k \times_{\operatorname{Spec} R[Q]} \operatorname{Spec} R[P] & \longrightarrow & \operatorname{Spec} R[P] \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} R[Q]. \end{array}$$

Define a log structure $\mathcal{M}$ on X by the pull–back of the canonical log structure on $\operatorname{Spec} R[P]$. Then we have a morphism

$$f : (X, \mathcal{M}) \longrightarrow (\operatorname{Spec} k, Q)$$

of fine saturated log schemes. This morphsim f is log smooth by Proposition 3.8 and Proposition 3.9. We denote this log scheme $(X, \mathcal{M})$ simply by (X, P).

Example 4.5 (Toric varieties.) In this and the following examples, we use the notation appearing in Example 4.4. Let σ be a cone in $N_{\mathbf{R}} = \mathbf{R}^d$ and $\sigma^\vee$ be its dual cone in $M_{\mathbf{R}} = \mathbf{R}^d$. Set $P = M \cap \sigma^\vee$ and $Q = \{0\} \subset P$. Then, $\operatorname{Spec} k \times_{\operatorname{Spec} \mathbf{Z}[Q]} \operatorname{Spec} \mathbf{Z}[P]$ is k–isomorphic to $\operatorname{Spec} k[P]$ which is nothing but an affine toric variety. Let $X \to \operatorname{Spec} k[P]$ be a smooth morphism. Then $(X, P) \to \operatorname{Spec} k$ is log smooth.

Example 4.6 (Variety with normal crossings.) Let σ be the cone in $M_{\mathbf{R}} = \mathbf{R}^d$ generated by $e_1, \ldots, e_d$, where $e_i = (0, \ldots, 0, 1, 0, \ldots 0)$ (1 at the i–th entry), $1 \le i \le d$. Let τ be the subcone generated by $a_1 e_1 + \cdots + a_d e_d$ with positive integers a_j for $j = 1, \ldots, d$. We assume that $\mathrm{GCD}(a_1, \ldots, a_d)(= N)$ is invertible in k. Set $R = \mathbf{Z}[1/N]$. Then, by setting $P = M \cap \sigma$ and $Q = M \cap \tau$, we see that $\operatorname{Spec} k \times_{\operatorname{Spec} R[Q]} \operatorname{Spec} R[P]$ is k–isomorphic to $\operatorname{Spec} k[z_1, \ldots, z_d]/(z_1^{a_1} \cdots z_d^{a_d})$ and f is induced by

$$\begin{array}{ccc} \mathbf{N}^d & \longrightarrow & k[z_1, \ldots, z_d]/(z_1^{a_1} \cdots z_d^{a_d}) \\ {\scriptstyle\varphi}\uparrow & & \uparrow \\ \mathbf{N} & \longrightarrow & k, \end{array}$$

where the morphism in the first row is defined by $e_i \mapsto z_i$, $(1 \le i \le d)$, and φ is defined by $\varphi(1) = a_1 e_1 + \cdots + a_d e_d$. Let $X \to \operatorname{Spec} k[z_1, \ldots, z_d]/(z_1^{a_1} \cdots z_d^{a_d})$ be a smooth morphism. Then, $(X, \mathbf{N}^d) \to (\operatorname{Spec} k, \mathbf{N})$ is log smooth.

The following theorem is an application of Theorem 4.1. We prove it in §7.

Theorem 4.7 *Let X be an algebraic scheme over a field k, and $\mathcal{M} \to \mathcal{O}_X$ a fine saturated log structure on X. Then, the log scheme $(X, \mathcal{M})$ is log smooth over* Spec k *with trivial log structure if and only if there exist an open étale covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X and a divisor D of X such that:*

1. *there exists a smooth morphism*
$$h_i : U_i \longrightarrow V_i$$
where V_i is a affine toric variety over k for each $i \in I$,

2. *the divisor $U_i \cap D$ of U_i is the pull–back of the union of the closure of codimension 1 torus orbits of V_i by h_i for each $i \in I$,*

3. *the log structure $\mathcal{M} \to \mathcal{O}_X$ is equivalent to the log structure $\mathcal{O}_X \cap j_* \mathcal{O}_{X-D}^{\times} \hookrightarrow \mathcal{O}_X$ where $j : X - D \hookrightarrow X$ is the inclusion.*

Corollary 4.8 *Let X be a smooth algebraic variety over a field k, and $\mathcal{M} \to \mathcal{O}_X$ a fine saturated log structure on X. Then, the log scheme $(X, \mathcal{M})$ is log smooth over* Spec k *with trivial log structure if and only if there exists a reduced normal crossing divisor D of X such that the log structure $\mathcal{M} \to \mathcal{O}_X$ is equivalent to the log structure $\mathcal{O}_X \cap j_* \mathcal{O}_{X-D}^{\times} \hookrightarrow \mathcal{O}_X$ where $j : X - D \hookrightarrow X$ is the inclusion.*

5 Log differentials and log derivations

In this section, we are going to discuss the log differentials and log derivations. These objects are closely related with log smoothness, and play important roles in the sequel. To begin with, we introduce a useful notation which we often use in the sequel. Let $(X, \mathcal{M})$ be a log scheme. If we like to omit writing the log structure $\mathcal{M}$, we write this log scheme by $\underline{X}$ to distinguish from the underlying scheme X.

Definition 5.1 (cf. [5], in different notation) Let $\underline{X} = (X, \mathcal{M})$ and $\underline{Y} = (Y, \mathcal{N})$ be fine log schemes, and $f = (f, \varphi) : \underline{X} \to \underline{Y}$ a morphism, where $\varphi : f^{-1}\mathcal{N} \to \mathcal{M}$ is a homomorphism of sheaves of monoids.

1. Let $\mathcal{E}$ be an $\mathcal{O}_X$–module. The sheaf of *log derivations* $\mathcal{D}er_{\underline{Y}}(\underline{X}, \mathcal{E})$ of $\underline{X}$ to $\mathcal{E}$ over $\underline{Y}$ is the sheaf of germs of couples $(D, D\log)$, where $D \in \mathcal{D}er_Y(X, \mathcal{E})$ and $D\log : \mathcal{M} \to \mathcal{E}$, such that the following conditions are satisfied:
   (a) $D\log(ab) = D\log(a) + D\log(b)$, for $a, b \in \mathcal{M}$,
   (b) $\alpha(a) D\log(a) = D(\alpha(a))$, for $a \in \mathcal{M}$,
   (c) $D\log(\varphi(c)) = 0$, for $c \in f^{-1}\mathcal{N}$.

2. The sheaf of *log differentials* of $\underline{X}$ over $\underline{Y}$ is the $\mathcal{O}_X$–module defined by
$$\Omega^1_{\underline{X}/\underline{Y}} = [\Omega^1_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbf{Z}} \mathcal{M}^{\mathrm{gp}})]/\mathcal{K},$$
   where $\mathcal{K}$ is the $\mathcal{O}_X$–submodule generated by
$$(d\alpha(a), 0) - (0, \alpha(a) \otimes a) \text{ and } (0, 1 \otimes \varphi(b)),$$
   for all $a \in \mathcal{M}$, $b \in f^{-1}\mathcal{N}$.

These are coherent $\mathcal{O}_X$–modules if Y is locally noetherian and X locally of finite type over Y (cf. [3]). The proofs of the following three propositions are found in [5, §3].

Proposition 5.2 *Let $\underline{X}$, $\underline{Y}$, f, and $\mathcal{E}$ be the same as in Definition 5.1. Then there is a natural isomorphism*

$$\mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{\underline{X}/\underline{Y}}, \mathcal{E}) \xrightarrow{\sim} \mathcal{D}er_{\underline{Y}}(\underline{X}, \mathcal{E}),$$

by $u \mapsto (u \circ d, u \circ d\log)$, where d and $d\log$ are defined by

$$d : \mathcal{O}_X \to \Omega^1_{X/Y} \to \Omega^1_{\underline{X}/\underline{Y}}$$

and

$$d\log : \mathcal{M} \to \mathcal{O}_X \otimes_{\mathbf{Z}} \mathcal{M}^{\mathrm{gp}} \to \Omega^1_{\underline{X}/\underline{Y}}.$$

Proposition 5.3 *Let $\underline{X} \xrightarrow{f} \underline{Y} \xrightarrow{g} \underline{Z}$ be morphisms of fine log schemes.*

1. *There exists an exact sequence*
$$f^*\Omega^1_{\underline{Y}/\underline{Z}} \to \Omega^1_{\underline{X}/\underline{Z}} \to \Omega^1_{\underline{X}/\underline{Y}} \to 0.$$
2. *If f is log smooth, then*
$$0 \to f^*\Omega^1_{\underline{Y}/\underline{Z}} \to \Omega^1_{\underline{X}/\underline{Z}} \to \Omega^1_{\underline{X}/\underline{Y}} \to 0 \tag{4}$$
is exact.
3. *If $g \circ f$ is log smooth and (4) is exact and splits locally, then f is log smooth.*

Proposition 5.4 *If $f : \underline{X} \to \underline{Y}$ is log smooth, then $\Omega^1_{\underline{X}/\underline{Y}}$ is a locally free $\mathcal{O}_X$–module of finite type.*

Example 5.5 (cf. [10, Chap. 3, §(3.1)]) Let X_Σ be a toric variety over a field k determined by a fan Σ on $N_{\mathbf{R}}$ with $N = \mathbf{Z}^d$. Consider the log scheme (X_Σ, Σ) (Example 2.6) over $\operatorname{Spec} k$. Then we have isomorphisms of O_X–modules

$$\mathcal{D}er_k(\underline{X}, \mathcal{O}_X) \cong \mathcal{O}_X \otimes_{\mathbf{Z}} N \quad \text{and} \quad \Omega^1_{\underline{X}/k} \cong \mathcal{O}_X \otimes_{\mathbf{Z}} M,$$

where $M = \operatorname{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$.

Example 5.6 For $X = \operatorname{Spec} k[z_1, \ldots, z_n]/(z_1 \cdots z_l)$, let $f : (X, \mathcal{M}) \to (\operatorname{Spec} k, \mathbf{N} \to k)$ be the log smooth morphism defined in Example 4.6. Then $\mathcal{D}er_{\underline{k}}(\underline{X}, \mathcal{O}_X)$ is a free $\mathcal{O}_X$–module generated by

$$z_1\frac{\partial}{\partial z_1}, \ldots, z_l\frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_{l+1}}, \ldots, \frac{\partial}{\partial z_n}$$

with a relation

$$z_1\frac{\partial}{\partial z_1} + \cdots + z_l\frac{\partial}{\partial z_l} = 0.$$

The sheaf $\Omega^1_{\underline{X}/\underline{k}}$ is a free $\mathcal{O}_X$–module generated by the *logarithmic differentials*:

$$\frac{dz_1}{z_1}, \ldots, \frac{dz_l}{z_l}, dz_{l+1}, \ldots, dz_n$$

with a relation

$$\frac{dz_1}{z_1} + \cdots + \frac{dz_l}{z_l} = 0.$$

In the complex analytic case, the sheaf $\Omega^1_{\underline{X}/\underline{k}}$ is nothing but the sheaf of *relative logarithmic differentials* introduced in, for example, [1, §3], and [6, §2].

6 The proof of Theorem 4.1

In this section, we give a proof of Theorem 4.1 due to Kato [5]. Before proving the general case, we prove the following proposition.

Proposition 6.1 *Let A be a commutative ring and $h : Q \to P$ a homomorphism of finitely generated integral monoids. The homomorphism h induces the morphism of log schemes*

$$f : \underline{X} = (\operatorname{Spec} A[P], P) \longrightarrow \underline{Y} = (\operatorname{Spec} A[Q], Q).$$

We set $K = \operatorname{Ker}(h^{\mathrm{gp}} : Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ and $C = \operatorname{Coker}(h^{\mathrm{gp}} : Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$, and denote the torsion part of C by C_{tor}. If both K and C_{tor} are finite groups of order invertible in A, then f is log smooth.

Proof. Suppose we have a commutative diagram

$$\begin{array}{ccccc} (T', \mathcal{L}') & \xrightarrow{s'} & \underline{X} & = & (\operatorname{Spec} A[P], P) \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle f} & & \\ (T, \mathcal{L}) & \underset{s}{\longrightarrow} & \underline{Y} & = & (\operatorname{Spec} A[Q], Q) \end{array}$$

in $\mathbf{LSch}^{\mathrm{f}}$, where the morphism t is a thickening of order 1. Since we may work étale locally, we may assume that T is affine. Set

$$\mathcal{I} = \operatorname{Ker}(\mathcal{O}_T \to \mathcal{O}_{T'}).$$

Since the morphism t is a thickening of order 1, by Lemma 3.6, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & 1+\mathcal{I} & \hookrightarrow & \mathcal{L} & \xrightarrow{t^*} & \mathcal{L}' & \longrightarrow & 1 \\ & & \| & & \cap & & \cap & & \\ 1 & \longrightarrow & 1+\mathcal{I} & \longrightarrow & \mathcal{L}^{\mathrm{gp}} & \underset{(t^*)^{\mathrm{gp}}}{\longrightarrow} & \mathcal{L}'^{\mathrm{gp}} & \longrightarrow & 1. \end{array}$$

Note that the right square of the above commutative diagram is cartesian.

First, consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & K & \longrightarrow & Q^{\mathrm{gp}} & \xrightarrow{h^{\mathrm{gp}}} & P^{\mathrm{gp}} & \longrightarrow & C & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & & & \\ 1 & \longrightarrow & 1+\mathcal{I} & \longrightarrow & \mathcal{L}^{\mathrm{gp}} & \underset{(t^*)^{\mathrm{gp}}}{\longrightarrow} & \mathcal{L}'^{\mathrm{gp}} & \longrightarrow & 1. & & \end{array}$$

The multiplicative monoid $1+\mathcal{I}$ is isomorphic to the additive monoid $\mathcal{I}$ by $1+x \mapsto x$ since $\mathcal{I}^2 = 0$. If the order of K is invertible in A, then we have $u = 1$, and hence there exists a morphism $a' : R \to \mathcal{L}^{\mathrm{gp}}$ with $R = \operatorname{Image}(h^{\mathrm{gp}} : Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ such that $a' \circ h^{\mathrm{gp}} = v$ and $(t^*)^{\mathrm{gp}} \circ a' = w$.

Next, we consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & 1 & \longrightarrow & R & \xrightarrow{i} & P^{\mathrm{gp}} & \longrightarrow & C & \longrightarrow & 1 \\ & & & & {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle w} & & & & \\ 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{L}^{\mathrm{gp}} & \underset{(t^*)^{\mathrm{gp}}}{\longrightarrow} & \mathcal{L}'^{\mathrm{gp}} & \longrightarrow & 1. & & \end{array}$$

We shall show that there exists a homomorphism $a'' : P^{\mathrm{gp}} \to \mathcal{L}^{\mathrm{gp}}$ such that $a'' \circ t = a'$ and $(t^*)^{\mathrm{gp}} \circ a'' = w$. The obstruction of existence of a'' lies in $\mathrm{Ext}^1(C, \mathcal{I})$. In general, if a positive integer n is invertible in A then we have $\mathrm{Ext}^1(\mathbf{Z}/n\mathbf{Z}, \mathcal{I}) = 0$. Combining this with $\mathrm{Ext}^1(\mathbf{Z}, \mathcal{I}) = 0$, we have $\mathrm{Ext}^1(C, \mathcal{I}) = 0$ since the order of the torsion part of C is invertible in A. Hence a homomorphism a'' exists. Since the diagram

$$\begin{array}{ccc} \mathcal{L} & \stackrel{t^*}{\longrightarrow} & \mathcal{L}' \\ \cap & & \cap \\ \mathcal{L}^{\mathrm{gp}} & \underset{(t^*)^{\mathrm{gp}}}{\longrightarrow} & \mathcal{L}'^{\mathrm{gp}}. \end{array}$$

is cartesian, we found a homomorphism

$$a : P \longrightarrow \mathcal{L}$$

such that $t^* \circ a = (s')^*$ and $a \circ h = s^*$. Using this a, we can construct a morphism of log schemes

$$g : (T, \mathcal{L}) \longrightarrow \underline{X} = (\mathrm{Spec}\, A[P], P)$$

such that $g \circ t = s'$ and $s \circ g = f$. $\square$

Now, let us prove Theorem 4.1. First, we prove the implication $2 \Rightarrow 1$. Let $R = \mathbf{Z}[1/(N_1 \cdot N_2)]$ where N_1 is the order of $\mathrm{Ker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ and N_2 is the order of the torsion part of $\mathrm{Coker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$. By the assumption (a), we have

$$Y \times_{\mathrm{Spec}\, \mathbf{Z}[Q]} \mathrm{Spec}\, \mathbf{Z}[P] \cong Y \times_{\mathrm{Spec}\, R[Q]} \mathrm{Spec}\, R[P].$$

Since $X \to Y \times_{\mathrm{Spec}\, R[Q]} \mathrm{Spec}\, R[P]$ is smooth by (b), f is log smooth due to Proposition 3.8, Proposition 3.9 and Proposition 6.1.

Next, let us prove the converse. Assume the morphism f is log smooth. Then, the sheaf $\Omega^1_{\underline{X}/\underline{Y}}$ is a locally free $\mathcal{O}_X$–module of finite type (Proposition 5.4). Take any point $x \in X$. We denote by $\bar{x}$ a separable closure of x.

Step 1 Consider the morphism of $\mathcal{O}_X$–modules

$$1 \otimes d\log : \mathcal{O}_X \otimes_{\mathbf{Z}} \mathcal{M}^{\mathrm{gp}} \longrightarrow \Omega^1_{\underline{X}/\underline{Y}},$$

which is surjective by the definition of $\Omega^1_{\underline{X}/\underline{Y}}$. Then we can take elements $t_1, \ldots, t_r \in \mathcal{M}_{\bar{x}}$ such that the system $\{d\log t_i\}_{1 \le i \le r}$ is a $\mathcal{O}_{X,\bar{x}}$–base of $\Omega^1_{\underline{X}/\underline{Y},\bar{x}}$. Consider the homomorphism $\psi : \mathbf{N}^r \to \mathcal{M}_{\bar{x}}$ defined by

$$\mathbf{N}^r \ni (n_1, \ldots, n_r) \mapsto t_1^{n_1} \cdots t_r^{n_r} \in \mathcal{M}_{\bar{x}}.$$

Combining this ψ with the homomorphism $Q \to f^{-1}(\mathcal{N})_{\bar{x}} \to \mathcal{M}_{\bar{x}}$, we have a homomorphism $\varphi : H = \mathbf{N}^r \oplus Q \to \mathcal{M}_{\bar{x}}$.

Step 2 Let $k(\bar{x})$ denote the residue field at $\bar{x}$. We have a homomorphism

$$k(\bar{x}) \otimes_{\mathbf{Z}} \mathbf{Z}^r \longrightarrow k(\bar{x}) \otimes_{\mathbf{Z}} \mathrm{Coker}(f^{-1}(\mathcal{N}^{\mathrm{gp}}/\mathcal{O}_Y^{\times})_{\bar{x}} \to \mathcal{M}_{\bar{x}}^{\mathrm{gp}}/\mathcal{O}_{X,\bar{x}}^{\times}) \tag{5}$$

by $k(\bar{x}) \otimes_{\mathbf{Z}} \psi^{\mathrm{gp}} : k(\bar{x}) \otimes_{\mathbf{Z}} \mathbf{Z}^r \to k(\bar{x}) \otimes_{\mathbf{Z}} \mathcal{M}_{\bar{x}}^{\mathrm{gp}}$ and canonical projectiones $\mathcal{M}_{\bar{x}}^{\mathrm{gp}} \to \mathcal{M}_{\bar{x}}^{\mathrm{gp}}/\mathcal{O}_{X,\bar{x}}^{\times} \to \mathrm{Coker}(f^{-1}(\mathcal{N}^{\mathrm{gp}}/\mathcal{O}_Y^{\times})_{\bar{x}} \to \mathcal{M}_{\bar{x}}^{\mathrm{gp}}/\mathcal{O}_{X,\bar{x}}^{\times})$. We claim that this morphism (5) is surjective. In fact, this morphism coincides with the composite morphism

$$k(\bar{x}) \otimes_{\mathbf{Z}} \mathbf{Z}^r \to k(\bar{x}) \otimes_{\mathcal{O}_{X,\bar{x}}} \Omega^1_{\underline{X}/\underline{Y},\bar{x}} \to k(\bar{x}) \otimes_{\mathbf{Z}} \mathrm{Coker}(f^{-1}(\mathcal{N}^{\mathrm{gp}}/\mathcal{O}_Y^{\times})_{\bar{x}} \to \mathcal{M}_{\bar{x}}^{\mathrm{gp}}/\mathcal{O}_{X,\bar{x}}^{\times}),$$

where the first morphism is induced by $d\log \circ \psi$ and the second one by the canonical projection, and these morphisms are clearly surjective. Hence the morphism (5) is surjective. On the other hand, the homomorphism

$$Q^{\mathrm{gp}} \longrightarrow f^{-1}(\mathcal{N}/\mathcal{O}_Y^{\times})_{\bar{x}}$$

is surjective since $Q \to \mathcal{N}$ is a chart of $\mathcal{N}$. Hence, the homomorphism

$$k(\bar{x}) \otimes_{\mathbf{Z}} Q^{\mathrm{gp}} \longrightarrow k(\bar{x}) \otimes_{\mathbf{Z}} f^{-1}(\mathcal{N}/\mathcal{O}_Y^{\times})_{\bar{x}}$$

is surjective, and then, the homomorphism

$$1 \otimes_{\mathbf{Z}} \varphi^{\mathrm{gp}} : k(\bar{x}) \otimes_Z H^{\mathrm{gp}} \longrightarrow k(\bar{x}) \otimes_{\mathbf{Z}} (\mathcal{M}_{\bar{x}}/\mathcal{O}_{X,\bar{x}}^{\times})$$

is surjective. This shows that the cokernel $C = \mathrm{Coker}(\varphi^{\mathrm{gp}} : H^{\mathrm{gp}} \to \mathcal{M}_{\bar{x}}^{\mathrm{gp}}/\mathcal{O}_{X,\bar{x}}^{\times})$ is annihilated by an integer N invertible in $\mathcal{O}_{X,\bar{x}}$.

Step 3 Take elements $a_1, \ldots, a_d \in \mathcal{M}_{\bar{x}}^{\mathrm{gp}}$ which generates C. Then we can write $a_i^n = u_i\varphi(b_i)$ for $u_i \in \mathcal{O}_{X,\bar{x}}^{\times}$ and $b_i \in H^{\mathrm{gp}}$, for $i = 1, \ldots, d$. Since $\mathcal{O}_{X,\bar{x}}^{\times}$ is N–divisible, we can write $u_i = v_i^N$ for $v_i \in \mathcal{O}_{X,\bar{x}}^{\times}$, for $i = 1, \ldots, d$, and hence we may suppose $a_i^N = \varphi(b_i)$, replacing a_i by a_i/v_i, for $i = 1, \ldots, d$. Let G be the push–out of the diagram

$$H^{\mathrm{gp}} \longleftarrow \mathbf{Z}^d \longrightarrow \mathbf{Z}^d,$$

where $\mathbf{Z}^d \to H^{\mathrm{gp}}$ is defined by $e_i \mapsto b_i$, and $\mathbf{Z}^d \to \mathbf{Z}^d$ is defined by $e_i \mapsto Ne_i$ for $i = 1, \ldots, d$. Then $\varphi^{\mathrm{gp}} : H^{\mathrm{gp}} \to \mathcal{M}_{\bar{x}}^{\mathrm{gp}}$ and $\mathbf{Z}^d \to \mathcal{M}_{\bar{x}}^{\mathrm{gp}}$, defined by $e_i \mapsto a_i$ for $i = 1, \ldots, d$, induce the homomorphism

$$\phi : G \longrightarrow \mathcal{M}_{\bar{x}}^{\mathrm{gp}}$$

which maps G surjectively onto $\mathcal{M}_{\bar{x}}^{\mathrm{gp}}/\mathcal{O}_{X,\bar{x}}^{\times}$. Define $P = \phi^{-1}(\mathcal{M}_{\bar{x}})$, then P defines a chart of $\mathcal{M}$ on some neighborhood of $\bar{x}$ ([5, Lemma 2.10]). If $\mathcal{M}$ is saturated, P is also saturated. There exists an induced map $Q \to P$ which defines a chart of f on some neighborhood of $\bar{x}$. Since $H^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is injective, $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is injective. The cokernel $\mathrm{Coker}(H^{\mathrm{gp}} \to P^{\mathrm{gp}})$ is annihilated by N, hence $\mathrm{Coker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}})_{\mathrm{tor}}$ is finite and annihilated by N.

Step 4 Set $X' = Y \times_{\mathrm{Spec}\,\mathbf{Z}[Q]} \mathrm{Spec}\,\mathbf{Z}[P]$ and $g : X \to X'$. We need to show that the morphism g is smooth in the usual sense. Since $\underline{X}$ has the log structure induced by g from $\underline{X}' = (X', P)$, it suffice to show that g is log smooth (Proposition 3.8). Since $k(\bar{x}) \otimes_{\mathbf{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}}) \cong k(\bar{x}) \otimes_{\mathbf{Z}} \mathbf{Z}^d \cong k(\bar{x}) \otimes_{\mathcal{O}_{X,\bar{x}}} \Omega^1_{\underline{X}/\underline{Y},\bar{x}}$ and $\Omega^1_{\underline{X}/\underline{Y}}$ is locally free, we have $\Omega^1_{\underline{X}/\underline{Y}} \cong \mathcal{O}_X \otimes_{\mathbf{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}})$ on some neighborhood of $\bar{x}$. On the other hand, by direct calculations, one sees that $\Omega^1_{\underline{X}'/\underline{Y}} \cong \mathcal{O}_{X'} \otimes_{\mathbf{Z}[P]} \Omega^1_{\mathbf{Z}[P]/\mathbf{Z}[Q]} \cong \mathcal{O}_{X'} \otimes_{\mathbf{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}})$. Hence we have $g^*\Omega^1_{\underline{X}'/\underline{Y}} \cong \Omega^1_{\underline{X}/\underline{Y}}$. This implies g is log smooth due to Proposition 5.3 (in fact, g is *log étale* (cf. [5])).

This completes the proof of the theorem. □

7 The proof of Theorem 4.7

In this section, we give a proof of Theorem 4.7. If $V = \operatorname{Spec} k[P]$ is an affine toric variety, then it is easy to see that the log structure associated to $P \to k[P]$ is equivalent to the log structure $\mathcal{O}_X \cap j_*\mathcal{O}^\times_{V-D} \hookrightarrow \mathcal{O}_X$ where D is the union of the closure of codimention 1 torus orbits of V and $j : V - D \hookrightarrow V$ is the inclusion. Hence, the "if" part of Theorem 4.7 is easy to see. Let us prove the converse. Let $(X, \mathcal{M})$ be as in the assumption of Theorem 4.7 and $f : (X, \mathcal{M}) \to \operatorname{Spec} k$ the structure morphism. The key–lemma is the following.

Lemma 7.1 *We can take étale locally a chart $P \to \mathcal{M}$ of $\mathcal{M}$ such that*

1. *the chart $(P \to \mathcal{M}, 1 \to k^\times, 1 \to P)$ of f satisfies the conditions* (a) *and* (b) *in Theorem 4.1,*

2. *P is a finitely generated integral saturated monoid, and has no torsion element.*

Here, by a torsion element, we mean an element $x \neq 1$ such that $x^n = 1$ for some positive integer n.

First, we are going to show that the theorem follows from the above lemma. Since the monoid P has no torsion element, P is the saturated submonoid of a finitely generated free abelian group P^{gp}. Hence, X is étale locally smooth over affine toric varieties, and the log structure $\mathcal{M}$ on X is étale locally equivalent to the pull–back of the log structure induced by the union of the closure of codimension 1 torus orbits. Since, these log structure glue to the log structure $\mathcal{M}$ on X, the pull–back of the union of the closure of codimension 1 torus orbits glue to a divisor on X. In fact, this divisor is the compliment of the largest open subset U such that $\mathcal{M}|_U$ is trivial with the reduced scheme structure. Hence our assertion is proved.

Now, we are going to prove Lemma 7.1. We may work étale locally. Take a chart $(P \to \mathcal{M}, 1 \to k^\times, 1 \to P)$ of f as in Theorem 4.1. We may assume that P is saturated. Define

$$P_{\mathrm{tor}} = \{x \in P \mid x^n = 1 \,(\text{for some } n)\}.$$

P_{tor} is a subgroup in P. Take a decomposition $P^{\mathrm{gp}} = G_{\mathrm{f}} \oplus G_{\mathrm{tor}}$ of the finitely generated abelian group P^{gp}, where G_{f} (resp. G_{tor}) is a free (resp. torsion) subgroup of P^{gp}. Then we have the equalities $P_{\mathrm{tor}} = P \cap G_{\mathrm{tor}} = G_{\mathrm{tor}}$ since P is saturated. Define a submonoid P_{f} by $P_{\mathrm{f}} = P \cap G_{\mathrm{f}}$.

Claim 1 $P = P_{\mathrm{f}} \oplus P_{\mathrm{tor}}$.

Proof. Take $x \in P$. Decompose $x = yz$ in P^{gp} such that $y \in G_{\mathrm{f}}$ and $z \in G_{\mathrm{tor}} = P_{\mathrm{tor}}$. Since $y^n = (xz^{-1})^n = x^n \in P$ for a large n, we have $y \in P$. Hence $y \in P_{\mathrm{f}}$. □

Define $\alpha_{\mathrm{f}} : P_{\mathrm{f}} \to \mathcal{O}_X$ by $P_{\mathrm{f}} \hookrightarrow P \xrightarrow{\alpha} \mathcal{O}_X$.

Claim 2 *The homomorphism $\alpha_{\mathrm{f}} : P_{\mathrm{f}} \to \mathcal{O}_X$ defines a log structure equivalent to $\mathcal{M}$.*

Proof. If $x \in P_{\mathrm{tor}}$, then $\alpha(x) \in \mathcal{O}^\times_X$ since $\alpha(x)^n = 1$ for a large n. Hence $\alpha(P_{\mathrm{tor}}) \subset \mathcal{O}^\times_X$. This implies that the associated log structure of P_{f} is equivalent to that of P. □

Hence, the morphism f is equivalent to the morphsim induced by the diagram

$$\begin{array}{ccc} P_{\mathrm{f}} & \xrightarrow{\alpha_{\mathrm{f}}} & \mathcal{O}_X \\ {\scriptstyle\varphi_{\mathrm{f}}}\uparrow & & \uparrow \\ 1 & \xrightarrow[\lambda]{} & k, \end{array}$$

Then we have to check the conditions (a) and (b) in Theorem 4.1. The condition (a) is easy to verify. Let us check the condition (b). We need to show that the morphism

$$X \longrightarrow \operatorname{Spec} k[P_{\mathrm{f}}]$$

induced by $X \to \operatorname{Spec} \mathbf{Z}[P] \to \operatorname{Spec} \mathbf{Z}[P_{\mathrm{f}}]$ is smooth.

Claim 3 *The morphism*

$$\operatorname{Spec} k[P] \longrightarrow \operatorname{Spec} k[P_{\mathrm{f}}] \tag{6}$$

induced by $P_{\mathrm{f}} \hookrightarrow P$ is étale.

Proof. Since $P = P_{\mathrm{f}} \oplus P_{\mathrm{tor}}$, we have $k[P] = k[P_{\mathrm{f}}] \otimes_k k[P_{\mathrm{tor}}]$. Since every element in P_{tor} is roots of 1, and the order of P_{tor} is invertible in k, the morphism

$$k \longhookrightarrow k[P_{\mathrm{tor}}]$$

is a finite separable extension of the field k. This shows that the morphism (6) is étale. □

Now we have proved Lemma 7.1, and hence, Theorem 4.7.

8 Formulation of log smooth deformation

From now on, we fix the following notation. Let k be a field and Q a finitely generated integral saturated monoid having no invertible element other than 1. Then we have a logarithmic point (Definition 4.3) $\underline{k} = (\operatorname{Spec} k, Q)$. Let $f = (f, \varphi) : \underline{X} = (X, \mathcal{M}) \to \underline{k} = (\operatorname{Spec} k, Q)$ be a log smooth morphism in $\mathbf{LSch}^{\mathrm{fs}}$.

Let Λ be a complete noetherian local ring with the residue field k. For example, $\Lambda = k$ or $\Lambda =$ the Witt vector ring of k when k is perfect. We denote, by $\Lambda[[Q]]$, the completion of the monoid ring $\Lambda[Q]$ along the maximal ideal $\mu + \Lambda[Q \setminus \{1\}]$ where μ denotes the maximal ideal of Λ. The completion $\Lambda[[Q]]$ is a complete local Λ–algebra and is noetherian since Q is finitely generated. If the monoid Q is isomorphic to $\mathbf{N}$ then the ring $\Lambda[[Q]]$ is isomorphic to $\Lambda[[t]]$ as local Λ–algebras. Let $\mathcal{C}_{\Lambda[[Q]]}$ be the category of artinian local $\Lambda[[Q]]$–algebras with the residue field k, and $\widehat{\mathcal{C}}_{\Lambda[[Q]]}$ be the category of pro–objects of $\mathcal{C}_{\Lambda[[Q]]}$ (cf. [11]). For $A \in \operatorname{Obj}(\widehat{\mathcal{C}}_{\Lambda[[Q]]})$, we define a log structure on the scheme $\operatorname{Spec} A$ by the associated log structure

$$Q \oplus A^{\times} \longrightarrow A$$

of the homomorphism $Q \to \Lambda[[Q]] \to A$. We denote, by $(\operatorname{Spec} A, Q)$, the log scheme obtained in this way.

Definition 8.1 For $A \in \mathrm{Obj}(\mathcal{C}_{\Lambda[[Q]]})$, a *log smooth lifting* of $f : (X, \mathcal{M}) \to (\mathrm{Spec}\, k, Q)$ on A is a morphism $\widetilde{f} : (\widetilde{X}, \widetilde{\mathcal{M}}) \to (\mathrm{Spec}\, A, Q)$ in $\mathbf{LSch}^{\mathrm{fs}}$ together with a cartesian diagram

$$\begin{array}{ccc} (X, \mathcal{M}) & \longrightarrow & (\widetilde{X}, \widetilde{\mathcal{M}}) \\ f\downarrow & & \downarrow\widetilde{f} \\ (\mathrm{Spec}\, k, Q) & \longrightarrow & (\mathrm{Spec}\, A, Q) \end{array}$$

in $\mathbf{LSch}^{\mathrm{fs}}$. Two liftings are said to be *isomorphic* if they are isomorphic in $\mathbf{LSch}^{\mathrm{fs}}_{(\mathrm{Spec}\, A, Q)}$.

Note that $(\mathrm{Spec}\, k, Q) \to (\mathrm{Spec}\, A, Q)$ is an exact closed immersion, and hence, the above diagram is cartesian in $\mathbf{LSch}^{\mathrm{fs}}$ if and only if so is in $\mathbf{LSch}$ (Lemma 3.4). In particular, the underlying morphisms of log smooth liftings are (not necessarily flat) liftings in the usual sense. Moreover, since exact closed immersions are stable under base changes, $(X, \mathcal{M}) \to (\widetilde{X}, \widetilde{\mathcal{M}})$ is also an exact closed immersion. If either $Q = \{1\}$ or $Q = \mathbf{N}$, the underlying morphisms of any log smooth liftings of f are flat since these morphisms of log schemes are *integral* (cf. [5]). Hence, in this case, the underlying morphisms of log smooth liftings of f are flat liftings of f.

Take a local chart $(P \to \mathcal{M}, Q \to Q \oplus k^{\times}, Q \to P)$ of f extending the given $Q \to k$ as in Theorem 4.1 such that $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is injective (Remark 4.2). Then, f factors throught $\mathrm{Spec}\, k \times_{\mathrm{Spec}\, \mathbf{Z}[Q]} \mathrm{Spec}\, \mathbf{Z}[P]$ by the smooth morphism $X \to \mathrm{Spec}\, k \times_{\mathrm{Spec}\, \mathbf{Z}[Q]} \mathrm{Spec}\, \mathbf{Z}[P]$ and the natural projection étale locally. For $A \in \mathrm{Obj}(\mathcal{C}_{\Lambda[[Q]]})$, a smooth lifting

$$\widetilde{X} \longrightarrow \mathrm{Spec}\, A \times_{\mathrm{Spec}\, \mathbf{Z}[Q]} \mathrm{Spec}\, \mathbf{Z}[P] \tag{7}$$

of $X \to \mathrm{Spec}\, k \times_{\mathrm{Spec}\, \mathbf{Z}[Q]} \mathrm{Spec}\, \mathbf{Z}[P]$, with the naturally induced log structure, gives a local log smooth lifting of f. Note that this local lifting $(\widetilde{X}, \widetilde{\mathcal{M}}) \to (\mathrm{Spec}\, A, Q)$ is log smooth. Conversely, suppose $\widetilde{f} : (\widetilde{X}, \widetilde{\mathcal{M}}) \to (\mathrm{Spec}\, A, Q)$ is a local log smooth lifting of f on A.

Lemma 8.2 *The local chart $(P \to \mathcal{M}, Q \to Q \oplus k^{\times}, Q \to P)$ of f lifts to the local chart $(P \to \widetilde{\mathcal{M}}, Q \to Q \oplus A^{\times}, Q \to P)$ of $\widetilde{f}$.*

Proof. The proof is done by the induction with respect to the length of A. Take $A' \in \mathrm{Obj}(\mathcal{C}_{\Lambda[[Q]]})$ with surjective morphism $A \to A'$ such that $I = \mathrm{Ker}(A \to A') \neq 0$ and $I^2 = 0$. Let $\widetilde{f}' : (\widetilde{X}', \widetilde{\mathcal{M}}') \to (\mathrm{Spec}\, A', Q)$ be a pull–back of $\widetilde{f}$. Then, $\widetilde{f}'$ is a log smooth lifting of f to A'. By the induction, we have the lifted local chart $(P \to \widetilde{\mathcal{M}}', Q \to Q \oplus A'^{\times}, Q \to P)$ of $\widetilde{f}'$. Since $(\widetilde{X}', \widetilde{\mathcal{M}}') \to (\widetilde{X}, \widetilde{\mathcal{M}})$ is a thickening of order 1, by Lemma 3.6, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{I} & \hookrightarrow & \widetilde{\mathcal{M}} & \to & \widetilde{\mathcal{M}}' & \to & 1 \\ & & \| & & \cap & & \cap & & \\ 0 & \to & \mathcal{I} & \to & \widetilde{\mathcal{M}}^{\mathrm{gp}} & \to & \widetilde{\mathcal{M}}'^{\mathrm{gp}} & \to & 1, \end{array}$$

where $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{\widetilde{X}'})$. The right square of this diagram is cartesian. Consider the

following commutative diagram of abelian groups with exact rows and columns:

$$\begin{array}{ccccccc}
\mathrm{Hom}(P^{\mathrm{gp}},\mathcal{I}) & \to & \mathrm{Hom}(P^{\mathrm{gp}},\widetilde{\mathcal{M}}^{\mathrm{gp}}) & \to & \mathrm{Hom}(P^{\mathrm{gp}},\widetilde{\mathcal{M}}'^{\mathrm{gp}}) & \to & \mathrm{Ext}^1(P^{\mathrm{gp}},\mathcal{I}) \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Hom}(Q^{\mathrm{gp}},\mathcal{I}) & \to & \mathrm{Hom}(Q^{\mathrm{gp}},\widetilde{\mathcal{M}}^{\mathrm{gp}}) & \to & \mathrm{Hom}(Q^{\mathrm{gp}},\widetilde{\mathcal{M}}'^{\mathrm{gp}}) & & \\
\downarrow & & & & & & \\
\mathrm{Ext}^1(C,\mathcal{I}), & & & & & &
\end{array}$$

where $C = \mathrm{Coker}(Q^{\mathrm{gp}} \hookrightarrow P^{\mathrm{gp}})$. By our assumption, we have $\mathrm{Ext}^1(P^{\mathrm{gp}},\mathcal{I}) = 0$ and $\mathrm{Ext}^1(C,\mathcal{I}) = 0$, since the order of the torsion part of each P^{gp} and C is invertible in A. Then, by an easy diagram chasing, we can show that the given morphism $P^{\mathrm{gp}} \to \widetilde{\mathcal{M}}'^{\mathrm{gp}}$ can be lifted to a morphism $P^{\mathrm{gp}} \to \widetilde{\mathcal{M}}^{\mathrm{gp}}$ such that

$$\begin{array}{ccc}
P^{\mathrm{gp}} & \longrightarrow & \widetilde{\mathcal{M}}^{\mathrm{gp}} \\
\uparrow & & \uparrow \\
Q^{\mathrm{gp}} & \longrightarrow & Q^{\mathrm{gp}} \oplus A^{\times}
\end{array}$$

is commutative. Then, we have the morphism $P \to \widetilde{\mathcal{M}}$ and we see that the diagram

$$\begin{array}{ccc}
\mathcal{P} & \longrightarrow & \widetilde{\mathcal{M}} \\
\uparrow & & \uparrow \\
Q & \longrightarrow & Q \oplus A^{\times}
\end{array}$$

is commutative. Since $\widetilde{\mathcal{M}}/\mathcal{O}^{\times}_{\widetilde{X}} \xrightarrow{\sim} \widetilde{\mathcal{M}}'/\mathcal{O}^{\times}_{\widetilde{X}'}$, we can easily show that the morphism $P \to \widetilde{\mathcal{M}}$ defines a chart (Lemma 3.3). $\square$

Then, $\tilde{f}$ is factors throught $\mathrm{Spec}\, A \times_{\mathrm{Spec}\,\mathbf{Z}[Q]} \mathrm{Spec}\,\mathbf{Z}[P]$ by the induced morphism $\widetilde{X} \to \mathrm{Spec}\, A \times_{\mathrm{Spec}\,\mathbf{Z}[Q]} \mathrm{Spec}\,\mathbf{Z}[P]$ and the natural projection, and we have the following commutative diagram

$$\begin{array}{ccc}
X & \longrightarrow & \widetilde{X} \\
\downarrow & & \downarrow \\
\mathrm{Spec}\, k \times_{\mathrm{Spec}\,\mathbf{Z}[Q]} \mathrm{Spec}\,\mathbf{Z}[P] & \longrightarrow & \mathrm{Spec}\, A \times_{\mathrm{Spec}\,\mathbf{Z}[Q]} \mathrm{Spec}\,\mathbf{Z}[P] \\
\downarrow & & \downarrow \\
\mathrm{Spec}\, k & \longrightarrow & \mathrm{Spec}\, A,
\end{array}$$

such that the each square is cartesian. Hence, $\widetilde{X} \to \mathrm{Spec}\, A \times_{\mathrm{Spec}\,\mathbf{Z}[Q]} \mathrm{Spec}\,\mathbf{Z}[P]$ is smooth since it is a smooth lifting of the smooth morphism $X \to \mathrm{Spec}\, k \times_{\mathrm{Spec}\,\mathbf{Z}[Q]} \mathrm{Spec}\,\mathbf{Z}[P]$, and is unique up to isomorphisms by the classical theory. Therefore, we have proved the following proposition.

Proposition 8.3 (cf. [5, (3.14)]) *For $A \in \mathcal{C}_{\Lambda[[Q]]}$, a log smooth lifting of $f : (X,\mathcal{M}) \to (\mathrm{Spec}\, k, Q)$ on A exists étale locally, and is unique up to isomorphisms. In particular, log smooth liftings are log smooth.*

Let $\widetilde{f} : (\widetilde{X}, \widetilde{\mathcal{M}}) \to (\mathrm{Spec}\, A, Q)$ be a log smooth lifting of f to A, and $u : A' \to A$ a surjective homomorphism in $\mathcal{C}_{\Lambda[[Q]]}$ such that $I^2 = 0$ where $I = \mathrm{Ker}(u)$. Suppose $\widetilde{f}' : (\widetilde{X}', \widetilde{\mathcal{M}}') \to (\mathrm{Spec}\, A', Q)$ is a log smooth lifting of f to A' which is also a lifting of $\widetilde{f}$. Let $(P \to \widetilde{\mathcal{M}}', Q \to Q \oplus A'^{\times}, Q \to P)$ be a local chart of $\widetilde{f}'$ which is a lifting of $(P \to \mathcal{M}, Q \to Q \oplus k^{\times}, Q \to P)$. Define a local chart $(P \to \widetilde{\mathcal{M}}, Q \to Q \oplus A^{\times}, Q \to P)$ of $\widetilde{f}$ by $P \to \mathcal{M}' \to \mathcal{M}$ and $Q \to Q \oplus A'^{\times} \to Q \oplus A^{\times}$. An automorphism $\Theta : (\widetilde{X}', \widetilde{\mathcal{M}}') \xrightarrow{\sim} (\widetilde{X}', \widetilde{\mathcal{M}}')$ over $(\mathrm{Spec}\, A', Q)$ which is identity on $(\widetilde{X}, \widetilde{\mathcal{M}})$ induces an automorphism $\theta : \widetilde{\mathcal{M}}'^{\mathrm{gp}} \xrightarrow{\sim} \widetilde{\mathcal{M}}'^{\mathrm{gp}}$. Consider the diagram

$$\begin{array}{ccccccccc} & & & & P^{\mathrm{gp}} & = & P^{\mathrm{gp}} & & \\ & & & & {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \alpha} & & \\ 1 & \to & 1+\mathcal{I} & \to & \widetilde{\mathcal{M}}'^{\mathrm{gp}} & \to & \widetilde{\mathcal{M}}^{\mathrm{gp}} & \to & 1. \end{array}$$

For $a \in P^{\mathrm{gp}}$, the element $\alpha'(a) \cdot [\theta \circ \alpha'(a)]^{-1}$ is in $1 + \mathcal{I}$. Then, we have a morphism $\Delta : P^{\mathrm{gp}} \to \mathcal{I} = I \cdot \mathcal{O}_{\widetilde{X}'} \cong I \otimes_A \mathcal{O}_{\widetilde{X}}$ by $\Delta(a) = \alpha'(a) \cdot [\theta \circ \alpha'(a)]^{-1} - 1$. The morphism Δ lifts to the morphism $\Delta : P^{\mathrm{gp}}/Q^{\mathrm{gp}} \to I \otimes_A \mathcal{O}_{\widetilde{X}}$ and defines a morphism of $\mathcal{O}_{\widetilde{X}}$–modules

$$\mathcal{O}_{\widetilde{X}} \otimes_{\mathbf{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}}) \to I \otimes_A \mathcal{O}_{\widetilde{X}}.$$

Since $\Omega^1_{\widetilde{X}/\underline{A}} \cong \mathcal{O}_{\widetilde{X}} \otimes_{\mathbf{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}})$ étale locally, this defines a local section of

$$\mathcal{H}om_{\mathcal{O}_{\widetilde{X}}}(\Omega^1_{\widetilde{X}/\underline{A}}, I \otimes_A \mathcal{O}_{\widetilde{X}}) \cong \mathcal{D}er(\underline{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}) \otimes_A I.$$

Conversely, for a local section $(D, D\mathrm{log}) \in \mathcal{D}er(\underline{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}) \otimes_A I$, D induces an automorphism of $O_{\widetilde{X}'}$ and $D\mathrm{log}$ induces an automorphism of $\widetilde{\mathcal{M}}'$, and then, indues an automorphism of $(\widetilde{X}', \widetilde{\mathcal{M}}')$. By this, applying the arguement in SGA I [2] Exposé 3, we get the following proposition.

Proposition 8.4 (cf. [5, (3.14)]) *Let $\widetilde{f} : (\widetilde{X}, \widetilde{\mathcal{M}}) \to (\mathrm{Spec}\, A, Q)$ be a log smooth lifting of f to A, and $u : A' \to A$ a surjective homomorphism in $\mathcal{C}_{\Lambda[[Q]]}$ such that $I^2 = 0$ where $I = \mathrm{Ker}(u)$ (i.e., $(\mathrm{Spec}\, A, Q) \to (\mathrm{Spec}\, A', Q)$ is a thickening of order ≤ 1).*

1. *The sheaf of germs of lifting automorphisms of $\widetilde{X}$ to A' is*
$$\mathcal{D}er_{\underline{A}}(\underline{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}) \otimes_A I.$$
2. *The set of isomorphism classes of log smooth liftings of $\widetilde{X}$ to A' is isomorphic to*
$$\mathrm{H}^1(\widetilde{X}, \mathcal{D}er_{\underline{A}}(\underline{\widetilde{X}}, \mathcal{O}_{\widetilde{X}})) \otimes_A I.$$
3. *The lifting obstructions of $\widetilde{X}$ to A' are in*
$$\mathrm{H}^2(\widetilde{X}, \mathcal{D}er_{\underline{A}}(\underline{\widetilde{X}}, \mathcal{O}_{\widetilde{X}})) \otimes_A I.$$

Define the *log smooth deformation functor* $\mathbf{LD} = \mathbf{LD}_{\underline{X}/\underline{k}}$ by

$$\mathbf{LD}_{\underline{X}/\underline{k}}(A) = \{\text{isomorphism class of log smooth lifting of } f : \underline{X} \to \underline{k} \text{ on } A\}$$

for $A \in \mathrm{Obj}(\mathcal{C}_{\Lambda[[Q]]})$. This is a covariant functor from $\mathcal{C}_{\Lambda[[Q]]}$ to **Ens**, the category of sets, such that $\mathbf{LD}_{\underline{X}/\underline{k}}(k)$ consists of one point. We shall prove the following theorem in the next section.

Theorem 8.5 *If the underlying scheme X is proper over k, then the log deformation functor $\mathbf{LD}_{\underline{X}/\underline{k}}$ has a representable hull* (cf. [11]).

9 The proof of Theorem 8.5

We are going to prove Theorem 8.5 by checking M. Schlessinger's criterion ([11, Theorem 2.11]) for **LD**. Let $u_1 : A_1 \to A_0$ and $u_2 : A_2 \to A_0$ be morphisms in $\mathcal{C}_{\Lambda[[Q]]}$. Consider the map

$$\mathbf{LD}(A_1 \times_{A_0} A_2) \longrightarrow \mathbf{LD}(A_1) \times_{\mathbf{LD}(A_0)} \mathbf{LD}(A_2). \tag{8}$$

Then we shall check the following conditions.

(H1) The map (8) is a surjection whenever $u_2 : A_2 \to A_0$ is a surjection.

(H2) The map (8) is a bijection when $A_0 = k$ and $A_2 = k[\epsilon]$, where $k[\epsilon] = k[E]/(E^2)$.

(H3) $\dim_k(t_{\mathbf{LD}}) < \infty$, where $t_{\mathbf{LD}} = \mathbf{LD}(k[\epsilon])$.

By Proposition 8.4, we have

$$t_{\mathbf{LD}} \cong \mathrm{H}^1(X, \mathcal{D}er_{\underline{k}}(\underline{X}, \mathcal{O}_X)).$$

Our assumption implies that $t_{\mathbf{LD}}$ is finite dimensional since $\mathcal{D}er_{\underline{k}}(\underline{X}, \mathcal{O}_X)$ is a coherent $\mathcal{O}_X$–module. Hence we need to check (H1) and (H2). Set $B = A_1 \times_{A_0} A_2$. Let $v_i : B \to A_i$ be the natural map for $i = 1, 2$. We denote the morphisms of schemes associated to u_i and v_i also by $u_i : \operatorname{Spec} A_0 \to \operatorname{Spec} A_i$ and $v_i : \operatorname{Spec} A_i \to \operatorname{Spec} B$ for $i = 1, 2$, respectively.

Proof of (H1). Suppose the homomorphism $u_2 : A_2 \to A_0$ is surjective. Take an element $(\eta_1, \eta_2) \in \mathbf{LD}(A_1) \times_{\mathbf{LD}(A_0)} \mathbf{LD}(A_2)$ where η_i is an isomorphism class of a log smooth lifting $f_i : (X_i, \mathcal{M}_i) \to (\operatorname{Spec} A_i, Q)$ for each $i = 1, 2$. The equality $\mathbf{LD}(u_1)(\eta_1) = \mathbf{LD}(u_2)(\eta_2)(= \eta_0)$ implies that there exists an isomorphism $(u_2)^*(X_2, \mathcal{M}_2) \xrightarrow{\sim} (u_1)^*(X_1, \mathcal{M}_1)$ over $(\operatorname{Spec} A_0, Q)$. Here, $(u_i)^*(X_i, \mathcal{M}_i)$ is the pull–back of $(X_i, \mathcal{M}_i)$ by $u_i : \operatorname{Spec} A_0 \to \operatorname{Spec} A_i$ for $i = 1, 2$. Set $(X_0, \mathcal{M}_0) = (u_1)^*(X_1, \mathcal{M}_1)$. We denote the induced morphism of log schemes $(X_0, \mathcal{M}_0) \to (X_i, \mathcal{M}_i)$ by $\underline{u_i}$ for $i = 1, 2$. Then we have the following commutative diagram:

$$\begin{array}{ccccc}
(X_1, \mathcal{M}_1) & \xleftarrow{\underline{u_1}} & (X_0, \mathcal{M}_0) & \xrightarrow{\underline{u_2}} & (X_2, \mathcal{M}_2) \\
f_1 \downarrow & & f_0 \downarrow & & \downarrow f_2 \\
(\operatorname{Spec} A_1, Q) & \xleftarrow{u_1} & (\operatorname{Spec} A_0, Q) & \xrightarrow{u_2} & (\operatorname{Spec} A_2, Q).
\end{array}$$

We have to find an element $\xi \in \mathbf{LD}(B)$, which represents a lifting of f to B, such that $\mathbf{LD}(v_i)(\xi) = \eta_i$ for $i = 1, 2$. Consider a scheme $Z = (|X|, \mathcal{O}_{X_1} \times_{\mathcal{O}_{X_0}} \mathcal{O}_{X_2})$ over $\operatorname{Spec} B$. Define a log structure on Z by the natural homomorphism

$$\mathcal{N} = \mathcal{M}_1 \times_{\mathcal{M}_0} \mathcal{M}_2 \longrightarrow \mathcal{O}_Z = \mathcal{O}_{X_1} \times_{\mathcal{O}_{X_0}} \mathcal{O}_{X_2}.$$

It is easy to verify that this homomorphism is a log structure. Since the diagram

$$\begin{array}{ccc}
\mathcal{N} & \longrightarrow & \mathcal{O}_Z \\
\uparrow & & \uparrow \\
Q \cong Q \times_Q Q & \longrightarrow & B
\end{array}$$

is commutative, we have a morphism $g : (Z, \mathcal{N}) \to (\operatorname{Spec} B, Q)$ of log schemes. By the construction, we have the morphism $v_i : (X_i, \mathcal{M}_i) \to (Z, \mathcal{N})$ for $i = 1, 2$ such that the diagram

$$\begin{array}{ccc} (X_1, \mathcal{M}_1) & \xrightarrow{\underline{v}_1} & (Z, \mathcal{N}) \\ {\scriptstyle \underline{u}_1}\uparrow & & \uparrow{\scriptstyle \underline{v}_2} \\ (X_0, \mathcal{M}_0) & \xrightarrow[\underline{u}_2]{} & (X_2, \mathcal{M}_2) \end{array}$$

is commutative. Since $u_2 : A_2 \to A_0$ is surjective, the underlying morphism $X_1 \to Z$ of $\underline{v}_1$ is a closed immersion in the classical sense. We have to show that the morphism $\underline{v}_1$ is an exact closed immersion. Take a local chart $(P \to \mathcal{M}, Q \to Q \oplus k^{\times}, Q \to P)$ of f as in Theorem 4.1 such that $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is injective. By Lemma 8.2, this local chart lifts to the local chart of f_i for each $i = 0, 1, 2$. Since $u_2 : A_2 \to A_0$ is surjective, we have the following isomorphism

$$\mathcal{N}/\mathcal{O}_Z^{\times} \xrightarrow{\sim} (\mathcal{M}_1/\mathcal{O}_{X_1}^{\times}) \times_{(\mathcal{M}_0/\mathcal{O}_{X_0}^{\times})} (\mathcal{M}_2/\mathcal{O}_{X_2}^{\times}).$$

By this, one sees that $P \cong P \times_P P \to \mathcal{N}$ is a local chart of $\mathcal{N}$. This shows that $(Z, \mathcal{N})$ is a fine saturated log scheme, and $\underline{v}_1$ is an exact closed immersion. Then, the exact closed immersion $(X, \mathcal{M}) \to (X_1, \mathcal{M}_1) \to (Z, \mathcal{N})$ gives g a structure of log smooth lifting of f to $(\operatorname{Spec} B, Q)$. Hence, g represents an element $\xi \in \mathbf{LD}(B)$. It is easy to verify that $\mathbf{LD}(v_i)(\xi) = \eta_i$ for $i = 1, 2$ since the morphism f_1, f_2 and g have the common local chart. Thus (H1) is now proved. $\square$

Proof of (H2). We continue to use the same notation as above. First, we prepare the following lemma.

Lemma 9.1 *Let $g' : (Z', \mathcal{N}') \to (\operatorname{Spec} B, Q)$ be a log smooth lifting of f with a commutative diagram*

$$\begin{array}{ccc} (X_1, \mathcal{M}_1) & \longrightarrow & (Z', \mathcal{N}') \\ {\scriptstyle \underline{u}_1}\uparrow & & \uparrow \\ (X_0, \mathcal{M}_0) & \xrightarrow[\underline{u}_2]{} & (X_2, \mathcal{M}_2) \end{array}$$

of liftings such that $(v_i)^*(Z', \mathcal{N}') \xleftarrow{\sim} (X_i, \mathcal{M}_i)$ over $(\operatorname{Spec} A_i, Q)$ for $i = 1, 2$. Then, the natural morphism $(Z, \mathcal{N}) \to (Z', \mathcal{N}')$ is an isomorphism.

Proof. We may work étale locally. By Lemma 8.2, the local chart $(P \to \mathcal{M}, Q \to Q \oplus k^{\times}, Q \to P)$ of f lifts to a local chart of g'. Take a local chart $(P \to \mathcal{N}, Q \to Q \oplus B^{\times}, Q \to P)$ of g by $P \to \mathcal{N}' \to \mathcal{N}$. Then, the schemes Z and Z' are smooth liftings of $X \to \operatorname{Spec} k \times_{\operatorname{Spec} \mathbf{Z}[Q]} \operatorname{Spec} \mathbf{Z}[P]$ to $\operatorname{Spec} B \times_{\operatorname{Spec} \mathbf{Z}[Q]} \operatorname{Spec} \mathbf{Z}[P]$. Hence, we have only to show that the natural morphism $Z \to Z'$ of underlying schemes is an isomorphism. But this follows from the classical theory [11, Corollary 3.6] since each X_i is a smooth lifting of $X \to \operatorname{Spec} k \times_{\operatorname{Spec} \mathbf{Z}[Q]} \operatorname{Spec} \mathbf{Z}[P]$ to $\operatorname{Spec} A_i \times_{\operatorname{Spec} \mathbf{Z}[Q]} \operatorname{Spec} \mathbf{Z}[P]$ for $i = 0, 1, 2$. $\square$

Let $g' : (Z', \mathcal{N}') \to (\operatorname{Spec} B, Q)$ be a log smooth lifting of f which represents a class $\xi' \in \mathbf{LD}(B)$. Suppose that the class ξ' is mapped to (η_1, η_2) by (8). Then,

$$(X_0, \mathcal{M}_0) \xrightarrow{\sim} (v_1 \circ u_1)_*(Z', \mathcal{N}') \cong (v_2 \circ u_2)_*(Z', \mathcal{N}') \xleftarrow{\sim} (X_0, \mathcal{M}_0)$$

defines an automorphism θ of the lifting $(X_0, \mathcal{M}_0)$. If this automorphism θ lifts to an automorphism θ' of the lifting $(X_1, \mathcal{M}_1)$ such that $\theta' \circ \underline{u}_1 = \underline{u}_1 \circ \theta'$, then, replacing $(X_1, \mathcal{M}_1) \to (Z', \mathcal{N}')$ by $(X_1, \mathcal{M}_1) \stackrel{\theta'}{\to} (X_1, \mathcal{M}_1) \to (Z', \mathcal{N}')$, we have a commutative diagram as in Lemma 9.1, and then, we have $\xi = \xi'$. Now if $A = k$ (so that $(X_0, \mathcal{M}_0) = (X, \mathcal{M})$, $\theta = \mathrm{id}$), θ' exists. Thus (H2) is proved. $\square$

Thus we have proved Theorem 8.5.

10 Example 1: Log smooth deformation over trivial base

As we have seen in Theorem 4.7, any log scheme $(X, \mathcal{M})$ which is log smooth over $\operatorname{Spec} k$ with the trivial log structure is smooth over an affine torus embedding étale locally.

Example 10.1 (Usual smooth deformations.) Let X be a smooth algebraic variety over a field k. Then X with the trivial log structure is log smooth over $\operatorname{Spec} k$ (this is the case $D = 0$ in Corollary 4.8), and our log smooth deformation of X is nothing but the usual smooth deformation of X.

Example 10.2 (Generalized relative deformations.) Let X be an algebraic variety over a field k. Assume that there exists a fine saturated log structure $\mathcal{M}$ on X such that $f : (X, \mathcal{M}) \to \operatorname{Spec} k$ is log smooth. Then, by Theorem 4.7, X is covered by étale open sets which are smooth over affine toric varieties, and the log structure $\mathcal{M} \to \mathcal{O}_X$ is equivalent to the log structure defined by

$$\mathcal{M} = j_* \mathcal{O}_{X-D}^{\times} \cap \mathcal{O}_X \tag{9}$$

for some divisor D of X, where j is the inclusion $X - D \hookrightarrow X$. In this situation, our log smooth deformation of f is the deformation of the pair (X, D). If X is smooth over k, then D is a reduced normal crossing divisor (Corollary 4.8). Assume X is smooth over k, then we have the exact sequence

$$0 \to \mathcal{D}er_k(\underline{X}, \mathcal{O}_X) \to \mathcal{D}er_k(X, \mathcal{O}_X)(= \Theta_X) \to \mathcal{N} \to 0,$$

where $\mathcal{N}$ is an $\mathcal{O}_X$–module locally written by

$$(\mathcal{N}_{D_1|X} \otimes \mathcal{O}_{D_1}) \oplus \cdots \oplus (\mathcal{N}_{D_d|X} \otimes \mathcal{O}_{D_d}),$$

where $D_1, \ldots, D_d$ are local components of D, and $\mathcal{N}_{D_i|X}$ is the normal bundle of D_i in X for $i = 1, \ldots, d$. Then, we have an exact sequence

$$\mathrm{H}^0(D, \mathcal{N}) \to t_{\mathbf{LD}} \to \mathrm{H}^1(X, \Theta_X) \to \mathrm{H}^1(D, \mathcal{N}).$$

In this sequence, $\mathrm{H}^0(D, \mathcal{N})$ is viewed as the set of isomorphism classes of locally trivial deformations of D in X, and $\mathrm{H}^1(D, \mathcal{N})$ is viewed as a set of obstructions of deformations of D in X. Hence this sequence explains the relation between the log smooth deformation and the usual smooth deformation. Note that, if D is a smooth divisor on X, the log smooth deformation is nothing but the *relative deformation* of the pair (X, D) studied by Makio [9].

Example 10.3 (Toric varieties.) Let X_Σ be a complete toric variety over a field k defined by a fan Σ in $N_{\mathbf{R}}$, and consider the log scheme $\underline{X} = (X_\Sigma, \Sigma)$ (Example 2.6) over $\operatorname{Spec} k$. We have seen in Example 5.5 that

$$\mathcal{D}er_k(\underline{X}, \mathcal{O}_X) \cong \mathcal{O}_X \otimes_{\mathbf{Z}} N,$$

and hence is a globally free $\mathcal{O}_X$–module. Since $\mathrm{H}^1(X, \mathcal{D}er_k(\underline{X}, \mathcal{O}_X)) = 0$, any toric varieties are infinitesimally rigid with respect to our log smooth deformation. Note that toric varieties without log structures are not necessarily rigid with respect to the usual smooth deformation.

11 Example 2: Smoothings of normal crossing varieties

Let k be a field. A *normal crossing variety* over k is a seperated, connected, and geometrically reduced scheme X of finite type over k which is covered by an étale open covering $\{X_\lambda\}_{\lambda\in\Lambda}$ such that each X_λ is isomorphic to $\operatorname{Spec} k[z_1, \ldots, z_n]/(z_1 \cdots z_{d_\lambda})$ over k where $n - 1 = \dim_k X$. Let $(\operatorname{Spec} k, \mathbf{N})$ be a standard log point (Definition 4.3).

Definition 11.1 (cf. [4, (2.6)]) A log structure $\alpha : \mathcal{M} \to \mathcal{O}_X$ on a normal crossing variety X is called a log structure of *semistable type* over $(\operatorname{Spec} k, \mathbf{N})$ if the following conditions are satisfied:

1. there exists an étale open covering $\{X_\lambda\}_{\lambda\in\Lambda}$ of X such that, for each $\lambda \in \Lambda$, X_λ is isomorphic to $\operatorname{Spec} k[z_1, \ldots, z_n]/(z_1 \cdots z_{d_\lambda})$ over k and the log structure $\mathcal{M}_\lambda = \mathcal{M}|_{X_\lambda}$ has a chart $\beta_\lambda : \mathbf{N}^{d_\lambda} \to \mathcal{M}_\lambda$ such that $\alpha \circ \beta_\lambda(e_i) = z_i$ for $i = 1, \ldots, d_\lambda$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (1 at the i–th entry),
2. there exists a morphism $f : (X, \mathcal{M}) \to (\operatorname{Spec} k, \mathbf{N})$ which has a local chart $(\beta_\lambda : \mathbf{N}^{d_\lambda} \to \mathcal{M}, \mathbf{N} \to \mathbf{N} \oplus k^\times, \varphi_\lambda : \mathbf{N} \to \mathbf{N}^{d_\lambda})$ where φ_λ is the diagonal homomorphism, for each $\lambda \in \Lambda$.

The morphism $f : (X, \mathcal{M}) \to (\operatorname{Spec} k, \mathbf{N})$ defined as above is called a *logarithmic semistable reduction*. Note that a logarithmic semistable reduction is log smooth (Example 4.6). Let $f : (X, \mathcal{M}) \to (\operatorname{Spec} k, \mathbf{N})$ be a logarithmic semistable reduction. Then, étale locally, X is isomorphic to $k[z_1, \ldots, z_n]/(z_1 \cdots z_d)$, and f is induced by the diagram

$$\begin{array}{ccc} \mathbf{N}^d & \longrightarrow & k[z_1, \ldots, z_n]/(z_1 \cdots z_d) \\ {\scriptstyle\varphi}\uparrow & & \uparrow \\ \mathbf{N} & \longrightarrow & k. \end{array}$$

For $A \in \operatorname{Obj}(\mathcal{C}_{\Lambda[[\mathbf{N}]]})$, the log structure on A is defined by $\gamma : \mathbf{N} \to A$, $\gamma(1) = \pi \in m_A$, where m_A is the maximal ideal of A. Then the log smooth lifting of f on A is locally equivalent to the morphism induced by the diagram

$$\begin{array}{ccc} \mathbf{N}^d & \longrightarrow & A[z_1, \ldots, z_n]/(z_1 \cdots z_d - \pi) \\ {\scriptstyle\varphi}\uparrow & & \uparrow \\ \mathbf{N} & \underset{\gamma}{\longrightarrow} & A. \end{array}$$

Hence, our log smooth deformation carries out the smoothing of the normal crossing variety X.

Now we assume that the singular locus $X_{\mathrm{sing}} = D$ of X is connected. Then our deformation theory coincides with the *logarithmic deformation theory of normal crossing varieties* introduced by Kawamata and Namikawa [6] in the complex analytic situation. The following theorem is essentially due to Kawamata and Namikawa [6], and we prove it in the next section.

Theorem 11.2 *Let X be a normal crossing variety over k. Then, X has a log structure of semistable type over $(\operatorname{Spec} k, \mathbf{N})$ if and only if X is d–semistable, i.e., $\mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X) \cong \mathcal{O}_D$* (cf. [1]).

12 The proof of Theorem 11.2

Let X be a normal crossing variety over a field k of dimension n. Then the scheme X is covered by an étale open covering $\{X_\lambda\}_{\lambda\in\Lambda}$ such that each X_λ is isomorphic to the divisor in the $(n+1)$–affine space over k, $V_\lambda = \operatorname{Spec} k[z_0^{(\lambda)}, \ldots, z_n^{(\lambda)}] \cong \mathbf{A}_k^{(n+1)}$, defined by $z_0^{(\lambda)} \cdots z_{d_\lambda}^{(\lambda)} = 0$ for some $0 \leq d_\lambda \leq n$. We call this covering $\mathcal{U} = \{X_\lambda \hookrightarrow V_\lambda\}$ a *coordinate covering* of X. If $\mathcal{U} = \{X_\lambda \hookrightarrow V_\lambda\}$ is a coordinate covering, the *singular locus* of X, denoted by D, is the closed subscheme of X defined locally by the equations $z_0^{(\lambda)} \cdots z_{i-1}^{(\lambda)} \cdot z_{i+1}^{(\lambda)} \cdots z_{d_\lambda}^{(\lambda)} = 0$ for $0 \leq i \leq d_\lambda$ in V_λ. We write the defining ideals of X_λ and D in V_λ as

$$\begin{aligned} \mathcal{I}_{X_\lambda} &= (z_0^{(\lambda)} \cdots z_{d_\lambda}^{(\lambda)}), \\ \mathcal{I}_{D_\lambda} &= (z_0^{(\lambda)} \cdots z_{i-1}^{(\lambda)} \cdot z_{i+1}^{(\lambda)} \cdots z_{d_\lambda}^{(\lambda)} \mid 0 \leq i \leq d_\lambda). \end{aligned}$$

These are ideals in $\mathcal{O}_{V_\lambda}$. In the sequel, we fix these notation and conventions, sometimes omitting the index λ.

Let us describe $\mathcal{T}^1_X = \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ locally. Consider the following exact sequence,

$$0 \to \mathcal{I}_X/\mathcal{I}_X^2 \to \Omega^1_V \otimes \mathcal{O}_X \to \Omega^1_X \to 0.$$

Here, we omitted the index λ. Its dual is

$$0 \to \mathcal{T}^0_X \to \Theta_V \otimes \mathcal{O}_X \to \mathcal{N}_{X|V}.$$

Then, we have an isomorphism $\mathcal{T}^1_X \cong \operatorname{Coker}(\Theta_V \otimes \mathcal{O}_X \to \mathcal{N}_{X|V})$.

Lemma 12.1 *$\mathcal{T}^1_X$ is an invertible $\mathcal{O}_D$–module. More precisely, we have an isomorphism of $\mathcal{O}_D$–modules*

$$\mathcal{T}^1_X \cong (\mathcal{I}_X/\mathcal{I}_X^2)^\vee \otimes \mathcal{O}_D.$$

Proof. First, note that $\mathcal{I}_X/\mathcal{I}_X^2 = \mathcal{O}_X \cdot d(z_0 \cdots z_d)$, and $\Omega^1_V \otimes \mathcal{O}_X = \oplus_{i=1}^n \mathcal{O}_X \cdot dz_i$. Here, we omited the index λ. Let us denote the inclusion $\mathcal{I}_X/\mathcal{I}_X^2 \hookrightarrow \Omega^1_V \otimes \mathcal{O}_X$ by ι. For each $f \in \Theta_V \otimes \mathcal{O}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_V \otimes \mathcal{O}_X, \mathcal{O}_X)$, set $f_i = f(dz_i) \in \mathcal{O}_X$. Then, $\iota^* f(d(z_0 \cdots z_d)) = \sum_{i=0}^d f_i z_0 \cdots z_{i-1} \cdot z_{i+1} \cdots z_d \in \mathcal{I}_D \otimes \mathcal{O}_X$. Hence, $\operatorname{Image} \iota^* \subseteq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{I}_D \otimes \mathcal{O}_X)$. Easy to see the converse. Then, we have $\operatorname{Image} \iota^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{I}_D \otimes \mathcal{O}_X)$. Hence, we get $\mathcal{T}^1_X \cong \operatorname{Coker} \iota^* \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{O}_D) \cong (\mathcal{I}_X/\mathcal{I}_X^2)^\vee \otimes \mathcal{O}_D$. □

Let X be a reduced normal crossing variety over a field k and $\mathcal{U} = \{X_\lambda \hookrightarrow V_\lambda\}$ a coordinate covering of X. A system $\{(\zeta_0^{(\lambda)}, \ldots, \zeta_n^{(\lambda)})\}_{\lambda\in\Lambda}$, where $\zeta_i^{(\lambda)} \in \mathrm{H}^0(X_\lambda, \mathcal{O}_X)$ for $\lambda \in \Lambda$ and $i = 0, \ldots, n$, is said to be a *log system* on X with respect to $\mathcal{U} = \{X_\lambda \hookrightarrow V_\lambda\}$ if the following conditions are satisfied:

1. for $0 \leq i \leq d_\lambda$, we have $\zeta_i^{(\lambda)} = u_i^{(\lambda)} \cdot z_i^{(\lambda)}$ for some $u_i^{(\lambda)} \in \mathrm{H}^0(X_\lambda, \mathcal{O}_X^\times)$,
2. for $d_\lambda < j \leq n$, $\zeta_j^{(\lambda)}$ is invertible on X_λ.

Then we have the following lemma.

Lemma 12.2 *On each $X_{\lambda\mu} = X_\lambda \cap X_\mu \neq \emptyset$, there exists a transition relation such as*

$$\zeta_i^{(\lambda)} = u_i^{(\lambda\mu)} \zeta^{(\mu)}_{\sigma^{(\lambda\mu)}(i)} \ (0 \leq i \leq n), \tag{10}$$

for some invertible function $u_i^{(\lambda\mu)}$ on $X_{\lambda\mu}$ and a permutation $\sigma^{(\lambda\mu)}$. (As is seen in the following proof, these $u_i^{(\lambda\mu)}$'s and $\sigma^{(\lambda\mu)}$'s are *not* unique.)

Proof. Set $E^{(\lambda)} = \{i \mid (\zeta_i^{(\lambda)} \mid_{X_{\lambda\mu}}) \notin \mathrm{H}^0(X_{\lambda\mu}, \mathcal{O}_X^\times)\}$, and set $E^{(\mu)}$ similarly. There is a one to one correspondence between elements in $E^{(\lambda)}$ and components of $X_{\lambda\mu}$ by $i \leftrightarrow \{\zeta_i^{(\lambda)} = 0\}$. This implies that there exists a bijection $\sigma : E^{(\lambda)} \xrightarrow{\sim} E^{(\mu)}$ such that $\{\zeta_i^{(\lambda)} = 0\} = \{\zeta_{\sigma(i)}^{(\mu)} = 0\}$. Hence, for $i \in E^{(\lambda)}$, we can write $\zeta_i^{(\lambda)} = u_i^{(\lambda\mu)} \cdot \zeta_{\sigma(i)}^{(\mu)}$ for some $u_i^{(\lambda\mu)} \in \mathrm{H}^0(X_{\lambda\mu}, \mathcal{O}_X^\times)$. On the other hand, if i is not in $E^{(\lambda)}$, we can write as (10) for these i's, since $\zeta_i^{(\lambda)}$ is invertible on $X_{\lambda\mu}$ (note that the number of these i's coincides with that of those j's which are not in $E^{(\mu)}$). □

Proposition 12.3 (cf. [6]) *The following conditions are equivalent.*

1. *X is d–semistable.*
2. *There exist a log system $\{(\zeta_0^{(\lambda)}, \ldots, \zeta_n^{(\lambda)})\}$ on X and its transition system $\{(u_i^{(\lambda\mu)}, \sigma^{(\lambda\mu)})\}$ as in Lemma 12.2 such that the equality*

$$u_0^{(\lambda\mu)} \cdots u_n^{(\lambda\mu)} = 1 \tag{11}$$

holds on each $X_\lambda \cap X_\mu \neq \emptyset$.

Proof. (1 ⇒ 2) Assume that the normal crossing variety X is d–semistable. Due to Lemma 12.1, the invertible $\mathcal{O}_X$–module $(\mathcal{I}_X/\mathcal{I}_X^2) \otimes \mathcal{O}_D \cong \mathcal{I}_X/\mathcal{I}_X\mathcal{I}_D$ is trivial, i.e.,

$$\mathcal{I}_X/\mathcal{I}_X\mathcal{I}_D \quad \cong \quad \mathcal{O}_D.$$

Let us denote the natural projection $\mathcal{I}_X/\mathcal{I}_X^2 \to \mathcal{I}_X/\mathcal{I}_X\mathcal{I}_D$ by p. The sheaf $\mathcal{I}_X/\mathcal{I}_X\mathcal{I}_D$ is a free $\mathcal{O}_D$-module and $p(z_0 \cdots z_d)$ is a $\mathcal{O}_D$-free base of it. By the above isomorphism, the global section $1 \in \mathcal{O}_D$ correspondes to $p(v \cdot z_0 \cdots z_d)$, for some invertible function v on X. Set

$$\zeta_i = \begin{cases} v \cdot z_0 & (i = 0), \\ z_i & (1 \leq i \leq d), \\ 1 & (d < i \leq n). \end{cases}$$

Then, the system $\{(\zeta_0^{(\lambda)}, \ldots, \zeta_n^{(\lambda)})\}$ is a log system on X. Due to Lemma 12.2, we can take a transition system $\{(u_i^{(\lambda\mu)}, \sigma^{(\lambda\mu)})\}$ such that the transition relation (10) holds. On each

$X_{\lambda\mu} \neq \emptyset$, we have $p_\lambda(\zeta_0^{(\lambda)} \cdots \zeta_n^{(\lambda)}) = p_\mu(\zeta_0^{(\mu)} \cdots \zeta_n^{(\mu)})$. Then, we have $u_0^{(\lambda\mu)} \cdots u_n^{(\lambda\mu)} p_\mu(\zeta_0^{(\mu)} \cdots \zeta_n^{(\mu)}) = p_\mu(\zeta_0^{(\mu)} \cdots \zeta_n^{(\mu)})$. Since, $p_\mu(v^{(\mu)} z_0^{(\mu)} \cdots z_{d_\mu}^{(\mu)})$ is a $\mathcal{O}_D$–free base of $\mathcal{I}_X/\mathcal{I}_X\mathcal{I}_D$, we have $u_0^{(\lambda\mu)} \cdots u_n^{(\lambda\mu)} = 1$ on D. Hence, we can write

$$u_0^{(\lambda\mu)} \cdots u_n^{(\lambda\mu)} = 1 + \sum_{j=0}^{n} a_j z_0^{(\mu)} \cdots z_{j-1}^{(\mu)} \cdot z_{j+1}^{(\mu)} \cdots z_n^{(\mu)}.$$

Replacing every $u_j^{(\lambda\mu)}$ with

$$u_j^{(\lambda\mu)} - a_j z_0^{(\mu)} \cdots z_{j-1}^{(\mu)} \cdot z_{j+1}^{(\mu)} \cdots z_n^{(\mu)} (u_0^{(\lambda\mu)} \cdots u_{j-1}^{(\lambda\mu)} \cdot u_{j+1}^{(\lambda\mu)} \cdots u_n^{(\lambda\mu)})^{-1},$$

we get $u_0^{(\lambda\mu)} \cdots u_n^{(\lambda\mu)} = 1$ on $X_{\lambda\mu}$ as desired.

$(2 \Rightarrow 1)$ The local section $\zeta_0^{(\lambda)} \otimes \cdots \otimes \zeta_n^{(\lambda)}$ is a local generator of $(\mathcal{I}_X/\mathcal{I}_X^2) \otimes \mathcal{O}_D$. By (11), these local generators glue to a global section. Hence, the invertible $\mathcal{O}_D$–module $(\mathcal{I}_X/\mathcal{I}_X^2) \otimes \mathcal{O}_D$ is trivial, and its dual $\mathcal{T}_X^1$ is also trivial. $\square$

Let $\mathcal{U} = \{X_\lambda \hookrightarrow V_\lambda\}$ be a coordinate covering on X, and take a log system $\{(\zeta_0^{(\lambda)}, \ldots, \zeta_n^{(\lambda)})\}$ with respect to $\mathcal{U} = \{X_\lambda \hookrightarrow V_\lambda\}$. On each X_λ, consider a pre–log structure $\alpha_\lambda : \mathbf{N}^{n+1} \longrightarrow \mathcal{O}_{X_\lambda}$, defined by $\alpha_\lambda(e_i) = \zeta_i^{(\lambda)}$ for $0 \leq i \leq n$. Then, the associate log structure of this pre–log structure is equivalent to the log structure $\mathcal{M}_\lambda$ of semistable type on the normal crossing variety X_λ. We write this log structure, according to §1, by

$$\mathcal{M}_\lambda = \mathbf{N}^{n+1} \oplus_{\alpha_\lambda^{-1}(\mathcal{O}_{X_\lambda}^\times)} \mathcal{O}_{X_\lambda}^\times \xrightarrow{\bar{\alpha}_\lambda} \mathcal{O}_{X_\lambda}, \tag{12}$$

and $\bar{\alpha}_\lambda(e_i, u) = u \cdot \zeta_i^{(\lambda)}$, for $0 \leq i \leq n$. Due to Lemma 12.2, on each $X_{\lambda\mu} = X_\lambda \cap X_\mu \neq \emptyset$, there exists a non canonical isomorphism

$$\begin{array}{ccc} \mathbf{N}^{n+1} \oplus_{\alpha_\lambda^{-1}(\mathcal{O}_{X_{\lambda\mu}}^\times)} \mathcal{O}_{X_{\lambda\mu}}^\times & \xrightarrow{\phi_{\lambda\mu}} & \mathbf{N}^{n+1} \oplus_{\alpha_\mu^{-1}(\mathcal{O}_{X_{\lambda\mu}}^\times)} \mathcal{O}_{X_{\lambda\mu}}^\times \\ \downarrow & & \downarrow \\ \mathcal{O}_{X_{\lambda\mu}} & = & \mathcal{O}_{X_{\lambda\mu}} \end{array}$$

defined by

$$\phi_{\lambda\mu}(e_i, u) = (e_{\sigma^{(\lambda\mu)}(i)}, u \cdot u_i^{(\lambda\mu)}), \tag{13}$$

for $0 \leq i \leq n$.

Lemma 12.4 *Any isomorphism $\phi_{\lambda\mu}$ which makes the above diagram commute is written in the form of (13).*

Proof. Changing indicies suitably, we may assume that

- for $0 \leq i \leq d$, $\alpha_\lambda(e_i)$ and $\alpha_\mu(e_i)$ are not invertible on $X_{\lambda\mu}$,
- for $d < i \leq n$, $\alpha_\lambda(e_i)$ and $\alpha_\mu(e_i)$ are invertible $X_{\lambda\mu}$.

Set $\phi_{\lambda\mu}(e_i, 1) = (\sum_j a_i^j e_j, u_i^{(\lambda\mu)})$, for $0 \leq i \leq n$.

Case $0 \leq i \leq d$: Since $\bar{\alpha}_\mu \circ \phi_{\lambda\mu} = \bar{\alpha}_\lambda$, the matrix $(a_i^j)_{0 \leq i,j \leq d}$ is a permutation matrix (Note that $A, B \in \mathrm{M}_d(\mathbf{N})$ and $AB = 1$ implies that A and B are permutation matrices).

Then we may assume $a_i^j = \delta_i^j \quad (0 \leq i, j \leq d)$. Hence, we can write $\phi_{\lambda\mu}(e_i, 1) = (e_i, u_i^{(\lambda\mu)}) \cdot (b_i, 1)$, where $\alpha_\mu(b_i) \in \mathrm{H}^0(X_{\lambda\mu}, \mathcal{O}_X^\times)$. Since we have the equality

$$(e_i, u_i^{(\lambda\mu)} \cdot \alpha_\mu(b_i)) \cdot (b_i, \alpha_\mu(b_i)^{-1}) = (e_i, u_i^{(\lambda\mu)}) \cdot (b_i, 1),$$

we get $\phi_{\lambda\mu}(e_i, 1) = (e_i, u_i^{(\lambda\nu)} \cdot \alpha_\mu(b_i))$ in the quotient monoid $\mathrm{N}^{n+1} \oplus_{\alpha_\mu^{-1}(\mathcal{O}_{X_{\lambda\mu}}^\times)} \mathcal{O}_{X_{\lambda\mu}}^\times$.

Case $d < i \leq n$: Since $\bar{\alpha}_\mu \circ \phi_{\lambda\mu}(e_i, 1)$ is invertible, we have $a_i^j = 0$, for $d < i \leq n$ and $0 \leq j \leq d$. This implies $\phi_{\lambda\mu}(e_i, 1) = (c_i, u_i^{(\lambda\mu)})$, where $\alpha_\mu(c_i) \in \mathrm{H}^0(X_{\lambda\mu}, \mathcal{O}_X^\times)$. Since

$$(c_i, u_i^{(\lambda\mu)}) \cdot (e_i, \alpha_\mu(e_i)^{-1}) = (e_i, \alpha_\mu(c_i) u_i^{(\lambda\mu)} \alpha_\mu(e_i)^{-1}) \cdot (c_i, \alpha_\mu(c_i)^{-1}),$$

we get $\phi_{\lambda\mu}(e_i, 1) = (e_i, \alpha_\mu(c_i) \cdot u_i^{(\lambda\nu)} \cdot \alpha_\mu(e_i)^{-1})$ in the quotient monoid $\mathrm{N}^{n+1} \oplus_{\alpha_\mu^{-1}(\mathcal{O}_{X_{\lambda\mu}}^\times)} \mathcal{O}_{X_{\lambda\mu}}^\times$.

Hence, for $0 \leq i \leq n$, we have the equality $\phi_{\lambda\mu}(e_i, 1) = (e_i, \widetilde{u_i^{(\lambda\mu)}})$, for some invertible function $\widetilde{u_i^{(\lambda\mu)}}$ on $X_{\lambda\mu}$, in the quotient monoid $\mathrm{N}^{n+1} \oplus_{\alpha_\mu^{-1}(\mathcal{O}_{X_{\lambda\mu}}^\times)} \mathcal{O}_{X_{\lambda\mu}}^\times$. Combining this with $\phi_{\lambda\mu}(0, u) = (0, u)$, we have the desired result. $\square$

Now, let us prove Theorem 11.2. Assume that X is d–semistable. Take a coordinate covering $\mathcal{U} = \{X_\lambda \hookrightarrow V_\lambda\}$, a log system $\{(\zeta_0^{(\lambda)}, \ldots, \zeta_n^{(\lambda)})\}$ and a transition system $\{(u_i^{(\lambda\mu)}, \sigma^{(\lambda\mu)})\}$ such that (11) holds. Then, each X_λ has a log structure by (12). Moreover, there exists a system of isomorphisms $\{\phi_{\lambda\mu}\}$ defined by (13). Now we are going to show that $\{\mathcal{M}_\lambda\}$ glues to a log structure $\mathcal{M}$ on X of the desired type.

Step 1 Let us prove that $\phi_{\mu\nu} \circ \phi_{\lambda\mu} = \phi_{\lambda\nu}$ on each $X_{\lambda\mu\nu} = X_\lambda \cap X_\mu \cap X_\nu \neq \emptyset$, i.e., $\{\phi_{\lambda\mu}\}$ satisfies the 1-cocycle condition. Set

$$\begin{aligned} \phi_{\mu\nu} \circ \phi_{\lambda\mu}(e_i, 1) &= (e_{\tau\circ\sigma(i)}, v_{\sigma(i)} u_i) \\ \phi_{\lambda\nu}(e_i, 1) &= (e_{\rho(i)}, w_i), \end{aligned}$$

where we put $\sigma = \sigma^{(\lambda\mu)}$, $\tau = \sigma^{(\mu\nu)}$, $\rho = \sigma^{(\lambda\nu)}$, $u = u^{(\lambda\mu)}$, $v = u^{(\mu\nu)}$, and $w = u^{(\lambda\nu)}$. Changing indicies suitably, we may assume

- for $0 \leq i \leq d$, $\zeta_i^{(\lambda)} |_{X_{\lambda\mu\nu}}$ is not invertible on $X_{\lambda\mu\nu}$,
- for $d < i \leq n$, $\zeta_i^{(\lambda)} |_{X_{\lambda\mu\nu}}$ is invertible on $X_{\lambda\mu\nu}$.

Since $\bar{\alpha}_\nu \circ \phi_{\mu\nu} \circ \phi_{\lambda\mu} = \bar{\alpha}_\lambda = \bar{\alpha}_\nu \circ \phi_{\lambda\nu}$, we have

$$v_{\sigma(i)} u_i \zeta_{\tau\circ\sigma(i)}^{(\nu)} = \zeta_i^{(\lambda)} = w_i \zeta_{\rho(i)}^{(\nu)} \quad (0 \leq i \leq n). \tag{14}$$

Case $d < i \leq n$: Since the equality (14) holds, both $\zeta_{\tau\circ\sigma(i)}^{(\nu)}$ and $\zeta_{\rho(i)}^{(\nu)}$ are invertible on $X_{\lambda\mu\nu}$ and we have

$$(e_{\tau\circ\sigma(i)}, v_{\sigma(i)} u_i) \cdot (e_{\rho(i)}, (\zeta_{\rho(i)}^{(\nu)})^{-1}) = (e_{\rho(i)}, w_i) \cdot (e_{\tau\circ\sigma(i)}, (\zeta_{\tau\circ\sigma(i)}^{(\nu)})^{-1}).$$

Hence, we have $(e_{\tau\circ\sigma(i)}, v_{\sigma(i)} u_i) = (e_{\rho(i)}, w_i)$ in the quotient monoid $\mathbf{N}^{n+1} \oplus_{\alpha_\nu^{-1}(\mathcal{O}_{X_{\lambda\mu\nu}}^\times)} \mathcal{O}_{X_{\lambda\mu\nu}}^\times$.

CASE $0 \leq i \leq d$: Since the components $\{\zeta^{(\nu)}_{\tau\circ\sigma(i)} = 0\}$ and $\{\zeta^{(\nu)}_{\rho(i)} = 0\}$ of $X_{\lambda\mu\nu}$ coincides due to (14), we have $\tau \circ \sigma(i) = \rho(i)$. Hence, we have $\zeta^{(\nu)}_{\rho(i)} = w_i^{-1} v_{\sigma(i)} u_i \zeta^{(\nu)}_{\rho(i)}$. This imples

$$w_i^{-1} v_{\sigma(i)} u_i = 1 + a_i z^{(\nu)}_{\rho(0)} \cdots z^{(\nu)}_{\rho(i-1)} \cdot z^{(\nu)}_{\rho(i+1)} \cdots z^{(\nu)}_{\rho(d)}, \tag{15}$$

for some $a_i \in \mathcal{O}_{X_{\lambda\mu\nu}}$. Since

$$\begin{aligned} \rho(\{d+1, \ldots, n\}) &= \tau \circ \sigma(\{d+1, \ldots, n\}) \\ &= \{j \mid (\zeta^{(\nu)}_j \mid_{X_{\lambda\mu\nu}}) \in \mathrm{H}^0(X_{\lambda\mu\nu}, \mathcal{O}_X^\times)\}, \end{aligned}$$

and $\zeta^{(\nu)}_{\rho(j)} = w_j^{-1} v_{\sigma(j)} u_j \zeta^{(\nu)}_{\tau\circ\sigma(j)}$ for $d < j \leq n$, we have

$$\prod_{d<j\leq n} w_j^{-1} v_{\sigma(j)} u_j = 1.$$

On the other hand, by our assumptions $u_1 \cdots u_n = 1$, etc., we have

$$\prod_{0\leq i\leq n} w_j^{-1} v_{\sigma(j)} u_j = 1.$$

Hence, we get the equality

$$\prod_{0\leq i\leq d} w_j^{-1} v_{\sigma(j)} u_j = 1.$$

By this and (15), we have

$$\sum_{0\leq i\leq d} a_i z^{(\nu)}_{\rho(0)} \cdots z^{(\nu)}_{\rho(i-1)} \cdot z^{(\nu)}_{\rho(i+1)} \cdots z^{(\nu)}_{\rho(d)} = 0,$$

and consequently we get $a_i = 0$, that is, $w_i^{-1} v_{\sigma(i)} u_i = 1$. Hence, also in this case, we have $(e_{\tau\circ\sigma(i)}, v_{\sigma(i)} u_i) = (e_{\rho(i)}, w_i)$. Therefore, we have proved that $\phi_{\mu\nu} \circ \phi_{\lambda\mu} = \phi_{\lambda\nu}$ holds and $\{\mathcal{M}_\lambda\}$ glues to a log structure $\mathcal{M}$ on X.

Step 2 The system of morphisms $\{f_\lambda\}$, where $f_\lambda : (X_\lambda, \mathcal{M}_\lambda) \to (\operatorname{Spec} k, \mathbf{N})$ is defined as in Definition 2.5, glues to the morphism f if and only if the following diagram commutes:

$$\begin{array}{ccc} \mathbf{N}^{n+1} \oplus_{\alpha_\lambda^{-1}(\mathcal{O}^\times_{X_{\lambda\mu}})} \mathcal{O}^\times_{X_{\lambda\mu}} & \xrightarrow{\phi_{\lambda\mu}} & \mathbf{N}^{n+1} \oplus_{\alpha_\mu^{-1}(\mathcal{O}^\times_{X_{\lambda\mu}})} \mathcal{O}^\times_{X_{\lambda\mu}} \\ \uparrow & & \uparrow \\ \mathbf{N} \oplus k^\times & = & \mathbf{N} \oplus k^\times, \end{array}$$

where $\mathbf{N} \oplus k^\times \to k$ is the associated log structure of $\mathbf{N} \to k$, and the homomorphism $\mathbf{N} \oplus k \to \mathbf{N}^{n+1} \oplus_{\alpha_\lambda^{-1}(\mathcal{O}^\times_{X_{\lambda\mu}})} \mathcal{O}^\times_{X_{\lambda\mu}}$ is defined by $(1,1) \mapsto (e_0 + \cdots + e_n, 1)$. The commutativity of this diagram is equivalent to the equality

$$\begin{aligned} (e_0 + \cdots + e_n, 1) &= \phi_{\lambda\mu}(e_0 + \cdots + e_n, 1) \\ &= (e_0 + \cdots + e_n, u_0^{(\lambda\mu)} \cdots u_n^{(\lambda\mu)}). \end{aligned}$$

It is easy to see that this is equivalent to $u_0^{(\lambda\mu)} \cdots u_n^{(\lambda\mu)} = 1$. Hence, $\{f_\lambda\}$ glues to the morphism f due to Proposition 12.3. Thus, we have proved that $\mathcal{M}$ is the log structure on X of the desired type.

Step 3 Let us prove the converse. Assume that X has a log structure of semistavle type. Then, by Lemma 12.4, Step 2 above and Proposition 12.3, it is easy to show that X is d–semistable.

Thus, the proof of Theorem 11.2 is completed.

References

[1] R. Friedman, *Global smoothings of varieties with normal crossings*, Annals of Math. **118** (1983), 75–114.

[2] A. Grothendieck, *Revêtement étale et groupe fondamental*, Springer Lecture Note **224** (1971).

[3] L. Illusie, *Introduction à la géométrie logarithmique*, Seminary note.

[4] T. Kajiwara, *Logarithmic compactifications of the generalized Jacobian variety*, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. **40** (1993), 473–502.

[5] K. Kato, *Logarithmic structures of Fontaine–Illusie*, in Algebraic Analysis, Geometry and Number Theory, J.–I. Igusa ed., 1988, Johns Hopkins Univ., 191–224.

[6] Y. Kawamata and Y. Namikawa, *Logarithmic Deformations of Normal Crossing Varieties and Smoothings of Degenerate Calabi–Yau Varieties*, preprint, 1992.

[7] K. Kodaira and D.C. Spencer, *On deformation of complex analytic structures, I–II*, Annals of Math. **67** (1958), 328–466.

[8] S. Lichtenbaum and M. Schlessinger, *The Cotangent Complex of a Morphism*, Trans. A.M.S. **128** (1967), 41–70.

[9] K. Makio, *On the Relative Pseudo–Rigidity*, Proc. Japan Acad., **49** (1973), 6–9.

[10] T. Oda, *Convex Bodies and Algebraic Geometry: An Introduction to the Theory of Toric Varieties*, Ergebnisse der Math. (3) **15**, Springer–Verlag, 1988.

[11] M. Schlessinger, *Functors of Artin rings*, Trans. A.M.S. **130** (1968), 208–222.

[12] J. H. M. Steenbrink, *Logarithmic Embeddings of Varieties with Normal Crossings and Mixed Hodge Structures*, Preprint, 1993.