On the Hamiltonian Structure of Non-Conservative Linear Flows on $\Re^2$.

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Abstract

We determine local Hamiltonians, Poisson structures and conserved measures for the linear flows on $\Re^2$. We study flows around critical points as deformations of the harmonic oscillator (HO) and the free particle. In particular the damped harmonic oscillator (DHO) is Hamiltonian with respect to non-standard Poisson structures for all non-zero values of the damping constant. Treating the DHO as a deformation of the HO, its qualitative phases can be related to the geometry of the corresponding deformed Poisson structures.

1 Introduction

Although linear dynamical systems are mathematically trivial, we know from the Hartman-Grobman theorem that they are homeomorphic to generic flows in finite but sufficiently small neighbourhoods of hyperbolic fixed points. In this sense the properties of local Hamiltonians and Poisson structures associated with the linear flows are representative of a much larger category of dynamical system. Motivated by a desire to understand the quantisation of dissipative systems, we will study the case of linear flows on $\Re^2$ paying particular attention to the stable equilibria which can be considered deformations of familiar conservative systems such as the HO and the free particle.

Despite their elementary nature the DHO and the driven damped harmonic oscillator (DDHO) arise naturally as effective classical models in a number of physically important contexts. Horace Lamb in 1900 [1] used the model of an infinite string attached to a particle suspended between two springs to explain a

phenomenon of radiation damping. The reduced equation of motion for the particle becomes that of the DHO. Lewis and Thomas [2] adopted a generalisation of this in their work on heat baths. They showed their model to be equivalent to one due to Ford, Kac and Mazur [3], and in further work they established its equivalence to the IO model [4]. The effective motion of the HO in the heat bath is always governed by a generalised Langevin equation, and the resevoir can be tuned so that this becomes either that of a DHO, or a DDHO driven by a stochastic force. They explicitly solve the quantum IO model and the quantum mechanics of the immersed particle is obtained in a physically appealing way by integrating out the resevoir degrees of freedom.

Caldeira and Leggett [5] emphasise the importance of the DHO and DDHO which arise as effective models in many natural systems. In particular they mention the LCR circuit at low temperatures as a candidate for observing quantum corrections to an effective DHO. Using the Feynman–Vernon influence functional formalism and embedding an HO in a specific model of a resevoir they too solve the quantum mechanics of the reduced system - a DHO - and compute the time evolution of the wavefunction of the DHO from certain initial states.

More recently a formalism inspired by conceptual problems in quantum cosmology, developed by Gell-Mann & Hartle [6], and applied to a variety of dissipative systems by T. Brun [7], specifically addresses the problem of how effective classical laws emerge from quantum mechanical systems, and provides a similar strategy for the quantisation of dissipative systems.

These works provide compelling and physically appealing arguments that the concepts of quantum mechanics are meaningful in the context of non-conservative dynamical systems. As strategies for quantisation however, the methods involve the solution of many body problems, integration over an infinite number of unwanted degrees of freedom, and they raise the question of the existence of general principles which allow one to quantise directly simple dynamical systems without interpreting them as components of a complex universe.

Some people have approached the problem of the DHO using time-dependent Hamiltonians. Although this seems well suited to computing such physical quantities as transitions between states [8], it is not obvious what the states actually are. Colegrave and Abdulla [9] determined the states of a time dependent Hamiltonian for the DHO. By this method the spectrum of states of the under-damped HO is discrete, and is given by $\hbar\omega(n+1/2)$. The overdamped case has a continuous spectrum given by $(-\infty, +\infty)$. In later work Colegrave et al. [10] quantised the DHO by considering a time-dependent energy function instead of a time dependent Hamiltonian and got different results. They found a discrete spectrum of states $\exp(-2\gamma\, t)\hbar\omega(n+1/2)$, valid for in both the under-damped and over-damped cases.

It has been claimed that there can be no quantisation without a Hamiltonian (Lagrangian) [11]. It therefore seems reasonable that a first step in a direct approach to the quantisation of a general dynamical system should consist of

understanding its classical Hamiltonian structure. Having understood this, one can formulate the quantisation problem in the spirit of Bayen et al [12] - as that of determining deformations of the classical algebra of observables generated by the appropriate Poisson structure.

In recent work [13], we demonstrated the existence of local Hamiltonians and local Poisson structures in the neighbourhood of non critical points of arbitrary vector fields. In principle this allows one to formulate more precisely the problem of quantisation. However our results are local in nature and there is a need to understand the global geometry of these unfamiliar structures. The purpose of this paper is to provide examples of the application of our previous results, as well as insight into the quantisation problem for general dynamical systems.

2 Linear Flows on $\Re^2$

Linear flows on $\Re^2$ can be divided into four broad categories which correspond up to equivalence under linear transformation to one of the following [14].

$$(i) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda_x & 0 \\ 0 & \lambda_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (ii) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$(iii) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (iv) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The critical points at the origin are known as (i) proper nodes, (ii) star nodes, (iii) improper nodes and (iv) foci. These are further classified as stable, unstable, star, saddle or center depending on simple properties of the eigenvalues [14]. Almost all of the ten qualitatively distinct flows are non-conservative in the sense that they do not preserve the standard volume measure on $\Re^2$. The conservative cases are the centers or foci with $\alpha = 0$, the improper nodes with $\lambda = 0$, and the proper nodes with $\lambda_x + \lambda_y = 0$. Physically these correspond to the harmonic oscillators, the free particles and to particles under the action of a central force. They are Hamiltonian with respect to the standard Poisson structure on $\Re^2$, the Hamiltonians being given by $(x^2 + y^2)/2$, $y^2/2$ and xy respectively.

The DHO naturally assumes the form of a deformation of the HO.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - \epsilon y \end{aligned} \tag{1}$$

For $0 < \epsilon < 2$ the system is damped and the critical point corresponds to a focus. For $\epsilon = 2$ the phase flow is that of an improper node and is said to be critically damped, whereas for $\epsilon > 2$ the system is overdamped and the critical point becomes a proper node.

From (iii) above we can see that the improper node which arises in the case of critical damping, can arise from a deformation in λ of the free particle, as well as a deformation in ϵ of the DHO.

Duarte and Mendes [15] made some interesting attempts to apply the concepts of Hamiltonian mechanics to dissipative systems. Based on a result due to Roels [16] which showed that on two dimensional symplectic manifolds an arbitrary vector field can be locally decomposed as the sum of a Hamiltonian vector field and a gradient field, they considered non-conservative systems as non-Hamiltonian deformations of conservative systems [17]. In our approach all deformations will be Hamiltonian in a sense which will soon become clear.

3 Local structure of vector fields

In [13] we proved that a vector field continuous on $\Re^{n+1}$, in the neighbourhood of a non-critical point locally has a multiple Hamiltonian structure. More precisely we proved that in a finite but sufficiently small neighbourhood U of a non-critical point of a smooth vector field $v = v^i \partial_i$, there exist functions $\Phi^{(k)} \in C^2(U)$ and independent differentiable rank 2 Poisson structures $\Lambda_{(k)}$ $k = 1, \ldots, n$ so that

$$v^i \quad = \quad \Lambda^{ij}_{(k)} \, \partial_j \Phi^{(k)} \qquad\qquad k = 1, \ldots, n$$

where

$$\Lambda^{ij}_k \quad = \quad \rho^{-1} \, \epsilon^{ij s_1 \ldots s_{k-1} s_{k+1} \ldots s_m} \, \partial_{s_1} \Phi^{(1)} \ldots \partial_{s_{k-1}} \Phi^{(k-1)} \partial_{s_{k+1}} \Phi^{(k+1)} \ldots \partial_{s_m} \Phi^{(m)}$$

for each $k = 1, \ldots, n$. In this sense v is locally Hamiltonian with Hamiltonian $\Phi^{(k)}$ with respect to the Poisson structure $\Lambda_{(k)}$ for any $k = 1, \ldots, n$.

We further showed that the function $\rho \in C^1(U)$ defines a local conserved measure according to $d\mu = \rho \; dx^0 \wedge \ldots \wedge dx^n$, and that the conserved measures are those functions which satisfy the first order linear PDE

$$\rho(\nabla \cdot v) + v \cdot (\nabla \rho) \quad = \quad 0. \tag{2}$$

On $\Re^2$ this means that for every smooth vector field $v = v_x \partial_x + v_y \partial_y$, there exist smooth function ρ and Φ in the neighbourhood of a non-critical point, such that

$$\begin{aligned} v_x &= \rho \partial_y \Phi, \\ -v_y &= \rho \partial_x \Phi, \end{aligned} \tag{3}$$

where ρ is a solution of equation (2). In general Φ and the corresponding ρ are not unique. In the case of $\Re^2$ flows we can take any $H = H(\Phi)$ as Hamiltonian, as long as H is a smooth non-constant function and we replace ρ by $\rho H'(\Phi)$.

4 Local Hamiltonians, Poisson strucures and invariant measures for linear flows

Each of the four classes of flows (i), (ii), (iii) and (iv) can be written in the form $\dot{x}^i = \Lambda^{ij}\partial_j\Phi$ where $\Lambda^{ij} = \rho^{-1}\epsilon^{ij}$. The Hamiltonians Φ can be determined either directly by finding some conserved quantity for the flow, or by first solving equation (2), substituting this into equation (3) and integrating. Possible choices for ρ and Φ corresponding to the four classes of linear system are provided by

$$\begin{array}{lll} (i) & \Phi = \lambda_x \ln|y| - \lambda_y \ln|x| & \rho = 1/xy \\ (ii) & \Phi = \lambda(\ln|y| - \ln|x|) & \rho = 1/xy \\ (iii) & \Phi = \ln|y| - \lambda xy^{-1} & \rho = 1/y^2 \\ (iv) & \Phi = \alpha \arctan(yx^{-1}) - \beta/2 \ln|x^2+y^2| & \rho = 1/(x^2+y^2) \end{array}$$

All solutions for Φ and ρ of eqn(3) are now of the form $H(\Phi)$ and $\rho\, H'(\Phi)$ for smooth non-constant H.

5 The improper node as a deformation of the free particle

In its most familiar form the free particle is Hamiltonian with respect to the standard Poisson structure $\rho = 1$ with Hamiltonian $\Phi = y^2/2$. Considering the phase flow around an improper node as a deformation of that associated with the free particle, we denote the deformed Hamiltonian and Poisson structure by Φ^I and ρ^I respectively. Using the results of the previous section we easily determine the unique choice which satisfies $\lim_{\lambda\to 0}\ \rho^I = 1$ and $\lim_{\lambda\to 0}\Phi^I = y^2/2$. This is given by

$$\begin{aligned} \Phi^I &= \frac{y^2}{2}\ \exp\left(-2\lambda\frac{x}{y}\right), \\ \rho^I &= \exp\left(-2\lambda\frac{x}{y}\right). \end{aligned}$$

There is an infinite discontinuity along the line $y = 0$ for all non-zero values of λ. As λ goes to zero the value of all points off this line tend to 1 uniformly. Close to the discontinuity the slope of ρ^I gets steeper, vanishing just as ρ^I becomes a vertical sheet on one side of $y = 0$. In the next section we will see a slightly softer limit as we approach the improper node through deformations of the HO.

6 The DHO as a deformed HO.

In a direct attempt to quantise the DHO it seems natural to try to maintain contact with the familiar territory of the HO by considering Poisson structures and Hamiltonians associated with the DHO which are deformations of those normally associated with the HO - namely $\rho^H = 1$ and $\Phi^H = (x^2+y^2)/2$. With this in mind we denote the Hamiltonians for each of the qualitative phases of the DHO, undamped, damped, critically damped and over damped by Φ^H, Φ^D, Φ^C and Φ^O respectively. Likewise the corresponding Poisson structures will be denoted ρ^H, ρ^D, ρ^C and ρ^O. All of these are uniquely determined by the following requirements.

$$\begin{array}{lcllcl} \lim^+_{\epsilon\to 2} \Phi^O &=& \Phi^C & \lim^+_{\epsilon\to 2} \rho^O &=& \rho^C \\ \lim^-_{\epsilon\to 2} \Phi^D &=& \Phi^C & \lim^-_{\epsilon\to 2} \rho^D &=& \rho^C \\ \lim^+_{\epsilon\to 2} \Phi^D &=& \Phi^H & \lim^+_{\epsilon\to 2} \rho^D &=& \rho^H \end{array}$$

We get

$$\begin{aligned} \Phi^H &= \frac{1}{2}(x^2+y^2) \\ \Phi^D &= \frac{1}{2}(x^2+y^2+\epsilon xy)\exp\left(2\frac{\epsilon}{\kappa}\arctan\left(\frac{\kappa x}{\epsilon x+2y}\right)\right) \\ \Phi^C &= \frac{1}{2}(x+y)^2\exp\left(\frac{2x}{x+y}\right) \\ \Phi^O &= \frac{1}{2}(x^2+y^2+\epsilon xy)\left|\frac{\epsilon x+2y+\kappa x}{\epsilon x+2y-\kappa x}\right|^{\frac{\epsilon}{\kappa}} \end{aligned}$$

where for damped motion $\kappa = +(4-\epsilon^2)^{1/2}$ and for overdamped motion $\kappa = +(\epsilon^2-4)^{1/2}$. On $\Re^2$ the invariant volume element provided by $\rho\, dx\wedge dy$ is also the symplectic structure corresponding to the relevant Poisson structure. These are derived from the following.

$$\begin{aligned} \rho^H &= 1 \\ \rho^D &= \exp\left(2\frac{\epsilon}{\kappa}\arctan\left(\frac{\kappa x}{\epsilon x+2y}\right)\right) \\ \rho^C &= \exp\left(\frac{2x}{x+y}\right) \\ \rho^O &= \left|\frac{\epsilon x+2y+\kappa x}{\epsilon x+2y-\kappa x}\right|^{\frac{\epsilon}{\kappa}}. \end{aligned}$$

In the undamped case ρ is the constant function 1, and the critical point at the origin is a center. When damping is switched on ρ^D develops a discontinuity.

Along the line $\epsilon x + 2y = 0$ it jumps from $\exp(-\pi\epsilon/\kappa)$ to $\exp(+\pi\epsilon/\kappa)$ and the origin is now a focus.

Fig(1) depicts the phase flow of the DHO for $\epsilon = 1$. The corresponding ρ^D is shown in Fig(2). In polar co-ordinates the value of ρ^D depends only on θ. Plotting ρ as a function of θ one can see the discontinuity along the line $x + 2y = 0$. This is due to the multivaluedness of ρ^D.

Fig(3) shows the flow in the phase plane for the case of critical damping or $\epsilon = 2$. The stable equilibrium at the origin is no longer a focus, it is an improper node. Fig(4) is a plot of ρ^C as a function of θ and now we can see that the finite jump in ρ^D across the line $\epsilon x + 2y = 0$ has become a jump in ρ^C from zero to infinity along $x + y = 0$.

Fig(5) depicts the phase flow of the DHO in the overdamped regime when $\epsilon = 3$. Fig(6) is a plot of the corresponding Poisson structure ρ^O. The improper node at the origin has become a proper node and the infinite discontinuity which appeared in ρ^C has resolved itself into a ramp from zero to infinity between the two lines $(\epsilon + \kappa)x + 2y = 0$ and $(\epsilon - \kappa)x + 2y = 0$ respectively.

This phase of the DHO corresponds to a stable proper-node, When the proper node is a saddle-point similar results hold. However in this case the two lines turn out to be the stable and unstable manifolds associated with the singularity.

7 Discussion

Motivated by the desire for a direct approach to the quantisation of non-conservative systems, we studied the Hamiltonian structure of the linear flows on $\Re^2$. Based on previous work in which we showed that generic flows are locally Hamiltonian [13], we computed all of the Hamiltonians and their corresponding Poisson structures. In particular we studied the DHO as a deformation of the HO in each of its qualitative phases. By constructing the unique family of Poisson structures for the DHO which depend continuously on the damping constant ϵ, and which becomes the Poisson structure of standard Hamiltonian dynamics in the limit $\epsilon \to 0$, it is possible to relate the qualitatively different phases of the flow to changes in the global geometry of the deformed Poisson structure. The passage from the undamped to the damped phase co-incides with the appearance of a jump-discontinuity along a line through the critical point at the origin. This is due to the multivalued nature of the deformed Poisson structure. In the critically damped phase, the finite jump becomes infinite. In the overdamped phase the jump from zero to infinity along a single line through the origin becomes a ramp from zero to infinity between a pair of lines through the origin. We remarked that in the case of the saddle point these two lines are none other than the stable and the unstable manifolds. We expect that this behaviour is general in the sense that by treating a generic flow in the neighbourhood of a critical point as a continuous Hamiltonian deformation of a conservative flow, a

similar 'unfolding' of the deformed Poisson structure will be observed.

We have not yet fully investigated the issues around the application of Moyal bracket techniques to the quantisation of systems which are Hamiltonian with respect to general Poisson structures. It will be interesting to see if and how the presence of discontinuities in the deformed Hamiltonians and associated deformed associated Poisson structures will play a significant role in the quantum theory. In particular the occurrence of singularities in the Poisson structure along the stable and unstable manifolds is interesting as these form the separatrices between classical regions of the flow corresponding to qualitatively different modes of the dyanamical system. The most obvious example is the simple pendulum in which case the singularity in the Poisson structure lies along the line seperating the vibrations from the rotations, phases which asymptotically correspond to the HO and the free rotator respectively.

Duarte and Mendes were able to apply their non-Hamiltonian deformation theory to study the nature of attractors of classical dissipative systems such as the Van Der Pol and Lorenz flows [17]. A direct application of our Hamiltonian deformation theory would require a full knowledge of the Hamiltonians of the system. This is no easier than acquiring a full set of solutions of the system, but it should be possible in these cases to gain an understanding numerically of the nature of the local Hamiltonians. This could be done by a study of solutions to equation (2) or its higher dimensional analogues.

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