N-HOMOCLINIC BIFURCATIONS FOR HOMOCLINIC ORBITS CHANGING ITS TWISTING

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ABSTRACT. We study bifurcations, called N-homoclinic bifurcations, producing homoclinic orbits which round N times ($N \geq 2$) in some tubular neighborhood of original homoclinic orbit. A family of vector fields undergoes such a bifurcation when it is perturbed from a vector field with a homoclinic orbit. N-homoclinic bifurcations are divided into two cases; one is that the linearization at the equilibrium has only real principal eigenvalues, and the other is that it has complex principal eigenvalues. We treat the former case, especially linearization has only one unstable eigenvalue. As main tools we use topological method –Conley Index and Connection Matrix Theory–, which enables us to treat more degenerate cases than those studied by analytical methods.

keywords Homoclinic Orbit, Bifurcation, Conley Index, Connection Matrix

1. INTRODUCTION

Let $X_\mu, \mu \in \mathbb{R}^2$, be a 2-parameter family of smooth vector fields on $\mathbb{R}^d$ ($d \geq 3$) with a hyperbolic equilibrium at the origin $\mathbf{O}$. Assume that there exists a curve in the parameter space such that there exists a homoclinic orbit Γ_μ associated to $\mathbf{O}$ if μ belongs to the curve.

We study the bifurcation which X_μ undergoes when it is perturbed from a vector field having a homoclinic orbit Γ_μ, and existence of homoclinic orbits to $\mathbf{O}$ which rounds N times ($N \geq 3$) in a tubular neighborhood of Γ_μ is proved. These homoclinic orbits are referred to as N-homoclinic orbits and this kind of bifurcations as N-homoclinic bifurcations.

N-homoclinic bifurcation has been investigated in two cases, one is that the linearization at the origin $DX_\mu(\mathbf{O})$ has only real principal eigenvalues, the other is that $DX_\mu(\mathbf{O})$ has complex principal eigenvalues, we treat former case.

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Evans *et al.* (Evans *et al.* 1982) proved, under suitable non-degenerate conditions, N- homoclinic orbits never bifurcate when the principal eigenvalues of $DX_\mu(\mathbf{O})$ are real. Accordingly Yanagida (Yanagida 1987) classified degeneracy into following three cases and asserted existence of $2-$homoclinic bifurcations for each case.

(1) A homoclinic orbit with resonance.
(2) An inclination-flip homoclinic orbit.
(3) An orbit-flip homoclinic orbit.

Yanagida's proof was incomplete, however, and several investigation have been carried out with analytic method to recover it under the weakest degeneracy.

As for first case *i.e.* a homoclinic orbit with resonance, Chow *et al.* (Chow *et al.* 1990) showed $2-$homoclinic orbits exist if and only if Γ_μ satisfies a certain Twist condition. Moreover N-homoclinic orbits never bifurcate for $N \geq 3$ (Kisaka *et al.* 1993).

As for second one, *i.e.* an inclination-flip homoclinic orbit, under some technical assumptions on non-principal eigenvalues, existence of $2-$homoclinic orbits has been discussed. In this case the situation differs depending on the proportion of principal eigenvalues (Kisaka *et al.* 1993). In particular, for systems on $\mathbb{R}^3$, N-homoclinic bifurcation for all $N \geq 2$ and suspension of the Smale's horseshoe exist under some conditions (Homburg *et al.*).

About the last an orbit-flip homoclinic orbit, the situation is almost the same as an inclination-flip above (Sandstede 1993, Kokubu *et al.*).

In this paper existence of N-homoclinic orbits shall be proven following the Yanagida's original idea. For this purpose we employ topological method, namely Conley Index and Connection Matrix Theory; this enables us to treat at the same time three cases above and more degenerate cases such that two or three of above occur simultaneously, or degeneracy persists on some interval in parameter space and so on. However as a limit of topological argument, we cannot get analytic information such as order of tangency of bifurcation curves *et cetera*.

The rest of this paper consists of eight sections. In Section 2 and Section 3 basic assumptions and definitions for setting shall be explained. We state the Main Theorems in Section 4. Section 5 and Section 7 are for introduction, without proofs, to the tools for later use –the Sil'nikov variables and exponential expansion in Section 5, Conley Index and Connection Matrix Theory in Section 7–. Inspection of essential mechanism of bifurcations shall be given in Section 6 and we complete the proof of the Theorems in Section 8. In the last section, Section 9, we introduce an Index which detects existence of $2-$homoclinic bifurcation.

2. Setting

First, we put some assumptions to set the problem.

Let $X_\mu\big(\,\mu=(\mu_1,\mu_2)\in\mathbb{R}^2\big)$ be a 2-parameter family of smooth vector fields on $\mathbb{R}^d$ ($d\geq 3$), which have a hyperbolic equilibrium at the origin $\mathbf{O}$.

(B.A.) Basic Assumptions:

(i) There exist homoclinic orbits Γ_μ for $\mu=(\mu_1,0)$ such that

$$\Gamma_\mu=\{\gamma_\mu(t)|t\in\mathbb{R}\}\ \text{ with }\ \lim_{t\to\pm\infty}\gamma_\mu(t)=\mathbf{O}, \tag{2.1}$$

where $\gamma_\mu(t)$ is a solution of $\dot{x}=X_\mu(x)$.

(ii) Consider an extended system with the trivial equation $\dot{\mu}=0$

$$\begin{cases}\dot{x}=X_\mu(x)\\ \dot{\mu}=0\end{cases}\qquad(\,\dot{}=\tfrac{d}{dt}). \tag{2.2}$$

This system has $\mathbf{M}=\{\mathbf{O}\}\times\mathbb{R}^2$ as a normally hyperbolic invariant set. Assume that the stable manifold and the unstable manifold of $\mathbf{M}$ intersect transversely along the homoclinic orbit Γ_μ.

(iii) The linearization $DX_\mu(\mathbf{O})$ at $\mathbf{O}$ has real simple eigenvalues $-\lambda_s(\mu)$ and $\lambda_u(\mu)$ such that

$$-\lambda_s(\mu)<0<\lambda_u(\mu). \tag{2.3}$$

Any other eigenvalues $\lambda(\mu)$ satisfy

$$\sup_\mu\,[Re\lambda(\mu)-\{-\lambda_s(\mu)\}]<\ -gap_s\ <\ 0 \tag{2.4}$$

for some positive constant gap_s. $-\lambda_s$ and λ_u are called principal eigenvalues. The inequality Eq(2.4) implies that the real part of each eigenvalue other than λ_u is negative, this means $\dim\mathbf{W}^u(\mathbf{O};\mu)=1$, where $\mathbf{W}^u(\mathbf{O};\mu)$ is the unstable manifold of $\mathbf{O}$.

Remark 1. (**B**.**A**.)(**2**) *implies that for $\mu_2\neq 0$ there is no homoclinic orbit associated to the origin* $\mathbf{O}$ *which rounds only once in some tubular neighborhood of Γ_μ for $\mu=(\mu_1,0)$.*

3. Variational Equation and Twisting

In this section we treat homoclinic orbits Γ_μ which satisfy next condition.

(P.H.) Principal Homoclinic Orbit:

$$\lim_{t\to-\infty}\frac{\gamma_\mu(t)}{|\gamma_\mu(t)|}=\mathbf{e}_u\quad\text{and}\quad\lim_{t\to+\infty}\frac{\gamma_\mu(t)}{|\gamma_\mu(t)|}=\mathbf{e}_s. \tag{3.1}$$

Here, $\mathbf{e}_u$ and $\mathbf{e}_s$ are the unit eigenvectors corresponding to the principal eigenvalues $\lambda_u(\mu)$ and $-\lambda_s(\mu)$, respectively.

Consider the variational equation along $\gamma_\mu(t)$

$$\dot{P} = DX_\mu\left(\gamma_\mu(t)\right)P. \tag{3.2}$$

We introduce a new variable τ via

$$t = \frac{1}{2\kappa}\log\left(\frac{1+\tau}{1-\tau}\right). \tag{3.3}$$

Using τ as a dependent variable Eq(3.2) turns into an autonomous system

$$\begin{cases} \dot{P} = DX_\mu(\tau)P \\ \dot{\tau} = \kappa(1-\tau^2), \end{cases} \qquad (\dot{} = \tfrac{d}{dt}) \tag{3.4}$$

where we write $DX_\mu(\tau) = DX_\mu\left(\gamma_\mu\left(t(\tau)\right)\right)$ using the relation Eq(3.3). By definition, Eq(3.4) is a system on $\mathbb{R}^d \times (-1,+1)$ — product of $\mathbb{R}^d$ and the open interval $(-1,+1)$. We augment it to a system on $\mathbb{R}^d \times [-1,+1]$ — product of $\mathbb{R}^d$ and the closed interval $[-1,+1]$ — by setting

$$DX_\mu(\pm 1) = DX_\mu(\mathbf{O}). \tag{3.5}$$

We write this system as

$$\begin{cases} \dot{P} = DX_\mu(\tau)P \\ \dot{\tau} = \kappa(1-\tau^2). \end{cases} \tag{3.6}$$

Lemma 1. *For $\kappa > 0$ small, the system Eq(3.6) is of class C^1 on $\mathbb{R}^d \times [-1,+1]$. Namely it can be extended to that on $\mathbb{R}^d \times [-1-\epsilon, +1+\epsilon]$ in a C^1 fashion for some $\epsilon > 0$.*

Proof. We shall check for $\tau = +1$; the argument for $\tau = -1$ is similar.

The condition (**P**.**H**.) implies that there is a positive constant $0 < a < \lambda_u(\mu)$ and C such that

$$|\gamma_\mu(t)| < Ce^{-at} \text{ and } |\dot{\gamma}_\mu(t)| < Ce^{-at} \qquad \text{for large } t. \tag{3.7}$$

To show $DX_\mu(\tau)$ is differentiable at $\tau = +1$, it suffices to prove that $\frac{\partial}{\partial\tau}DX_\mu(+1) = 0$. From Eq(3.7) there exists a positive constant C_1 such that

$$|DX_\mu\left(\gamma_\mu(t)\right) - DX_\mu(\mathbf{O})| < C_1 e^{-at}. \tag{3.8}$$

From this and Eq(3.3) we obtain a constant C_2 such that

$$\left|\frac{DX_\mu\left(\gamma_\mu((t(\tau))\right) - DX_\mu(\mathbf{O})}{\tau - 1}\right| \le C_2(1-\tau)^{\frac{a}{2\kappa}-1} \qquad \text{for } 0 < \tau < 1. \tag{3.9}$$

Taking $\kappa < \frac{a}{2}$ leads to $\frac{\partial}{\partial\tau}DX_\mu(+1) = 0$, and $DX_\mu(\tau)$ can be extended by reflection.

Next we need to show that $\frac{\partial}{\partial\tau}DX_\mu(\tau)$ is continuous. If $\tau < 1$, then

$$\frac{\partial}{\partial\tau}DX_\mu(\tau) = D^2X_\mu\left(\gamma_\mu\left(t(\tau)\right)\right)\cdot\dot{\gamma}_\mu(t)\cdot\frac{dt}{d\tau}\ . \tag{3.10}$$

From $\frac{dt}{d\tau} = \frac{1}{\kappa}\cdot\frac{1}{1-\tau^2}$ and Eq(3.7),

$$\left|\frac{\partial}{\partial\tau}DX_\mu(\tau)\right| < C_3(1-\tau)^{\frac{a}{2\kappa}-1} \qquad \text{for } 0<\tau<1 \text{ and some } C_3>0, \tag{3.11}$$

which implies $\frac{\partial}{\partial\tau}DX_\mu(\tau) \longrightarrow 0$ as $\tau \longrightarrow +1$ for $\kappa < \frac{a}{2}$. $\square$

Because Eq(3.6) is linear in P, it induces a vector field on $\mathbb{S}^{d-1}\times[-1,+1]$ of class C^1 :

$$\begin{cases} \dot{\tilde{P}} = \tilde{X}_\mu(\tilde{P},\tau) \\ \dot{\tau} = \kappa(1-\tau^2) \end{cases} \qquad (\tilde{P}\in\mathbb{S}^{d-1}). \tag{3.12}$$

From the assumption on $DX_\mu(\mathbf{O})$, points $(u_\pm,\pm1)$ and $(s_\pm,\pm1)$ on $\mathbb{S}^{d-1}\times[-1,+1]$ are equilibriums of Eq(3.12). Here $u_\pm = \pi(\pm\mathbf{e}_u)$, $s_\pm = \pi(\pm\mathbf{e}_s)$ for the projection $\pi\colon \mathbb{R}^d\setminus\{\mathbf{O}\}\to\mathbb{S}^{d-1}$. Moreover $(u_\pm,\pm1)$ are attracting equilibriums of the system restricted to invariant subspaces $\tau=\pm1$ and $(s_\pm,\pm1)$ lie in the complementary repeller. Besides, if $\tau\neq\pm1$, then $\dot{\tau}>0$ so that $(u_\pm,+1)$ are attracting equilibriums of the system Eq(3.12). As a result, generically the system Eq(3.4) has a heteroclinic orbit from $(s_+,-1)$ to $(u_+,+1)$ or to $(u_-,+1)$.

Definition 1. A homoclinic orbit Γ_μ is called of cylinder type iff the system Eq(3.12) corresponding to Γ_μ has a heteroclinic orbit from $(s_+,-1)$ to $(u_+,+1)$. Similarly it is called of Möbius type iff it has one from $(s_+,-1)$ to $(u_-,+1)$

The next Lemma immediately follows from the definition.

Lemma 2. *The definition above is well-defined. Namely Γ_μ cannot be of cylinder type and of Möbius type at the same time.*

4. Main Results

The main theorems of this paper shall be stated.

Theorem 1. *Suppose that the 2-parameter family of vector fields X_μ satisfies* (**B**.**A**.) *and the assumption* (**P**.**H**.) *holds for the homoclinic orbits $\Gamma_{\mu=(\mu_1,0)}$ at $\mu_1=m_1$ and $\mu_1=m_2$. In addition, we assume that Γ_μ is of cylinder type or $1<\nu$ at $\mu_1=m_1$,and at $\mu_1=m_2$ of Mö type with $0<\nu<1$, where $\nu=\nu(\mu):=\frac{\lambda_s(\mu)}{\lambda_u(\mu)}$ Under this condition we have double homoclinic orbits bifurcating from the original single homoclinic orbit Γ_μ at some μ_1 with $m_1\le\mu_1\le m_2$ ($m_2\le\mu_1\le m_1$) (see Figure 1).*

Here a double homoclinic orbit (or a 2-homoclinic orbit) is a homoclinic orbit which rounds twice in a tubular neighborhood of the original single homoclinic orbit (or 1-homoclinic orbit) Γ_μ (see Figure 2).

Moreover the following can be proved by similar argument.

Theorem 2. *If the 2-homoclinic orbits bifurcating from 1-homoclinic orbits are also of Möbius type with $0 < \nu < 1$, then a similar argument shows that 3- or 4- homoclinic orbits must also bifurcate.*

Inductively if N-homoclinic orbits bifurcating from Γ_μ are of Möbius type with $0 < \nu < 1$, then k-homoclinic orbits for some k with $N+1 \leq k \leq 2N$ must bifurcate (see Figure 3).

5. The Sil'nikov Variables and The Exponential Expansion

Proving the Theorems requires to "linearize" X_μ around the origin. We employ the Sil'nikov variables and the exponential expansions due to Deng (Deng 1989) and Sil'nikov (Sil'nikov 1967). A suitable change of coordinates leads to the following for some neighborhood of $\mu = (m_i, 0)$ $(i = 1, 2)$ in the parameter space.

$$\mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R} = \left\{(x, y) | x \in \mathbb{R}^{d-1}, y \in \mathbb{R}\right\}. \tag{5.1}$$

$$\mathbf{W}^s_{loc}(\mathbf{O}; \mu) = \{y = 0\}, \qquad \mathbf{W}^u_{loc}(\mathbf{O}; \mu) = \{x = 0\}, \tag{5.2}$$

where $\mathbf{W}^s_{loc}(\mathbf{O}; \mu)$ and $\mathbf{W}^u_{loc}(\mathbf{O}; \mu)$ are the local stable and unstable manifolds in some neighborhood of the origin.

$$DX_\mu(\mathbf{O}) = \begin{pmatrix} -\lambda_s(\mu) & 0 & 0 \\ 0 & A(\mu) & 0 \\ 0 & 0 & \lambda_u(\mu) \end{pmatrix}, \tag{5.3}$$

where $A(\mu)$ is a $(d-2) \times (d-2)$ matrix satisfying

$$\sup_\mu [Re\{specA(\mu)\} - \{-\lambda_s(\mu)\}] < -gap_s < 0. \tag{5.4}$$

We shall write $x = (x_0, x_1) \in \mathbb{R} \times \mathbb{R}^{d-2}$ and choose directions of coordinate axes so that $\mathbf{e}_s$ and $\mathbf{e}_u$ point positive side of x_0-axis and y-axis.

Next we set local sections Σ^s and Σ^u as follows.

$$\begin{array}{l} \Sigma^s = \{(x, y) | x_0 = \delta_0,\ |x| \leq \delta_0,\ |y| \leq \delta_0\} \\ \Sigma^u = \{(x, y) |\ |x| \leq \delta_0,\ y = \delta_0\} \end{array}, \tag{5.5}$$

where $\delta_0 > 0$ is a small constant and $|\cdot|$ stands for the maximum norm. Notations (x^s, y^s) and (x^u, y^u) are used to represent points on Σ^s and Σ^u, and put

$$\Sigma^s_x = \left\{x^s = (\delta_0, x^s_1) |\ |x^s_1| \leq \delta_0\right\}. \tag{5.6}$$

Proposition 1. *For $\delta_0 > 0$ small enough and for any $(\tau, x^s, y^u) \in \mathbb{R}_+ \times \Sigma^s_x \times \{y^u\}$ there exists $t_0 = t_0(\tau, x^s, y^u) > 0$ such that X_μ has a unique solution $(x, y)(t) = (x, y)(t; \tau, x^s, y^u; \mu)$ $(0 \le t \le t_0)$ satisfying $x(0) = x^s$ and $y(\tau) = y^u (= \delta_0)$. Moreover the solution (x, y) is smooth with respect to $(t; \tau, x^s, y^u; \mu)$. (see Figure 4)*

We refer to (τ, x^s, y^u) as Sil'nikov variables according to (Deng 1989).
Let

$$\begin{aligned} x^u &:= x(\tau; \tau, x^s, y^u; \mu) \\ y^s &:= y(0; \tau, x^s, y^u; \mu) \\ r &:= e^{-\lambda_u(\mu)\tau}, \end{aligned} \tag{5.7}$$

then next holds.

Proposition 2. *For a sufficiently small $\delta_0 > 0$ and constants ω_1, ω_2 satisfying*

$$\begin{aligned} &0 < \omega_1 < \inf_\mu\{\nu(\mu), 1\} \\ &0 < \omega_2 < \inf_\mu\{\nu(\mu), 1, \tfrac{gap_s}{\lambda_u(\mu)}\} \end{aligned} \qquad \text{where } \nu(\mu) = \frac{\lambda_s(\mu)}{\lambda_u(\mu)} \tag{5.8}$$

we have a constant C independent of $\tau > 0, x^s \in \Sigma^s_x, y^u (= \delta_0)$ and $\mu \in \mathbb{R}^2$, such that y^s and x^u can be expanded into the following forms respectively.

$$y^s(r, x^s, y^u; \mu) = r\{\psi(x^s, y^u; \mu) + R(r, x^s, y^u; \mu)\} \qquad (r \searrow 0) \tag{5.9}$$

where the leading term ψ satisfies

$$\psi(x^s, 0; \mu) \equiv 0 \qquad D_{y^u}\psi(0, 0; \mu) = 1 \tag{5.10}$$

and the remainder term R satisfies

$$\|D^{k_1}_{x^s} D^{k_2}_{y^u} D^{k_3}_{\mu} R\| \le C r^{\omega_1}, \qquad \|D^{k_4}_r R\| \le C r^{\omega_1 - k_4} \qquad \text{for } k_i = 0, 1. \tag{5.11}$$

$$x^u(r, x^s, y^u; \mu) = r^{\nu(\mu)}\{\varphi(x^s, y^u; \mu) + Q(r, x^s, y^u; \mu)\} \qquad (r \searrow 0) \tag{5.12}$$

where the leading term $\varphi = (\varphi_0, \varphi_1) \in \mathbb{R} \times \mathbb{R}^{d-2}$ satisfies

$$\varphi_0(0, y^u; \mu) \equiv 0 \qquad \varphi_1(x^s, 0; \mu) \equiv 0 \tag{5.13}$$

$$D_{x^s}\varphi(0, 0; \mu) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \tag{5.14}$$

and the remainder term Q satisfies

$$\|D^{k_1}_{x^s} D^{k_2}_{y^u} D^{k_3}_{\mu} Q\| \le C r^{\omega_2}, \qquad \|D^{k_4}_r Q\| \le C r^{\omega_2 - k_4} \qquad \text{for } k_i = 0, 1. \tag{5.15}$$

The expressions above are called the exponential expansions of y^s and x^u respectively.

Through applying another suitable coordinate transformation, we can normalize φ as follows.

Proposition 3. *There exists a change of coordinate through which we have*

$$\varphi = (\varphi_0, \varphi_1) \in \mathbb{R} \times \mathbb{R}^{d-2} \text{ with } \varphi_1 \equiv 0. \tag{5.16}$$

6. Twisting and Unstable Manifold

Consider the vector fields X_μ where parameter μ is in a neighborhood of μ_0 at which X_{μ_0} has a homoclinic orbit of Möbius type. We shall write $(\delta_0, \bar{x}_1^s, 0) := \Gamma_{\mu_0} \cap \Sigma^s$ the intersection point of the homoclinic orbit Γ_μ and the local section Σ^s. For each μ near μ_0, let $(\delta_0, \tilde{x}_1^s(\mu), \tilde{y}^s(\mu)) \in \mathbf{W}^u(\mathbf{O}; \mu) \cap \Sigma^s$ be the first intersection point of Σ^s and the branch of the $\mathbf{e}_u$ direction of $\mathbf{W}^u(\mathbf{O}; \mu)$.

Proposition 2 (1) implies that $\mathbf{W}^u(\mathbf{O}; \mu)$ intersects Σ^u if $\tilde{y}^s(\mu) > 0$. We refer this point as $(\tilde{x}_0^u(\mu), \tilde{x}_1^u(\mu), \delta_0) \in \mathbf{W}^u(\mathbf{O}; \mu) \cap \Sigma^u$(see Figure 5).

The transversality condition (**B**.**A**.) (2) guarantees existence of a smooth path through μ_0 in the parameter space parameterized by ε satisfying

$$\left(\delta_0, \tilde{x}_1^s\left(\mu(\varepsilon)\right), \tilde{y}^s\left(\mu(\varepsilon)\right)\right) = (\delta_0, \bar{x}_1^s, \varepsilon). \tag{6.1}$$

Lemma 3.

$$\left(\tilde{x}_0^u\left(\mu(\varepsilon)\right), \tilde{x}_1^u\left(\mu(\varepsilon)\right), \delta_0\right) = (C_1 \varepsilon^{\nu(\mu_0)}, 0, \delta_0) + \textit{higher order term} \qquad (\varepsilon \searrow 0)$$

for some positive constant C_1.

Proof. When ε is positive, from Proposition 2 we get following expressions:

$$\varepsilon = r\left\{\psi(\delta_0, \bar{x}_1^s, \delta_0; \mu) + R(r, \delta_0, \bar{x}_1^s \delta_0; \mu)\right\} \tag{6.2}$$

$$(\tilde{x}_0^u(\mu), \tilde{x}_1^u(\mu)) = r^{\nu(\mu)}\left\{\varphi(\delta_0, \bar{x}_1^s, \delta_0; \mu) + Q(r, \delta_0, \bar{x}_1^s, \delta_0; \mu)\right\}, \tag{6.3}$$

where $\mu = \mu(\varepsilon)$ and δ_0 is substituted for x_0^s and y^u.

$$\varepsilon \approx r \qquad (r \searrow 0) \tag{6.4}$$

holds because

$$\begin{array}{ll} \psi(\delta_0, \bar{x}_1^s, \delta_0; \mu) \approx \delta_0 & \\ \varepsilon_0(\delta_0, \bar{x}_1^s, \delta_0; \mu) \approx \delta_0 & (\delta_0 \searrow 0), \\ \varepsilon_1(\delta_0, \bar{x}_1^s, \delta_0; \mu) = \textit{ higher order term} & \end{array} \tag{6.5}$$

which means

$$\begin{array}{l} \bar{x}_0^u\left(\mu(\varepsilon)\right) \approx \varepsilon^{\nu(\mu_0)} \\ \bar{x}_1^u\left(\mu(\varepsilon)\right) = \ \textit{higher order term.} \end{array} \tag{6.6}$$

□

Because $\mathbf{W}^s(\mathbf{O};\mu)$ has codimension one, it divides a neighborhood of the origin into two region. One side of these two regions is referred to as the positive side if it is the side which $\mathbf{e}_u$ points, and the other side as the negative side. Let $u(t;\varepsilon) = (u_0(t;\varepsilon), u_1(t;\varepsilon), u_y(t;\varepsilon))$ be the solution of $X_{\mu(\varepsilon)}$ which starts $\left(\tilde{x}_0^u\left(\mu(\varepsilon)\right), \tilde{x}_1^u\left(\mu(\varepsilon)\right), \delta_0\right)$ at $t = 0$, and $\gamma_{\mu_0}(t) = (\gamma_0(t), \gamma_1(t), \gamma_y(t))$ be the homoclinic solution which passes through $(0, 0, \delta_0)$ when $t = 0$. Let $\gamma_{\mu_0}(T) = \Gamma_{\mu_0} \cap \Sigma^s = (\delta_0, \bar{x}_1^s, 0)$.

Proposition 4. *If Γ_{μ_0} is a Möbius type homoclinic orbit and $0 < \nu < 1$, then $u_y(T;\varepsilon) < 0$ for small $\varepsilon > 0$. This means that $u(T;\varepsilon)$ lies in the negative side of $\mathbf{W}^s(\mathbf{O};\mu)$. Thus $u(t;\varepsilon)$ leaves a tubular neighborhood of Γ_{μ_0} immediately (see Figure 6).*

Proof. Consider the variational equation along $\gamma_{\mu_0}(t)$

$$\dot{P} = DX_{\mu_0}\left(\gamma_{\mu_0}(t)\right) P. \tag{6.7}$$

If $P(t;\varepsilon)$ is a solution of Eq(6.7) satisfying the initial condition

$$P(0,\varepsilon) = u(0;\varepsilon) - \gamma_{\mu_0}(0) \tag{6.8}$$

then next estimate holds.

$$\begin{aligned} &\left|\frac{d}{dt}\left|u(t;\varepsilon) - \gamma_{\mu_0}(t) - P(t;\varepsilon)\right|\right| \\ &\le \left|X_{\mu(\varepsilon)}\left(u(t;\varepsilon)\right) - X_{\mu_0}\left(\gamma_{\mu_0}(t)\right) - DX_{\mu_0}\left(\gamma_{\mu_0}(t)\right) P(t;\varepsilon)\right| \\ &\le \left|DX_{\mu_0}\left(\gamma_{\mu_0}(t)\right)\left(u(t;\varepsilon) - \gamma_{\mu_0}(t) - P(t;\varepsilon)\right) + \left.\frac{\partial X_\mu}{\partial \mu}\right|_{\mu=\mu_0}\left(\mu(\varepsilon) - \mu_0\right)\right| \\ &\qquad\qquad + O\left(\left|u(t;\varepsilon) - \gamma_{\mu_0}(t)\right|^2\right) \\ &\le M_1\left|u(t;\varepsilon) - \gamma_{\mu_0}(t) - P(t;\varepsilon)\right| + M_2\left[\sup_{0\le t\le T}\left|u(t;\varepsilon) - \gamma_{\mu_0}(t)\right|^2 + \left\{\mu(\varepsilon) - \mu_0\right\}\right], \end{aligned} \tag{6.9}$$

where M_1 and M_2 are some positive constants.

In comparison with the solution $r = \frac{M'}{M}(1 - e^{-Mt})$ of the equation $\frac{dr}{dt} = Mr + M'$, we get

$$\left|u(T;\varepsilon) - \gamma_{\mu_0}(T) - P(T;\varepsilon)\right| \le \frac{M_2}{M_1}\left[\sup_{0\le t\le T}\left|u(t;\varepsilon) - \gamma_{\mu_0}(t)\right|^2 + \left\{\mu(\varepsilon)\right\}\right]. \tag{6.10}$$

Now that we are assuming Γ_{μ_0} is of Möbius type, the system Eq(3.12) has a heteroclinic orbit from $(s_+, -1)$ to $(u_-, +1)$. Let $\tilde{P}(t;\varepsilon)$ be the image of $P(t;\varepsilon)$ under the projection $\pi : \mathbb{R}^d \setminus \{\mathbf{O}\} \to \mathbb{S}^{d-1}$. When ε is small, $\tilde{P}(0;\varepsilon) = \pi(P(0;\varepsilon))$ is near to s_+ so that $\tilde{P}(t;\varepsilon)$ tends to u_- as t goes to $+\infty$.

As a result $\tilde{P}(T;\epsilon)$ is near u_- when δ_0 is sufficiently small, which means

$$P_y(T;\varepsilon) < 0 \quad \text{and} \quad \left|\frac{P_i(T;\varepsilon)}{P_y(T;\varepsilon)}\right| < A \tag{6.11}$$

for some $A > 0$ independent of ε, where $P(T;\varepsilon) = (P_0(T;\varepsilon), P_1(T;\varepsilon), P_y(T;\varepsilon))$ and $i = 0,\ 1$.

From Eq(6.10)

$$u(T;\varepsilon) = \gamma_{\mu_0}(T) + P(T;\varepsilon) + O(\varepsilon^{\min\{2\nu,1\}}) \tag{6.12}$$

together with $\gamma_y(T) = 0$ and $P(T;\varepsilon) \approx \varepsilon^\nu$, so from the definition we get the Proposition. □

Similar argument applied to cylinder type homoclinic orbits or homoclinic orbits with $1 < \nu$ leads to next.

Proposition 5. *If Γ_{μ_0} is of cylinder type or $1 < \nu$ then for small $\varepsilon > 0$, $u_y(T;\varepsilon) > 0$. Thus $u(t;\varepsilon)$ stays in some tubular neighborhood of Γ_{μ_0} for a while (see Figure 7).*

7. The Connection Matrix Theory

We briefly recall the connection matrix theory for Morse decompositions. The details and their proofs can be found in Franzosa (Franzosa 1986,1988,1989), see also Conley (Conley 1978) for isolated invariant set and Morse index.

In this section P denotes a finite indexing set with p elements. A partial order on P is a relation, $<$, on the elements of P satisfying

(1) $\pi < \pi$ never holds for $\pi \in P$.
(2) $\pi < \pi'$ and $\pi' < \pi''$ imply $\pi < \pi''$.

An extension of $<$ is a partial order $<'$ P for which $\pi < \pi'$ implies $\pi <' \pi'$. An interval in $<$ is a subset $I \subset P$ for which $\pi, \pi' \in I$ and $\pi < \pi'' < \pi'$ imply $\pi'' \in I$. The set of intervals in $<$ is denoted $I(<)$.

For $\pi, \pi' \in P$ we say π and π' are adjacent if $\{\pi, \pi'\} \in I(<)$ and noncomparable if $\pi \not< \pi'$ and $\pi' \not< \pi$.

Assume S is an isolated invariant set of a local flow in a locally compact metric space with flow, that is S is the maximal invariant set in a compact neighborhood of itself. If S_1 and S_2 are compact invariant subsets of S then $C(S_2, S_1) := \{\gamma \in S | \omega(\gamma) \subset S_1 \text{ and } \alpha(\gamma) \subset S_2\}$ where $\omega(\gamma)$ and $\alpha(\gamma)$ are ω and α limit sets of γ.

Definition 2. A ($<$-ordered) Morse decomposition of S is a collection $M(S) = \{M(\pi)\}_{\pi \in P}$ of mutually disjoint compact invariant subsets of S such that if $\gamma \in S \setminus \cup_{\pi \in P} M(\pi)$, then there exist $\pi < \pi'$ with $\gamma \in C\left(M(\pi'), M(\pi)\right)$.

Assume that $M = \{M(\pi)\}_{\pi \in P}$ is a $<$-ordered Morse decomposition of S. The partial order $<$ on P induces a partial order on M called an admissible ordering of the Morse decomposition. The flow defines an "extremal" admissible ordering of M called the flow ordering of M, denoted $<_F$ and such that $\pi <_F \pi'$ if and only if there exists a sequence of distinct elements of $P : \pi = \pi_0, \ldots \pi_n = \pi'$ with $C\left(M(\pi_j), M(\pi_{j-1})\right) \neq \emptyset$ for each $j = 1, \ldots, n$. Every admissible ordering of M is an extension of $<_F$.

Associated to the admissible ordering $<$ of M there is a distinguished collection of subsets of S, called the Morse sets of $<$ and defined for each $I \in I(<)$ by

$$M(I) = \left(\bigcup_{\pi \in I} M(\pi) \right) \cup \left(\bigcup_{\pi, \pi' \in I} C\left(M(\pi'), M(\pi)\right) \right).$$

S being an isolated invariant set implies that each Morse set too. Here we define the Conley index of an isolated invariant set S.

Definition 3. A pair of compact sets (N_1, N_2) $\left(N_1 \supset N_2\right)$ is called an index pair for S if

(1) $S \subset int(N_1 \setminus N_2)$ and S is the maximal invariant set in $\overline{(N_1 \setminus N_2)}$
(2) $\gamma \in N_2,\ \gamma \cdot [0,t] \subset N_1 \implies \gamma \cdot [0,t] \subset N_2$
(3) $\gamma \in N_1,\ \gamma \cdot \mathbb{R}^+ \not\subset N_1 \implies \exists t \geq 0$ s.t. $\gamma \cdot [0,t] \subset N_1$ and $\gamma \cdot t \in N_2$.

Here $\gamma \cdot [0,t]$ means the orbit of γ from 0 till t and $\gamma \cdot t$ is the point γ reaches at t.

Definition 4. The homotopy type of the pointed quotient space N_1/N_2 is called the Conley index of S and denoted $h(S)$.

For each Morse set $M(I)$ let $C(I; G)$ denotes the singular chains of the index space for $M(I)$ with coefficients in G. Passing to homology one obtains $H_*\left(h\left(M(I)\right); G\right)$ the singular homology with coefficients in G of $h\left(M(I)\right)$ the Conley index of $M(I)$, and for simplicity we denote it by $H(I)$ which is called homology index of $M(I)$.

Let $C = \{C\Delta(\pi)\}_{\pi\in I}$ be a collection of graded modules indexed by $I \subset P$, then a map

$$\Delta(I)\colon \bigoplus_{\pi\in I} C\Delta(\pi) \to \bigoplus_{\pi\in I} C\Delta(\pi) \tag{7.1}$$

can be regarded as a matrix

$$\begin{pmatrix} \vdots & \\ \cdots \ \Delta_{\pi,\pi'} \ \cdots & \\ \vdots & \end{pmatrix}_{\pi,\pi'\in I} : \begin{pmatrix} \vdots \\ C\Delta(\pi') \\ \vdots \end{pmatrix}_{\pi'\in I} \to \begin{pmatrix} \vdots \\ C\Delta(\pi) \\ \vdots \end{pmatrix}_{\pi\in P} \tag{7.2}$$

where each $\Delta_{\pi,\pi'}$ is a map from $C\Delta(\pi')$ to $C\Delta(\pi)$.

Definition 5.

(1) $\Delta(I)$ is strictly uppertriangular if $\Delta_{\pi,\pi'} \neq 0$ implies $\pi < \pi'$.
(2) $\Delta(I)$ is a boundary map if each $\Delta_{\pi,\pi'}$ is of degree -1 and $\Delta^2 = 0$.

Proposition 6. *Let $M = \{M(\pi)\}_{\pi\in P}$ be a Morse decomposition of the isolated invariant set S with admissible ordering $<$ and $\{H(\pi)\}_{\pi\in P}$ be the set of homology indices of Morse sets $\{M(\pi)\}_{\pi\in P}$. Suppose each $H(\pi)$ is free.*

Then there exists a strictly upper triangular boundary map

$$\Delta\colon \bigoplus_{\pi\in P} H(\pi) \longrightarrow \bigoplus_{\pi\in P} H(\pi) \tag{7.3}$$

such that for each $I \in I(<)$ the boundary map

$$\Delta(I)\colon \bigoplus_{\pi\in I} H(\pi) \longrightarrow \bigoplus_{\pi\in I} H(\pi) \tag{7.4}$$

induces the homology group isomorphic to the homology index $H(I)$ of $M(I)$.

Definition 6. Δ above is called a connection matrix of the admissible ordering $<$, the set of connection matrices of the admissible ordering $<$ is denoted $\mathcal{CM}(<; G)$.

If $<$ is flow ordering $\mathcal{CM}(<; G)$ is called the set of the connection matrices of the Morse decomposition M and is denoted $\mathcal{CM}(M; G)$.

For the connection matrices next holds.

Proposition 7. *If $\Delta \in \mathcal{CM}(M; G)$, π and π' are adjacent in the flow ordering, and $\Delta(\pi', \pi) := \Delta_{\pi,\pi'} \neq 0$, then $C\left(M(\pi'), M(\pi)\right) \neq \emptyset$.*

Let Λ be a locally path connected parameter space and $X_\lambda, \lambda \in \Lambda$ be a family of local flow called product parameterization of a local flow, which means continuous dependence on λ. Then we have

Proposition 8. *There exists a neighborhood W of λ in Λ such that if $\mu \in W$, then M_λ is related by continuation to a Morse decomposition M_μ of an isolated invariant set in X_μ, and for such M_μ, $\mathcal{CM}(M_\mu) \subset \mathcal{CM}(M_\lambda)$.*

Here "related by continuation" roughly means M_λ can be identified with M_μ.

8. Proof of The Theorems

In this section we give a proof of the main theorems with connection matrix theory. First suppose that $\mathbf{W}^u(\mathbf{O};\mu)$ crosses $\mathbf{W}^s(\mathbf{O};\mu)$ from the negative side of $\mathbf{W}^s(\mathbf{O};\mu)$ to the positive side of it as μ_2 increases, in other words the path $\mu(\varepsilon)$ in Eq(6.1) would be taken so that $\frac{d\mu_2}{d\varepsilon} > 0$ for every μ_1. This can be done from the next Proposition.

Proposition 9. *The direction to which $\mathbf{W}^u(\mathbf{O};\mu)$ crosses $\mathbf{W}^s(\mathbf{O};\mu)$ when μ_2 increases through $\mu = (\mu_1, 0)$ is independent of μ_1.*

Proof. If it were not, then a branch of 1-homoclinic orbit other than μ_1-axis must exist in the parameter space. This contradicts to (**B**.**A**.). Details are similar to the proof of the Theorems itself but easier, so that it shall be omitted. □

First we prove the Theorem 1. From the figure 6 and figure 7, the existence of a double homoclinic orbit is intuitively clear. Yanagida (Yanagida 1987) discussed the existence under the condition that in some neighborhood of the origin the vector fields X_μ is linearizable uniformly with respect to μ and proved using intermediate value theorem. Generally the condition for linearization is not the case, however. To prove the existence of a double homoclinic orbit for general case, we employ Conley Index and Connection Matrix Theory as a generalization of the intermideate value theorem. But here we face another problem, that is, the Connection Matrix Theory is not applicable for detecting recurrent orbits, so that we cannot use it for homoclinic orbit directly.

Proof of the Theorems

To overcome the difficulty above, we take universal covering of a tubular neighborhood of original homoclinic orbits in the line $\mu = (\mu_1, 0)$ and let vector fields $\tilde{X}_\mu$ be the lift of X_μ. The local flow which $\tilde{X}_\mu$ defines can be trivially extended to a flow on a wider space, so that we can apply Conley index and connection matrix theory.

Consider three successive lifts of the origin, which are equilibria in $\tilde{X}_\mu$. We modify $\tilde{X}_\mu$ by putting an additional attracting equilibrium as in Figure 8, and put label for each equilibrium. We write the homology index with coefficients in $\mathbb{Z}_2$ of each equilibrium as $H_*(i)$, and that for each Morse set consisting of two equilibriums and the connecting orbits between them shall be written as $H_*(ij)$ $(i, j = 1, 2, 3, 4)$. From

the phase portraits we have next both for $\mu_1 \approx m_1$ and $\mu_1 \approx m_2$.

$$\begin{aligned} H_n(1) &= \begin{cases} \mathbb{Z}_2 & (n=0) \\ 0 & (n \neq 0) \end{cases} \\ H_n(i) &= \begin{cases} \mathbb{Z}_2 & (n=1) \\ 0 & (n \neq 1) \end{cases} \quad 2 \leq i \leq 4 \\ H_*(12) &= 0 \quad H_*(13) = 0 \end{aligned} \tag{8.1}$$

Then Δ_{12} and Δ_{13} are isomorphism by next diagram.

$$\begin{array}{ccccccc} & & & n=1 & & n=0 & \\ 0 & \longrightarrow & 1 & \begin{pmatrix} 0 \\ \mathbb{Z}_2 \end{pmatrix} & \longrightarrow & \begin{pmatrix} \mathbb{Z}_2 \\ 0 \end{pmatrix} & \longrightarrow 0. \\ & & 2 & & & & \end{array} \tag{8.2}$$

Moreover equilibria 2 and 3, 3 and 4 are noncomparable. Then connection matrices of both must be of next form.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & \cong & \cong & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{l} \text{where } \cong \text{ means isomorphism} \\ \text{and } * \text{ means undecided.} \end{array} \tag{8.3}$$

As for Δ_{24} we have $\Delta_{24} = 0$ from the diagram

$$\begin{array}{ccccccc} & & & n=1 & & n=0 & \\ 0 & \longrightarrow & 2 & \begin{pmatrix} \mathbb{Z}_2 \\ \mathbb{Z}_2 \end{pmatrix} & \longrightarrow & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \longrightarrow 0. \\ & & 4 & & & & \end{array} \tag{8.4}$$

So we conclude that in the parameter space $\mu = (\mu_1, \mu_2)$ with $0 < \mu_2 \ll 1$, the connection matrices are written as follows.

$$\begin{pmatrix} 0 & \cong & \cong & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{8.5}$$

Now the last entry shall be determined. When $\mu_1 \approx m_1$ we have $H_*(14) = 0$. This requires that Δ_{14} is an isomorphism. On the other hand if $\mu_1 \approx m_2$, 1 and 4 are noncomparable and $\Delta_{14} = 0$.

Let

$$M_I = \{(\mu_1, \mu_2) | \mu_2 > 0 \text{ and } \mathcal{CM} = \left\{ \begin{pmatrix} 0 & \cong & \cong & \cong \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}\} \tag{8.6}$$

and

$$M_{II} = \{(\mu_1, \mu_2) | \mu_2 > 0 \text{ and } \mathcal{CM} = \left\{ \begin{pmatrix} 0 & \cong & \cong & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}\} \ . \tag{8.7}$$

Then M_I and M_{II} include parameter regions $\{\mu_1 \approx m_1, 0 < \mu_2 \ll 1\}$ and $\{\mu_1 \approx m_2, 0 < \mu_2 \ll 1\}$ respectively. Here we remark Proposition 8 implies M_I and M_{II} are open. In the region $\{0 < \mu_2 \ll 1\} \setminus (M_I \cup M_{II})\Big(\neq \emptyset\Big)$ the set of connection matrices has two elements(Prop. 6 and 8).

$$\mathcal{CM} = \left\{ \begin{pmatrix} 0 & \cong & \cong & \cong \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \cong & \cong & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}. \tag{8.8}$$

This means $1 <_F 4$ and $H_*(14) \neq 0$, so there exists a heteroclinic orbit from 4 to 2 in $\tilde{X}_\mu$ (where $<_F$ means flow ordering). This heteroclinic orbit corresponds to a 2-homoclinic orbit in X_μ.

Theorem2 can be proved by same argument. In this case the connection matrices are of the following form.

$$\begin{array}{c} \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccccc} & & & & N+2 & & & 2N+2 \end{array} \\ \begin{pmatrix} 0 & \cong & \cdots & \cong & * & \cdots & * \\ 0 & & & \cdots & & & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & & & \cdots & & & 0 \end{pmatrix} \end{array} . \tag{8.9}$$

Q.E.D.

9. Twisting Index

In this section we introduce an Index which detects existence of 2−homoclinic bifurcation.

In the Theorem1, consider the case where $0 < \nu < 1$. The family of homoclinic orbits existing on μ_1−axis has following two ways of changing its types: from cylinder type to Möbius type (see figure 9).

(i) $\mathbf{e}_s$ does not change its direction.

(ii) At $\mu_1 = m_1$, $\mathbf{e}_s$ points opposite direction to the one it points at $\mu_1 = m_1$.

The case(i) corresponds to the case where changes take place via inclination-flip homoclinic orbits, and the case(ii) does to via orbit-flip homoclinic orbits, which we mentioned in section 1.

As for the case(i), take Eq(3.12) as a system which depends on a parameter μ_1, then $s_+(\mu_1) = \pi(\pm\mathbf{e}_s(\mu_1))$ is continuous as a map on the interval $[m_1, m_2]$.

Now, we write the heteroclinic orbit of Eq(3.12) connecting $(s_+, -1)$ to $(u_+, +1)$ when $\mu_1 = m_1$ as $\{(P_1(\tau), \tau)\}$ and the one connecting $(s_+, -1)$ to $(u_-, +1)$ when $\mu_1 = m_2$ as $\{(P_2(\tau), \tau)\}$, and define a map $H\colon \partial([m_1, m_2] \times [-1, 1]) \cong \mathbb{S}^1 \to \mathbb{RP}^{d-1}$ as follows.

$$H(\mu_1, \tau) = \begin{cases} \tilde{\pi}(s_\pm) & (\mu_1 \in [m_1, m_2], \tau = -1) \\ \tilde{\pi}(P_1(\tau)) & (\mu_1 = m_1, \tau \in [-1, +1]) \\ \tilde{\pi}(P_2(\tau)) & (\mu_1 = m_2, \tau \in [-1, +1]) \\ \tilde{\pi}(u_\pm) & (\mu_1 \in [m_1, m_2], \tau = +1) \end{cases} \tag{9.1}$$

where $\tilde{\pi}\colon \mathbb{S}^{d-1} \to \mathbb{RP}^{d-1}$ is the projection.

This map H defines a non-trivial element of the fundamental group of $\mathbb{RP}^{d-1}$,

$$\mathbb{Z}_2 \cong \pi_1(\mathbb{RP}^{d-1}) \ni [H] \neq 0 \tag{9.2}$$

As for the case(ii), $s_+(m_1)$ continues to $s_-(m_2)$ along $\mu_1 \in [m_1, m_2]$. This time, for the tangent vector $\{\dot{\gamma}_i(t)\}$ to the homoclinic orbits $\Gamma_{\mu=(m_i,0)} = \{\gamma_i(t)\}$, $(\pi(\dot{\gamma}_i(\tau(t))), \tau)$ define heteroclinic orbits of Eq(3.12) from $(u_+(m_i), -1)$ to $(s_+(m_i), +1)$. Define $G\colon \partial([m_1, m_2] \times [-1, +1]) \to \mathbb{RP}^{d-1}$ as,

$$G(\mu_1, \tau) = \begin{cases} \tilde{\pi}(u_\pm) & (\mu_1 \in [m_1, m_2], \tau = -1) \\ \tilde{\pi} \circ \pi(\dot{\gamma}_1(t(\tau))) & (\mu_1 = m_1, \tau \in [-1, +1]) \\ \tilde{\pi} \circ \pi(\dot{\gamma}_2(t(\tau))) & (\mu_1 = m_2, \tau \in [-1, +1]) \\ \tilde{\pi}(s_\pm) & (\mu_1 \in [m_1, m_2], \tau = +1) \end{cases} \tag{9.3}$$

and the element $[G]$ is non-trivial.

$$\mathbb{Z}_2 \cong \pi_1(\mathbb{RP}^{d-1}) \ni [G] \neq 0 \tag{9.4}$$

Theorem1 implies next theorem.

Theorem 3. *Suppose that the 2-parameter family of vector fields X_μ satisfies* (**B**.**A**.) *and $0 < \nu < 1$, and the assumption* (**P**.**H**.) *holds for the homoclinic orbits $\Gamma_{\mu=(\mu_1,0)}$ at $\mu_1 = m_1$ and $\mu_1 = m_2$.*

Under this condition the system undergoes $2-$homoclinic bifurcation if

$$\pi_1(\mathbb{RP}^{d-1}) \ni [H] + [G] \neq 0 \tag{9.5}$$

Definition 7. We call $[H]+[G]$ as Twisting Index for a family $\left\{\Gamma_{\mu=(\mu_1,0)}\right\}_{m_1\leq\mu_1\leq m_2}$ of homoclinic orbits.

Remark 2. *If* $[H]=1$ *and* $[G]=1$ *simultaneously then* $\Gamma_{(m_1,0)}$ *and* $\Gamma_{(m_2,0)}$ *are same type concerning their twistings.*

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