ON CONFORMAL FIELD THEORY

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Introduction

Belavin-Polyakov-Zamodolochikov ([BPZ]) initiated conformal field theory as a certain limit of the theory of two-dimensional lattice model. The theory has deep relationship with string theory and it has rich mathematical structure. It is a two-dimensional quantum field theory invariant by conformal transformations. Actually, as we shall see below, it is invariant under much bigger transformations. This gives relationship with moduli space of algebraic curves ([FS], [EO]).

A typical example of conformal field theory is abelian conformal field theory, the theory of free fermions over a compact Riemann surface. For mathematical rigorous treatment on abelian conformal field theory we refer the reader to [KNTY]. The theory has deep relationship with different fields of mathematics, such as the moduli theory of algebraic curves, KP hierarchy, theta functions, complex cobordism ring and formal groups ([KNTY], [KSU2], [KSU3]).

For non-abelian conformal field theory the first mathematically rigorous treatment was given by Tsuchiya-Kanie ([TK]). They constructed conformal field theory over $\mathbf{P}^1$. Later Tsuchiya-Ueno-Yamada ([TUY]) generalized it over algebraic curves of arbitrary genus.

Let us explain briefly the main ideas of conformal field theory. Conformal field theory can be decomposed into two part, holomorphic part and anti-holomorphic part. Therefore, in the following we only consider holomorphic conformal field theory. Holomorphic conformal field theory is often called chiral conformal field theory by physicists.

Let ξ be a complex coordinate. An infinitesimal conformal transformation is expressed in the form

$$\xi \mapsto \xi + \epsilon f(\xi) \tag{1}$$

where ϵ is the dual number, $x \mod (x^2)$ in $\mathbf{C}[\,x\,]/(x^2)$ and $f(\xi)\frac{d}{d\xi}$ is a local holomorphic vector field. Let us choose special vector fields

$$\ell_n = \xi^{n+1}\frac{d}{d\xi}, \quad n = -1, 0, 1, 2, \ldots, n, \ldots . \tag{2}$$

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Then, these vector fields satisfy the following commutation relations.

$$[\,\ell_m, \ell_n\,] = (m-n)\ell_{m+n}.$$

Thus, $\{\ell_n\}_{n\geq -1}$ forms an infinite dimensional Lie algebra.

Note that $\{\ell_{-1}, \ell_0, \ell_1\}$ forms a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbf{C})$. This is the Lie algebra of linear fractional transformations of the Riemann sphere $\mathbf{P}^1$. Hence, conformal field theory is invariant under such transformations.

The most important fact in conformal field theory is that physicists imposed the condition that the theory should be invariant not only by the transformation (1), but also the transformations where $f(\xi)\frac{d}{d\xi}$ is a local meromorphic vector field ([BPZ]). Then, of course this is no more infinitesimal conformal transformation. But, we may interpret such a transformation as an infinitesimal change of a complex structure. In this way conformal field theory is related to the moduli theory of algebraic curves. We need to generalize the Lie algebra $\{\ell_n\}$ by adding negative powers of ξ in (2).

As we said at the beginning conformal field theory is a kind of quantum field theory. This means that the theory has infinite degrees of freedom and divergence appears. Therefore, we need to renormalize the theory. Renormalization is very simple in our case. It can be done by using the normal ordering of operators, which we shall explain below. Renormalization induces a central extension of the infinite Lie algebra $\{\ell_n\}$, which is called Virasoro algebra.

The Virasoro algebra $\{L_n\}$ has the following commutation relations.

$$[\,L_n, L_m\,] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\cdot id.$$

The number c is called the central charge of the Virasoro algebra. Then, conformal invariance of the theory is expressed by the commutation relation such as

$$[L_n, X(z)] = z^n(z\frac{d}{dz} + n + 1)X(z).$$

This means that the field $X(z)$ behaves like a meromorphic one-form on a Riemann surface. (More precise formulation will be given in Chapter I §1.2 c), below).

Conformal field theory can be formulated not only over a compact Riemann surface but also on a semi-stable algebraic curve, that is, a complete curve with at worst ordinary double points as singularities. These singular curves correspond points on the boundary of the moduli space of smooth curves. Precisely speaking, we need to consider pointed stable curves.

Let C be a semi-stable curve of genus g. To formulate conformal field theory we also need the set Λ of labels which describe basic particles (or fields, physicists call them primary fields) on the curve C. There is a special

label $0 \in \Lambda$ which corresponds to the state without particles. Also we assume that the set Λ has the involution

$$\dagger : \Lambda \to \Lambda,$$

where $\lambda^\dagger$ describe the anti-particle of the particle corresponding to $\lambda \in \Lambda$.

Usually the label Λ is related to certain representations of the infinite dimensional Lie algebra or more general algebras describing symmetry of the theory. In conformal field theory with gauge symmetry (for simplicity we call it non-abelian conformal field theory) the Lie algebra is an affine Lie algebra associated with a simple Lie algebra. In the following we also give a sketch of the theory of abelian conformal field theory, which is a generalization of the usual abelian conformal field described in [KNTY]. In this case as the algebra we take the one generated by vertex operators and an affine Lie algebra attached to the one-dimensional abelian Lie algebra.

Let us choose N smooth points $Q_1,\ Q_2, \dots, Q_N$ on the curve C with local coordinates ξ_j with center Q_j. Choose an element λ_j of the label Λ to each point Q_j. We also fix local coordinates ξ_j with centre Q_j.

Conformal field theory is a theory to attach a certain finite dimensional vector space "conformal block" to the data

$$\mathfrak{X} = (C; Q_1, Q_2, \dots, Q_N; \xi_1, \xi_2, \dots, \xi_N)$$

and $\vec{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_N) \in \Lambda^{\oplus N}$. The conformal block $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$ satisfies the following conditions.

1) Let P be a smooth point on the curve C with coordinate η. Put

$$\widetilde{\mathfrak{X}} = (C; Q_1, Q_2, \dots, Q_N, P; \xi_1, \xi_2, \dots, \xi_N, \eta).$$

Then, there is a canonical isomorphism

$$\mathcal{V}^\dagger_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}) \simeq \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}).$$

2) Let $\nu : \widetilde{C} \to C$ be a normalization of the curve C and P_+ and P_- be the inverse image of the double point of C with local coordinates η_+, η_-, respectively. Then, there exists a canonical isomorphism

$$\bigoplus_{\mu \in \Lambda} \mathcal{V}^\dagger_{\vec{\lambda},\mu\mu^\dagger}(\widetilde{\mathfrak{X}}) \simeq \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$$

where we put

$$\widetilde{\mathfrak{X}} = (\widetilde{C}; Q_1, Q_2, \dots, Q_N, P_+, P_-; \zeta_1, \eta_2, \dots, \zeta_N, \eta_+, \eta_-).$$

3) $\dim \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$ depends only on the genus of the curve C , the number N of the points Q_j and $\vec{\lambda}$.

4) The collection $\bigoplus_{[\mathfrak{X}]} \mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X})$ over the moduli space of N-pointed curves forms a vector bundle and it carries a projectively flat connection with regular singularities along the boundary consisting of the locus of N-pointed singular curves.

The conformal block is also called a space of vacua ([TUY]) or modular functor ([S]). The above property 2) is often called factorization of conformal blocks and physically it is used to construct conformal field theory. The property 2) gives a way to calculate the dimensions of conformal blocks by reducing to the case of three-pointed $\mathbf{P}^1$. This is a combinatorial problem and not easy to calculate. The best way is to use Verlinde formula, first conjectured by Verlinde ([Ve]) and proved in ([MS1]).

The above property 4) is a vague statement. Conformal block $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X})$ depends on the choice of local coordinates ξ_j at the point Q_j and we can show that it only depends on the first order infinitesimal neighbourhood of the points. Therefore, as the moduli space we need to take the one of N-pointed stable curves with the first oder infinitesimal neighbourhoods. We can show that the

the conformal block

In the present notes we shall briefly explain a method to construct conformal block for conformal field theory with gauge symmetries with simple Lie algebra as gauge group and abelian conformal field theory. We shall follow [TK] and [TUY] to construct conformal blocks. The method is often called operator formalism. It is a primitive but direct method to construct the theory by following physicists' description of conformal field theory. It also gives a mathematically rigorous interpretation of physical theory. There are other approaches to conformal field theory ([H], [BL], [BF], [Fe], [KNR], [Ku]). These approaches give other description of conformal blocks different from ours. Also it is an important problem to generalize integral representation of conformal blocks of N-pointed $\mathbf{P}^1$ given in [BF] and [Ku] to those of N-pointed curve of genus $g \geq 2$.

The details of our arguments on non-abelian conformal field theory can be found in [U].

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Chapter I Conformal Field Theory with Gauge Symmetries

§1.1 Affine Lie algebras and integrable highest weight modules

In this section we recall basic facts on integrable highest weight representations of affine Lie algebras. Then we shall define the energy momentum tensor which plays an important role in conformal field theory. For the details of integrable highest weight representations of affine Lie algebras we refer the reader to Kac's book [Ka].

a) Affine Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra over the complex numbers $\mathbf{C}$ and $\mathfrak{h}$ its Cartan subalgebra. By Δ we denote the root system of $(\mathfrak{g}, \mathfrak{h})$. We have the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

Let $\mathfrak{h}^*_{\mathbf{R}}$ be the linear span of Δ over $\mathbf{R}$. Fix a lexicographic ordering of $\mathfrak{h}^*_{\mathbf{R}}$ once for all. This gives the decomposition $\Delta = \Delta_+ \sqcup \Delta_-$ of the root system into the positive roots and the negative roots. Put $\mathfrak{h}^* = \mathfrak{h}^*_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$, the linear span of Δ over $\mathbf{C}$. Let $(\ ,\ )$ be a constant multiple of the Cartan-Killing form of the simple Lie algebra $\mathfrak{g}$. For each element of $\lambda \in \mathfrak{h}^*$, there exists a unique element $H_\lambda \in \mathfrak{h}^*$ such that

$$\lambda(H) = (H_\lambda, H)$$

for all $H \in \mathfrak{h}$. For $\alpha \in \Delta$, H_α is called the *root vector* corresponding to the root α.

On $\mathfrak{h}^*$ we introduce an inner product by

$$(\lambda, \mu) = (H_\lambda, H_\mu). \tag{1.1-1}$$

Let us normalize the inner product $(\quad , \quad)$ and call it the normalized Cartan-Killing form. Let θ be the highest (or longest) root, that is,

$$\theta \in \Delta_+, \quad \theta + \alpha_i \notin \Delta$$

for all simple roots α_i. For example, the highest root of the simple Lie algebra of type A_ℓ is given by

$$\theta = \sum_{i=1}^{\ell} \alpha_i.$$

The normalized Cartan-Killing form is given by the condition

$$(\theta, \theta) = 2. \tag{1.1-2}$$

Note that the Cartan-Killing form has the following property.

$$([X, Y], Z) + (Y, [X, Z]) = 0. \tag{1.1-3}$$

By $\mathbf{C}[[\xi]]$ and $\mathbf{C}((\xi))$ we mean the ring of formal power series in ξ and the field of formal Laurent power series in ξ, respectively.

Definition 1.1.1. The affine Lie algebra $\widehat{\mathfrak{g}}$ over $\mathbf{C}((\xi))$ associated with $\mathfrak{g}$ is defined to be

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}((\xi)) \oplus \mathbf{C}c$$

where c is an element of the center of $\widehat{\mathfrak{g}}$ and the Lie algebra structure is given by

$$[X \otimes f(\xi), Y \otimes g(\xi)] = [X, Y] \otimes f(\xi)g(\xi) + c \cdot (X, Y) \operatorname*{Res}_{\xi=0}(g(\xi)df(\xi)) \tag{1.1-4}$$

for

$$X, Y \in \mathfrak{g}, \quad f(\xi), g(\xi) \in \mathbf{C}((\xi)).$$

Put

$$\widehat{\mathfrak{g}}_+ = \mathfrak{g} \otimes \mathbf{C}[[\xi]]\xi, \quad \widehat{\mathfrak{g}}_- = \mathfrak{g} \otimes \mathbf{C}[\xi^{-1}]\xi^{-1}. \tag{1.1-5}$$

We regard $\widehat{\mathfrak{g}}_+$ and $\widehat{\mathfrak{g}}_-$ as Lie subalgebras of $\widehat{\mathfrak{g}}$. Also in the following we often identify $\mathfrak{g}$ and $\mathfrak{g} \otimes 1$ so that we may regard $\mathfrak{g}$ as a Lie subalgebra of $\widehat{\mathfrak{g}}$. We have a decomposition

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \oplus \mathfrak{g} \oplus \mathbf{C}c \oplus \widehat{\mathfrak{g}}_-. \tag{1.1-6}$$

In what follows we use the following nation freely.

$$X(n) := X \otimes \xi^n, \quad X \in \mathfrak{g}, \quad n \in \mathbf{Z}$$
$$X = X(0) = X \otimes 1$$

Remark 1.1.2. Note that usually the affine Lie algebra is defined by using $\mathbf{C}[\xi, \xi^{-1}]$. Namely, put

$$\mathfrak{g}_{aff} := \mathfrak{g} \otimes \mathbf{C}[\xi, \xi^{-1}] \oplus \mathbf{C} \cdot c$$

with commutation relation (1.1-4). Put

$$\mathfrak{g}_+ = \mathfrak{g} \otimes \mathbf{C}[\,\xi\,]\xi.$$

This is a Lie subalgebra and we have

$$\mathfrak{g}_{aff} = \mathfrak{g}_+ \oplus \mathfrak{g} \oplus \mathbf{C} \cdot c \oplus \widehat{\mathfrak{g}}_-.$$

Moreover, the Lie algebra $\widehat{\mathfrak{g}}_+$ is contained in the ξ-adic completion of $\mathfrak{g}_+$.

b) Integrable highest weight modules

Let us recall briefly the representation theory of a simple Lie algebra $\mathfrak{g}$. An irreducible left $\mathfrak{g}$-module V_λ is called a highest weight module with highest weight λ, if there exists a non-zero vector $e \in V_\lambda$ (called a highest weight vector) such that

$$He = \lambda(H)e, \quad Xe = 0 \quad \text{for all } X \in \mathfrak{g}_\alpha, \quad \alpha \in \Delta_+.$$

It is well-known that a finite dimensional irreducible left $\mathfrak{g}$-module is a highest weight module and two irreducible left $\mathfrak{g}$-modules are isomorphic if and only if they have the same highest weight. A weight $\lambda \in \mathfrak{h}^*_{\mathbf{R}}$ is called an integral weight, if

$$2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbf{Z}$$

for any $\alpha \in \Delta$. A weight $\lambda \in \mathfrak{h}^*_{\mathbf{R}}$ is called a dominant weight, if

$$w(\lambda) \leq \lambda$$

for any element w of the Weyl group W of $\mathfrak{g}$. By P_+ we denote the set of dominant integral weights of $\mathfrak{g}$. A weight λ is the highest weight of an irreducible left $\mathfrak{g}$-module if and only if $\lambda \in P_+$.

Let us fix a positive integer ℓ (called the *level*) and put

$$P_\ell = \{\, \lambda \in P_+ \,|\, 0 \leq (\theta, \lambda) \leq \ell \,\}.$$

Let us introduce the involution

$$\begin{aligned} \dagger : P_\ell &\to P_\ell \\ \lambda &\mapsto \lambda^\dagger \end{aligned}$$

by

$$\lambda^\dagger = -w(\lambda)$$

where w is the longest element of the Weyl group of the simple Lie algebra $\mathfrak{g}$. Note that $\lambda^\dagger$ is also characterized by the fact that $-\lambda^\dagger$ is the lowest weight of the $\mathfrak{g}$-module V_λ.

For each element $\lambda \in P_\ell$ the Verma module $\mathcal{M}_\lambda$ is defined as follows. Put

$$\widehat{\mathfrak{p}}_+ := \widehat{\mathfrak{g}}_+ \oplus \mathfrak{g} \oplus \mathbf{C} \cdot c.$$

Then $\widehat{\mathfrak{p}}_+$ is a Lie subalgebra of $\widehat{\mathfrak{g}}$. Let V_λ is the irreducible left $\mathfrak{g}$-module of highest weight λ. The action of $\widehat{\mathfrak{p}_+}$ on V_λ is defined as

$$\begin{aligned} cv &= \ell v \quad \text{for all } v \in V_\lambda \\ av &= 0 \quad \text{for all } a \in \widehat{\mathfrak{g}}_+ \text{ and } v \in V_\lambda \end{aligned}$$

Put

$$\mathcal{M}_\lambda := U(\widehat{\mathfrak{g}}) \otimes_{\widehat{\mathfrak{p}}_+} V_\lambda. \tag{1.1-8}$$

Then $\mathcal{M}_\lambda$ is a left $\widehat{\mathfrak{g}}$-module and is called a *Verma module*. The Verma module $\mathcal{M}_\lambda$ is not irreducible and contains the maximal proper submodule $\mathcal{J}_\lambda$. The quotient module $\mathcal{H}_\lambda := \mathcal{M}_\lambda/\mathcal{J}_\lambda$ has the following properties.

Theorem 1.1.3. *For each $\lambda \in P_\ell$, the left $\widehat{\mathfrak{g}}$-module $\mathcal{H}_\lambda$ is the unique left $\widehat{\mathfrak{g}}$-module (called the* integrable highest weight $\widehat{\mathfrak{g}}$-module*) satisfying the following properties.*

(1) *$V_\lambda = \{\, |v\rangle \in \mathcal{H}_\lambda \mid \widehat{\mathfrak{g}}_+|v\rangle = 0 \,\}$ is the irreducible left $\mathfrak{g}$-module with highest weight λ.*
(2) *The central element c acts on $\mathcal{H}_\lambda$ as $\ell \cdot \mathrm{id}$.*
(3) *$\mathcal{H}_\lambda$ is generated by V_λ over $\widehat{\mathfrak{g}}_-$ with only one relation*

$$(X_\theta \otimes \xi^{-1})^{\ell-(\theta,\lambda)+1}|\lambda\rangle = 0 \tag{1.1-9}$$

where $X_\theta \in \mathfrak{g}$ is the element corresponding to the maximal root θ and $|\lambda\rangle \in V_\lambda$ is the highest weight vector.

The theorem says that the maximal proper submodule $\mathcal{J}_\lambda$ is given by

$$\mathcal{J}_\lambda = U(\widehat{\mathfrak{p}}_-)|J_\lambda\rangle \tag{1.1-10}$$

where we put

$$|J_\lambda\rangle = (X_\theta \otimes \xi^{-1})^{\ell-(\theta,\lambda)+1}|\lambda\rangle. \tag{1.1-11}$$

For the details see Kac [Ka, (10.4.6)].

Similarly we can define the integrable lowest weight right $\widehat{\mathfrak{g}}$-module $\mathcal{H}_\lambda^\dagger$ which will be discussed in below.

c) Energy momentum tensor and Virasoro algebra

Next we shall define the energy momentum tensor and the Virasoro algebra. For that purpose we introduce the following notation.

$$X(z) = \sum_{n \in \mathbf{Z}} X(n) z^{-n-1}$$

where z is a variable. The normal ordering $\overset{\circ}{\circ}\ \ \overset{\circ}{\circ}$ is defined by

$$\overset{\circ}{\circ} X(n)Y(m) \overset{\circ}{\circ} = \begin{cases} X(n)Y(m), & n < m, \\ \frac{1}{2}(X(n)Y(m) + Y(m)X(n)) & n = m, \\ Y(m)X(n) & n > m. \end{cases}$$

Note that by (1.1-4), if $n > m$ and $X = Y$, we have

$$\overset{\circ}{\circ} X(n)X(m) \overset{\circ}{\circ} = X(n)X(m) - n\delta_{n+m,0}(X,Y) \cdot c. \tag{1.1-12}$$

Definition 1.1.4. The *energy-momentum tensor* $T(z)$ of level ℓ is defined by

$$T(z) = \frac{1}{2(g^* + \ell)} \sum_{a=1}^{\dim \mathfrak{g}} \overset{\circ}{\circ} J^a(z) J^a(z) \overset{\circ}{\circ}$$

where $\{J^1, J^2, \dots\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the Cartan-Killing form $(\ \ ,\ \ )$ and g^* is the dual Coxeter number of $\mathfrak{g}$.

Put

$$L_n = \frac{1}{2(g^* + \ell)} \sum_{m \in \mathbf{Z}} \sum_{a=1}^{\dim \mathfrak{g}} \overset{\circ}{\circ} J^a(m) J^a(n-m) \overset{\circ}{\circ}. \tag{1.1-13}$$

Then we have the expansion

$$T(z) = \sum_{n \in \mathbf{Z}} L_n z^{-n-2}.$$

The operator L_n is called the Virasoro operator which acts on $\mathcal{H}_\lambda$. By (1.1-12), if $n \neq 0$, in the definition of L_n, we need not use the normal ordering, that is, we have

$$L_n = \frac{1}{2(g^* + \ell)} \sum_{m \in \mathbf{Z}} \sum_{a=1}^{\dim \mathfrak{g}} J^a(m) J^a(n-m).$$

For $n = 0$ we need the normal ordering to define L_0. The operator

$$\frac{1}{2(g^* + \ell)} \sum_{m \in \mathbf{Z}} \sum_{a=1}^{\dim \mathfrak{g}} J^a(m) J^a(-m)$$

cannot operate on $\mathcal{H}_\lambda$. The operator L_0 is a generalization of the Casimir element of the simple Lie algebra $\mathfrak{g}$

$$\Omega = \sum_{a=1}^{\dim \mathfrak{g}} J^a J^a$$

to the affine lie algebra $\widehat{\mathfrak{g}}$. The Casimir element Ω belongs to the center of $U(\mathfrak{g})$, hence acts on V_λ by

$$2(g^* + \ell)\Delta_\lambda \cdot \mathrm{id}$$

where

$$\Delta_\lambda = \frac{(\lambda, \lambda) + 2(\lambda, \rho)}{2(g^* + \ell)}, \quad \rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha \tag{1.1-14.}$$

The following lemma can be proved by direct calculations. It plays an important role in the theory.

Lemma 1.1.5. *The set $\{L_n\}$ forms a Virasoro algebra and we have*

$$[L_n,\, X(m)] = -mX(n+m), \quad \text{for } X \in \mathfrak{g}$$
$$[L_n,\, L_m] = (n-m)L_{n+m} + \frac{c_v}{12}(n^3 - n)\delta_{n+m,0}$$

where

$$c_v = \frac{\ell \dim \mathfrak{g}}{g^* + \ell}$$

is the central charge of the Virasoro algebra.

Corollary 1.1.6.

$$[L_n, X(z)] = z^n (z\frac{d}{dz} + n + 1)X(z).$$

For $X \in \mathfrak{g}$, $f = f(z) \in \mathbf{C}((z))$ and $\underline{\ell} = \ell(z)\dfrac{d}{dz} \in \mathbf{C}((z))\dfrac{d}{dz}$ we use the following notation.

$$X[f] = \operatorname*{Res}_{z=0}(X(z)f(z)dz)$$
$$T[\underline{\ell}] = \operatorname*{Res}_{z=0}(T(z)\ell(z)dz).$$

In particular, we have

$$L_0 = T[\xi \frac{d}{d\xi}]. \tag{1.1-15}$$

Lemma 1.1.7. *$X[f]$ and $T[\underline{\ell}]$ act on $\mathcal{H}_\lambda$ and we have*

$$\begin{aligned} X[f] &= X \otimes f(\xi), \\ [T[\underline{\ell}],\, X[f]] &= -X[\underline{\ell}(f)], \\ [T[\underline{\ell}],\, T[\underline{m}]] &= -T[[\underline{\ell},\, \underline{m}]] + \frac{c_v}{12} \operatorname*{Res}_{z=0}(\ell''' m dz). \end{aligned} \tag{1.1-16}$$

Remark 1.1.8. In the last formula of (1.1-16) we can use other expressions based on the following equalities.

$$\operatorname*{Res}_{z=0}(\ell'''(z)m(z)dz) = -\operatorname*{Res}_{z=0}(m'''(z)\ell(z)dz) = \frac{1}{2}\operatorname*{Res}_{z=0}\left(\begin{vmatrix} \ell'(z)m'(z) \\ \ell''(z)m''(z) \end{vmatrix} dz\right).$$

d) Filtration on $\mathcal{H}_\lambda$ and right $\widehat{\mathfrak{g}}$-modules

Next Let us introduce the filtration $\{F_\bullet\}$ on $\mathcal{H}_\lambda$. For that purpose first define the subspace $\mathcal{H}_\lambda(d)$ of $\mathcal{H}_\lambda$ for a non-negative integer d by

$$\mathcal{H}_\lambda(d) = \{\, |v\rangle \in \mathcal{H}_\lambda \mid \ L_0|v\rangle = (d + \Delta_\lambda)|v\rangle \,\} \tag{1.1-17}$$

where Δ_λ is defined in (1.1-14). Note that by (1.1-13), on V_λ the operator L_0 acts as

$$L_0|v\rangle = \frac{1}{2(g^* + \ell)} \sum_{a=1}^{\dim \mathfrak{g}} J^a J^a |v\rangle = \frac{1}{2(g^* + \ell)} \Omega|v\rangle = \Delta_\lambda |v\rangle$$

for $|v\rangle \in \mathcal{H}_\lambda$. For a positive integer m and $|v\rangle \in V_\lambda$, we have

$$\begin{aligned} L_0 X(-m)|v\rangle &= L_0 X(-m)|v\rangle + mX(-m)|v\rangle \\ &= (\Delta_v + m)X(-m)|v\rangle. \end{aligned}$$

Hence, we have $X(-m)|v\rangle \in F_m\mathcal{H}_\lambda$.

Similarly, for positive integers $m_1, \ldots, m_k$, we have

$$\begin{aligned} &L_0 X_1(M_1)\cdots X_k(-m_k)|v\rangle \\ &\qquad = (\Delta_\lambda + m_1 + \cdots + m_k) X_1(M_1)\cdots X_k(-m_k)|v\rangle, \quad |v\rangle \in V_\lambda. \end{aligned}$$

From this, it is easy to show that $\mathcal{H}_\lambda(d)$ is a finite dimensional vector space and we have

$$\mathcal{H}_\lambda = \bigoplus_{d=0}^{\infty} \mathcal{H}_\lambda(d).$$

For a negative integer $-d$ we put

$$\mathcal{H}_\lambda(-d) = \{0\}.$$

Now we define the filtration $\{F_p\mathcal{H}_\lambda\}$ by

$$F_p\mathcal{H}_\lambda = \sum_{d=0}^{p} \mathcal{H}_\lambda(d). \tag{1.1-18}$$

Note that all the filtrations defined above are the increasing ones.

Put

$$\mathcal{H}_\lambda^\dagger(d) = \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}_\lambda(d), \mathbf{C}). \tag{1.1-19}$$

Then the dual space $\mathcal{H}_\lambda^\dagger$ of $\mathcal{H}_\lambda$ is defined to be

$$\mathcal{H}_\lambda^\dagger = \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}_\lambda, \mathbf{C}) = \prod_{d=0}^{\infty} \mathcal{H}_\lambda^\dagger(d). \tag{1.1-20}$$

By our definition $\mathcal{H}_\lambda^\dagger$ is a right $\widehat{\mathfrak{g}}$-module. A decreasing filtration $\{F^p\mathcal{H}_\lambda^\dagger\}$ is defined by

$$F^p\mathcal{H}_\lambda^\dagger = \prod_{d\geq p} \mathcal{H}_\lambda^\dagger(d). \tag{1.1-21}$$

There is a canonical perfect bilinear pairing

$$\langle \quad | \quad \rangle : \mathcal{H}_\lambda^\dagger \times \mathcal{H}_\lambda \longrightarrow \mathbf{C}, \tag{1.1-22}$$

which satisfies the following equality for each $a \in \widehat{\mathfrak{g}}$.

$$\langle u|av\rangle = \langle ua|v\rangle, \quad \text{for all } \langle u| \in \mathcal{H}_\lambda^\dagger \text{ and } |v\rangle \in \mathcal{H}_\lambda\,.$$

Note that the filtrations $\{F_p\}$ and $\{F^p\}$ define a uniform topology on $\mathcal{H}_\lambda$ and $\mathcal{H}_\lambda^\dagger$, respectively. With respect to this topology $\mathcal{H}_\lambda^\dagger$ is complete and is the integrable highest weight right $\widehat{\mathfrak{g}}$-module with the lowest weight λ. Put

$$V_\lambda^\dagger = \{\, \langle v| \in \mathcal{H}_\lambda^\dagger \mid \ \langle v|\widehat{\mathfrak{g}}_- = 0 \,\}.$$

It is easy to show that $V_\lambda^\dagger = \mathcal{H}_\lambda^\dagger(0)$ and $V_\lambda^\dagger$ is the irreducible right $\mathfrak{g}$-module with lowest weight λ. The integrable highest weight right $\widehat{\mathfrak{g}}$-module with lowest weight λ is generated by $V_\lambda^\dagger$ over $\widehat{\mathfrak{g}}_+$ with only one relation

$$\langle\lambda|(X_{-\theta}\otimes\xi)^{\ell-(\theta,\lambda)+1} = 0.$$

Lemma 1.1.9.

$$\begin{aligned} X(m)\mathcal{H}_\lambda(d) &\subset \mathcal{H}_\lambda(d-m) \\ L_m\mathcal{H}_\lambda(d) &\subset \mathcal{H}_\lambda(d-m) \\ \mathcal{H}_\lambda^\dagger(d)X(m) &\subset \mathcal{H}_\lambda^\dagger(d+m) \\ \mathcal{H}_\lambda^\dagger(d)L_m &\subset \mathcal{H}_\lambda^\dagger(d+m). \end{aligned}$$

In the similar manner we can introduce a filtration on the Verma module $\mathcal{M}_\lambda$.

Proposition 1.1.10. *For a root vector $X_\alpha \in \mathfrak{g}_\alpha$ of the simple Lie algebra $\mathfrak{g}$ and any element $f(\xi) \in \mathbf{C}((\xi))$ the actions of $X_\alpha \otimes f(\xi)$ on $\mathcal{H}_\lambda$ and $\mathcal{H}_\lambda^\dagger$ are locally nilpotent. That is, for elements $|\phi\rangle \in \mathcal{H}_\lambda$ and $\langle\psi| \in \mathcal{H}_\lambda^\dagger$ there exists a positive integer m such that for each $n \geq m$ we have*

$$(X_\alpha \otimes f(\xi))^n|\phi\rangle = 0 \quad \langle\psi|(X_\alpha \otimes f(\xi))^n = 0.$$

In particular

$$\sum \frac{1}{n!}(X_\alpha \otimes f(\xi))^n$$

acts on $\mathcal{H}_\lambda$ and $\mathcal{H}_\lambda^\dagger$.

Now let us introduce the left $\widehat{\mathfrak{g}}$-module structure on $\mathcal{H}_\lambda^\dagger$ by

$$X(n)\langle\Phi| := -\langle\Phi|X(-n). \tag{1.1-23}$$

It is easy to check that this indeed defines the left $\widehat{\mathfrak{g}}$-module structure on $\mathcal{H}_\lambda^\dagger$.

Now we give the relationship between the *left* $\widehat{\mathfrak{g}}$-modules $\mathcal{H}_\lambda^\dagger$ and $\mathcal{H}_{\lambda^\dagger}$.

Lemma 1.1.11. *There exists a bilinear pairing*

$$(\quad|\quad) : \mathcal{H}_\lambda \times \mathcal{H}_{\lambda^\dagger} \to \mathbf{C}$$

unique up to a constant multiple such that we have

$$(X(n)u|v) + (u|X(-n)v) = 0$$

for any $X \in \mathfrak{g}$, $n \in \mathbf{Z}$, $|u\rangle \in \mathcal{H}_\lambda$, $|v\rangle \in \mathcal{H}_{\lambda^\dagger}$ and $(\quad|\quad)$ is zero on $\mathcal{H}_\lambda(d) \times \mathcal{H}_{\lambda^\dagger}(d')$, if $d \neq d'$.

Proof. Since $V_\lambda \otimes V_{\lambda^\dagger}$, considered as a $\mathfrak{g}$-module by the diagonal action, contains only one-dimensional trivial $\mathfrak{g}$-module $\mathbf{C}|0_{\lambda,\lambda^\dagger}\rangle$, we have a bilinear form $(\quad|\quad) \in \mathrm{Hom}_{\mathfrak{g}}(V_\lambda \otimes V_{\lambda^\dagger}, \mathbf{C})$ unique up to the constant multiple. Assume that we have a bilinear form $(\quad|\quad) \in \mathrm{Hom}(F_p\mathcal{H}_\lambda \otimes F_p\mathcal{H}_{\lambda^\dagger}, \mathbf{C})$ with desired properties. For an element

$$X(-m)|u\rangle \in F_{p+1}\mathcal{H}_\lambda, \quad |u\rangle \in F_p\mathcal{H}_\lambda, \quad m > 0$$

and an element $|v\rangle \in F_{p+1}\mathcal{H}_{\lambda^\dagger}$ define

$$(X(-m)u|v) = -(u|X(m)v). \tag{1.1-24}$$

Note that since $X(m)|v\rangle \in F_{p+1-m}\mathcal{H}_{\lambda^\dagger}$, the right hand side is defined already. It is easy to show that in this way we can define the bilinear form $(\quad|\quad)$ satisfying the conditions of Lemma 1.1.11. QED.

Corollary 1.1.12. *There is a canonical left* $\mathfrak{g}$*-module isomorphism*

$$\mathcal{H}_\lambda^\dagger \simeq \widehat{\mathcal{H}}_{\lambda^\dagger}$$

where $\widehat{\mathcal{H}}_{\lambda^\dagger}$ *is the completion of* $\mathcal{H}_{\lambda^\dagger}$ *with respect to the filtration* $\{F_p\}$.

§1.2 Conformal blocks attached to N-pointed stable curves

a) N-pointed stable curves

Definition 1.2.1. Data $\mathfrak{X} = (C;\, Q_1, Q_2, \dots, Q_N)$ consisting of a curve C and points $Q_1, \dots, Q_N$ on C are called an *N-pointed stable curve*, if the following conditions are satisfied.

(1) The curve C is a reduced connected complete algebraic curve defined over the complex numbers $\mathbf{C}$. The singularities of the curve C are at worst ordinary double points. That is, C is a semi-stable curve.

(2) $Q_1, Q_2, \dots, Q_N$ are non-singular points of the curve C.

(3) If an irreducible component C_i is a projective line (i.e. Riemann sphere) $\mathbf{P}^1$ (resp. a rational curve with one double point, resp. an elliptic curve), the sum of the number of intersection points of C_i and other components and the number of Q_j's on C_i is at least three (resp. one).

(4) $\dim_{\mathbf{C}} H^1(C, \mathcal{O}_C) = g$.

Note that the above condition (3) is equivalent to saying that $\mathrm{Aut}(\mathfrak{X})$ is a finite group so that $\mathfrak{X}$ has no infinitesimal automorphisms. In the following we often add the following condition (Q) for an N-pointed stable curve $\mathfrak{X}$.

(Q) Each component C_i contains at least one Q_j.

The meaning of the condition (Q) will be clarified in the following Lemma 1.2.5 and Lemma 1.2.6.

Definition 1.2.2. Let C be a curve and Q a non-singular point on C. An *formal neighbourhood* s of C at the point Q is a $\mathbf{C}$-algebra isomorphism

$$s : \widehat{\mathcal{O}}_{C,Q} = \varinjlim \mathcal{O}_{C,Q}/\mathfrak{m}_Q^{n+1} \simeq \mathbf{C}[[\,\xi\,]]/(\xi^{n+1}) \tag{1.2-1}$$

where $\mathfrak{m}_Q$ is the maximal ideal of $\mathcal{O}_{C,Q}$ consisting of germs of holomorphic functions vanishing at Q.

Definition 1.2.3. Data

$$\mathfrak{X} = (C; Q_1, Q_2, \dots, Q_N;\ s_1, s_2, \dots, s_N)$$

are called an *N-pointed stable curve of genus g with formal neighbourhoods*, if

(1) $(C;\, Q_1, Q_2, \ldots, Q_N)$ is an N-pointed stable curve of genus g.
(2) s_j is a formal neighbourhood of C at Q_j.

Let C be a semi-stable curve, that is, C is a reduced curve with at most ordinary double points and proper over $\mathbf{C}$. Let Ω^1_C be a sheaf of Kähler differentials of the curve C and ω_C be the dualizing sheaf of the curve C. Near a singular point P, the curve C is analytically isomorphic to the variety defined by

$$xy = 0.$$

By these coordinates the sheaf Ω^1_C is expressed as

$$\Omega^1_C = (\mathcal{O}_C dx + \mathcal{O}_C dy)/\mathcal{O}_C(xdy + ydx). \tag{1.2-2}$$

On the other hand, near the singular point P the dualizing sheaf ω_C is an invertible sheaf generated by the differential ζ given by dx/x outside $x = 0$ and $-dy/y$ outside $y = 0$. Moreover, outside singular points of the curve C, the sheaves Ω^1_C and ω_C coincide. Thus, we have

$$\Omega^1_C = \mathfrak{m}\omega_C \tag{1.2-3}$$

where $\mathfrak{m}$ is the defining ideal sheaf of the singular points of C. Hence, we have the following exact sequence.

$$0 \to \Omega^1_C \to \omega_C \to \omega_C \otimes \mathcal{O}_{C_{Sing}} \to 0.$$

Let $\nu : \widetilde{C} \to C$ be the normalization of the curve C. We let $\{P_1, \ldots, P_q\}$ be the set of double points of the curve C and for each double point P_i, put $\nu^{-1}(P_i) = \{\, P_{i,+}, P_{i,-} \,\}$. Then, we have the following exact sequence.

$$0 \to \omega_C \to \nu_*\omega_{\widetilde{C}}(\sum_i^k (P_{i,+} + P_{i,-})) \xrightarrow{r} \bigoplus_i^q \mathbf{C} \to 0 \tag{1.2-4}$$

where at each double point P_i, the mapping r is given by

$$\mathrm{res}_{P_{i,+}}(\tau) - \mathrm{res}_{P_{i,-}}(\tau).$$

This means that a local holomorphic section of the dualizing sheaf ω_C is regarded as a local meromorphic section of one-form on $\widetilde{C}$ which has a pole of order one at $P_{i,+}$ and $P_{i,-}$ such that the sum of the residues is zero and holomorphic outside $P_{i,\pm}$'s. In the following we shall often use this interpretation. The following lemma is an easy consequence of this interpretation.

Lemma 1.2.4. *Let τ be a meromorphic section of the dualizing sheaf ω_C holomorphic at the double points. Then the sum of the residues of τ is zero.*

Lemma 1.2.5. *Assume that an N-pointed stable curve*

$$\mathfrak{X} = (C; Q_1, Q_2, \dots, Q_N;\ s_1, s_2, \dots, s_N)$$

with formal neighbourhoods satisfies the condition (Q). By t_j we denote the Laurent expansions at Q_j with respect to a formal parameter $\xi_j = s_j^{-1}(\xi)$. Then, the following homomorphisms are injective.

(1.2-5)

$$t = \oplus t_j : H^0(C, \mathcal{O}(*\sum_{j=1}^{N} Q_j)) \longrightarrow \bigoplus_{j=1}^{N} \mathbf{C}((\xi_j))$$

(1.2-6)

$$t = \oplus t_j : H^0(C, \omega_C(*\sum_{j=1}^{N} Q_j)) \longrightarrow \bigoplus_{j=1}^{N} \mathbf{C}((\xi_j))d\xi_j$$

where ω_C is the dualizing sheaf of the curve C.

By this Lemma $H^0(C, \mathcal{O}(*\sum_{j=1}^{N} Q_j))$ (resp. $H^0(C, \omega_C(*\sum_{j=1}^{N} Q_j))$) can be regarded as a subspace of $\oplus_{j=1}^{N} \mathbf{C}((\xi_j))$ (resp. $\oplus_{j=1}^{N} \mathbf{C}((\xi_j))d\xi_j$). There is the residue pairing

(1.2-7)

$$\begin{array}{lcl} \bigoplus_{j=1}^{N} \mathbf{C}((\xi_j)) \times \bigoplus_{j=1}^{N} \mathbf{C}((\xi_j))d\xi_j & \to & \mathbf{C} \\ ((f(\xi_1), .., f(\xi_N), g(\xi_1)d\xi_1, .., g(\xi_N)d\xi_N) & \mapsto & \sum_{j=1}^{N} \operatorname{Res}_{\xi_j=0}(f(\xi_j)g(\xi_j)d\xi_j). \end{array}$$

The following Lemma is well-known and plays an important role in our theory.

Lemma 1.2.6. *Under the residue pairing (1.2-7) the vector space*

$$H^0(C, \mathcal{O}(*\sum_{j=1}^{N} Q_j))$$

and the vector space

$$H^0(C, \omega_C(*\sum_{j=1}^{N} Q_j))$$

are the annihilators to each other.

b) Conformal blocks

For an N-pointed stable curve $\mathfrak{X} = (C; Q_1, Q_2, \dots, Q_N; \eta_1, \eta_2, \dots, \eta_N)$ with formal neighbourhoods let us define the conformal block (the space of vacua in [TUY]) $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X})$ and its dual space, the space of covacua $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X})$ where $\vec{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_N)$, $\lambda_j \in P_\ell$. For that purpose we first define a generalized affine Lie algebra $\widehat{\mathfrak{g}}_N$.

Definition 1.2.7. Let $\mathfrak{g}$ be a simple Lie algebra over the complex numbers. A Lie algebra $\widehat{\mathfrak{g}}_N$ is defined as

$$\widehat{\mathfrak{g}}_N = \bigoplus_{j=1}^{N} \mathfrak{g} \otimes_{\mathbf{C}} \mathbf{C}((\xi_j)) \oplus \mathbf{C}c$$

with the following commutation relations.

$$[(X_j \otimes f_j), (Y_j \otimes g_j)] = ([X_j, Y_j] \otimes f_j g_j) + c \sum_{j=1}^{N} (X_j, Y_j) \operatorname*{Res}_{\xi_j = 0} (g_j df_j) \tag{1.2-8}$$

where (a_j) means $(a_1, a_2, \ldots, a_N)$ and c belongs to the center of $\widehat{\mathfrak{g}}_N$.

We also put

$$\widehat{\mathfrak{g}}(\mathfrak{X}) = \mathfrak{g} \otimes_{\mathbf{C}} H^0(C, \mathcal{O}_C(* \sum_{j=1}^{N} Q_j)). \tag{1.2-9}$$

By Lemma 1.2.5 we have a natural embedding

$$t : H^0(C, \mathcal{O}_C(* \sum_{j=1}^{N} Q_j)) \hookrightarrow \bigoplus_{j=1}^{N} \mathbf{C}((\xi_j)).$$

We regard $H^0(C, \mathcal{O}_C(* \sum_{j=1}^{N} Q_j))$ as a subspace of $\bigoplus_{j=1}^{N} \mathbf{C}((\xi_j))$. By Lemma 1.2.4 we have the following lemma.

Lemma 1.2.8. *$\widehat{\mathfrak{g}}(\mathfrak{X})$ is a Lie subalgebra of $\widehat{\mathfrak{g}}_N$.*

Let us fix a non-negative integer ℓ. For each $\vec{\lambda} = (\lambda_1, \ldots, \lambda_N) \in (P_\ell)^N$, a left $\widehat{\mathfrak{g}}_N$-module $\mathcal{H}_{\vec{\lambda}}$ and a right $\widehat{\mathfrak{g}}_N$-module $\mathcal{H}^\dagger_{\vec{\lambda}}$ are defined by

$$\mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1} \otimes_{\mathbf{C}} \cdots \otimes_{\mathbf{C}} \mathcal{H}_{\lambda_N}$$
$$\mathcal{H}^\dagger_{\vec{\lambda}} = \mathcal{H}^\dagger_{\lambda_1} \widehat{\otimes}_{\mathbf{C}} \cdots \widehat{\otimes}_{\mathbf{C}} \mathcal{H}^\dagger_{\lambda_N}.$$

For each element $X_j \in \mathfrak{g}$, $f(\xi_j) \in \mathbf{C}((\xi_j))$, the action ρ_j of $X_j[f_j]$ on $\mathcal{H}_{\vec{\lambda}}$ is given by

$$\rho_j(X_j[f_j])|v_1 \otimes \cdots \otimes v_N\rangle = |v_1 \otimes \cdots \otimes v_{j-1} \otimes (X_j[f_j])v_j \otimes v_{j+1} \otimes \cdots v_N\rangle \tag{1.2-10}$$

where

$$|v_1 \otimes \cdots \otimes v_N\rangle$$

means

$$|v_1\rangle \otimes \cdots \otimes |v_N\rangle, \quad |v_j\rangle \in \mathcal{H}_{\lambda_j}.$$

The left $\widehat{\mathfrak{g}}_N$-action is given by

$$(1.2\text{-}11)\quad (X_1 \otimes f_1, \ldots, X_N \otimes f_N)|v_1 \otimes \cdots v_N\rangle = \sum_{j=1}^{N} \rho_j(X_j[f_j])|v_1 \otimes \cdots v_N\rangle.$$

Similarly, the right $\widehat{\mathfrak{g}}_N$-action on $\mathcal{H}^{\dagger}_{\vec{\lambda}}$ is defined by

$$(1.2\text{-}12)\quad \langle u_1 \otimes \cdots u_N|(X_1 \otimes f_1, \ldots, X_N \otimes f_N) = \sum_{j=1}^{N} \langle u_1 \otimes \cdots u_N|\rho_j(X_j[f_j]).$$

As a Lie subalgebra, $\widehat{\mathfrak{g}}(\mathfrak{X})$ operates on $\mathcal{H}_{\vec{\lambda}}$ and $\mathcal{H}^{\dagger}_{\vec{\lambda}}$ as

(1.2-13)

$$(X \otimes f)|v_1 \otimes \cdots \otimes v_N\rangle = \sum_{j=1}^{N} \rho_j(X \otimes t_j(f))|v_1 \otimes \cdots v_N\rangle$$

(1.2-14)

$$\langle u_1 \otimes \cdots \otimes u_N|(X \otimes f) = \sum_{j=1}^{N} \langle u_1 \otimes \cdots \otimes u_N|\rho_j(X \otimes t_j(f)).$$

The pairing $\langle \quad | \quad \rangle$ introduced in (1.1-22) induces a perfect bilinear pairing

$$(1.2\text{-}15)\qquad \langle \quad | \quad \rangle : \mathcal{H}^{\dagger}_{\vec{\lambda}} \times \mathcal{H}_{\vec{\lambda}} \to \mathbf{C}$$
$$(\langle u_1 \otimes \ldots \otimes u_N|, |v_1 \otimes \ldots \otimes u_N\rangle) \to \langle u_1|v_1\rangle\langle u_2|v_2\rangle \cdots \langle u_N|v_N\rangle$$

which is $\widehat{\mathfrak{g}}_N$-invariant.

$$\langle \Psi(X_j \otimes f_j)|\Phi\rangle = \langle \Psi|(X_j \otimes f_j)\Phi\rangle.$$

Now we are ready to define the conformal block attached to $\mathfrak{X}$.

Definition 1.2.9. Assume that $\mathfrak{X}$ enjoys the property (Q) in §1.1. Put

$$(1.2\text{-}16)\qquad \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}) = \mathcal{H}_{\vec{\lambda}}/\widehat{\mathfrak{g}}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}}.$$

The vector space $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X})$ is called the *space of covacua* attached to $\mathfrak{X}$. The *conformal block* (or *space of vacua*) attached to $\mathfrak{X}$ is defined as

$$(1.2\text{-}17)\qquad \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}) = \mathrm{Hom}_{\mathbf{C}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}), \mathbf{C}).$$

In case $\mathfrak{X}$ does not satisfy the property (Q), we use Theorem 1.2.14 below to define the conformal block.

From the definition (1.2-17) the following lemma follows easily.

Lemma 1.2.10.

$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}) = \{\ \langle\Psi| \in \mathcal{H}^{\dagger}_{\vec{\lambda}}\ |\langle\Psi|\ \widehat{\mathfrak{g}}(\mathfrak{X}) = 0\ \}. \tag{1.2-18}$$

Corollary 1.2.11. *Let* $\mathfrak{X}_1 = (C_1; Q_1, \ldots, Q_M; s_1, \ldots, s_M)$ *and* $\mathfrak{X}_2 = (C_2, Q_{M+1}, \ldots, Q_N; s_{M+1}, \ldots, s_N)$ *be* M*-pointed and* $(N-M)$*-pointed stable curves with formal neighbourhoods, respectively. Let*

$$\mathfrak{X} = (C_1 \sqcup C_2; Q_1, \ldots, Q_N; s_1, \ldots, s_N)$$

be the N*-pointed stable curve with formal neighbourhoods obtained from* $\mathfrak{X}_1$ *and* $\mathfrak{X}_2$*. Then, we have a canonical isomorphism*

$$\mathcal{V}^{\dagger}_{\vec{\lambda}_1,\vec{\lambda}_2}(\mathfrak{X}) \simeq \mathcal{V}^{\dagger}_{\vec{\lambda}_1}(\mathfrak{X}_1) \otimes \mathcal{V}^{\dagger}_{\vec{\lambda}_2}(\mathfrak{X}_2).$$

Let us study the conformal blocks when the underlying curve is the Riemann sphere $\mathbf{P}^1$. We regard $\mathbf{P}^1$ as $\mathbf{C} \cup \{\infty\}$ and let z be a global coordinate of $\mathbf{C}$. For a positive integer $N \geq 1$ let us choose N-points $z_1, z_2, \ldots z_N$ of $\mathbf{P}^1$ and put

$$\xi_j = \begin{cases} z - z_j & \text{if } z_j \neq \infty \\ 1/z & \text{if } z_j = \infty. \end{cases}$$

Then $\mathfrak{X} = (\mathbf{P}^1; z_1, \ldots, z_N; \xi_1, \ldots, \xi_N)$ is an N-pointed curve with formal coordinates. Choose $\vec{\lambda} = (\lambda_1, \ldots \lambda_N) \in (P_\ell)^N$ and put

$$V_{\vec{\lambda}} = V_{\lambda_1} \otimes_{\mathbf{C}} \cdots \otimes_{\mathbf{C}} V_{\lambda_N}.$$

We have the following result.

Proposition 1.2.12. *The natural restriction mapping*

$$\mathrm{Hom}_{\mathbf{C}}(\mathcal{H}_{\vec{\lambda}}, \mathbf{C}) \to \mathrm{Hom}_{\mathbf{C}}(V_{\vec{\lambda}}, \mathbf{C}).$$

induces an injective homomorphism

$$j : \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(V_{\vec{\lambda}}, \mathbf{C}).$$

The proposition implies that the space of vacua attached to the Riemann sphere $\mathbf{P}^1$ is finite dimensional. Gabber's theorem [Ga] implies this is true in general. (For details see [TUY] or [U].)

Theorem 1.2.13. $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X})$ *and* $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})$ *are finite-dimensional vector space.*

In §1.3 we shall show that $\dim_{\mathbf{C}} \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})$ depends only on the genus $g(C)$ of the curve C and $\vec{\lambda}$.

Next we shall discuss important properties of conformal blocks. For $\mathfrak{X} = (C; Q_1, \ldots, Q_N; \eta_1, \ldots, \eta_N)$ let P be a non-singular point of the curve C and t a formal parameter of C at P. Put

$$\widetilde{\mathfrak{X}} = (C; Q_1, \ldots, Q_N, Q_{N+1}; \eta_1, \ldots, \eta_N, \eta_{N+1})$$

where $Q_{N+1} = P$ and $\eta_{N+1} = \eta$.

In the following we fix a highest weight vector $|0\rangle$ of the integrable left $\widehat{\mathfrak{g}}$-module $\mathcal{H}_0$. Since there is a canonical inclusion

$$\begin{aligned} \mathcal{H}_{\vec{\lambda}} &\longrightarrow \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_0 \\ |v\rangle &\longrightarrow |v\rangle \otimes |0\rangle \end{aligned}$$

we have a canonical surjection

$$\widehat{\iota}^* \ : \ \mathcal{H}^\dagger_{\vec{\lambda}} \widehat{\otimes} \mathcal{H}^\dagger_0 \longrightarrow \mathcal{H}^\dagger_{\vec{\lambda}} \, .$$

Theorem 1.2.14. *The canonical surjection $\widehat{\iota}^*$ induces a canonical isomorphism*

$$\mathcal{V}^\dagger_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}) \simeq \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}).$$

Corollary 1.2.15. *There is a canonical isomorphism*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}) \simeq \mathcal{V}_{\vec{\lambda},0}(\widetilde{\mathfrak{X}})$$

Next let us consider a singular curve. For an N-pointed stable curve $\mathfrak{X} = (C; Q_1, \ldots, Q_N; \eta_1, \ldots, \eta_N)$ with formal neighbourhoods, assume that the curve C has a double point P. Let $\nu : \widetilde{C} \to C$ be the normalization at the point P. Put $\nu^{-1}(P) = \{P', P''\}$. Furthermore we introduce formal neighbourhoods η' and η'' at P' and P'', respectively.

Proposition 1.2.16. *Under the above notation, for an N-pointed stable curve $\mathfrak{X} = (C; Q_1, \ldots, Q_N; \eta_1, \ldots, \eta_N)$ with formal neighbourhoods, put*

$$\widetilde{\mathfrak{X}} = (\widetilde{C}; P', P'', Q_1, \ldots, Q_N; \eta', \eta'', \eta_1, \ldots, \eta_N).$$

Then there is a canonical isomorphism

$$\bigoplus_{\mu \in P_l} \mathcal{V}^\dagger_{\mu, \mu^\dagger, \vec{\lambda}}(\widetilde{\mathfrak{X}}) \xrightarrow{\sim} \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}).$$

c) Correlation functions

Let $\mathfrak{X} = (C; Q_1, \ldots, Q_N; \eta_1, \ldots, \eta_N)$ be an N-pointed stable curve with formal neighbourhoods. Put

$$\omega_j = \sum_{n \in \mathbf{Z}} \langle \Psi | \rho_j(X(n)) | u \rangle \xi_j^{-n-1} d\xi_j \, , \quad j = 1, 2, \ldots, N + M$$

where $\xi_j = \eta_j^{-1}(\xi)$. For an element $f \in H^0(*\sum_{j=1}^{N+M} Q_j))$ let $f_j(\xi_j) = \sum a_n^{(j)} \xi_j^n$ be the formal Laurent expansion of f at the point Q_j by the formal parameter $\xi_j = \eta_j^{-1}(\xi)$. Hence $t(f) = (f_1(\xi_1), \ldots, f_{N+M}(\xi_{N+M}))$. Then we have

$$\sum_{j=1}^{N+M} \operatorname*{Res}_{\xi_j=0}(f_j(\xi_j)\omega_j) = \sum_{j=1}^{N+M} \sum_{n\in\mathbf{Z}} \langle\Psi|\rho_j(X(n))|u\rangle a_n^{(j)}$$
$$= \langle\Psi|X \otimes t(f)|u\rangle = 0$$

since $\langle\Psi|X \otimes t(f) = 0$ by our assumption. Therefore, by Lemma 1.2.6 there exists an element $\omega \in H^0(C, \omega_C(*\sum_{j=1}^{N+M} Q_j))$ with

$$t(\omega) = (\omega_1, \ldots, \omega_{N+M}).$$

The meromorphic form ω is written as

$$\langle\Psi|X(z)|u\rangle dz$$

and called the *correlation function* or *one point function* of the current $X(z)$.

More generally we can show the following result.

Theorem 1.2.17. *Fix $\langle\Psi| \in \mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X})$. For each non-negative integer M the data*

$$X_1, X_2, \ldots, X_M \in \mathfrak{g}, \quad |\Phi\rangle \in \mathcal{H}_{\vec{\lambda}}$$

define an element

$$F = \langle\Psi|X_1(z_1)X_2(z_2)\ldots X_M(z_M)|\Phi\rangle dz_1 dz_2 \ldots dz_M$$

of

$$H^0\Big(C^M, \omega_{C^M}(\sum_{1\le i<j\le M} *\Delta_{ij} + \sum_{i=1}^{M}\sum_{j=1}^{N} *\pi_i^{-1}(Q_j))\Big),$$

where $\Delta_{ij} = \{(P_1, \ldots, P_N)|P_i = P_j\}$ is the (i,j) diagonal. The meromorphic form has the following properties.

0) *For $M = 0$, $F =< \Psi|\Phi >$ is the canonical pairing induced by the pairing* (1.1-22).

1) *F is linear with respect to $|\Phi\rangle$ and multi-linear with respect to X_i's.*

2) *For any permutation $\sigma \in \mathfrak{S}_M$, we have*

$$F = \langle\Psi|X_{\sigma(1)}(z_{\sigma(1)})X_{\sigma(2)}(z_{\sigma(2)})\cdots X_{\sigma(M)}(z_{\sigma(M)})|\Phi\rangle dz_1 dz_2 \ldots dz_M\,.$$

For example, for a transposition $(i, i+1)$ we have

$$F = \langle\Psi|X_1(z_1)\cdots X_{i-1}(z_{i-1})X_{i+1}(z_{i+1})X_i(z_i)$$
$$X_{i+2}(z_{i+2})\ldots X_M(z_M)|\Phi\rangle dz_1 dz_2 \ldots dz_M\,.$$

3) *For $k = 1, \dots, N$ and $\xi_k = \eta_k^{-1}(\xi)$, if ξ_k is a holomorphic coordinate, then we have the equality*

$$\operatorname*{Res}_{\xi_k=0}(\xi_k^n \langle \Psi | X(\xi_k) X_1(z_1) X_2(z_2) \dots X_M(z_M) | \Phi \rangle d\xi_k) dz_1 \cdots dz_M$$
$$= \langle \Psi | X_1(z_1) X_2(z_2) \cdots X_M(z_M) | \rho_k(X(n)) \Phi \rangle dz_1 \cdots dz_M.$$

In other words, we have an expansion

$$\langle \Psi | X(\xi_k) X_1(z_1) X_2(z_2) \dots X_M(z_M) | \Phi \rangle d\xi_k dz_1 \cdots dz_M$$
$$= \sum_{n \in \mathbf{Z}} \langle \Psi | X_1(z_1) X_2(z_2) \cdots X_M(z_M) | \rho_k(X(n)) \Phi \rangle \xi_k^{-n-1} d\xi_k dz_1 \cdots dz_M$$

4) *For a local holomorphic coordinate z at a nonsingular point P of the curve C, we have the following equality.*

$$\langle \Psi | X(z) Y(w) X_1(z_1) X_2(z_2) \cdots X_M(z_M) | \Phi \rangle$$
$$= \left\{ \frac{\ell \cdot (X, Y)}{(z-w)^2} \langle \Psi | X_1(z_1) X_2(z_2) \cdots X_M(z_M) | \Phi \rangle \right.$$
$$+ \frac{1}{z-w} \langle \Psi | [X, Y](w) X_1(z_1) X_2(z_2) \cdots X_M(z_M) | \Phi \rangle$$
$$\left. + \text{regular at } z = w \right\} dz_1 \cdots dz_M .$$

5) *For a local holomorphic coordinate z at Q_i and for $|v\rangle \in V_{\vec{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N} \subset \mathcal{H}_{\vec{\lambda}}$, we have an equality*

$$\langle \Psi | X(w) X_1(z_1) X_2(z_2) \cdots X_M(z_M) | v \rangle$$
$$= \left\{ \frac{1}{z - z(Q_i)} \langle \Psi | X_1(z_1) X_2(z_2) \cdots X_M(z_M) | \rho_i(X) v \rangle \right.$$
$$\left. + \text{ regular at } z = Q_i \right\} dz_1 \cdots dz_M .$$

The result 4) is often expressed as

$$X(z)Y(w) \sim \frac{\ell \cdot (X,\, Y)}{(z-w)^2} + \frac{[X,\, Y](w)}{z-w}$$

and called *operator product expansion.*

For energy momentum tensor we have similar correlation functions.

Theorem 1.2.18.

1) *Put*

$$\begin{aligned}&\langle\Psi|T(z)X_1(z_1)X_2(z_2)\dots X_M(z_M)|\Phi\rangle dz^2dz_1dz_2\cdots dz_M\\&=\frac{1}{2(g^*+l)}\lim_{w\to z}\left\{\sum_{a=1}^{\dim\mathfrak{g}}\langle\Psi|J^a(z)J^a(w)\right.\\&\qquad X_1(z_1)X_2(z_2)\dots X_M(z_M)|\Phi\rangle dzdwdz_1dz_2\cdots dz_M\\&\left.-\frac{\ell\dim\mathfrak{g}}{(z-w)^2}\langle\Psi|X_1(z_1)X_2(z_2)\dots X_M(z_M)|\Phi\rangle dzdwdz_1dz_2\cdots dz_M\right\}.\end{aligned}$$

Then, this is well-defined and for $k=1,\dots,N$, *we have*

$$\begin{aligned}&\operatorname*{Res}_{\xi_k=0}(\xi_k^{n+1}\langle\Psi|T(\xi_k)X_1(z_1)X_2(z_2)\dots X_M(z_M)|\Phi\rangle d\xi_k)dz_1dz_2\cdots dz_M\\&\qquad=\langle\Psi|X_1(z_1)X_2(z_2)\dots X_M(z_M)|\rho_k(L_n)\Phi\rangle dz_1dz_2\cdots dz_M\end{aligned}$$

where $\{J^1,\dots,J^{\dim\mathfrak{g}}\}$ *is an orthonormal basis of the Lie algebra* $\mathfrak{g}$. *Thus we have an expansion*

$$\begin{aligned}&\langle\Psi|T(\xi_k)X_1(z_1)X_2(z_2)\dots X_M(z_M)|\Phi\rangle d\xi_k^2)dz_1dz_2\cdots dz_M\\&=\sum_{n\in\mathbf{Z}}\langle\Psi|X_1(z_1)X_2(z_2)\dots X_M(z_M)|\rho_k(L_n)\Phi\rangle\xi_k^{-n-2}d\xi_k^2dz_1dz_2\cdots dz_M.\end{aligned}$$

2) *For a holomorphic coordinate transformation* $w=w(z)$ *we have*

$$\begin{aligned}&\langle\Psi|T(w)X_1(z_1)X_2(z_2)\dots X_M(z_M)|\Phi\rangle dw^2dz_1dz_2\cdots dz_M\\&\quad=\langle\Psi|T(z)X_1(z_1)X_2(z_2)\dots X_M(z_M)|\Phi\rangle dz^2dz_1dz_2\cdots dz_M\\&\quad-\frac{c_v}{12}\{w(z);z\}\langle\Psi|X_1(z_1)X_2(z_2)\dots X_M(z_M)|\Phi\rangle dz^2dz_1dz_2\cdots dz_M\end{aligned}$$

where $\{w(z);z\}$ *is the Schwarzian derivative.*

d) $\mathbf{P}^1$ and elliptic curves

In this subsection we shall discuss correlation functions of current and the energy momentum tensor in case of $\mathbf{P}^1$ and elliptic curves.

As above we regard $\mathbf{P}^1$ as $\mathbf{C}\cup\{\infty\}$ and let z be a global coordinate of $\mathbf{C}$. For a positive integer $N\geq 1$ let us choose N-points $z_1,z_2,\dots z_N$ of $\mathbf{P}^1$ and put

$$\xi_j=\begin{cases}z-z_j & \text{if } z_j\neq\infty\\ 1/z & \text{if } z_j=\infty.\end{cases}$$

Let us consider N-pointed $\mathbf{P}^1$ $\mathfrak{X} = (\mathbf{P}^1; z_1, \dots, z_N; \xi_1, \dots, \xi_N)$. formal coordinates. Since there are no holomorphic one-forms on $\mathbf{P}^1$, for $X \in \mathfrak{g}$ and $|v\rangle \in V_{\vec{\lambda}}$ by Theorem 1.2.17, 5), we have

$$\langle \Psi | X(z) | v \rangle dz = \sum_{j=1}^{N} \frac{1}{z - z_j} \langle \Psi | \rho_j(X) v \rangle dz. \tag{1.2-19}$$

Note that if one of z_j, say z_1 is the point at infinity ∞, then in (1.2-19) the term

$$\frac{1}{z - z_j} \langle \Psi | \rho_1(X) v \rangle dz$$

disappears. This is because the residue at the point at infinity ∞ of the form

$$\sum_{j=2}^{N} \frac{1}{z - z_j} \langle \Psi | \rho_j(X) v \rangle dz$$

is

$$-\sum_{j=2}^{N} \langle \Psi | \rho_j(X) v \rangle = \langle \Psi | \rho_1(X) v \rangle$$

by the gauge condition.

For the correlation function of energy momentum tensor, first consider $\langle \Psi | X(z) X(w) | v \rangle dz dw$ for $X \in \mathfrak{g}$. By Theorem 1.2.18 we have

(1.2-20)

$$\begin{aligned} &\langle \Psi | X(z) X(w) | v \rangle dz dw \\ &\quad = \frac{\ell \cdot (X, X)}{(z - w)^2} \langle \Psi | v \rangle dz dw + \sum_{j=1}^{N} \frac{1}{z - z_j} \langle \Psi | X(w) | \rho_j(X) v \rangle dz dw \end{aligned}$$

since there are non holomorphic two-forms on $\mathbf{P}^1 \times \mathbf{P}^1$. By (1.2-19) the equality (1.2-20) is rewritten in the form

(1.2-21)

$$\begin{aligned} \langle \Psi | X(z) X(w) | v \rangle dz dw &= \frac{\ell \cdot (X, X)}{(z - w)^2} \langle \Psi | v \rangle dz dw \\ &\quad + \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{1}{(z - z_j)(w - z_k)} \langle \Psi | \rho_j(X) \rho_k(X) v \rangle dz dw. \end{aligned}$$

Here, again, if $z_1 = \infty$, then the terms containing z_1 should be omitted. By (1.2-21) and Theorem 1.2.18, for a correlation function of the energy

momentum tensor we have the following expression.

(1.2-22)

$$\langle\Psi|T(z)|v\rangle(dz)^2$$
$$= \frac{1}{2(g^*+\ell)}\sum_{j,k}\frac{1}{(z-z_j)(z-z_k)}\cdot\sum_{a=1}^{\dim\mathfrak{g}}\langle\Psi|\rho_j(J^a)\rho_k(J^a)|v\rangle(dz)^2$$

for $\langle\Psi|\in\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X})$ and $|v\rangle\in V_{\vec{\lambda}}$. Put

$$\Omega_{jk}=\sum_{a=1}^{\dim\mathfrak{g}}\rho_j(J^a)\rho_k(J^a) \tag{1.2-23}$$

and

$$\langle\psi|=j(\langle\Psi|)\in\mathrm{Hom}_{\mathfrak{g}}(V_{\vec{\lambda}},\mathbf{C}).$$

Then we have the following expression which will be used below.

$$\langle\Psi|T(z)|v\rangle(dz)^2=\frac{1}{2(g^*+\ell)}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\langle\psi|\Omega_{jk}|v\rangle}{(z-z_j)(z-z_k)}(dz)^2. \tag{1.2-24}$$

If $z_1=\infty$ this formula should read

$$\langle\Psi|T(z)|v\rangle(dz)^2=\frac{1}{2(g^*+\ell)}\sum_{j=2}^{N}\sum_{k=2}^{N}\frac{\langle\psi|\Omega_{jk}|v\rangle}{(z-z_j)(z-z_k)}(dz)^2. \tag{1.2-25}$$

Next let us consider correlation functions for one pointed elliptic curves. Let E be an elliptic curve with period matrix $(1,\tau)$ with $\tau\in H$, where H is the upper half plane:

$$E=\mathbf{C}/(1,\tau).$$

Let us consider a one-pointed curve $\mathfrak{X}=(\,E\,;\,[0]\,;\,z\,)$ of genus 1 with formal coordinate where $[0]$ is the origin of the elliptic curve E and z is a global coordinate of $\mathbf{C}$. The conformal block $\mathcal{V}_{\lambda}^{\dagger}(\mathfrak{X})$ is given by the conditions

$$\langle\Psi|X\otimes\wp^{(n)}(z)=0\quad n=0,1,\dots$$
$$\langle\Psi|X\otimes 1=0$$

where $\wp(z)$ is the Weierstrass $\wp$ function. For an element $\langle\Psi|\in\mathcal{V}_{\lambda}^{\dagger}(\mathfrak{X})$, an element $X\in\mathfrak{g}$ and an element $|v\rangle\in V_{\lambda}$, the one form $\langle\Psi|X(z)|v\rangle dz$ has the expansion

$$\langle\Psi|X(z)|v\rangle dz$$
$$=\sum_{n\in\mathbf{Z}}\langle\Psi|X(n)v\rangle z^{-n-1}dz$$
$$=\sum_{n=0}^{\infty}\langle\Psi|X(-n)v\rangle z^{n-1}dz.$$

Since there is no meromorphic one form on the elliptic curve E which has a pole of order one at the origin and holomorphic outside the origin, the form $\langle\Psi|X(z)|v\rangle dz$ is holomorphic. Therefore, we have

$$\langle\Psi|X(z)|v\rangle dz = \langle\Psi|X(-1)|v\rangle dz \tag{1.2-26}$$

and if $n \neq 1$ we have

$$\langle\Psi|X(-n)|v\rangle = 0. \tag{1.2-27}$$

By using Theorem 1.2.17 we have

$$\langle\Psi|X(z)X(w)|v\rangle dzdw = \langle\Psi|X(-1)Y(-1)v\rangle dzdw$$

for $\langle\Psi| \in \mathcal{V}_\lambda^\dagger(\mathfrak{X})$, $X, \in \mathfrak{g}$ and an element $|v\rangle \in V_\lambda$. Hence, by Theorem 1.2.18 we also have the following result. For $|v\rangle \in V_\lambda$ we have

$$\langle\Psi|T(z)|v\rangle dz^2 = \frac{1}{2(g^* + \ell)} \sum_{a=1}^{\dim \mathfrak{g}} \langle\Psi|J^a(-1)J^a(-1)|v\rangle dz^2.$$

Again by Theorem 1.2.18, 1) we also have

$$\begin{aligned}
\langle\Psi|T(z)|v\rangle dz^2 &= \sum_{n\in\mathbf{Z}} \langle\Psi|L_n v\rangle z^{-n-2} dz^2 \\
&= \langle\Psi|L_0 v\rangle \wp(z) dz^2 + \langle\Psi|L_{-2} v\rangle dz^2 + \cdots \\
&= \Delta_\lambda \langle\Psi|v\rangle \wp(z) dz^2 + \langle\Psi|L_{-2} v\rangle dz^2 + \cdots \\
&= \langle\Psi|L_{-2} v\rangle dz^2 \qquad (1.2\text{-}28)
\end{aligned}$$

In particular

$$\langle\Psi|L_{-1} v\rangle = 0.$$

§1.3 Sheaf of conformal blocks

In this section we shall define the sheaf of conformal blocks attached to a family of N-pointed stable curves with formal coordinates and show that it is coherent and locally free.

a) Sheaf of conformal blocks

Definition 1.3.1. Data $(\pi : Y \to B; s_1, s_2, \ldots, s_N; \eta_1, \eta_2, \ldots, \eta_N)$ are called a (holomorphic) family of N-pointed stable curves of genus g with formal neighbourhoods, if the following conditions are satisfied.

(1) Y and B are connected complex manifolds, $\pi : Y \to B$ is a proper flat holomorphic map and $s_1, s_2, \ldots, s_N$ are holomorphic sections of π.

(2) For each point $b \in B$ the data $(Y_b := \pi^{-1}(b); s_1(b), s_2(b), \ldots, s_N(b))$ is an N-pointed stable curve of genus g.

(3) η_j is an $\mathcal{O}_B$-algebra isomorphism

$$\eta_j : \widehat{\mathcal{O}}_{/s_j} = \varprojlim_{n\to\infty} \mathcal{O}_Y/I_j^n \simeq \mathcal{O}_B[[\xi]],$$

where I_j is the defining ideal of $s_j(B)$ in Y.

Definition 1.3.2. A family $\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B};\ s_1, s_2, \ldots, s_N)$ of N-pointed stable curves of genus g is called to be *versal* (resp. *universal*) at a point $b \in \mathcal{B}$, if for any deformation $\mathfrak{Y} = (\pi : X \to Y; s_1, \ldots, s_N)$ of $\pi^{-1}(b) = (C; Q_1, \ldots, Q_N)$ with prescribed point $y \in Y$ there exists a holomorphic mapping (resp. unique holomorphic mapping) f from a neighbourhood of y in Y to $\mathcal{B}^{(n)}$ such that the pullback $f^*\mathfrak{X}$ is isomorphic to $\mathfrak{Y}$ in a neighbourhood of y in Y and that df is uniquely determined at the base point. If the family is versal (resp. universal) at each point of $\mathcal{B}$, the family $\mathfrak{X}$ is called a versal (resp. universal) family.

Let $\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_N; \eta_1, \ldots, \eta_N)$ be a family of N-pointed stable curves of genus g with formal coordinates. We assume that $\mathcal{B}$ is a finite dimensional complex manifold and that each fibre of the family $\mathfrak{F}$ satisfies the condition (Q) in §1.2, but we do *not* assume that the family is connected. The main purpose of the present section is to define the sheaf of conformal blocks $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F})$ attached to the family $\mathfrak{F}$ and show that it is a coherent $\mathcal{O}_{\mathcal{B}}$-module.

Definition 1.3.3. The sheaf $\widehat{\mathfrak{g}}_N(\mathcal{B})$ of affine Lie algebra over $\mathcal{B}$ is a sheaf of $\mathcal{O}_{\mathcal{B}}$-module

$$\widehat{\mathfrak{g}}_N(\mathcal{B}) = \mathfrak{g} \otimes_{\mathbf{C}} (\bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}((\xi_j))) \oplus \mathcal{O}_{\mathcal{B}} \cdot c$$

with the following commutation relation, which is $\mathcal{O}_{\mathcal{B}}$-bilinear.

$$\begin{aligned}[(X_1 \otimes f_1, \ldots, &X_N \otimes f_N), (Y_1 \otimes g_1, \ldots, Y_N \otimes g_N)] \\ &= ([X_1, Y_1] \otimes (f_1 g_1), \ldots, [X_N, Y_N] \otimes (f_N g_N)) \\ &\quad \oplus c \cdot \sum_{j=1}^{N} (X_j, Y_j) \operatorname*{Res}_{\xi_j = 0} (g_j df_j)\end{aligned}$$

$$c \in \text{Center}$$

where

$$X_j,\ Y_j \in \mathfrak{g}, \quad f_j,\ g_j \in \mathcal{O}_{\mathcal{B}}((\xi_j))\,.$$

Put

$$\widehat{\mathfrak{g}}(\mathfrak{F}) = \mathfrak{g} \otimes_{\mathbf{C}} \pi_*(\mathcal{O}_{\mathcal{C}}(*S)) \tag{1.3-1}$$

where we define

$$S = \sum_{j=1}^{N} s_j(\mathcal{B})$$
$$\pi_*(\mathcal{O}_{\mathcal{C}}(*S)) = \varinjlim_{k} \pi_*(\mathcal{O}_{\mathcal{C}}(kS)) .$$

There is a sheaf version of homomorphism defined in (1.2-5), by using the formal neighbourhoods η_j.

$$\tilde{t} : \pi_*(\mathcal{O}_{\mathcal{B}}(*S)) \to \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}((\xi_j))$$

and we may regard $\widehat{\mathfrak{g}}(\mathfrak{F})$ as a Lie subalgebra of $\widehat{\mathfrak{g}}_N(\mathcal{B})$.

Fix a non-negative integer ℓ. For any $\vec{\lambda} = (\lambda_1, \dots, \lambda_N) \in (P_\ell)^N$, put

$$\mathcal{H}_{\vec{\lambda}}(\mathcal{B}) = \mathcal{O}_{\mathcal{B}} \otimes_{\mathbf{C}} \mathcal{H}_{\vec{\lambda}}, \tag{1.3-2}$$

$$\mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B}) = \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{B}}}(\mathcal{H}_{\vec{\lambda}}(\mathcal{B}), \mathcal{O}_{\mathcal{B}}) = \mathcal{O}_{\mathcal{B}} \otimes_{\mathbf{C}} \mathcal{H}^{\dagger}_{\vec{\lambda}}. \tag{1.3-3}$$

The pairing (1.1-22) induces an $\mathcal{O}_{\mathcal{B}}$-bilinear pairing

$$\langle \quad | \quad \rangle : \mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B}) \times \mathcal{H}_{\vec{\lambda}}(\mathcal{B}) \to \mathcal{O}_{\mathcal{B}}. \tag{1.3-4}$$

The sheaf of affine Lie algebra $\widehat{\mathfrak{g}}_N(\mathcal{B})$ acts on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ and $\mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B})$ by

(1.3-5)

$$((X_1 \otimes \sum_{n\in\mathbf{Z}} a_n^{(1)} {\xi_1}^n), \dots, (X_N \otimes \sum_{n\in\mathbf{Z}} a_n^{(N)} {\xi_N}^n))(F \otimes |\Psi\rangle)$$
$$= \sum_{j=1}^{N} \sum_{n\in\mathbf{Z}} (a_n^{(j)} F) \otimes \rho_j(X_j(n))|\Psi\rangle$$

The action of $\widehat{\mathfrak{g}}_N(\mathcal{B})$ on $\mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B})$ is the dual action of $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$, that is,

$$\langle \Psi a | \Phi \rangle = \langle \Psi | a\Phi \rangle \quad \text{for any} \quad a \in \widehat{\mathfrak{g}}_N. \tag{1.3-6}$$

Definition 1.3.4. Put

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) = \mathcal{H}_{\vec{\lambda}}(\mathcal{B})/\widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$$
$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}) = \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{B}}}(\mathcal{V}_{\vec{\lambda}}(\mathcal{B}), \mathcal{O}_{\mathcal{B}}).$$

These are sheaves of $\mathcal{O}_{\mathcal{B}}$-modules on $\mathcal{B}$. The sheaf $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ is called the *sheaf of conformal blocks* attached to the family $\mathfrak{F}$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is called the *sheaf of covacua*. Note that we have

$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}) = \{\, \langle\Psi| \in \mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}) \mid \ \langle\Psi|a = 0 \quad \text{for any} \ \ a \in \widehat{\mathfrak{g}}(\mathfrak{F}) \,\}.$$

The pairing (1.3-4) induces an $\mathcal{O}_{\mathcal{B}}$-bilinear pairing

$$\langle \quad | \quad \rangle : \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}) \times \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \to \mathcal{O}_{\mathcal{B}}. \tag{1.3-7}$$

Lemma 1.3.5. *For a point $s \in \mathcal{B}$ put*

$$\mathfrak{F}_s := (\pi^{-1}(s); s_1(s), \ldots, s_N(s); \eta_1|_{\pi^{-1}(s)}, \ldots, \eta_N|_{\pi^{-1}(s)})$$
$$\mathbf{C}_s := \mathcal{O}_{\mathcal{B},s}/\mathfrak{m}_s$$

where $\mathfrak{m}_s$ is the maximal ideal of the stalk $\mathcal{O}_{\mathcal{B},s}$. Then, we have the following canonical isomorphisms.

$$\mathbf{C}_s \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{H}_{\vec{\lambda}}(\mathcal{B}) \simeq \mathcal{H}_{\vec{\lambda}}$$
$$\mathbf{C}_s \otimes_{\mathcal{O}_{\mathcal{B}}} \widehat{\mathfrak{g}}_N(\mathcal{B}) \simeq \widehat{\mathfrak{g}}_N$$
$$\widehat{\mathfrak{g}}(\mathfrak{F}) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathbf{C}_s \simeq \widehat{\mathfrak{g}}(\mathfrak{F}_s)$$
$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathbf{C}_s \simeq \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_s)$$
$$\mathbf{C}_s \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B}) \simeq \mathcal{H}^{\dagger}_{\vec{\lambda}}.$$

More generally, for a holomorphic mapping $f : Y \to \mathcal{B}$ we let $\mathfrak{F}_Y$ be the pull-back of the family $\mathfrak{F}$ by the morphism f. Then, we have the following canonical isomorphisms.

$$\mathcal{O}_Y \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{H}_{\vec{\lambda}}(\mathcal{B}) \simeq \mathcal{H}_{\vec{\lambda}}(Y)$$
$$\widehat{\mathfrak{g}}_N(\mathcal{B}) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_Y \simeq \widehat{\mathfrak{g}}_N(Y)$$
$$\widehat{\mathfrak{g}}(\mathfrak{F}) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_Y \simeq \widehat{\mathfrak{g}}(\mathfrak{F}_Y)$$
$$\mathcal{O}_Y \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \simeq \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_Y)$$
$$\mathcal{O}_Y \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B}) \simeq \mathcal{H}^{\dagger}_{\vec{\lambda}}(Y).$$

Moreover, the actions of $\widehat{\mathfrak{g}}_N(\mathcal{B})$ and $\widehat{\mathfrak{g}}_N(Y)$ on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$, $\mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B})$, $\mathcal{H}_{\vec{\lambda}}(Y)$ and $\mathcal{H}^{\dagger}_{\vec{\lambda}}(Y)$ defined in (1.3-5) *and* (1.3-6) *and the action of $\widehat{\mathfrak{g}}_N$ on $\mathcal{H}_{\vec{\lambda}}$ are compatible with respect to the above canonical isomorphisms.*

Note that a priori it is not clear that the natural mapping

$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathbf{C}_s \to \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}_s)$$

is isomorphic. This is the case if the sheaf $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is locally free. We shall show later this fact.

For our family, Theorem 1.2.11 takes the following form.

Theorem 1.3.6. *The sheaves $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ and $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ are coherent $\mathcal{O}_{\mathcal{B}}$-modules.*

b) Local freeness of sheaf of conformal blocks

In this subsection we shall prove local freeness of the sheaf of conformal blocks $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ over the locus $\mathcal{B} \setminus D$ of smooth curves.(For the notation see below.) For that purpose first we introduce a certain $\mathcal{O}_{\mathcal{B}}$-submodule $\mathcal{L}(\mathfrak{F})$ of $\bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}((\xi_j^{-1}))\dfrac{d}{d\xi_j}$ and an action of $\mathcal{L}(\mathfrak{F})$ on the sheaves of conformal blocks and covacua as first order twisted differential operators. In the next section this action will be used to define a projectively flat connection on the sheaf of conformal blocks.

Let $\mathfrak{F}^{(0)} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \dots, s_N)$ be a *versal family* of N-pointed stable curves of genus g. We let Σ be the locus of double points of the fibres of $\mathfrak{F}^{(0)}$ and D be $\pi(\Sigma)$. Note that Σ is a no-singular submanifold of codimension two in $\mathcal{C}$ and D is a divisor in $\mathcal{B}$ whose irreducible components D_i, $i = 1, 2, \dots, k$ are non-singular. Assume that formal coordinates

$$\eta_j : \widehat{\mathcal{O}}_{\mathcal{C}/s_j(\mathcal{B})} \simeq \mathcal{O}_{\mathcal{B}}[[\ \xi\ ]], \qquad j = 1, 2, \dots, N$$

are given. For simplicity, in the following we assume that $\eta_j^{-1}(\xi)$ is holomorphic in a neighbourhood of $s_j(\mathcal{B})$. The general case can be treated by approximating formal coordinates by holomorphic ones. We use the following notation freely.

$$S_j = s_j(\mathcal{B}), \qquad S = \sum_{j=1}^{N} S_j, \qquad \xi_j = \eta_j^{-1}(\xi).$$

Now $\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \dots, s_N; \eta_1, \dots, \eta_N)$ is a family of N-pointed stable curves of genus g with formal neighbourhoods. For each $\vec{\lambda} \in (P_\ell)^{\oplus N}$ we can define the sheaf of conformal blocks $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ and the sheaf of covacua $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$. These are a subsheaf of $\mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathfrak{B})$ and a quotient sheaf of $\mathcal{H}_{\vec{\lambda}}(\mathfrak{B})$, respectively.

First recall that we have the following exact sequence of $\mathcal{O}_{\mathcal{B}}$-modules.

$$0 \to \pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(*S)) \xrightarrow{b} \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}[\ \xi_j^{-1}\ ]\frac{d}{d\xi_j} \xrightarrow{\vartheta} R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(-S) \to 0 \tag{1.3-8}$$

More precisely, for any positive integer $m \geq 4g - 3$ we have the following exact sequence.

$$0 \to \pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(mS)) \xrightarrow{b_m} \bigoplus_{j=1}^{N}\bigoplus_{k=0}^{m} \mathcal{O}_{\mathcal{B}}\xi_j^{-k}\frac{d}{d\xi_j} \xrightarrow{\vartheta_m} R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(-S) \to 0 \tag{1.3-9}$$

Note that the mappings b and b_m correspond to the Laurent expansions with respect to ξ_j up to the zero-th order. To define the first order twisted differential operators acting on the sheaves of conformal blocks and covacua, we need to modify the exact sequence (1.3-8) in the following way.

There is an exact sequence

$$0 \to \Theta_{\mathcal{C}/\mathcal{B}} \to \Theta_{\mathcal{C}} \overset{d\pi}{\to} \pi^*\Theta_{\mathcal{B}} \to 0$$

where $\Theta_{\mathcal{C}/\mathcal{B}}$ is a sheaf of vector fields tangent to the fibres of π. Put

$$\Theta'_{\mathcal{C},\pi} = d\pi^{-1}(\pi^{-1}\Theta_{\mathcal{B}}(-\log D)).$$

Hence, $\Theta'_{\mathcal{C},\pi}$ is a sheaf of vector field on $\mathcal{C}$ tangent along Σ whose vertical components are constant along the fibres of π. That is, in a neighbourhood of a smooth point of a fibre $\Theta'_{\mathcal{C},\pi}$ consists of germs of holomorphic vector fields of the form

$$a(z,u)\frac{\partial}{\partial z} + \sum_{i=1}^{m} b_i(u)u_i\frac{\partial}{\partial u_i} + \sum_{i=m+1}^{n} b_i(u)\frac{\partial}{\partial u_i}$$

where $(z, u_1, \ldots, u_n)$ is a system of local coordinates such that the mapping π is expressed as the projection

$$\pi(z, u_1, \ldots, u_n) = (u_1, \ldots, u_n)$$

and $\pi(\Sigma) = D$ is given by the equation

$$u_1 \cdot u_2 \cdots u_m = 0.$$

More generally, we can define a sheaf $\Theta'_{\mathcal{C}}(mS)_\pi$ as the one consisting of germs of meromorphic vector fields of the form

$$A(z,u)\frac{\partial}{\partial z} + \sum_{i=1}^{m} B_i(u)u_i\frac{\partial}{\partial u_i} + \sum_{i=m+1}^{n} B_i(u)\frac{\partial}{\partial u_i}$$

where $A(z,u)$ has the poles of order at most m along S. Now we have an exact sequence

$$0 \to \Theta_{\mathcal{C}/\mathcal{B}}(mS) \to \Theta'_{\mathcal{C}}(mS)_\pi \overset{d\pi}{\to} \pi^{-1}\Theta_{\mathcal{B}}(-\log D) \to 0. \tag{1.3-10}$$

Note that $\Theta'_{\mathcal{C}}(mS)_\pi$ has a structure of a sheaf of Lie algebras by the usual bracket operation of vector fields and the above exact sequence is the one as sheaves of Lie algebras.

For $m > \frac{1}{N}(2g-2)$, by (1.3-10) we have an exact sequence of $\mathcal{O}_{\mathcal{B}}$-modules.

$$0 \to \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(mS) \to \pi_*\Theta'_{\mathcal{C}}(mS)_\pi \overset{d\pi}{\to} \Theta_{\mathcal{B}}(-\log D) \to 0 \tag{1.3-11}$$

which is also an exact sequence of sheave of Lie algebras. Taking $m \to \infty$ we obtain the exact sequence

$$0 \to \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*S) \to \pi_*\Theta'_{\mathcal{C}}(*S)_\pi \overset{d\pi}{\to} \Theta_{\mathcal{B}}(-\log D) \to 0. \tag{1.3-12}$$

The exact sequences (1.3-8) and (1.3-12)are related by the following commutative diagram.

$$\begin{array}{ccccccccc}
0 \to & \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*S) & \to & \pi_*\Theta'_{\mathcal{C}}(*S)_\pi & \overset{d\pi}{\to} & \Theta_{\mathcal{B}}(-\log D) & \to 0 \\
& \| & & \downarrow\ p & & \downarrow\ \rho & \\
0 \to & \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*S) & \to & \bigoplus_{j=1}^N \mathcal{O}_{\mathcal{B}}[\ \xi_j^{-1}]\dfrac{d}{d\xi_j} & \overset{\vartheta}{\to} & R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(-S) & \to 0
\end{array}$$

where ρ is the Kodaira-Spencer mapping of the family $\mathfrak{F}^{(0)}$ and p is given by taking the non-positive part of the $\dfrac{d}{d\xi_j}$ part of the Laurent expansions of the vector fields in $\pi_*\Theta_{\mathcal{C}}(mS)_\pi$ at $s_j(\mathcal{B})$. Since our family $\mathfrak{F}^{(0)}$ is versal, the Kodaira-Spencer mapping ρ is an isomorphism of $\mathcal{O}_{\mathcal{B}}$-modules. Therefore, p is isomorphic. By taking the Laurent expansion at the section $s_j(\mathcal{B})$ with respect the formal coordinate ξ_j, we have an $\mathcal{O}_{\mathcal{B}}$-homomorphism

$$\widetilde{p}\colon \pi_*\Theta_{\mathcal{C}}(*S)_\pi \to \bigoplus_{j=1}^N \mathcal{O}_{\mathcal{B}}((\ \xi_j\ ))\frac{d}{d\xi_j},$$

which is a natural extension of the homomorphism p. Put

$$\mathcal{L}(\mathfrak{F}) := \mathrm{Im}(\pi_*\Theta_{\mathcal{C}}(*S)_\pi) \subset \bigoplus_{j=1}^N \mathcal{O}_{\mathcal{B}}((\ \xi_j\ ))\frac{d}{d\xi_j}.$$

Then, we have the following exact sequence.

$$0 \to \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*S) \to \mathcal{L}(\mathfrak{F}) \overset{\theta}{\to} \Theta_{\mathcal{B}}(-\log D) \to 0 \tag{1.3-13}$$

of $\mathcal{O}_{\mathcal{B}}$-modules. The Lie bracket $[\ \ ,\ \ ]_d$ of $\mathcal{L}(\mathfrak{F})$ is obtained by that of $\pi_*\Theta'_{\mathcal{C}}(*S)_\pi)$ by the mapping p. Thus, for $\vec{\ell}$, $\vec{m} \in \mathcal{L}(\mathfrak{F})$ we have

$$[\vec{\ell}, \vec{m}]_d = [\vec{\ell}, \vec{m}]_0 + \theta(\vec{\ell})(\vec{m}) - \theta(\vec{m})(\vec{\ell}) \tag{1.3-14}$$

where $[\ ,\ ]_0$ is the usual bracket of formal vector fields and the action of $\theta(\vec{\ell})$ on

$$\vec{m} = (m_1 \frac{d}{d\xi_1}, \dots, m_N \frac{d}{d\xi_N})$$

is defined by

$$(\theta(\vec{\ell})(m_1)\frac{d}{d\xi_1}, \dots, \theta(\vec{\ell})(m_N)\frac{d}{d\xi_N}).$$

Then, the exact sequence (1.3-13) is also that of sheaves of Lie algebras.

Let us define an action of $\mathcal{L}(\mathfrak{F})$ on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$.

Definition 1.3.7. *For $\vec{\ell} = (\underline{l}_1, \dots, \underline{l}_N) \in \mathcal{L}(\mathfrak{F})$, the action $D(\vec{\ell})$ on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ is defined by*

$$D(\vec{\ell})(F \otimes |\Phi\rangle) = \theta(\vec{\ell})(F) \otimes |\Phi\rangle - F \cdot (\sum_{j=1}^{N} \rho_j(T[\underline{l}_j])|\Phi\rangle \tag{1.3-15}$$

where

$$F \in \mathcal{O}_{\mathcal{B}}, \quad |\Phi\rangle \in \mathcal{H}_{\vec{\lambda}}.$$

The following proposition is an easy consequence of the definition and (1.1-10).

Proposition 1.3.8. *The action $D(\vec{\ell})$ of $\vec{\ell} \in \mathcal{L}(\mathfrak{F})$ on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ defined above has the following properties.*

1) *For any $f \in \mathcal{O}_{\mathcal{B}}$ we have*

$$D(f\vec{\ell}) = fD(\vec{\ell}).$$

2) *For $\vec{\ell}$, $\vec{m} \in \mathcal{L}(\mathfrak{F})$ we have*

$$[\, D(\vec{\ell}), D(\vec{m})\,] = D([\,\vec{\ell}, \vec{m}\,]_d) + \frac{c_v}{12} \sum_{j=1}^{N} \operatorname*{Res}_{\xi_j=0} \left(\frac{d^3 \ell_j}{d\xi_j^3} m_j d\xi_j \right) \cdot id.$$

3) *For $f \in \mathcal{O}_{\mathcal{B}}$ and $|\phi\rangle \in \mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ we have*

$$D(\vec{\ell})(f|\phi\rangle) = (\theta(\vec{\ell})(f))|\phi\rangle + fD(\vec{\ell})|\phi\rangle.$$

Namely, $D(\vec{\ell})$ is a first order differential operator, if $\theta(\vec{\ell}) \neq 0$.

We define the dual action of $\mathcal{L}(\mathfrak{F})$ on $\mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B})$ by

$$D(\vec{\ell})(F \otimes \langle\Psi|) = (\theta(\vec{\ell})F) \otimes \langle\Psi| + \sum_{j=1}^{N} F \cdot \langle\Psi|\rho_j(T[\underline{l}_j]). \tag{1.3-16}$$

where

$$F \in \mathcal{O}_{\mathcal{B}}, \qquad \langle \Psi | \in \mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B}).$$

Then, for any $|\widetilde{\Phi}\rangle \in \mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ and $\langle \widetilde{\Psi}| \in \mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B})$, we have

$$\{D(\vec{\ell})\langle \widetilde{\Psi}|\}|\widetilde{\Phi}\rangle + \langle \widetilde{\Psi}|\{D(\vec{\ell})|\widetilde{\Phi}\rangle\} = \theta(\vec{\ell})\langle \widetilde{\Psi}|\widetilde{\Phi}\rangle. \tag{1.3-17}$$

This agrees with the usual definition of the dual connection. See also Proposition 1.3.14 and (1.3-21) below.

Now we shall show that the operator $D(\vec{\ell})$ acts on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$.

Proposition 1.3.9. *For any $\vec{\ell} \in \mathcal{L}(\mathfrak{F})$ we have*

$$D(\vec{\ell})(\widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathcal{B})) \subset \widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathcal{B}).$$

Hence, $D(\vec{\ell})$ operates on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$. Moreover, it is a first order differential operator, if $\theta(\vec{\ell}) \neq 0$.

Proof. An element of $\widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ is a linear combination of elements of the form

$$F \otimes (\sum_{j=1}^{N} \rho_j(X \otimes t_j(h))|\Phi\rangle)$$

where

$$F \in \mathcal{O}_{\mathcal{B}}, \quad X \in \mathfrak{g}, \quad h \in \pi_* \mathcal{O}_{\mathcal{C}}(*S), \quad |\Phi\rangle \in \mathcal{H}_{\vec{\lambda}}$$

and $t_j(h)$ is the Laurent expansion of h at $S_j = s_j(\mathcal{B})$ with respect to the parameter ξ_j. First we shall show the following equality as operators on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$.

$$[\, D(\vec{\ell}), \sum_{j=1}^{N} \rho_j(X \otimes t_j(h))\,] = \sum_{j=1}^{N} \rho_j\left(X \otimes \left\{\theta(\vec{\ell})(t_j(h)) + \underline{\ell}_j(t_j(h))\right\}\right) \tag{1.3-18}$$

where $\theta(\vec{\ell})$ operates on the coefficients of $t_j(h)$. By Proposition 1.3.8, 3) it is enough to show the equality (1.3-18) as operators on $\mathcal{H}_{\vec{\lambda}}$. For $|\Phi\rangle \in \mathcal{H}_{\vec{\lambda}}$, by (1.1-16) we have

$$\begin{aligned} D(\vec{\ell})(\sum_{j=1}^{N} \rho_j(X \otimes t_j(h))|\Phi\rangle) &- \sum_{j=1}^{N} \rho_j(X \otimes t_j(h))(D(\vec{\ell})|\Phi\rangle) \\ &= \sum_{j=1}^{N} \{\rho_j(X \otimes \theta(\vec{\ell})(t_j(h)) - T[\underline{\ell}_j]\rho_j(X \otimes t_j(h))\}|\Phi\rangle \\ &\qquad + \sum_{j=1}^{N} \rho_j(X \otimes t_j(h))T[\underline{\ell}_j]|\Phi\rangle \\ &= \sum_{j=1}^{N} \{\rho_j(X \otimes \theta(\vec{\ell})(t_j(h)) + \rho_j(X \otimes \underline{\ell}_j(t_j(h)))\}|\Phi\rangle. \end{aligned}$$

But $\theta(\vec{\ell})(t_j(h)) + \underline{\ell}_j(t_j(h))$ is nothing but a Laurent expansion at $s_j(\mathcal{B})$ of a meromorphic function $\tau(h)$ where $\tau = p^{-1}(\vec{\ell}) \in \pi_*(\Theta'_{\mathcal{C}}(*S)_\pi$. Hence, we have the desired result. QED.

Similarly we can show that $D(\vec{\ell})$ acts on $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F})$.

Corollary 1.3.10. *The $\mathcal{O}_{\mathcal{B}}$-module $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is locally free on $\mathcal{B} \setminus D$.*

Proof. By Lemma 1.3.5, for any point $x \in \mathcal{B} \setminus D$ we have an isomorphism

$$\mathbf{C}_x \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \simeq \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_x)$$

where we put

$$\mathfrak{F}_x := (\, C_x = \pi^{-1}(x), s_1(x), \ldots, s_N(x); \eta_1|_{C_x}, \ldots, \eta_N|_{C_x}\,).$$

Let $v_1, \ldots, v_m$ be local holomorphic sections of $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ in a neighbourhood of x such that $\{v_1(x), \ldots, v_m(x)\}$ is a basis of $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_x)$. Suppose that there is a non-trivial relation

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0 \tag{1.3-19}$$

where a_i's are holomorphic function in a neighbourhood of x. By our assumption

$$a_1(x) = a_2(x) = \cdots = a_m(x) = 0.$$

Changing the order of suffices if necessary , we may assume that there is a positive integer k such that

$$\begin{aligned} &a_1 \in \mathfrak{m}_x^k \setminus \mathfrak{m}_x^{k+1}, \\ &a_i \in \mathfrak{m}_x^l, \quad l \geq k, \quad i = 2, 3, \ldots, m. \end{aligned}$$

We choose a_i's in such a way that the positive number k is the smallest among the relations (1.3-19). Let τ be a nowhere vanishing local holomorphic vector field in a neighbourhood of x such that $\tau(a_1) \in \mathfrak{m}_x^{k-1}$. There exists $\vec{\ell} \in \mathcal{L}(\mathfrak{F})$ with $\theta(\vec{\ell}) = \tau$. Applying $\tau = \theta(\vec{\ell})$ to the equality (1.3-19) we obtain the equality

$$\sum_{i=1}^m (\tau(a_i) + \sum_{j=1}^m \alpha_{ji}) v_i = 0 \tag{1.3-20}$$

where

$$D(\vec{\ell})(v_i) = \sum_{j=1}^m \alpha_{ij} v_j.$$

Then, the relation (1.3-20) is non trivial and

$$\tau(a_1) + \sum_{j=1}^{m} \alpha_{j1} \in \mathfrak{m}_x^{k-1}.$$

This contradicts our assumption. Therefore, $v_1, \ldots, v_m$ are $\mathcal{O}_{\mathcal{B}}$-linearly independent. Hence, $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is a locally free $\mathcal{O}_{\mathcal{B}}$-module at x. QED.

For a coherent $\mathcal{O}_{\mathcal{B}}$-module $\mathcal{G}$, the locus M consisting of points at which $\mathcal{G}$ is not locally free is a closed analytic subset of $\mathcal{B}$ of codimension at least 2. Therefore, we have the following corollary.

Corollary 1.3.11. *Let W be the maximal subset of $\mathcal{B}$ over which $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is not locally free. Then, W is an analytic subset of $\mathcal{B}$ and*

$$W \subsetneqq D.$$

Since we defined

$$\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F}) = \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{B}}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}), \mathcal{O}_{\mathcal{B}})$$

we have the following corollary.

Corollary 1.3.12. *$\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F})|_{\mathcal{B}\setminus D}$ is a locally free $\mathcal{O}_{\mathcal{B}}$-module and for any subvariety Y of $\mathcal{B} \setminus D$ we have an $\mathcal{O}_Y$-module isomorphism*

$$\mathcal{O}_Y \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F}) \simeq \mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F}|_Y)$$

These two corollaries play crucial role to prove local freeness in general. The above corollaries imply the following theorem.

Theorem 1.3.13. *If $\mathfrak{F}$ is a family of N-pointed smooth curves with formal neighbourhoods such that the induced family $\mathfrak{F}^{(0)}$ of N-pointed smooth curves is not necessarily versal, then $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ and $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F})$ are locally free $\mathcal{O}_{\mathcal{B}}$-modules and they are dual to each other.*

Another important consequence of the above discussions is the following.

Proposition 1.3.14. *For each element $\vec{\ell} \in \mathcal{L}(\mathfrak{F})$, $D(\vec{\ell})$ acts on $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F})$. Moreover, if $\theta(\vec{\ell}) \neq 0$, then $D(\vec{\ell})$ acts on $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F})$ as a first order differential operator.*

Note that for the natural bilinear pairing $\langle \quad | \quad \rangle : \mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F}) \times \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \to \mathcal{O}_{\mathcal{B}}$, we have the equality

$$\{D(\vec{\ell})\langle\Psi|\}|\Phi\rangle + \langle\Psi|\{D(\vec{\ell})|\Phi\rangle\} = \theta(\vec{\ell})(\langle\Psi|\Phi\rangle). \tag{1.3-21}$$

§1.4 Projectively flat connection and flat sections

In this section, based on the arguments in §1.3, b) we shall define the projectively flat connection on the sheaf of conformal blocks. The connection is defined over the locus of smooth curves $\mathcal{B} \setminus D$ and it has regular singularities along the boundary D, the locus of the singular curves.

a) Projectively flat connection

Let $\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \dots, s_N; \eta_1, \dots, \eta_N)$ be a family of N-pointed curves with formal neighbourhoods such that $\mathfrak{F}^{(0)} := (\pi : \mathcal{C} \to \mathcal{B}; s_1, \dots, s_N)$ is a versal family of N-pointed curves. We shall use freely the notations in the previous sections.

As was shown in Proposition 1.3.9 and Proposition 1.3.14 the sheaf $\mathcal{L}(\mathfrak{F})$ acts on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ and $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ from the left. Moreover, for $\vec{\ell} \in \mathcal{L}(\mathfrak{F})$, if $\theta(\vec{\ell}) \neq 0$, the action defines a first order differential operator. In this section first we shall study the action of $\vec{\ell}$ more closely when $\theta(\vec{\ell}) = 0$. Note that by (1.3-14) we have an exact sequence

$$0 \to \pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(*S)) \xrightarrow{t} \mathcal{L}(\mathfrak{F}) \xrightarrow{\theta} \Theta_{\mathcal{B}}(-\log D) \to 0. \tag{1.4-1}$$

In the following for simplicity we shall discuss mainly a smooth family of N-pointed curves with formal neighbourhoods though we shall use general notation.

Lemma 1.4.1. *Assume that $\mathcal{B}$ is small enough so that we can find a symmetric bidifferential*

$$\omega \in H^0(C \times_{\mathcal{B}} C, \omega_{C \times_{\mathcal{B}} C/\mathcal{B}}(2\Delta))$$

satisfying $\mathrm{Res}^2(\omega) = 1$, *that is*

$$\omega = \left(\frac{1}{(z-w)^2} + \textit{holomorphic} \right) dzdw$$

at the diagonal Δ. Then there exists a unique $\mathcal{O}_{\mathcal{B}}$-module homomorphism

$$a : \pi_* \Theta_{\mathcal{C}/\mathcal{B}}(*S) \to \mathcal{O}_{\mathcal{B}}$$

independent of the choice of ω such that for any $\vec{\ell} \in \mathcal{L}(\mathfrak{F})$ which is the image of an element of $\pi_* \Theta_{\mathcal{C}/\mathcal{B}}(*S)$ by the exact sequence (1.4-1), *we have*

$$D(\vec{\ell}) = a(\vec{\ell}) \cdot \mathrm{id}$$

as a linear operator acting on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ from the left and

$$D(\vec{\ell}) = -a(\vec{\ell}) \cdot \mathrm{id}$$

as a operator acting on $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ *from the left .*

Proof. Since $\theta(\vec{\ell}) = 0$, by (1.3-15) for any $\langle\Psi| \in \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ and $|\Phi\rangle \in \mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ we have

$$\begin{aligned}\langle\Psi|\{D(\vec{\ell})|\Phi\rangle\} &= -\sum_{j=1}^{N}\langle\Psi|\rho_j(T[\underline{\ell}_j])|\Phi\rangle \\ &= -\sum_{j=1}^{N}\operatorname*{Res}_{\xi_j=0}(\ell_j(\xi_j)\langle\Psi|T(\xi_j)|\Phi\rangle d\xi_j)\end{aligned}$$

where $\underline{\ell}_j = \ell_j(\xi_j)\dfrac{d}{d\xi_j}$ and by Theorem 1.2.18 we have

$$\begin{aligned}&\langle\Psi|T(\xi_j)|\Phi\rangle(d\xi_j)^2 \\ &\quad= \lim_{\zeta_j\to\xi_j}\left\{\frac{1}{2(\ell+g^*)}\sum_{a=1}^{\dim\mathfrak{g}}\langle\Psi|J^a(\zeta_j)J^a(\xi_j)|\Phi\rangle d\zeta_j d\xi_j\right. \\ &\qquad\qquad\left.-\frac{c_v}{2(\zeta_j-\xi_j)^2}\langle\Psi|\Phi\rangle d\zeta_j d\xi_j\right\}\end{aligned}$$

where

$$c_v = \frac{\ell\cdot\dim\mathfrak{g}}{g^*+\ell}.$$

Choose a symmetric bidifferential

$$\omega \in H^0(\mathcal{C}\times_{\mathcal{B}}\mathcal{C}, \omega_{\mathcal{C}\times_{\mathcal{B}}\mathcal{C}/\mathcal{B}}(2\Delta))$$

with $\operatorname{Res}^2(\omega) = 1$. The existence of such a bidifferential is well-known for smooth curves. Put

(1.4-2)

$$\begin{aligned}&\langle\Psi|\widetilde{T}(z)|\Phi\rangle(dz)^2 \\ &\quad:= \lim_{w\to z}\left\{\frac{1}{2(\ell+g^*)}\sum_{a=1}^{\dim\mathfrak{g}}\langle\Psi|J^a(w)J^a(z)|\Phi\rangle dwdz\right. \\ &\qquad\qquad\left.-\frac{c_v}{2}\omega(w,z)\langle\Psi|\Phi\rangle dwdz\right\}.\end{aligned}$$

Also define $S_\omega(z)(dz)^2$ by

$$S_\omega(z)(dz)^2 = 6\lim_{w\to z}\left\{\omega(w,z)dwdz - \frac{dwdz}{(w-z)^2}\right\}.$$

The form $S_\omega(z)(dz)^2$ is called a projective connection. It depends not only on the choice of ω but also on the choice of local coordinates:

$$S_\omega(w)(dw)^2 = S_\omega(z)(dz)^2 + \{w;z\}(dz)^2.$$

Now we have

$$\langle\Psi|T(\xi_j)|\Phi\rangle(d\xi_j)^2 = \langle\Psi|\widetilde{T}(\xi_j)|\Phi\rangle(d\xi_j)^2 + \frac{c_v}{12}\langle\Psi|\Phi\rangle S_\omega(\xi_j)(d\xi_j)^2.$$

Thus we have

(1.4-3)

$$\langle\Psi|\{D(\vec{\ell})|\Phi\rangle\} = -\sum_{j=1}^{N}\operatorname*{Res}_{\xi_j=0}(\ell_j(\xi_j)\langle\Psi|\widetilde{T}(\xi_j)|\Phi\rangle\Phi d\xi_j) - \frac{c_v}{12}\langle\Psi|\Phi\rangle\sum_{j=1}^{N}\operatorname*{Res}_{\xi_j=0}(\ell_j(\xi_j)S_\omega(\xi_j)d\xi_j).$$

Since $\ell_j(z)\langle\Psi|\widetilde{T}(z)|\Phi\rangle dz$ is a global meromorphic one-form which has poles only at $s_j(\mathcal{B})$, the first term of the right hand side of the above equality is zero. Therefore, if we put

$$a_\omega(\vec{\ell}) = -\frac{c_v}{12}\sum_{j=1}^{N}\operatorname*{Res}_{\xi_j=0}(\ell_j(\xi_j)S_\omega(\xi_j)d\xi_j) \tag{1.4-4}$$

then we have

$$\langle\Psi|\{D(\vec{\ell})|\Phi\rangle\} = a_\omega(\vec{\ell})\langle\Psi|\Phi\rangle.$$

Now $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ and $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F})$ are locally free and dual to each other, we conclude

$$D(\vec{\ell})|\Phi\rangle = a_\omega(\vec{\ell})|\Phi\rangle.$$

Let us show that $a_\omega(\vec{\ell})$ is independent of the choice of ω. If $\omega' \in H^0(\mathcal{C}\times_{\mathcal{B}}\mathcal{C}, \omega_{\mathcal{C}\times_{\mathcal{B}}\mathcal{C}/\mathcal{B}}(2\Delta))$ is another symmetric bidifferential with $\operatorname{Res}^2(\omega) = 1$, then $\omega - \omega'$ is a holomorphic bidifferential on $\mathcal{C}\times_{\mathcal{B}}\mathcal{C}$. Hence, $S_\omega - S_{\omega'}$ is also a holomorphic section of $\omega^{\otimes 2}_{\mathcal{C}/\mathcal{B}}$ on $\mathcal{C}$. Let τ be an element of $\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*S)$ with $t(\tau) = \vec{\ell}$. Then, $\tau(z)(S_\omega(z) - S_{\omega'}(z))dz$ is a meromorphic one form on $\mathcal{C}$. Hence, we have

$$\sum_{j=1}^{N}\operatorname*{Res}_{\xi_j=0}(\ell(\xi_j)(S_\omega(\xi_j) - S_{\omega'}(\xi_j))d\xi_j) = 0.$$

Thus, we conclude

$$a_\omega(\vec{\ell}) = a_{\omega'}(\vec{\ell}).$$

This shows the first part of the lemma. By (1.3-21) similarly we can show the second part. QED.

The above proof shows also the following corollary.

Corollary 1.4.2. *Under the same assumption as in Lemma* 1.4.1 *for each* $\vec{\ell} \in \mathcal{L}(\mathfrak{F})$, *if we define* $a_\omega(\vec{\ell})$ *by* (1.4-4), *then*

$$a_\omega : \mathcal{L}(\mathfrak{F}) \to \mathcal{O}_{\mathcal{B}}$$

is an $\mathcal{O}_{\mathcal{B}}$*-module homomorphism.*

Remark 1.4.3. In the above corollary, since $\vec{\ell}$ is not necessarily an image of a global meromorphic vector field, $a_\omega(\vec{\ell})$ does depend on the choice of ω.

Now we are ready to define the connections on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ and $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F})$, if $\mathcal{B}$ is small enough. Let us fix a symmetric bidifferential $\omega \in H^0(\mathcal{C} \times_{\mathcal{B}} \mathcal{C}, \omega_{\mathcal{C}\times_{\mathcal{B}}\mathcal{C}/\mathcal{B}}(2\Delta))$ with $\mathrm{Res}^2(\omega) = 1$. For each element $X \in \Theta_{\mathcal{B}}(-\log D)$, there is an element $\vec{\ell} \in \mathcal{L}(\mathfrak{F})$ with $\theta(\vec{\ell}) = X$. Define an operator $\nabla^{(\omega)}_X$ acting on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ from the left by

$$\nabla^{(\omega)}_X([|\Phi\rangle]) = D(\vec{\ell})([|\Phi\rangle]) - a_\omega(\vec{\ell})([|\Phi\rangle]) \tag{1.4-5}$$

where by $[|\Phi\rangle]$ we denote the element in $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ corresponding to $|\Phi\rangle$.

Proposition 1.4.4. $\nabla^{(\omega)}_X$ *is well defined and enjoys the following property.*

$$\nabla^{(\omega)}_X([f|\Phi\rangle]) = X(f)[|\Phi\rangle] + f\nabla^{(\omega)}_X([|\Phi\rangle]) \tag{1.4-6}$$

where $f \in \mathcal{O}_{\mathcal{B}}$. *Hence, the correspondence* $X \mapsto \nabla^{(\omega)}_X$ *defines a connection on* $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ *with regular singularities along* D.

Proof. Choose another $\vec{\ell}' \in \mathcal{L}(\mathfrak{F})$ with $\theta(\vec{\ell}') = X$. Then, we have

$$a(\vec{\ell} - \vec{\ell}') = a_\omega(\vec{\ell}) - a_\omega(\vec{\ell}').$$

By Lemma 1.4.1 we have

$$(D(\vec{\ell}) - D(\vec{\ell}'))[|\Phi\rangle] = -\sum_{j=1}^{N}[|\Phi\rangle] = a(\vec{\ell} - \vec{\ell}')[|\Phi\rangle].$$

Thus, as operators on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ we have the equality

$$D(\vec{\ell}) - a_\omega(\vec{\ell}) \cdot \mathrm{id} = D(\vec{\ell}') - a_\omega(\vec{\ell}') \cdot \mathrm{id}.$$

Hence, $\nabla^{(\omega)}_X$ does not depend on the choice of $\vec{\ell}$. By Proposition 1.3.8, 3) the equality (1.4-6) holds. QED.

Theorem 1.4.5. *The connection $\nabla^{(\omega)}$ on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})|_{\mathcal{B}-D}$ is projectively flat.*

Remark 1.4.6. For $\mathbf{P}^1$, since the form

$$\omega = \frac{dwdz}{(w-z)^2}$$

is a global form, we have

$$a_\omega = 0.$$

Hence, in this case the connection is flat. For N-pointed $\mathbf{P}^1$, by (1.2-24) the connection is the inhomogeneous form of Knizhnik-Zamolodchikov equation ([KZ]).

b) Family of one-pointed stable curves of genus 1

In this subsection we shall show how to prove local freeness of the sheaf of conformal blocks over the locus of singular curves in case of the universal family of one-pointed stable curves of genus 1. The argument will also show how to construct multivalued flat sections from the data of a boundary. It is deeply related to monodromy representation of Teichmüller modular groups ([TK], [Ko], [W]) and the Verlinde conjecture ([Ve], [MS1]). For details we refer the reader to [U].

First let us construct a versal family of stable curves of genus one. Let z and w be coordinates of $\mathbf{P}^1$ with center 0 and ∞, respectively with

$$zw = 1.$$

Put

$$D = \{q \in \mathbf{C} \mid |q| < 1\}, \quad X = \{R \in \mathbf{P}^1 \mid |z(R)| < 1\},$$
$$Y = \{P \in \mathbf{P}^1 \mid |w(R)| < 1\}.$$

and

$$S = \{(x, y, q) \in \mathbf{C}^3 \mid xy = q,\ |x| < 1, |y| < 1, |q| < 1\}$$
$$Z = \{(R, q) \in \mathbf{P}^1 \times D \mid R \in \mathbf{P}^1 \setminus \{X \cup Y\} \quad \text{or} \quad R \in X \quad |z(R)| > |q|\}$$
$$W = \{(P, q) \in \mathbf{P}^1 \times D \mid P \in \mathbf{P}^1 \setminus \{X \cup Y\} \quad \text{or} \quad P \in X \quad |w(P)| > |q|\}$$

Let us introduce an equivalence relation on $S \cup Z \cup W$. The equivalence relation is generated by the following relations.

1) A point $(R, q) \in X \times D \cap Z$ and a point $(x, y, q') \in S$ are equivalent if

$$(x, y, q') = (z(R), \frac{q}{z(R)}, q).$$

2) A point $(P, q) \in Y \times D \cap W$ and a point $(x, y, q') \in S$ are equivalent if

$$(x, y, q') = (\frac{q}{w(P)}, w(P), q).$$

3) A point $(R, q) \in Z$ and a point $(P, q') \in W$ are equivalent if

$$q' = q, \quad z(R)w(P) = q.$$

Put

$$\mathcal{E} = (S \cup Z \cup W)/\sim .$$

Then $\mathcal{E}$ is a two-dimensional complex manifold. For points $(x, y, q) \in S$, $(R, q) \in Z$ and $(P, q) \in W$ we denote the corresponding points in $\mathcal{E}$ by $[(x, y, q)]$, $[(R, q)]$ and $[(P, q)]$, respectively. We have a natural holomorphic mapping

$$\pi : \mathcal{E} \to D.$$

For $q \in D$ put

$$\begin{aligned} S(q) &= \{(x, y) \in \mathbf{C}^2 \mid xy = q \}, \\ Z(q) &= \{\mathbf{P}^1 - X \cup Y\} \cup \{R \in X \mid |z(R)| > |q| \}, \\ W(q) &= \{\mathbf{P}^1 - X \cup Y\} \cup \{P \in Y : \mid |w(P)| > |q| \}. \end{aligned}$$

By the reason which will become clear later we use mainly the coordinate w so that

$$\mathbf{P}^1 = \{(w : 1) \mid w \in \mathbf{C}\} \cup \{(1 : 0)\}.$$

Assume that $q \neq 0$. We can rewrite

$$\begin{aligned} \mathbf{P}^1 - X \cup Y &= \{(w : 1) \mid |w| = 1 \} \\ Z(q) &= \{(w : 1) \mid 1 \leq |w| < 1/|q| \} \\ W(q) &= \{(w : 1) \mid |q| < |w| \leq 1 \} \end{aligned}$$

On $S(q) \cup Z(q) \cup W(q)$ the above equivalence relation induces the following one.

$(w : 1) \in X \times q \cap Z(q) = \{ w| \ 1 \leq |w| < 1/|q|\}$ and $(x, y, q) \in S(q)$ is equivalent if

$$(x, y, q) = (1/w, qw, q);$$

$(w' : 1) \in Y \times q \cap W(q) = \{ w'| \ |q| < |w'| \leq 1\}$ $(x, y, q) \in S(q)$ is equivalent if

$$(x, y, q) = (q/w', w', q).$$

Note that if $q \neq 0$, a point $(x, y, q) \in S(q)$ is either equivalent to the point $(w : 1) \in X \times q \cap Z(q)$ (when $|x| \leq 1$) or to the point $(w' : 1) \in Y \times q \cap W(q)$

(when $|y| \leq 1$). Therefore, the quotient manifold $\mathcal{E}(q) = (S(q) \cup Z(q) \cup W(q))/\sim$ is obtained by identifying $1 \leq |w| < 1/|q|$ and $|q| < |w'| \leq 1$ by $w' = qw$. In other word, $\mathcal{E}(q)$ is the quotient manifold $E_{\langle q \rangle} = \mathbf{C}^*/\langle q \rangle$ where $\langle q \rangle$ is the infinite cyclic group of analytic automorphism generated by

$$w \mapsto qw.$$

Hence, $\mathcal{E}(q)$ is the elliptic curve $E_{\langle q \rangle}$, if $q \neq 0$ and a rational curve with an ordinary double point for $q = 0$. Note that $\mathcal{E}(q)$ is the fibre $\pi^{-1}(q)$ of $\pi : \mathcal{E} \to D$ and that the point $[1]$ of $\mathcal{E}(q)$ corresponding to $w = 1$ is the origin of the elliptic curve. Moreover, if $q = 0$, the fibre $\pi^{-1}(0)$ of π is a rational curve with ordinary double point obtained by identifying the points $\infty = (0 : 1)$ and $0 = (1 : 0)$.

It is easy to show that $\dfrac{dw}{w}$ is a holomorphic one-form on $\mathcal{E}(q)$ for $q \neq 0$. and the family $\pi : \mathcal{E} \to D$ is versal at each point $q \in D$.

The mapping

$$\begin{aligned} o : D &\to \quad \mathcal{E} \\ q &\mapsto [(q, 1, q)], \quad \text{where } (q, 1, q) \in S \end{aligned}$$

defines a holomorphic section o of the family $\pi : \mathcal{E} \to D$.

Let us consider the family $\mathfrak{F} = (\mathcal{E} \to D; o; u)$ of one-pointed stable curves of genus one with coordinate, where $u = w - 1$. Let us consider the sheaf of conformal blocks $\mathcal{V}_0^\dagger(\mathfrak{F})$. First we shall construct formal sections of $\mathcal{V}_0^\dagger(\mathfrak{F})$ starting from the data on the fibre $\mathcal{E}(0)$ at the origin.

Let

$$\nu : \widetilde{\mathcal{E}}(0) = \mathbf{P}^1 \to \mathcal{E}(0)$$

be the normalization of $\mathcal{E}(0)$. Then, we have

$$\begin{aligned} &\nu(0) = \nu(\infty) = \text{the double point} \\ &\nu(1) = o(0). \end{aligned}$$

We know that $\mathcal{V}_0^\dagger((\mathcal{E}(0); o(0), u))$ is canonically isomorphic to

$$\bigoplus_{\lambda \in \Lambda} \mathcal{V}^\dagger_{\lambda^\dagger, \lambda, 0}((\mathbf{P}^1; \infty, 0, 1; \eta, \xi, \xi - 1)).$$

For simplicity, put

$$\mathcal{V}^\dagger_{\lambda^\dagger, \lambda, 0} = \mathcal{V}^\dagger_{\lambda^\dagger, \lambda, 0}((\mathbf{P}^1; \infty, 1, 0; \eta, \xi - 1, \xi)).$$

The vector space $\mathcal{V}^\dagger_{\lambda^\dagger, \lambda, 0}$ is one-dimensional and there is the element $\langle \Psi | \in \mathcal{V}^\dagger_{\lambda^\dagger, \lambda, 0}$ such that

$$\langle \Psi | v^\dagger \otimes v \otimes 0 \rangle = (v^\dagger | v)$$

for any $|v^\dagger\rangle \in \mathcal{H}_{\lambda^\dagger}$ and $|v\rangle \in \mathcal{H}_\lambda$ where $|0\rangle$ is the highest weight vector of $\mathcal{H}_0$ and $(\ \ |\ \ )$ is the bilinear pairing

$$\mathcal{H}_{\lambda^\dagger} \otimes \mathcal{H}_\lambda \to \mathbf{C}$$

defined in §2.2. Let us construct an element

$$\langle \Psi^{(\lambda)}| \in \mathcal{V}_0^\dagger(\mathfrak{F})$$

from $\langle \Psi|$.

For any non-negative integer d, by Lemma 1.1.11 there exist bases $\{v^k(d)\}$ and $\{v_k(d)\}$, $k = 1, 2, \ldots, m_d$ of $\mathcal{H}_{\lambda^\dagger}(d)$ and $\mathcal{H}_\lambda(d)$, respectively such that

$$(v^j(d)|v_k(d)) = \delta_k^j.$$

For each element $|\phi\rangle \in \mathcal{H}_0$ $\langle \Psi_d| \in \mathcal{H}_0^\dagger$ is defined by

$$\langle \Psi_d|\phi\rangle = \sum_{i=1}^{m_d} \langle \Psi|v^i(d) \otimes v_i(d) \otimes \phi\rangle.$$

Now $\langle \Psi^{(\lambda)}|$ is defined by

$$\langle \Psi^{(\lambda)}|\phi\rangle = \sum_{d=0}^{\infty} \langle \Psi_d|\phi\rangle q^d.$$

This is a formal power series in q and from our construction it is not clear that it converges. Later we shall show that $q^{\Delta_\lambda}\langle \Psi^{(\lambda)}|\phi\rangle$ is a formal solution of a differential equation of Fuchsian type so that it converges near the origin. If we put $|\phi\rangle = |0\rangle$, then

$$q^{\Delta_\lambda}\langle \Psi^{(\lambda)}|0\rangle = \sum_{d=0}^{\infty}\sum_{i=1}^{m_d}(v^i)d)|v_i(d))q^d = \sum_{d=0}^{\infty} \dim \mathcal{H}_\lambda(d) q^{\Delta_\lambda + d}. \tag{1.4-7}$$

This formal power series is the character of the integrable $\mathfrak{g}$-module $\mathcal{H}_\lambda$ and can be described by using theta functions ([Ka]). It is a multivalued holomorphic function on D^*.

At the moment we only know that $\langle \Psi^{(\lambda)}|$ is an element of $\mathcal{H}_0[[q]]$. We shall show that it satisfies the formal gauge condition so that it can be regarded as an element of the completion of $\mathcal{V}_0^\dagger(\mathfrak{F})$ at the origin.

To explain the formal gauge condition first we shall consider meromorphic functions on our elliptic curve $\mathcal{E}(q)$ having only poles at the origin [1]. Put

$$x(q,w) = -2\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} + \sum_{n=-\infty}^{+\infty} \frac{wq^n}{(1-wq^n)^2} \tag{1.4-8}$$

$$y(q,w) = \sum_{n=-\infty}^{+\infty} \frac{(1+wq^n)wq^n}{(1-wq^n)^3} \tag{1.4-9}$$

Then, if we put $x = \exp(2\pi\sqrt{-1}z)$, we have

$$x(q,w) = -\frac{1}{12} - \frac{1}{4\pi^2}\wp(z)$$
$$y(q,w) = \frac{\sqrt{-1}}{8\pi^3}\wp'(z)$$

where $\wp(z)$ is the Weierstrass $\wp$ function. Moreover, we have

$$x(q,qw) = x(q,w)$$
$$y(q,qw) = y(q,w)$$
$$x(q,w^{-1}) = x(q,w)$$
$$y(q,w^{-1}) = -y(q,w)$$

and they satisfy the following equation:

$$y^2 - 4x^3 - x^2 + g_2(q)x + g_3(q) = 0, \tag{1.4-10}$$

where we put

$$x = x(q,w)$$
$$y = y(q,w)$$
$$g_2(q) = 20\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}$$
$$g_3(q) = \frac{1}{3}\sum_{n=1}^{\infty}\frac{(7n^5+5n^3)q^n}{1-q^n}.$$

Note that $x(q,w)$ has pole of order 2 at [1] and $y(q,w)$ has pole of order 3 at [1]. They are holomorphic except [1]. By the theory of elliptic functions we know that $H^0(\mathcal{E}(q), \mathcal{O}(*[1]))$ is spanned by monomials of $x(q,w)$ and monomials of $x(q,w)$ times $y(q,w)$. Note that $x(q,w)$ has an expansion

$$x(q,w) = \frac{w}{(1-w)^2} + \sum_{\ell=1}^{\infty}\left\{\sum_{m|\ell}(mw^m + mw^{-m} - 2)\right\}q^\ell.$$

This formula shows that each coefficient of q^ℓ can be regarded as a meromorphic function of $\mathbf{P}^1$ having only poles at the points ∞, 1 and 0. The same is true if we expand polynomials $P(x,y)$ of $x(q,w)$ and $y(q,w)$ in q. That is, if we have an expansion

$$\begin{aligned} P(x(q,w), y(q,w)) &= Q_1(x(q,w)) + Q_2(x(q,w))y(q,w) \\ &= \sum_{k=0}^{\infty} f_k(w)q^k, \end{aligned} \tag{1.4-11}$$

all coefficients $f_k(w)$ are in $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(*(\infty+1+0)))$.

Now the gauge condition can be calculated formally as follows.

$$\langle\Psi^{(\lambda)}|X\otimes P(x(q,w),y(q,w))\phi\rangle = \sum_{d=0}^{\infty}\langle\Psi_d|\sum_{k=0}^{\infty}(X\otimes f_k(w))q^k|\phi\rangle q^d$$

$$= \sum_{d,k=0}^{\infty}(\sum_{i=1}^{m_d}\langle\Psi|(X\otimes f_k(w))|v^i(d)\otimes v_i(d)\otimes\phi\rangle)q^{d+k} \tag{1.4-12}$$

By the gauge condition on $\langle\Psi| \in \mathcal{V}_{\lambda^\dagger,0,\lambda}$, we have

$$\langle\Psi|(X\otimes f_k(w))|v^i(d)\otimes v_i(d)\otimes\phi\rangle = 0.$$

This proves formally

$$\langle\Psi^{(\lambda)}|X\otimes P(x(q,w),y(q,w))$$

$$= \sum_{d,k=0}^{\infty}(\sum_{i=1}^{m_d}\langle\Psi|(X\otimes f_k(w))|v^i(d)\otimes v_i(d)\otimes\phi\rangle)q^{d+k}. \tag{1.4-13}$$

Since we do not know the convergence of $\langle\Psi^{(\lambda)}|$, the above argument does not prove the fact

$$\langle\Psi^{(\lambda)}| \in \mathcal{V}_0^\dagger(\mathfrak{F}).$$

To go further we need some facts from algebraic geometry. First we take the completion $\widehat{\mathcal{V}}_0^\dagger(\mathfrak{F})_{/0}$ of our sheaf $\mathcal{V}_0^\dagger(\mathfrak{F})$ at the origin. Let $\mathfrak{m}$ be the defining ideal sheaf of the point $q=0$ in D. Then, the completion is defined by

$$\widehat{\mathcal{V}}_0^\dagger(\mathfrak{F})_{/0} = \varprojlim_{n\to\infty}\mathcal{V}_0^\dagger(\mathfrak{F})/\mathfrak{m}^n\mathcal{V}_0^\dagger(\mathfrak{F}).$$

Let $\widehat{\mathcal{O}}_{D/0}$ be the completion of the structure sheaf $\mathcal{O}_D$ of D at the origin. Since $\widehat{\mathcal{O}}_{D/0}$ is faithfully flat over $\mathcal{O}_D$, we have

$$\widehat{\mathcal{V}}_0^\dagger(\mathfrak{F})_{/0} = \mathcal{V}_0^\dagger(\mathfrak{F})\otimes_{\mathcal{O}_D}\widehat{\mathcal{O}}_{D/0}.$$

On the other hand, we can show that $\widehat{\mathcal{V}}_0^\dagger(\mathfrak{F})_{/0}$ is characterized as a subsheaf of $\mathcal{H}_0\otimes_{\mathcal{O}_D}\widehat{\mathcal{O}}_{D/0}$ satisfying the formal gauge condition (1.4-13). Hence, we have

$$\langle\Psi^{(\lambda)}| \in \widehat{\mathcal{V}}_0^\dagger(\mathfrak{F})_{/0}.$$

It is also easy to show that $\langle\Psi^{(\lambda)}|$, $\lambda\in P_\ell$ are linearly independent over $\widehat{\mathcal{O}}_{D/0}$. This means that

$$\dim_K\widehat{\mathcal{V}}_0^\dagger(\mathfrak{F})_{/0}\otimes K \geq \dim\mathcal{V}_0^\dagger((\mathcal{E}(0),o(0),u)), \tag{1.4-14}$$

where K is the field of fraction of $\widehat{\mathcal{O}}_{D/0}$. Note that we already proved that the sheaf $\mathcal{V}_0^\dagger(\mathfrak{F})$ is locally free over D^* and we can show that

$$\text{rank}\mathcal{V}_0^\dagger(\mathfrak{F})|_{D^*} = \dim_K \widehat{\mathcal{V}}_0^\dagger(\mathfrak{F})_{/0} \otimes K. \tag{1.4-15}$$

Now we need to consider the sheaf $\mathcal{V}_0(\mathfrak{F})$. In Lemma 1.3.3 we showed the isomorphism

$$\mathcal{V}_0(\mathfrak{F}) \otimes \mathbf{C}_0 \simeq \mathcal{V}_0((\mathcal{E}(0), o(0), u)). \tag{1.4-16}$$

The similar isomorphism is indeed true for $\mathcal{V}_0^\dagger(\mathfrak{F}) \otimes \mathbf{C}_0$, but this can be proved after we prove that $\mathcal{V}_0^\dagger(\mathfrak{F})$ is locally free at the origin. This is one of the main reasons we need to introduce the sheaf of covacua. From (1.4-14), (1.4-15) and (1.4-16) we conclude

$$\begin{aligned}\text{rank}\mathcal{V}_0(\mathfrak{F})|_{D^*} &= \text{rank}\mathcal{V}_0^\dagger(\mathfrak{F})|_{D^*} \\ &\geq \dim \mathcal{V}_0^\dagger((\mathcal{E}(0), o(0), u)) = \dim \mathcal{V}_0((\mathcal{E}(0), o(0), u)) \\ &= \dim \mathcal{V}_0(\mathfrak{F}) \otimes \mathbf{C}_0.\end{aligned}$$

But, since our sheaf $\mathcal{V}_0(\mathfrak{F})$ is coherent, we have the opposite inequality:

$$\dim \mathcal{V}_0(\mathfrak{F}) \otimes \mathbf{C}_0 \geq \text{rank}\mathcal{V}_0(\mathfrak{F})|_{D^*}. \tag{1.4-17}$$

Therefore in (1.4.17) the equality holds. This means that our sheaf $\mathcal{V}_0(\mathfrak{F})$ is a locally free $\mathcal{O}_D$-module, hence so is $\mathcal{V}_0^\dagger(\mathfrak{F})$. This proves local freeness.

The essential ideas of proving the local freeness for general case are the same as above.

Next we write down explicitly the connection on the sheaf $\mathcal{V}_0^\dagger(\mathfrak{F})$.

Let us consider the exact sequence given in (1.4-1)

$$0 \to \pi_* \Theta_{\mathcal{E}/D}(*o(D)) \xrightarrow{t} \mathcal{L}(\mathfrak{F}) \xrightarrow{\theta} \Theta_D(-\log 0) \to 0.$$

The sheaf $\Theta_D(-\log 0)$ is free $\mathcal{O}_D$-module generated by $q\dfrac{d}{dq}$. We can show that we have

$$\theta(u^{-1}\frac{d}{du}) = -q\frac{d}{dq}.$$

Introducing the connection on $\mathcal{V}_0^\dagger(\mathfrak{F})$ with regular singularity at the origin is equivalent to giving the action of $\Theta_D(-\log 0)$ on $\mathcal{V}_0^\dagger(\mathfrak{F})$. For that purpose it is enough to show that the image of the mapping t acts trivially on $\mathcal{V}_0^\dagger(\mathfrak{F})$.

By (1.3-17) for any local section $\langle\Psi|$ of $\mathcal{V}_0^\dagger(\mathfrak{F})|_{D^*}$ and $\underline{\ell} \in \mathcal{L}(\mathfrak{F})$, we have

$$\{D(\underline{\ell})|\langle\Psi|\}\phi\rangle = -\langle\Psi|\{D(\underline{\ell})\phi\rangle\} + \theta(\underline{\ell})\langle\Psi|\phi\rangle, \tag{1.4-18}$$

for any local section $|\phi\rangle \in \mathcal{H}_0 \otimes \mathcal{O}_D$. By (1.3-15) we also have

$$D(\underline{\ell})|\phi\rangle = \theta(\underline{\ell})|\phi\rangle - T[\underline{\ell}]|\phi\rangle.$$

If $\underline{\ell}$ is in the image of t we have $\theta(\underline{\ell}) = 0$. Hence, it is enough to show

$$\langle\Psi|T[\underline{\ell}]|\phi\rangle = 0.$$

That is

$$\operatorname*{Res}_{u=0}\{\ell(u)\langle\Psi|T[u]|\phi\rangle du\} = 0 \tag{1.4-19}$$

where $\underline{\ell} = \ell(u)\dfrac{d}{du}$.

As was shown in §1.2, c) the form

$$\langle\Psi|T[z]|\phi\rangle dz^2$$

is a single valued meromorphic form. In our coordinate it can be expressed in the form

$$\langle\Psi|T[w]|\phi\rangle(dw/w)^2$$

where

$$w = \exp(2\pi\sqrt{-1}z).$$

Now in our case, since we have

$$\underline{\ell} = \theta(\tau)$$

for a relative vector field $\tau \in \pi_*\Theta_{\mathcal{E}/D}(*o(D))$ having pole only along $o(D)$, the form $\ell(u)\langle\Psi|T[u]|\phi\rangle du$ is a global meromorphic one form

$$\iota_\tau(\langle\Psi|T[w]|\phi\rangle(dw/w)^2)$$

which has pole only at $o(D)$, where ι is the interior product. Hence (1.4-19) holds. Thus we can define the connection on $\mathcal{V}_0^\dagger(\mathfrak{F})$. We can show that the connection is flat. If the genus of a curve is greater than one, the form $\langle\Psi|T[z]|\phi\rangle dz^2$ is no more global form on the curve. Therefore, in general we can only show that the image of t acts on $\mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{F})$ as the multiplication of a certain holomorphic functions. This is the reason why we only have a projectively flat connection in general.

Finally we need to show that our formal solutions actually converge so that we have

$$\langle\Psi^{(\lambda)}| \in \mathcal{V}_0(\mathfrak{F}).$$

Put

$$\langle\widetilde{\Psi}^{(\lambda)}| = q^{\Delta_\lambda}\langle\Psi^{(\lambda)}|.$$

We can show that $\langle\widetilde{\Psi}^{(\lambda)}|$ satisfies the differential equation

$$q\frac{d}{dq}\langle\widetilde{\Psi}^{(\lambda)}|\phi\rangle - \langle\widetilde{\Psi}^{(\lambda)}|L_{-2}\phi\rangle = 0 \tag{1.4-20}$$

for any $|\phi\rangle \in V_0 \otimes \mathcal{O}_D$, where $V_0 = \mathcal{H}_0(0)$.

It is rather messy to prove this by direct calculation. A general method to prove this fact can be found in [U, §5.3]. Here we only prove the equation (1.4.20) when $|\phi\rangle = |0\rangle$. In this case, we have

$$\begin{aligned}\langle\widetilde{\Psi}^{(\lambda)}|0\rangle &= \sum_{d=0}^{\infty}(\sum_{i=1}^{m_d}\langle\Psi|v^i(d)\otimes v_i(d)\otimes 0\rangle)q^{\Delta_\lambda+d}\\ &= \sum_{d=0}^{\infty}(\dim\mathcal{H}_\lambda(d))q^{\Delta_\lambda+d}.\end{aligned}$$

First let us calculate

$$\langle\widetilde{\Psi}^{(\lambda)}|L_{-2}0\rangle.$$

For that purpose it is enough to calculate

$$\begin{aligned}&\langle\Psi|v^i(d)\otimes v_i(d)\otimes L_{-2}0\rangle\\ &\quad= \frac{1}{2(g^*+\ell)}\sum_{a=1}^{\dim\mathfrak{g}}\sum_{m=-\infty}^{\infty}\langle\Psi|v^i(d)\otimes v_i(d)\otimes J^a(m)J^a(-2-m)0\rangle\end{aligned}$$

For $m \neq -1$, since m or $-2-m$ is non-negative, we have

$$J^a(m)J^a(-2-m)|0\rangle = J^a(-2-m)J^a(m)|0\rangle = 0.$$

For $m = -1$ the gauge condition implies

$$\begin{aligned}&\langle\Psi|v^i(d)\otimes v_i(d)\otimes J^a(-1)J^a(-1)0\rangle\\ &\qquad= \langle\Psi|v^i(d)\otimes v_i(d)\otimes J^a[\frac{1}{\xi-1}]J^a[\frac{1}{\xi-1}]0\rangle\\ &\qquad= \langle\Psi|J^a[\frac{\eta}{1-\eta}]J^a[\frac{\eta}{1-\eta}]v^i(d)\otimes v_i(d)\otimes 0\rangle\\ &\qquad\quad+ 2\langle\Psi|J^a[\frac{\eta}{1-\eta}]v^i(d)\otimes J^a[\frac{1}{\xi-1}]v_i(d)\otimes 0\rangle\\ &\qquad\quad+ \langle\Psi|v^i(d)\otimes J^a[\frac{1}{\xi-1}]J^a[\frac{1}{\xi-1}]v_i(d)\otimes 0\rangle\end{aligned}$$

We have

$$\begin{aligned}&\langle\Psi|J^a[\frac{\eta}{1-\eta}]J^a[\frac{\eta}{1-\eta}]v^i(d)\otimes v_i(d)\otimes 0\rangle\\ &\qquad= \sum_{m,n=1}^{\infty}\langle\Psi|J^a[\eta^m]J^a[\eta^n]v^i(d)\otimes v_i(d)\otimes 0\rangle\\ &\qquad= \sum_{m,n=1}^{\infty}(J^a(m)J^a(n)v^i(d)|v_i(d)) = 0,\end{aligned}$$

since $J^a(m)J^a(n)v^i(d) \notin \mathcal{H}_{\lambda^\dagger}(d)$. Similarly we have

$$\langle \Psi | v^i(d) \otimes J^a[\frac{1}{\xi-1}]J^a[\frac{1}{\xi-1}]v_i(d) \otimes 0\rangle = \sum_{m,n=0}^{\infty} (v^i(d)|J^a(m)J^a(n)v_i(d)) = (v^i(d)|J^aJ^av_i(d)).$$

Also we have

$$\begin{aligned}\langle \Psi | J^a[\frac{\eta}{1-\eta}]v^i(d) \otimes J^a[\frac{1}{\xi-1}]v_i(d) \otimes 0\rangle &= -\sum_{m,n=0}^{\infty} (J^a(m+1)v^i(d)|J^a(n)v_i(d)) \\ &= -\sum_{n=1}^{\infty}(J^a(n)v^i(d)|J^a(n)v_i(d)) \\ &= \sum_{n=1}^{\infty}(v^i(d)|J^a(-n)J^a(n)v_i(d))\end{aligned}$$

Thus we conclude

$$\langle \Psi | v^i(d) \otimes v_i(d) \otimes L_{-2}0\rangle = (v^i(d)|L_0v_i(d)) = (\Delta_\lambda + d)(v^i(d)|v_i(d)).$$

Thus we obtain

$$\langle \widetilde{\Psi}^{(\lambda)} | L_{-2}0\rangle = \sum_{d=0}^{\infty}(\Delta_\lambda + d)\dim \mathcal{H}_\lambda(d)q^{\Delta_\lambda + d}.$$

The right hand side of the formula is equal to

$$q\frac{q}{dq}\langle \widetilde{\Psi}^{(\lambda)}|0\rangle.$$

Thus we obtain the desired result. The argument can be generalized for general $|\phi\rangle \in \mathcal{H}_0 \otimes \mathcal{O}_D$.

Remark1.4.6. In Physics literature, $\langle \widetilde{\Psi}^{(\lambda)}|\phi\rangle$ is written as

$$\mathrm{tr}_{\mathcal{H}_\lambda}(q^{L_0}\langle \Psi| * \otimes * \phi\rangle). \tag{1.4-21}$$

Usually physicists add the term $q^{-\frac{c_v}{24}}$:

$$\mathrm{tr}_{\mathcal{H}_\lambda}(q^{L_0 - \frac{c_v}{24}}\langle \Psi| * \otimes * \phi\rangle). \tag{1.4-22}$$

This only changes the differential equation (1.4.15) to

$$q\frac{d}{dq}\langle \widetilde{\Psi}^{(\lambda)}|\phi\rangle - \langle \widetilde{\Psi}^{(\lambda)}|L_{-2}\phi\rangle - \frac{c_v}{24}\langle \widetilde{\Psi}^{(\lambda)}|\phi\rangle = 0.$$

The expressions (1.4-21) and (1.4-22) can be interpreted as follows. If we fix an element $|\phi\rangle \in \mathcal{H}_0$, then

$$\langle\Psi| * \otimes * \phi\rangle : u\otimes \mapsto \langle\Psi|u^\dagger \otimes u \otimes \phi\rangle \tag{1.4-23}$$

defines an endomorphism of $\mathcal{H}_\lambda(d)$, where $u^\dagger$ is the dual of u with respect to the inner product defined in Lemma 1.1.11. Hence,

$$\sum_{i=1}^{m_d} \langle\Psi|v^i(d) \otimes v_i(d) \otimes \phi\rangle$$

can be interpreted as the trace on $\mathcal{H}_\lambda(d)$ of the mapping

$$\langle\Psi| * \otimes * \phi\rangle \in \mathrm{Hom}(\mathcal{H}_{\lambda,d}, \mathcal{H}_\lambda(d)).$$

Moreover, since L_0 acts on $\mathcal{H}_\lambda(d)$ by the multiplication of $\Delta_\lambda + d$, we may justify the notation

$$\mathrm{tr}_{\mathcal{H}_\lambda(d)}(q^{L_0}\langle\Psi| * \otimes * \phi\rangle) = q^{\Delta_\lambda + d}\sum_{i=1}^{m_d}\langle\Psi|v^i(d) \otimes v_i(d) \otimes \phi\rangle.$$

Hence, in (1.4-21) and (1.4-22) we may regard $\mathrm{tr}_{\mathcal{H}_\lambda}$ as

$$\sum_{d=0}^{\infty} \mathrm{tr}_{\mathcal{H}_\lambda(d)}.$$

Remark 1.4.7. A rational mapping

$$\begin{aligned} \mathcal{E} &\to \mathbf{P}^2 \times D \\ [(q,(w:1)] &\mapsto ((1 : x(q,w) : y(q,w)), q) \end{aligned}$$

gives a projective imbedding of $\mathcal{E}$ as a submanifold in $\mathbf{P}^2 \times D$ defined by the equation (1.4-10).

Chapter II Abelian Conformal Field Theory

In this chapter we shall briefly explain a generalization of abelian conformal field theory. This is a joint work with A. Tsuchiya. Abelian conformal field theory is usually discussed from the view point of the universal Grassmann manifold and Krichever maps ([KNTY]). Here, we consider it from the view point of non-abelian conformal field theory discussed in the previous chapter and generalize it. We take the Vertex operator algebra constructed from Heisenberg algebra as a gauge group. In the following we shall show

that the main ideas of the non-abelian conformal field theory can be applied to our situation and the relationship between conformal blocks and theta functions of higher level will be clarified. The results in this section are joint work with A. Tsuchiya.

§2.1 Vertex operators

For a positive even integer M we let H_M be an algebra generated by operators $a(n)$, $n \in \mathbf{Z}$ with commutation relation

$$[a(n), a(m)] = Mn\delta_{n+m,0} \cdot id. \tag{1.1}$$

The algebra H_M is called the Heisenberg algebra. It is a universal enveloping algebra of an affine Lie algebra $\{a(n)\}$ associated with a one-dimensional abelian Lie algebra $\mathbf{C}$ with commutation relation (1.1). For each $p \in \mathbf{C}$, by $\mathcal{F}(p)$ we denote an irreducible highest weight module of H_M determined by

$$\begin{aligned} a(0)|p\rangle &= p|p\rangle \\ a(n)|p\rangle &= 0, \quad \text{if} \quad n \geq 1, \end{aligned}$$

where $|p\rangle$ is a highest weight vector. Let $t_0, t_1, t_2, \ldots$ be independent variables. Put

$$\begin{aligned} a(m) &= \frac{\partial}{\partial t_m}, \quad m = 0, 1, 2, \ldots \\ a(-n) &= nMt_n, \quad n = 1, 2, 3, \ldots \end{aligned}$$

Then, the Heisenberg algebra H_M and its irreducible module $\mathcal{F}(p)$ are realized as

$$\begin{aligned} H_M &= \mathbf{C}[t_1, t_2, \ldots t_n, \ldots, \frac{\partial}{\partial t_0}, \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_m}, \ldots, \frac{\partial}{\partial t_m}, \ldots] \\ \mathcal{F}(p) &= \mathbf{C}[t_1, t_2, \ldots, t_n, \ldots]e^{pt_0}, \end{aligned}$$

where the highest weight vector $|p\rangle$ corresponds to e^{pt_0}. Using this realization, we introduce an operator $\widehat{q}$ by

$$\widehat{q} = Mt_0.$$

Put

$$\begin{aligned} \phi(z) &= \widehat{q} + a(0)\log z - \sum_{n \neq 0} \frac{a(n)}{n} z^{-n} \\ a(z) &= \sum_{n \in \mathbf{Z}} a(n) z^{-n-1} \end{aligned}$$

Then we have

$$d\phi(z) = a(z)dz$$

For each integer k, the Vertex operator $V_k(z)$ is defined as

$$V_k(z) = {}^{\circ}_{\circ} e^{k\phi(z)} {}^{\circ}_{\circ}$$

where ${}^{\circ}_{\circ}\ \ {}^{\circ}_{\circ}$ is a normal ordering defined by putting $a(n)$, $n \geq 0$ the right hand side, and $\widehat{q}$ and $a(-n)$,$n \geqslant 1$ the left hand side. Hence, we have

$$V_k(z) = \exp\left\{k\sum_{n=1}^{\infty}\frac{a(-n)}{n}z^n\right\}e^{k\widehat{q}}e^{ka(0)\log z}\exp\left\{-k\sum_{n=1}^{\infty}\frac{a(n)}{n}z^{-n}\right\}$$

The Vertex operator $V_k(z)$ is an intertwiner between the representations $\mathcal{F}(p)$ and $\mathcal{F}(kM+p)$. The energy-momentum tensor $T(z)$ is defined as

$$T(z) = \frac{1}{2M}{}^{\circ}_{\circ} a(z)a(z) {}^{\circ}_{\circ}$$

There is a formal expansion

$$T(z) = \sum_{n\in\mathbf{Z}} L_n z^{-n-2},$$

and $\{L_n\}$ is a Virasoro algebra. These operators have the following operator product expansions.

$$\text{(2.1-1)}\qquad \begin{cases} a(z)a(w) & \sim \dfrac{M}{(z-w)^2} \\ T(z)a(w) & \sim \dfrac{1}{(z-w)^2}a(w) + \dfrac{1}{z-w}\partial_w a(w) \\ T(z)V_k(w) & \sim \dfrac{\frac{k^2}{2}M}{(z-w)^2}V_k(w) + \dfrac{1}{z-w}\partial_w V_k(w) \\ a(z)V_k(w) & \sim \dfrac{k}{z-w}V_k(w) \end{cases}$$

These formulas show that $a(z)$ behaves as a one-form and $V_k(z)$ behaves as a $\frac{k^2}{2}M$-form. Also we have

$$\text{(2.1-2)}\qquad V_{k_1}(z)V_{k_2}(w) = (z-w)^{Mk_1k_2}\,{}^{\circ}_{\circ} V_{k_1}(z)V_{k_2}(w) {}^{\circ}_{\circ}.$$

In the following we only consider irreducible highest weight representations of H_M with highest weight vectors $|p\rangle$ where p's are *integers*.

Let $\Lambda = \{\overline{0}, \overline{1}, \ldots, \overline{M-1}\}$ be representatives of the module $\mathbf{Z}/M\mathbf{Z}$. For each $\overline{p} \in \{\overline{0}, \overline{1}, \ldots, \overline{M-1}\}$, put

$$\mathcal{H}(\overline{p}) := \bigoplus_{\overline{p}=p \mod M} \mathcal{F}(p).$$

Let $\mathfrak{X} = (C; Q_1, \ldots, Q_N; \xi_1, \ldots, \xi_N)$ be an N-pointed stable curve of genus g with formal neighbourhoods. To each point Q_j we associate an element $\overline{p_j} \in \Lambda$ and put

$$\vec{\overline{p}} = (\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_N),$$
$$\mathcal{H}(\vec{\overline{p}}) = \mathcal{H}(\overline{p}_1) \otimes \mathcal{H}(\overline{p}_2) \otimes \cdots \otimes \mathcal{H}(\overline{p}_N)$$

Put also

$$\mathcal{H}^\dagger(\vec{\overline{p}}) = \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}(\vec{\overline{p}}), \mathbf{C}).$$

We have a natural pairing

$$\begin{aligned} \mathcal{H}^\dagger(\vec{\overline{p}}) \times \mathcal{H}(\vec{\overline{p}}) &\to \mathbf{C} \\ (\langle\psi|, |\phi\rangle) &\mapsto \langle\psi|\phi\rangle \end{aligned}$$

where $\langle\psi|\phi\rangle$ means $\psi(|\phi\rangle)$.

§2.2 Conformal blocks and theta functions

Definition 2.2.1. Conformal block (or space of vacua) $\mathcal{V}^\dagger_{\vec{\overline{p}}}(\mathfrak{X})$ attached to the N-pointed stable curve $\mathfrak{X} = (C; Q_1, \ldots, Q_N; \xi_1, \ldots, \xi_N)$ with formal neighbourhoods $\mathfrak{X}$ is a subspace of $\mathcal{H}^\dagger(\vec{\overline{p}})$ consisting of vectors $\langle\psi|$ satisfying the following conditions.

(1) For each $|\phi\rangle \in \mathcal{H}(\vec{\overline{p}})$, the data

$$\langle\psi|\rho_j(a(\xi_j))|\phi\rangle d\xi_j, \quad j = 1, 2, \ldots, N$$

are the Laurent expansions of an element

$$\omega \in H^0(C, \omega_C(* \sum Q_j))$$

at Q_j's with respect to the formal coordinates ξ_j's,

(2) For each $|\phi\rangle \in \mathcal{H}(\vec{\overline{p}})$, the data

$$\langle\psi|\rho_j(V_{\pm 1}(\xi_j))|\phi\rangle (d\xi_j)^{\frac{M}{2}}, \quad j = 1, 2, \ldots, N$$

are the Laurent expansions of an element $\tau \in H^0(C, \omega_C^{\otimes \frac{M}{2}}(* \sum Q_j))$ at Q_j's with respect to the formal coordinates ξ_j.

Theorem 2.2.2. *We have*

$$\dim_{\mathbf{C}} \mathcal{V}^{\dagger}_{\overline{\overline{p}}}(\mathfrak{X}) = \begin{cases} M^g, & \text{if } \ \overline{p}_1 + \cdots + \overline{p}_N = \overline{0} \\ 0, & \text{otherwise} \end{cases}$$

where g is the genus of the stable curve C.

For a proof of the above theorem we first rewrite the conditions (1), (2) in Definition 2.2.1. Analogue of Lemma 1.2.5 shows that the condition (1) is equivalent to the condition

$$(1^*) \qquad \sum_{j=1}^{n} \operatorname*{Res}_{\xi_j=0}(\langle\psi|\rho_j(a(\xi_j))|\phi\rangle g(\xi_j)d\xi_j) = 0$$

for every $g \in H^0(C, \mathcal{O}_C(*\sum Q_j))$, where $g(\xi_j)$ is the Laurent expansion of g at Q_j. The condition (2) is equivalent to the condition

$$(2^*) \qquad \sum_{j=1}^{N} \operatorname*{Res}_{\xi_j=0}(\langle\psi|\rho_j(V_{\pm 1}(\xi_j))|\phi\rangle h(\xi_j)d\xi_j) = 0$$

for every $h \in H^0(C, \omega_C^{\otimes(1-\frac{M}{2})}(*\sum Q_j))$, where $h(\xi_j)(d\xi_j)^{\frac{M}{2}}$ is the Laurent expansion of h at Q_j. Thus, the conditions (1^*) and (2^*) are analogue of the gauge condition (1.2-8).

In the following we choose integers p_j such that $\overline{p}_j = p_j \mod M$, and put

$$|p_1, p_2, \ldots, p_N\rangle = |p_1\rangle \otimes |p_2\rangle \otimes \cdots \otimes |p_N\rangle.$$

Since we have

$$\operatorname*{Res}_{\xi_j=0}\{(a(\xi_j)|p_j\rangle d\xi_j\} = a(0)|p_j\rangle = p_j|p_j\rangle,$$

applying the condition (1^*) to $\langle\psi| \in \mathcal{V}^{+}_{\overline{\overline{p}}}(\mathfrak{X})$ and $1 \in H^0(C, \mathcal{O}_C(*\sum Q_j))$, we have

$$(\sum_{j=1}^{N} p_j)\langle\psi|p_1, p_2, \ldots, p_N\rangle = 0.$$

Hence, if $\langle\psi|p_1, p_2, \ldots, p_N\rangle \neq 0$, then $\sum_{j=1}^{N} p_j = 0$.

Let us consider an N-pointed projective line $(\mathbf{P}^1; z_1, z_2, \ldots, z_N)$ with $z_1 = 0, z_2 = 1, z_N = \infty$. Let z (resp. w) be a coordinate of an affine line in $\mathbf{P}^1$ containing 0 (resp.∞) with $z \cdot w = 1$. Put

$$(2.2\text{-}1) \qquad \xi_j = \begin{cases} z - z_j, & j = 1, 2, \cdots, N-1 \\ w, & j = N, \end{cases}$$

and

$$\mathfrak{X} = (\mathbf{P}^1; z_1, z_2, \ldots, z_N; \xi_1, \xi_2, \ldots, \xi_N).$$

First we shall prove the following proposition.

Proposition 2.2.3.

$$\dim_{\mathbf{C}} \mathcal{V}^{\dagger}_{\vec{\overline{p}}}(\mathfrak{X}) = \begin{cases} 1, & \text{if} \quad \overline{p}_1 + \overline{p}_2 + \cdots + \overline{p}_N = 0 \\ 0, & \text{otherwise} \end{cases}$$

Let $F_0\mathcal{H}(\overline{p}_j)$ be a subspace of $\mathcal{H}(\overline{p}_j)$ spanned by the highest weight vectors $|lM + p_j\rangle$, $l \in \mathbf{Z}$ over $\mathbf{C}$. Put

$$F_0\mathcal{H}(\vec{\overline{p}}) = F_0\mathcal{H}(\overline{p}_1) \otimes F_0\mathcal{H}(\overline{p}_2) \otimes \cdots \otimes F_0\mathcal{H}(\overline{p}_N).$$

To prove the above proposition we need the following lemma.

Lemma 2.2.4. *Under a natural mapping*

$$j : \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}(\vec{\overline{p}}), \mathbf{C}) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(F_0\mathcal{H}(\vec{\overline{p}}), \mathbf{C}),$$

the conformal block $\mathcal{V}^{\dagger}_{\vec{\overline{p}}}(\mathfrak{X})$ of the N-pointed projective line with coordinates (2.2-1) is mapped injectively.

The lemma and the above consideration imply

$$\mathcal{V}^{\dagger}_{\vec{\overline{p}}}(\mathfrak{X}) = 0$$

if $\overline{p}_1 + \overline{p}_2 + \ldots + \overline{p}_N \neq 0$. Therefore, assume $\overline{p}_1 + \overline{p}_2 + \ldots + \overline{p}_N = 0$. Choose p_j's in such a way that

$$p_1 + p_2 + \ldots + p_N = 0.$$

For an element $\langle\psi| \in \mathcal{V}^{\dagger}_{\vec{\overline{p}}}(\mathfrak{X})$, put

$$\psi_{l_1, l_2, \ldots, l_N} = \langle\psi|(|l_1M + p_1\rangle \otimes |l_2M + p_2\rangle \otimes \cdots \otimes |l_NM + p_N\rangle).$$

If $\psi_{l_1, l_2, \ldots, l_N} \neq 0$, then $l_1 + l_2 + \ldots l_N = 0$. The condition (1*) implies that $\psi_{l_1, l_2, \ldots, l_N}$ determines uniquely the values

$$\langle\psi|(a(-n_1^{(1)}) \ldots a(-n_{k_1}^{(1)})|l_1M + p_1\rangle \otimes a(-n_1^{(2)}) \ldots a(-n_{k_2}^{(2)})|l_2M + p_2\rangle \otimes$$
$$\ldots \otimes a(-n_1^{(N)}) \ldots a(-n_{k_N}^{(N)})|l_NM + p_M\rangle),$$

for any positive integers $n_j^{(i)}$. Also,the condition (2*) implies that $\psi_{l_1, l_2, \ldots, l_N}$ can be uniquely determined by the value $\psi_{0,0,\ldots,0}$. Thus, we conclude that

$$\dim_{\mathbf{C}} \mathcal{V}^{\dagger}_{\vec{\overline{p}}}(\mathfrak{X}) = 1$$

This proves Proposition 2.2.3.

In the following we assume

$$\overline{p}_1 + \overline{p}_2 + \ldots + \overline{p}_N = 0$$

Let us define a subspace $\mathcal{V}^{\dagger}_{\vec{\overline{p}}}(n)$ of $\mathcal{H}^{\dagger}(\vec{\overline{p}})$. An element $\langle\psi|$ is in $\mathcal{V}^{\dagger}_{\vec{\overline{p}}}(n)$, if $\langle\psi|$ satisfies the following two conditions (1^{**}_n) and (2^{**}_n).

$$(1^{**}_n) \qquad \sum_{j=1}^{N} \operatorname{Res}_{\xi_j=0}(\langle\psi|a(\xi_j)|\phi\rangle g_j(\xi_j)d\xi_j) = 0$$

for all

$$(g_j(\xi_j)d\xi_j) \in \mathbf{C} \bigoplus \oplus_{j=1}^{N}(\mathbf{C}[\xi_j]\xi_j^{-n}), \quad \text{and} \quad |\phi\rangle \in \mathcal{H}(\vec{\overline{p}}),$$

where an element $c \in \mathbf{C}$ in the right hand side can be considered as $(c, c, \ldots, c)$.

$$(2^{**}_n) \qquad \sum_{j=1}^{N} \operatorname{Res}_{\xi_j=0}(\langle\psi|V_{\pm 1}(\xi_j)|\phi\rangle g_j(\xi_j)d\xi_j) = 0$$

for all

$$(h_j(\xi_j)(d\xi_j)^{\frac{M}{2}}) \in \bigoplus_{j=1}^{N}(\mathbf{C}[\xi_j]\xi_j^{-n})(d\xi_j)^{(1-\frac{M}{2})}, \quad \text{and} \quad |\phi\rangle \in \mathcal{H}(\vec{\overline{p}}).$$

Key Lemma. *Under the above notation we have*

$$\dim \mathcal{V}^{\dagger}_{\vec{\overline{p}}}(n) < \infty.$$

To prove the Key Lemma we first prove Theorem 2.2.2 for non-singular curve. In this case we can prove a stronger theorem.

Theorem 2.2.5. *Let $\mathfrak{X}$ be an N-pointed* smooth *curve of genus g with formal neighbourhoods. If we have*

$$\overline{p}_1 + \overline{p}_2 + \cdots + \overline{p}_N = 0,$$

then there is a canonical isomorphism

$$\mathcal{V}^{\dagger}_{\vec{\overline{p}}}(\mathfrak{X}) \simeq H^0(J(C), \mathcal{O}(M\Theta)).$$

where Θ is the theta divisor.

Since the vector space $H^0(J(C), \mathcal{O}(M\Theta))$ of the theta functions of level M is of dimension M^g, this theorem implies Theorem 2.2.2 for non-singular underlying curve C.

To prove Theorem 2.2.5 we need to consider the correlation functions of Vertex operators. For $\vec{k} = (k_1, k_2, \dots, k_m)$, $\vec{p} = (p_1, p_2, \dots, p_N)$ with $\overline{p}_j = p_j \mod M$ and $\langle\Psi| \in \mathcal{V}^{\dagger}_{\overline{\vec{p}}}(\mathfrak{X})$ put

(2.2-1)

$$\Psi_{\vec{k}\vec{p}}(z)(dz_1)^{\frac{k_1^2}{2}M}\cdots(dz_N)^{\frac{k_N^2}{2}M}$$
$$= \langle\Psi|V_{k_1}(z_1)V_{k_2}(z_2)\cdots V_{k_m}(z_m)|\vec{p}\rangle(dz_1)^{\frac{k_1^2}{2}M}\cdots(dz_N)^{\frac{k_N^2}{2}M}.$$

Then, by the operator product expansions (2.1-1) we can show the following Lemma.

Lemma 2.2.6. 1) *If $\Psi_{\vec{k}\vec{p}}(z) \not\equiv 0$, then we have*

$$\sum_{i=1}^{m} p_i + M\sum_{a=1}^{N} k_a = 0.$$

2) *Near $z_a = z_b$ we have the expansion*

$$\Psi_{\vec{k}\vec{p}}(z) = (z_a - z_b)^{Mk_ak_b}\Psi_{\vec{k}'\vec{p}}(z') + \quad \textit{higher order terms}$$

where

$$\vec{k}' = (k_1, \cdots, \overset{\vee}{k_a}, \cdots, k_m)$$
$$z' = (z_1, \cdots, \overset{\vee}{z_a}, \cdots, z_m).$$

3) *Near $z_a = Q_j$ we have the expansion*

$$\Psi_{\vec{k}\vec{p}}(z) = (z_a - z_b)^{k_ap_j}\Psi_{\vec{k}_{\widehat{a}}\vec{p}_{\widehat{a}}}(z) + \quad \textit{higher order terms}$$

where

$$\vec{k}_{\widehat{a}} = (k_1, \cdots, \overset{\vee}{k_a}, \cdots, k_m)$$
$$\vec{p}_{\widehat{a}} = (p_1, \cdots, p_a + k_a, \cdots, p_N).$$

Conversely, for $\langle\Psi| \in \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}(\vec{p}), \mathbf{C})$ if we define

$$\Psi_{\vec{k}\vec{p}}(z)(dz_1)^{\frac{k_1^2}{2}M}\cdots(dz_N)^{\frac{k_N^2}{2}M}$$

by (2.2-1) and if it has the above properties 1), 2), 3) for all $\vec{k}$ and $\vec{p}$, then

$$\langle\Psi| \in \mathcal{V}_{\overline{\vec{p}}}(\mathfrak{X}).$$

Furthermore, $\langle\Psi|$ is uniquely determined by

$$\Psi_{\vec{k}\vec{p}}(z)(dz_1)^{\frac{k_1^2}{2}M}\cdots(dz_N)^{\frac{k_N^2}{2}M}$$

for all $\vec{k}$ and $\vec{p}$.

Let

$$E(z,w)(dz)^{-1/2}(dw)^{-1/2}$$

be the prime form of the non-singular curve C. Put

$$\psi_{\vec{k}\vec{p}}(z)=\prod_{1\le a<b\le m}E(z_a,z_b)^{Mk_ak_b}\prod_{a=1}^{m}\prod_{j=1}^{N}E(z_a,Q_j)^{k_ap_j}.$$

Then, by Lemma 2.2.6 it is easy to show that

$$\phi_{\vec{k}\vec{p}}=\frac{\Psi_{\vec{k}\vec{p}}(z)}{\psi_{\vec{k}\vec{p}}(z)}$$

is a multivalued holomorphic functions on C^m. Multivaluedness comes from that of the prime form. Let $\{\alpha_1,\cdots,\alpha_g,\beta_1,\cdots,\beta_g\}$ and $\{\omega_1,\cdots,\omega_g\}$ are a basis of the first homology group of the curve C and a basis of holomorphic one-forms of C with

$$\int_{\alpha_i}\omega_j=\delta_{ij}\qquad\int_{\beta_i}\omega_j=\tau_{ij},$$

respectively. Let

$$\begin{aligned} j:C &\to J(C)\\ P &\mapsto [(\int_{P_0}^{P}\omega_1,\cdots,\int_{P_0}^{P}\omega_g)]\end{aligned}$$

be a natural mapping of the curve C into its Jacobian. Then, quasi periodicity of the prime form is given by

$$E(P+\alpha_i,Q)=E(P,Q)$$

$$E(P+\beta_i,Q)=\exp(-2\pi\sqrt{-1}(\frac{\tau_{ii}}{2}+\int_Q^P\omega_i))E(P,Q)$$

where $P+\alpha_i$ means the analytic continuation along the cycle α_i. Then, $\phi_{\vec{k}\vec{p}}(z)$ has the following quasi periodicity.

$$\phi_{\vec{k}\vec{p}}(z_1,\cdots,z_a+\alpha_i,\cdots z_m)=\phi_{\vec{k}\vec{p}}(z_1,\cdots,z_a,\cdots,z_m)$$

$$\begin{aligned}\phi_{\vec{k}\vec{p}}(z_1,\cdots,z_a+\beta_i,\cdots z_m)&=\exp(\frac{k_a^2}{2}M\tau_{ii}\\ &+k_a\sum_{b+1}^{m}Mk_b\int_{P_0}^{z_b}\omega_i+k_a\sum_{j=1}^{N}p_j\int_{P_0}^{Q_j}\omega_i)\phi_{\vec{k}\vec{p}}(z_1,\cdots,z_a,\cdots,z_m)\end{aligned}$$

Let us define a holomorphic mapping of C^m to the Jacobian $J(C)$ by

$$\begin{aligned} \varpi_n : \qquad C^m \qquad &\to \qquad J(C) \\ (P_1, \cdots, P_n) &\mapsto [(\sum_{b=1}^{m} k_b \int_{P_0}^{P_b} \vec{\omega} + (\sum_{j=1}^{N} \frac{p_j}{M} \int_{P_0}^{Q_j} \vec{\omega})]. \end{aligned}$$

It is well-known that if m is sufficiently large, then ϖ is surjective. Moreover, we can show the following lemma.

Lemma 2.2.7. *The multivalued holomorphic function $\phi_{\vec{k}\vec{p}}$ is a holomorphic section of the line bundle $\varpi_m^*[\Theta]$, where Θ is the theta divisor. Moreover, if m is sufficiently large, we have*

$$H^0(C^m, \mathcal{O}(\varpi_m^*[M\Theta])) = \varpi_m^* H^0(J(C), \mathcal{O}(M\Theta)).$$

This implies Theorem 2.2.5.

To prove Key Lemma we also need the following lemma.

Lemma 2.2.8. *For positive integers n and N there exist a smooth curve D of genus g and points $Q_1, \cdots, Q_N$ on D with local coordinates $\xi_1, \xi_2, \cdots, \xi_N$ such that*

$$\begin{aligned} Gr_\bullet^F H^0(D, \mathcal{O}_D(* \sum Q_j)) &\subset \mathbf{C} \bigoplus \oplus_{j=1}^N \mathbf{C}[\xi_j^{-1}]\xi_j^{-n} \\ Gr_\bullet^F H^0(D, \omega_D^{\otimes(1-\frac{M}{2})}(* \sum Q_j)) &\subset \mathbf{C} \bigoplus \oplus_{j=1}^N \mathbf{C}[\xi_j^{-1}]\xi_j^{-n}(d\xi_j)^{1-\frac{M}{2}} \end{aligned}$$

where the filtration F can be defined by the order of poles at Q_j.

The first inclusion can be proved, if the divisor $n(Q_1 + Q_2 + \cdots + Q_N)$ is non-special on a curve D. The second inclusion is trivially true, if we have $(2g-2)(1-\frac{M}{2}) > nN$.

Now introducing the filtration on $\mathcal{H}(\vec{\overline{p}})$ and $\mathcal{H}^\dagger(\vec{\overline{p}})$ compatible with the filtration in Lemma 2.2.8, we can show Key Lemma which also implies *finite dimensionality* of $\mathcal{V}_{\vec{\overline{p}}}(\mathfrak{X})$ for all N-pointed stable curve with formal neighbourhoods.

Now let us consider a semi-stable curve C. For a double point $P \in C$ we let $\pi : \tilde{C} \to C$ be the normalization at the point P. Then, the inverse image $\pi^{-1}(P)$ of the point P consists of two points P_+, P_-. Let η_+, η_- be formal coordinates of P_+ and P_-respectively such that C is defined formally in a neighbourhood of the origin of $\mathbf{C}^2$ by an equation $\eta_+ \cdot \eta_- = 0$. Let $\mathfrak{X} = (C; Q_1, \ldots, Q_N; \xi_1, \ldots, \xi_N)$ be an N-pointed stable curve with formal neighbourhoods whose underling curve is the semi-stable curve C. Put

$$\tilde{\mathfrak{X}} = (\tilde{C}; Q_1, \ldots, Q_N, P_+, P_-; \xi_1, \ldots, \xi_N, \eta_+, \eta_-).$$

Then,we have the following theorem.

Theorem 2.2.9. *Under the above notation and assumptions, we have a canonical isomorphism.*

$$\bigoplus_{\overline{q}\in \mathbf{Z}/M\mathbf{Z}} \mathcal{V}^{\dagger}_{\overline{q},-\overline{q},\vec{\overline{p}}}(\tilde{\mathfrak{X}}) \simeq \mathcal{V}^{+}_{\vec{\overline{p}}}(\mathfrak{X}).$$

From this theorem and Proposition 2.2.3 we infer the following lemma.

Lemma 2.2.10. *Let $\mathfrak{X} = (C; Q_1, \ldots, Q_N; \xi_1, \ldots, \xi_N)$ be an N-pointed stable curve with formal neighbourhoods. Assume that all the irreducible component of the semi-stable curve C are $\mathbf{P}^1$ and the genus of C is g. Then, we have*

$$\dim_{\mathbf{C}} \mathcal{V}^{\dagger}_{\vec{\overline{p}}}(\mathfrak{X}) = \begin{cases} M^g, & \text{if } \ \overline{p}_1 + \cdots + \overline{p}_N = 0 \\ 0, & \text{otherwise.} \end{cases}.$$

Now we need to show that $\dim_{\mathbf{C}} \mathcal{V}^{+}_{\vec{\overline{p}}}(\mathfrak{X})$ depends only on the genus of the underlying curve C. For that purpose we need to consider the sheaf of conformal blocks attached to a family of N-pointed stable curves with formal neighbourhoods. Similar arguments as in the previous chapter gives a proof of Theorem 2.2.2.

The family $\mathcal{V}_{\vec{\overline{p}},N} = \bigcup_{\mathfrak{X}} \mathcal{V}^{\dagger}_{\vec{\overline{p}}}(\mathfrak{X})$ over the moduli space $\overline{\mathcal{M}}^{(\infty)}_{g,N}$ of N-pointed curves of genus g with formal neighbourhoods. By a similar method as the one in [TUY], we can show that $\overline{\mathcal{V}}_{\vec{\overline{p}},N}$ comes from a sheaf $\mathcal{V}^{(1)}_{\vec{\overline{p}},N}$ on $\mathcal{M}^{(1)}_{g,N}$, the moduli space of N-pointed curves of genus g with first order neighbourhoods. Then, by Key Lemma we can show that $\mathcal{V}^{(1)}_{\vec{\overline{p}},N}$ is a coherent $\mathcal{O}_{\overline{\mathcal{M}}^{(1)}_{g,N}}$ -module and it carries a logarithmic projectively flat connection. From these fact we infer that $\mathcal{V}^{(1)}_{\vec{\overline{p}},N}$ is locally free on the open part of $\overline{\mathcal{M}}^{(1)}_{g,N}$ corresponding to non-singular curves.

Again, using a similar arguments as in Chapter I §1.3 and §1.4, we can show that $\mathcal{V}^{(1)}_{\vec{\overline{p}},N}$ is locally free. By Lemma 2.2.11 this implies our Theorem 2.2.2.

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