LIPSCHITZ CONSTRUCTION OF INFINITE GROUPS AND HIGHER SIGNATURES

Tsuyoshi KATO

Department of Mathematics, Faculty of Science, Kyoto University.

Introduction

In this paper, we study Novikov conjecture. Novikov conjecture is the problem of homotopy invariance of higher signatures. So far several partial solutions have been given to geometric classes of discrete groups. For example, the problem has been solved for discrete groups which are word hyperbolic, fundamental groups of closed, non positively curved manifolds, etc.

In [CGM] , A.Connes , M.Gromov and H.Moscovici solved the problem in the case when discrete groups satisfy several geometrical conditions . These coditions are satisfied for large class of discrete groups .

In this paper, we will extend their method. Although essencially, they have required the existence of finite dimensional simplicial complex which is homotopically equivalent to $K(\Gamma, 1)$ over $\mathbb{Q}$, we no longer assume it. Our basic idea is to approximate $K(\Gamma, 1)$ by finite dimensional simplicial complexes.

To state the problem , let us recall the definition of higher signatures. A higher signature is a characteristic number of the form

$$< L(M)f^*(x), [M] >$$

where M is a closed manifold , $[M]$ the fundamental class, $L(M)$ its Hirzebruch's L genus , $f : M \mapsto K(\Gamma, 1)$ is a map and $x \in H^*(K(\Gamma, 1); \mathbb{Q})$.

We will introduce later a geometric class of discrete groups. Motivated by the property of non positively curved manifolds , which is the convexity on the universal covering space, a notion , bicombing for a discrete group Γ is defined in [G1]. The property , bicombing is the convexity on the Cayley graph of Γ for a fixed generator. In the study of discrete groups as a metric space , it is desirable if a notion on them is characterized independently of choice of a generator of Γ. For this reason , in the definition of the bicombing groups , a path connecting two points of Γ is not required to be geodesic in the Cayley

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

graph. In this paper , we introduce quasi geodesic bicombing groups which in addition to bicombing , have additional properties that geodesically bicombing groups have . In fact our proof is available for a product of quasi geodesic bicombing groups. For this reason, we only have to prove homotopy invariance of higher signatures for $4k$ dimensional closed manifolds.

Main Theorem. *Let Γ be a torsion free quasi geodesic bicombing group. Then the higher signatures of Γ are oriented homotopy invariants.*

In the case when $x = 1 \in H^0(K(\Gamma, 1)), \mathbb{Q})$, as is well known as the Hirzebruch's signature theorem , the characteristic number is the index for the Laplacian, on the other hand , the analytical index is the signature through the de Rham theorem. So to show the problem in general , it is natunal to try to find generalized signatures , or signatures with coefficients and their signature theorem. At this point , there is a remarkable work by A.S. Miscenko ([Mis2]). He defined a notion , Fredholm representation with respect to any discrete group Γ and constructed a map from the set of Fredholm representations to $K^*(K(\Gamma, 1))$.

Theorem (Miscenko) [Mis2]. *Let M_1 and M_2 be $4k$ dimensional homotopically equivalent manifolds , a homotopy equivalence $h : M_1 \mapsto M_2$ and a continuous map $f : M_2 \mapsto K(\Gamma, 1)$. Moreover let F be a Fredholm representation with respect to Γ and ξ_F a virtual bundle over $K(\Gamma, 1)$ constructed from F. Then*

$$< L(M_1)(h \circ f)^*(ch^*(\xi_F)), [M_1] > \quad = \quad < L(M_2) f^*(ch^*(\xi_F)), [M_2] > .$$

In the framework of Kasparov's KK theory , Miscenko 's map is written as follows,

$$\mu : \quad KK_\Gamma(\mathbb{C}, \mathbb{C}) \longmapsto K^*(K(\Gamma, 1)) \qquad \text{by} \quad \mu(F) = \xi_F.$$

In [CGM] , it is shown that if Γ is word hyperbolic, then $\mu \otimes \mathbb{Q}$ is surjective. It should be pointed out that when Γ is word hyperbolic, the rank of the cohomology with rational coefficient is finite. More strongly there is a finite dimensional contractible simplicial complex $P(\Gamma)$ with a proper cocompact action of Γ. This complex is called a Rips complex. For any discrete group Γ, there exists a family of Rips complexes $\{P_n(\Gamma)\}$ and when Γ is word hyperboic, $P_n(\Gamma)$ is contractible for sufficiently large n. We shall define them later. If Γ happens to be torsion free , then the action of Γ on $P(\Gamma)$ is also free . Thus in this case , $P(\Gamma)/\Gamma$ is nothing but a $K(\Gamma, 1)$ space. By passing a well controlled Poincaré duality , $\mu \otimes \mathbb{Q}$ is shown to be surjective when Γ is word hyperbolic.

Let us consider a fiberwise proper Lipschitz map

$$\alpha : \quad P \times_\Gamma \Gamma \mapsto \mathbb{R}^N$$

where P is Γ manifold of dimension $N - d$. Then we construct an element $c_\alpha \in H^*(\Gamma)$ from α as follows. Let $\Delta(\gamma_0 \dots \gamma_k)$ be a simplex spanned by $\gamma_0, \dots, \gamma_k \in \Gamma$ and extend linearly to $\alpha : P \times \Delta \mapsto \mathbb{R}^N$. Then as a k cochain on Γ ,

$$c_\alpha(\gamma_0, \dots, \gamma_k) = \int_{P \times \Delta} \alpha_\Delta^*(\omega)$$

where $\omega \in H_{cp}^N(\mathbb{R}^N)$ is the Bott element . The above construction extends to

$$\alpha\cap : \quad H_i(P : \Gamma) \mapsto H^{N-i}(\Gamma)$$

If $c \in H^*(\Gamma)$ lies in the image of $\alpha\cap$ for some (P, α) , then it is called a proper Lipschitz cohomology class. In [CGM], it is shown that a higher signature coming from a proper Lipschitz cohomology class satisfies Novikov conjecture for an arbitrary discrete group Γ. When Γ is word hyperbolic , it is shown in [CGM] that proper Lipschitz cohomology classes occupy $H^*(B\Gamma)$. For simplicity , let us assume Γ is torsion free. If we take $P(\Gamma)$ instead of P , then

$$\alpha\cap : \quad H_i(B\Gamma) \mapsto H^{N-i}(B\Gamma)$$

Moreover if α is induced from a (Γ invariant) Poincaré duality map

$$\alpha : \quad P \times P \mapsto TP$$

then $\alpha\cap$ is nothing but Poincaré duality. To obtain such α , we deform an ordinary Poincaré duality map to be a fiberwise proper Lipschitz map by a proper homotopy . Using the fact that $P(\Gamma)$ is contractible , there exists a proper Lipschitz map with small Lipschitz constant , which is fiberwise properly homotopic to identity,

$$F : \qquad P(\Gamma) \times_\Gamma P(\Gamma) \mapsto P(\Gamma) \times_\Gamma P(\Gamma)$$

Let $\alpha : \quad P(\Gamma) \times_\Gamma P(\Gamma) \mapsto TP$ be a Poincaré duality map . We want to deform α so that it becomes fiberwise proper Lipschitz. Take sufficiently large r and let $D = \{(x, y) \in P(\Gamma) \times_\Gamma P(\Gamma), d(x, y) \leq r\}$. By modifying α on $F^{-1}(D)$, we may assume $\mu^{-1}\alpha \circ F|_{\partial F^{-1}(D)} = \alpha|_{\partial F^{-1}(D)}$ for some $0 < \mu < 1$. Then letting $B_i = \{(x, y) \in P(\Gamma) \times_\Gamma P(\Gamma), F^i(x, y) \equiv F \circ \dots \circ F(x, y) \in D\}$, $D_i \equiv B_i \backslash B_{i-1}$,

we define another $\alpha_\infty : \quad P(\Gamma) \times_\Gamma P(\Gamma) \mapsto TP$ by $\alpha_\infty|_{D_i} = \mu^{-i+1}\alpha \circ F^{i-1}$. Then it is easy to see that α_∞ is continuous on $P(\Gamma) \times_\Gamma P(\Gamma)$. Because the Lipschitz constant of F can be taken to be arbitrarily small , it follows that α_∞ is fiberwise proper Lipschitz. Moreover , using that F is fiberwise properly homotopic to identity , α_∞ is fiberwise properly homotopic to α . Thus

$$\alpha_\infty : \quad P(\Gamma) \times_\Gamma P(\Gamma) \mapsto TP$$

induces Poincaré duality which is fiberwise proper Lipschitz. It should be pointed out that the existence of such F implies that $P(\Gamma)$ is contractible. It is unclear whether there might exist a Rips complex which is contractible on a class larger than word hyperbolic groups. Although the definition of our class is motivated by the fundamental groups of closed non positively curved manifolds, it is unclear whether there exists a Rips complex which is contractible for any discrete group in our class. In particular , we do not know whether $H^*(\Gamma; \mathbb{Q})$ might be of finite rank , or not. Thus we cannot expect the Poincaré duality on $K(\Gamma, 1)$. Considering these things , we will have reached to approximate $K(\Gamma, 1)$ by a family of finite dimensional simplicial complexes , Rips complexes.

Let Γ be an arbitrary finitely generated discrete group. We fix a generator of Γ. Then Γ has a metric with respect to the generator, so called word metric.

Definition. Let $n \geq 1$. The Rips complex $P_n(\Gamma)$ is a simplicial complex whose k simplices are spanned by $\{(\gamma_0, \ldots, \gamma_k), |\gamma_i - \gamma_j| \leq n, \gamma_i \in \Gamma\}$ where the metric $|\quad|$ is the word metric. $P_n(\Gamma)$ has the following properties,

(1) Γ acts on $P_n(\Gamma)$ simplicially.
(2) If Γ is torsion free , then the action of Γ on $P_n(\Gamma)$ is also free.
(3) $P_n(\Gamma)/\Gamma$ is compact.
(4) $P_n(\Gamma) \subset P_{n+1}(\Gamma) \ldots$.
(5) [Rips] If Γ is word hyperbolic, then for sufficiently large n , $P_n(\Gamma)$ is contractible.

Thus in the case of a torsion free word hyperbolic group Γ , we can take $P_n(\Gamma)/\Gamma$ as a $K(\Gamma, 1)$ space. Beyond word hyperbolic groups , we use a family of Rips complexes instead of a single one , which we regard as an approximation of $K(\Gamma, 1)$ space. Fortunately, we have the following theorem.

Theorem [ECHLPT]. *If Γ is a combable group, there is a CW complex of $K(\Gamma, 1)$ type, having a finite number of cells in each dimension.*

Combable group is a very large class and it contains word hyperbolic groups and our groups later defined. The theorem says that (co)homology rank on every fixed dimension is finite. Moreover it is well known

$$lim_{i \to \infty} H_*(P_i(\Gamma)/\Gamma) \quad = \quad H_*(P_\infty(\Gamma)/\Gamma).$$

If we take $P_\infty(\Gamma)/\Gamma$ as a $K(\Gamma,1)$ space, these tell us that as far as concerning (co)homology of $K(\Gamma,1)$, it can be possible to use approximation method. Now we summarize our context.

In §1 , we construct a Poincaré duality map

$$\alpha_0 : \quad \tilde{P}_0 \times_\Gamma \tilde{P}_0 \mapsto \mathbb{R}^N$$

where P_0 is a Rips complex $P_{d_0}(\Gamma)$ for some $d_0 \in \mathbb{N}$ and $\tilde{P}_0/\Gamma$ is a regular neighbourhood in an Euclidean space. Although in our case , we do not assume a Rips complex is contractible , we use the fact that any Rips complex $P_i(\Gamma)$ is contractible in $P_{n(i)}(\Gamma)$ for some $n(i)$. By taking a regular neighbourhood in $P_i(\Gamma)/\Gamma \subset P_{n(i)}(\Gamma) \subset \mathbb{R}^{N_i}$, it is possible to construct the above map. For our purpose , we need a family of maps

$$\alpha_i : \quad \tilde{P}_0 \times_\Gamma \tilde{P}_i \mapsto \mathbb{R}^{N_i}$$

where $P_i = P_{d_i}(\Gamma)$, some Rips complex , $d_0 \leq d_1 \leq \dots$, $N_0 \leq N_1 \leq \cdots \nearrow \infty$. Only α_0 is concerned with the Poincaré duality and each α_i has a well controlled Lipschitz constant on $N_i(r) \subset \tilde{P}_0 \times_\Gamma \tilde{P}_i$ where $N_i(r)$ is the set of points whose distance from the base point is less than r. In fact , the Lipschitz constant of α_i on $N_i(r)$ is bounded by some constant $\tilde{H}_i(r)$ which is a Lipschitz function of r. Concerning α_i , as we have to be careful at the boundary , we will discuss about it at the end of §2. Next using a family of maps $\{\alpha_i\}_{i\geq 0}$, we deform α_0 so that we can obtain a fiberwise proper Lipschitz map . In §3, §4 , for our groups , we construct a family of maps

$$G_i : \quad \tilde{P}_0 \times_\Gamma \tilde{P}_i \mapsto \tilde{P}_0 \times_\Gamma \tilde{P}_{i+1}.$$

Each G_i is a fiberwise proper contruction. Moreover it is properly homotopic to an embedding. In §2 , postponing the construction , we construct , using them , a fiberwise proper Lipschitz map

$$\alpha_\infty : \quad \tilde{P}_0 \times_\Gamma \tilde{P}_0 \mapsto \mathbb{R}^\infty$$

fiberwise properly homotopic to α_0. Let $\mathrm{pr} : \mathbb{R}^\infty \mapsto \mathbb{R}^{N_0}$ be the standard projection. Then we construct α_∞ so that

$$\mathrm{pr} \circ \alpha_\infty : \quad \tilde{P}_0 \times_\Gamma \tilde{P}_0 \mapsto \mathbb{R}^{N_0}$$

is also fiberwise proper Lipschitz . Note that although we cannot directly connect α_0 and $\mathrm{pr}\circ\alpha_\infty$ by a proper homotopy , it is possible by passing through

α_∞, which is a map to " tangent space of $P_\infty(\Gamma) = E\Gamma$ ". To construct α_∞, we choose $\{r_i\}_{i\geq 0}$ carefully and put $\tilde{\alpha}_i(x) \equiv \alpha_i(G_{i-1} \circ \dots G_0(x))$. Roughly speaking , α_∞ is constructed by modifying $\tilde{\alpha}_\infty$, where $\tilde{\alpha}_\infty(x) = \mu^{-i}\alpha_i(x)$ for $x \in D_i \equiv B_i \backslash B_{i-1}$, $B_i \equiv (G_{i-1} \circ \dots G_0)^{-1}(N_i(r_i))$. In §5, we shortly review Kasparov's KK theory. In §6, §7, using pr $\circ\alpha_\infty$, we construct the following commutative diagram. Let us regard P_∞/Γ as a $K(\Gamma, 1)$ space and r : $P_0/\Gamma \mapsto P_\infty/\Gamma$ be the inclusion. Then we have the following commutative diagram.

$$\begin{array}{ccc} KK_\Gamma(\tilde{P}_0,\ \mathrm{pt}\ ) & \xrightarrow{\phi\circ\mu} & K^*(B\Gamma) \\ {\scriptstyle ch_*}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{r}^*\circ ch^*} \\ H_*^{\inf}(\tilde{P}_0 : \Gamma) & \xrightarrow[\alpha_0\cap]{} & H^*(P_0/\Gamma) \end{array}$$

where $\phi : KK_\Gamma(\tilde{P}_0,\ \mathrm{pt}\ ) \mapsto KK_\Gamma(\ \mathrm{pt}\ ,\ \mathrm{pt}\ )$. One of the most particular feature of KK theory is the existence of the intersection product ([K], see [Bl]),

$$KK_\Gamma(X, Y) \times KK_\Gamma(Y, Z) \mapsto KK_\Gamma(X, Z).$$

Using pr$\circ\alpha_\infty$, we obtain $K(\alpha) \in KK_\Gamma(\ \mathrm{pt}\ , \tilde{P}_0)$ and ϕ is the intersection product with $K(\alpha)$. To construct this element , pr$\circ\alpha_\infty$ is required to be fiberwise proper Lipschitz.

From the above diagram , we will now obtain the main theorem. Take P_0 so that $H_*(P_0/\Gamma) \mapsto H_*(B\Gamma)$ is surjective , $0 \leq * \leq N$ for a fixed N . Then for any $x \in H^*(B\Gamma)$, $0 \leq * \leq N$, there exists a Fredholm representation F such that $x - ch^*(\mu(F)) \in H^*(B\Gamma)$, $* \geq N + 1$. Thus take a homotopy equivalence h : $M_1 \mapsto M_2$ and a map f : $M_2 \mapsto B\Gamma$. Then

$$\begin{aligned} < L(M_1)(h \circ f)^*(x), [M_1] > &= < L(M_1)(h \circ f)^*(ch^*(\mu(F))), [M_1] > \\ < L(M_2)f^*(x), [M_2] > &= < L(M_2)f^*(ch^*(\mu(F))), [M_2] > . \end{aligned}$$

Thus combined with the result of Miscenko , we obtain the main theorem.

The authur would like to thank to Professor Kenji Fukaya for his encouragement. He also wishes to express his gratitude to Professor Toshikazu Sunada for useful comments.

PART I

§ 1 Poincaré duality

Simplicial embedding.

We embed each Rips complex into an Euclidean space and take a regular neighbourhood to obtain a manifold with boundary. In the following, we write P_i in place of $P_{n_i}(\Gamma)$, $n_i \in \mathbb{N}$, which is some Rips complex. Let $\{P_i\}_{i\geq 0}$ be a family of Rips complexes for a fixed generator of Γ where Γ is torsion free and consider simplicial embeddings,

$$P_0 \subset P_1 \subset P_2 \subset \dots .$$

We assign to every vertices of P_0/Γ $e_i = (0, \dots, 1, \dots)$ (i-th) and extend to vertices of P_1 , P_2 , $\dots$. Moreover extending to all simplices of P_i simplicially, we obtain compatible embeddings , $P_i \subset \mathbb{R}^{N_i}$. In the following, we write $\tilde{P}_i/\Gamma$: a regular neighbourhood of P_i/Γ in $\mathbb{R}^{N_i}$ for some N_i and $\tilde{P}_i$ its Γ covering. So $\tilde{P}_i/\Gamma$ is a manifold with boundary. We do not specify the dimension of the Euclidean space into which the Rips complex is embedded , but always assume that it is even. We introduce a geodesic metric on $\tilde{P}_i/\Gamma$ induced from the embedding in $\mathbb{R}^{N_i}$. Notice that even if $i << i'$, we can take embeddings so that $N_i = N_{i'}$.

Next we define the notion of quasi geodesic bicombing on a discrete group.

Definition 1.1 ([G1]). Γ is bicombing if for every fixed generator, its associated Cayley graph $G(\Gamma)$ has the following property. There exists

$$S: \quad \Gamma \times \Gamma \times [0,1] \mapsto G(\Gamma)$$

continuous , Γ equivariant and $k \geq 1$, $C \geq 0$ such that

$$\begin{aligned} &S(0, \gamma_1, \gamma_2) = \gamma_1, \qquad S(1, \gamma_1, \gamma_2) = \gamma_2, \\ &d(S_t(\gamma_1, \gamma_2), S_t(\gamma_1', \gamma_2')) \quad \leq k(td(\gamma_2, \gamma_2') + (1-t)d(\gamma_1, \gamma_1')) + C. \end{aligned}$$

In the case when $k = 1$ and $C = 0$, this is precisely convexity in the usual sense.

Definition 1.2 ([G1]). Γ : *a* bicombing is bounded if there exists $k \geq 1$, $C \geq 0$ such that

$$d(S_t(\gamma_1, \gamma_2), S_{t'}(\gamma_1, \gamma_2)) \leq k|t - t'|d(\gamma_1, \gamma_2) + C.$$

Our class is quasi geodesic bicombing.

Definition 1.3. S : a bounded bicombing for Γ is said to be quasi geodesic if S is normal in the sense that S has no locally back tracking with respect to variable t and there exists $k \geq 1$, $C \geq 0$ such that if we write paths

$$\omega_{\gamma_i} : \quad [0, |S(\quad, e, \gamma_i)|] \mapsto G(\Gamma)$$

for $S(\quad, e, \gamma_i)$ where $d(\gamma_1, \gamma_2) \leq 1$, then

$$Ud(\omega_{\gamma_1}, \omega_{\gamma_2}) \equiv \sup_t d(\omega_{\gamma_1}(t), \omega_{\gamma_2}(t)) \leq C,$$
$$|S(\quad, e, \gamma)| \leq k|\gamma| + C.$$

Moreover there exists $A > 0$, $B \geq 0$ such that

$$d(\gamma, S(t, \gamma, \gamma')) \geq Atd(\gamma, \gamma') - B.$$

Sometimes , we write $S : \Gamma \times [0,1] \mapsto G(\Gamma)$ for $S(t, \gamma) = S(t, e, \gamma)$ where e is the identity.

Remark 1.4.

(1) $(\tilde{P}_i, d_i)$ and $(\tilde{P}_i, d_{i+1}|_{\tilde{P}_i})$ is Lipschitz equivalent, where d_i is a geodesic metric induced from the embedding into $\mathbb{R}^{N_i}$. i.e. $1 \leq d_i / d_{i+1}|_{\tilde{P}_i} \leq \nu_i$ for some $\nu_i \geq 1$.

(2) Γ is X_i net in $\tilde{P}_i$ for some $X_i \geq 0$. (i.e. for any $x \in \tilde{P}_i$, there exists $x' \in \Gamma$, $d_i(x, x') \leq X_i$.)

(3) If P is a Rips complex of Γ : quasi geodesic bicombing, there exists $f : P \times P \times [0,1] \mapsto P$, not required to be continuous , such that

(A) $f|_{\Gamma \times \Gamma \times [0,1]}$ is the continuous , quasi geodesic bicombing,

(B) there exists $k \geq 1$, $C \geq 0$, $A > 0$, $B \geq 0$ such that

a $d(f_t(x,y), f_t(x',y')) \leq k(td(y,y') + (1-t)d(x,x')) + C,$

b $d(f_t(x,y), f_{t'}(x,y)) \leq k|t - t'|d(x,y) + C,$

c $d(f_t(x,y), x) \geq Atd(x,y) - B,$

(C) $f_0(x,y) = x$, $f_1(x,y) = y$.

(4) It may happen that the product $\Gamma = \Gamma_1 \times \Gamma_2$ of two quasi geodesic bicombing groups Γ_1, Γ_2 is not quasi geodesic bicombing. But if we regard that the two bicombings S_1, S_2 are maps into $P_1(\Gamma_1), P_1(\Gamma_2)$ respectively , (notice $G(\Gamma) \subset P_1(\Gamma)$) there exists a continuous map

$$f : \quad \Gamma \times \Gamma \times [0,1] \mapsto P_1(\Gamma)$$

satisfying the above properties A, B, C. In fact our proof of Novikov conjecture is valid for torsion free dicrete groups with the above properties. In particular , it is valid for the product of two torsion free quasi geodesic bicombing groups. Thus , replacing the manifold by its product with $S^1 \dots S^1$, it is enough to prove homotopy invariance of higher signatures for $4k$ dimensional closed manifold because $\mathbb{Z}^i$ is a quasi geodesic bicombing group for $1 \leq i \leq 3$.

In the following , we are mainly concerned with proving Proposition 1.6, 1.7.

Lemma 1.5 ([Al]). *Let Γ be a quasi geodesic bicombing group. Then every Rips complex $P_i(\Gamma)$ is contractible in $P_n(\Gamma)$ for some $n = n(i) \in \mathbb{N}$.*

Proof. By definition , if ω_1 and ω_2 represent combings $S_t(\gamma_1)$ and $S_t(\gamma_2)$ respectively , $|\gamma_1 - \gamma_2| \leq i$, then there exists a constant $C = C(i)$ such that $|\omega_1(t) - \omega_2(t)| \leq C$. So $P_i(\Gamma)$ can be contructed along the combing in$P_{C+1}(\Gamma)$.

In the following , we use a Rips complex as a space approximating $E\Gamma$ and do the Poincaré duality on the space . Moreover as we need to use a proper Lipschitz map , it is not enough to use a single Rips complex, but a family of Rips complexes.

Propositon 1.6. *Let Γ be a torsion free quasi geodesic bicombing group. Then there exists a fiberwise proper map*

$$\alpha_0 : \quad \tilde{P}_0 \times \tilde{P}_0 \mapsto T\tilde{P}_0 = \tilde{P}_0 \times \mathbb{R}^{N_0}$$

which induces Poincaré duality, where $\tilde{P}_0$ is the covering of a regular neighbourhood of some Rips complex in $\mathbb{R}^{N_0}$ and we regard $\tilde{P}_0$ as an open manifold without boundary.

In the later section , we need not only the above α , but also another family of maps . So before proving Proposition 1.6 , we show the following Proposition.

Proposition 1.7. *There exists $\{P_i\}$; an increasing sequence of Rips complexes which satisfy the following. Let $\bar{P}_i$ be the Γ covering of a regular neighbourhood of $P_i/\Gamma \subset P_{i'}/\Gamma \subset \mathbb{R}^{n(i')}$ for some $i' = i'(i)$, $\epsilon' >> \epsilon > 0$ sufficiently small and put $\partial\partial\bar{P}_i = \{x \in \bar{P}_i, d(x, \partial\bar{P}_i) < \epsilon'\}$, $\tilde{P}_i = \bar{P}_i \backslash \partial\partial\bar{P}_i$. Then there exists*

$$F_i : \quad \tilde{P}_0 \times \tilde{P}_i \mapsto N^i_{\epsilon'}(\Delta) = \{(p, p'), p \in \tilde{P}_0, p' \in \bar{P}_i, d_i(p, p') \leq \epsilon'\}$$

such that

(1) Γ *equivariant and fiberwise Lipschitz* ,

(2) $F_i(\tilde{P}_0 \times \tilde{P}_i \setminus N^i_{\epsilon'}(\Delta)) \subset N^i_{\epsilon'}(\Delta) \setminus N^i_{\epsilon}(\Delta)$,

(3) $F_0|_{N^0_\epsilon(\Delta)} =$ *embedding* ,

(4) $F_0(\tilde{P}_0 \times_\Gamma \tilde{P}_0 \backslash N^0_\epsilon(\Delta)) \subset N^0_{\epsilon'}(\Delta) \backslash N^0_\epsilon(\Delta)$,

(5) *the following diagram commutes,*

$$\begin{array}{ccc} \tilde{P}_0 \times_\Gamma \tilde{P}_i & \xrightarrow{F_i} & N^i_{\epsilon'}(\Delta) \\ {\scriptstyle incl}\downarrow & & \downarrow{\scriptstyle incl} \\ \tilde{P}_0 \times_\Gamma \tilde{P}_{i+1} & \xrightarrow[F_{i+1}]{} & N^{i+1}_{\epsilon'}(\Delta) \end{array}$$

where in Proposition 1.7, *we regard* $\tilde{P}_i$ *as a manifold with boundary.*

Proof. Take $r >> 0$ sufficiently large and a combing $S : \Gamma \times [0,1] \mapsto G(\Gamma)$. First we fix $P_0 = P_{d_0}(\Gamma)$ for some d_0 and let $f_t : \Gamma \mapsto P_0$ by

$$f_t(\gamma) = \begin{cases} \omega_\gamma(|S(t,e,\gamma)| - t) & |S(t,e,\gamma)| > t + r, \\ \omega_\gamma(r) & r \le |S(t,e,\gamma)| \le t + r, \\ \gamma & |S(t,e,\gamma)| \le r. \end{cases}$$

where ω_γ is a path representing $S(t,e,\gamma)$. We choose $\{P_i\}$ arbitrarily , but assume every P_i is contractible in P_{i+1}. First f_t can be extended to a simplicial map $f_t : P_0 \mapsto P_1$. Notice that for any $x \in P_0$, $f_t(x)$ lies in a compact neighbourhood of $e =$ identity $\in \Gamma \subset P_1$ for sufficiently large t. Put $f = f_\infty$. Thus there exists

$$f : \quad P_0 \mapsto K \subset P_1$$

where K ($\subset$ [simplicial span of $\{\gamma | |\gamma| \le r\}$]) is compact. We may assume K is contractible in P_1 (replacing P_1 to another P_2 if necessarily). Extending to a simplicial map from P_i to P_{i+1}, we have a map

$$f : \quad P_i \mapsto P_{i+1}$$

such that $f(P_i) \subset K_i$ is contractible in P_{i+1}.

Next for $\gamma_0 \in \Gamma$, putting $f(\gamma_0, \gamma) = (\gamma_0, \gamma_0 f(\gamma))$, we have

$$(*) \qquad f : \quad \Gamma \times P_0 \mapsto \Gamma \times P_1.$$

Take a regular neighbourhood $\bar{P}_0/\Gamma$ of P_0/Γ in $P_0/\Gamma \subset P_1/\Gamma \subset \mathbb{R}^{N_1}$, ... $\bar{P}_i/\Gamma$ in $P_i/\Gamma \subset P_{i+1}/\Gamma \subset \mathbb{R}^{N_{i+1}}$ and so on. We may assume $\tilde{P}_i \subset \tilde{P}_{i+1}$ for each i. Because the retraction $\tilde{P}_i \mapsto P_i$ is uniformly continuous, using the homotopy extension theorem , we can extend $(*)$ to

$$F_i : \quad \tilde{P}_0 \times_\Gamma \tilde{P}_i \mapsto \tilde{P}_0 \times_\Gamma (\tilde{P}_i \cup P_{i+1})$$

such that

(1) on a large neighbourhood of diagonal , F_0, F_1 are embedding ,

(2) each F_i is uniformly continuous ,

(3) image $F_i \equiv \tilde{K}_i$ is compact and fiberwise contractible ,

(4) F_i and F_{i+1} commute with the inclusions,

(5) $F_i(\tilde{P}_0 \times_\Gamma \tilde{P}_i \backslash N^i_{\epsilon'}(\Delta)) \subset \tilde{K}_i \backslash N^i_\epsilon(\Delta)$.

Using the fact that for $i = 0, 1$, K_i is contractible , there exists

$$f_i : \quad \tilde{P}_0 \times_\Gamma \tilde{P}_i \mapsto N^i_{\epsilon'}(\Delta) \qquad \text{for } i = 0, 1$$

such that

(1) f_i factors through F_i, $\quad f_i|_{N^i_{\epsilon'}(\Delta)} \;=$ embedding ,

(2) f_1 is an extension of f_0,

(3) $f_i(\tilde{P}_i \times_\Gamma \tilde{P}_i \backslash N^i_{\epsilon'}(\Delta)) \subset \partial N^i_{\epsilon'}(\Delta)$.

Secondly , we inductively define $f_i : \tilde{P}_0 \times_\Gamma \tilde{P}_i \mapsto N^i_{\epsilon'}(\Delta)$. For $i \geq 2$, take another uniformly continuous map , factouring through each F_i ,

$$\tilde{F}_i : \quad \tilde{P}_0 \times_\Gamma \tilde{P}_i \mapsto N^i_{\epsilon'}(\Delta)$$

such that

$$\tilde{F}_i(\tilde{P}_0 \times_\Gamma \tilde{P}_i \backslash N^i_{\epsilon'}(\Delta)) \subset N^i_{\epsilon'}(\Delta) \backslash N^i_\epsilon(\Delta).$$

Let

$$\tilde{f}_i : \quad \tilde{P}_0 \times_\Gamma \tilde{P}_{i-1} \backslash N^{i-1}_{\epsilon'}(\Delta) \cup \partial N^i_{\epsilon'}(\Delta) \mapsto N^i_{\epsilon'}(\Delta) \backslash N^i_\epsilon(\Delta)$$

be

$$\tilde{f}_i|_{\tilde{P}_0 \times_\Gamma \tilde{P}_{i-1} \backslash N^{i-1}_{\epsilon'}(\Delta)} = f_{i-1} \qquad \tilde{f}_i|_{\partial N^i_{\epsilon'}(\Delta)} = \tilde{F}_i.$$

Then by the homotopy extension theorem , we can extend $\tilde{f}_i$ to

$$f_i : \quad \tilde{P}_0 \times_\Gamma \tilde{P}_i \backslash N^i_{\epsilon'}(\Delta) \mapsto N^i_{\epsilon'}(\Delta) \backslash N^i_\epsilon(\Delta).$$

Finally , we put $f_i|_{N^i_{\epsilon'}(\Delta)} = \tilde{F}_i$. We thus get an extension f_i of f_{i-1} such that $f_i(\tilde{P}_0 \times_\Gamma \tilde{P}_i \backslash N^i_{\epsilon'}(\Delta)) \subset N^i_{\epsilon'}(\Delta) \backslash N^i_\epsilon(\Delta)$ and uniformly continuous. We need f_i to be fiberwise Lipschitz. We approximate $exp^{-1} f_i$ (we write $exp^{-1} f_i = f_i$ for simplicity) so that they are fiberwise Lipschitz.

Sublemma 1.8. *Each uniformly continuous map $f_i : \tilde{P}_0 \times_\Gamma \tilde{P}_i \mapsto N^i_\epsilon(\Delta)$ can be approximated by a fiberwise Lipschitz map.*

Proof. We recall for $p \in \tilde{P}_0$, $p \times \tilde{P}_i \cap N^i_\epsilon(\Delta)$ can be identified with $B_\epsilon(0) \subset \mathbb{R}^{N_i}$ for some N_i. Take small $\delta > 0$ so that 2δ neighbourhood of $x \in \tilde{P}_0$, $B_{2\delta}(x) \subset \tilde{P}_0$ has a trivialization and

$$\psi_x : \quad B_{2\delta}(x) \cong B_{\delta'}(0) \text{ for some } \delta'.$$

Take a covering of $\tilde{P}_0$, $\{B_\delta(x_l)\}$ and a partition of unity $\{\varphi_{x_l}\}$ subordinate to the covering. We approximate

$$(\varphi_{x_l} f_i)\psi^{-1}_{x_l} : \quad E \equiv B_{\delta'}(0) \times \tilde{P}_i \mapsto B_{\epsilon+2\delta}(0)$$

by a Lipschitz map $(\tilde{f}_i)_l$ such that

(1) Lipschitz constant of $(\tilde{f}_i)_l$ is bounded independently of i. (this is possible because ψ_{x_l} is uniformly Lipschitz equivalent)

(2) $sup_{x \in E} |\varphi_{x_l} f_i \psi^{-1}_{x_i}(x) - (\tilde{f}_i)_l(x)| < \epsilon''$ for sufficiently small ϵ'' compared to ϵ.

Take

$$\Psi_l : \quad B_{2\delta}(x_l) \mapsto [0,1]$$

such that

(1) $\Psi_l|_{supp\varphi_{x_l}} = 1, \qquad \Psi_l|_{\partial B_{2\delta}(x_l)} = 0,$

(2) Lipschitz and its constant is bounded independently of i.

Then

$$sup_{x \in \tilde{P}_0 \times_\Gamma \tilde{P}_i} |\Psi_l \varphi_{x_l} f(x) - \Psi_l (\tilde{f}_i)_l \psi_l(x)| \\ \leq sup_{x \in E} |(\varphi_{x_l} f_i)\psi^{-1}_{x_l}(x) - (\tilde{f}_i)_l(x)| < \epsilon''.$$

Thus $(\Psi_l(\tilde{f}_i)_l)\psi_l$ can be regarded as a map $\tilde{P}_0 \times_\Gamma \tilde{P}_i \mapsto N^i_{\epsilon+\epsilon''}(\Delta)$. Put

$$\tilde{f} : \quad \tilde{P}_0 \times_\Gamma \tilde{P}_i \mapsto N^i_{\epsilon + M\epsilon''}(\Delta)$$

$$\tilde{f}(x) = \sum_l \Psi_l(x)((\tilde{f}_i)_l \psi_l)(x),$$

where M is the multiplicity of the covering. Then

$$|f(x)-\tilde{f}(x)| = |\sum_l \varphi_{x_l}(x)f(x) - \sum_l \Psi_l(x)(\tilde{f}_i)_l\psi_l(x)|$$
$$\leq \quad \sum_l |\Psi_l(x)\varphi_{x_l}(x)f_i(x) - \Psi_l(x)(\tilde{f}_i)_l(x)\psi_l(x)| \quad \leq M\epsilon''.$$

Proof of the Proposition 1.7. Take sufficiently small positive constants $\{\epsilon_i''\}$ so that $\sum_{i\geq 0}\epsilon_i'' \leq \tilde{\epsilon}''$ where $\tilde{\epsilon}''$ is sufficiently small , compared to ϵ. Then we approximate f_i by a Lipschitz map F_i' so that

$$F_1'|_{N^1_{\epsilon-\epsilon_1''}(\Delta)\cap \tilde{P}_0\times_\Gamma \tilde{P}_1} \equiv f_1,$$
$$|F_i' - f_i| \equiv \sup_{(x,y)\in \tilde{P}_0\times_\Gamma \tilde{P}_i} d_i(F_i'(x,y), f_i(x,y)) \leq \epsilon_i''.$$

Take a family of Lipschitz functions $\{\psi_i\}_{i\geq 0}$ where $\psi_i : \tilde{P}_0 \times_\Gamma \tilde{P}_i \mapsto [0,1]$ so that $\psi_i|_{\tilde{P}_0\times_\Gamma \tilde{P}_{i-1}} \equiv 1, \psi_i|_{\tilde{P}_0\times_\Gamma \partial\tilde{P}_i} \equiv 0$. Define the maps $\tilde{F}_i$ inductively by

$$\tilde{F}_0 = F_1'|_{\tilde{P}_0\times_\Gamma \tilde{P}_0},$$
$$\tilde{F}_1 = \psi_1(z)F_1'(z) + (1-\psi_1)(z)F_2',$$
$$\tilde{F}_i(z) = \psi_i(z)\tilde{F}_{i-1}(z) + (1-\psi_i(z))F_{i+1}'(z) \quad \text{for } i\geq 2.$$

Then by replacing $\epsilon - \tilde{\epsilon}''$ by ϵ , $\epsilon' + \tilde{\epsilon}''$ by ϵ' , it is easy to see that $\{\tilde{F}_i\}_{i\geq 0}$ have the desired properties.

Proof of the Proposition 1.6. We take a proper function to be determined later

$$H : \quad [0,\infty) \mapsto [1,\infty)$$

where H satisfies $H|_{[0,\epsilon]} = 1$. Define

$$\alpha_0(x,y) \equiv H(x,y)F_0(x,y) \equiv H(d_0(x,y))F_0(x,y).$$

To induce the Poincaré duality, we must make α_0 to be proper at the end. To do so , we multiply a function on α_0. The explicit construction is done at the end of the section §2 , [Modification at the boundary].

Corollary 1.9 of the Proposition 1.7. *We have the following commutative diagram, here we regard $\tilde{P}_i$ as a manifold with boundary.*

$$\begin{array}{ccc}
\tilde{P}_0 \times_\Gamma \tilde{P}_0 & \xrightarrow{\alpha_0} & \mathbb{R}^{N_0} \\
{\scriptstyle incl}\downarrow & & \downarrow{\scriptstyle incl} \\
\tilde{P}_0 \times_\Gamma \tilde{P}_1 & \xrightarrow{\alpha_1} & \mathbb{R}^{N_1} \\
{\scriptstyle incl}\downarrow & & \downarrow{\scriptstyle incl} \\
\cdots & \longrightarrow & \cdots
\end{array}$$

And there exist famillies of constants $\{C_i\}$, $\{a_i\}$ such that the Lipschitz constant of α_i on $N_i(r) = \{(x,y)|x \in \tilde{P}_0, y \in \tilde{P}_i, d_i(x,y) \leq r\}$, for sufficiently large r , is bounded by $C_i H(a_i r)$ where H is the above function on $[0,\infty)$.

Proof. Later, we see that H can be taken so that

(1) monotone increasing ,

(2) 1 Lipschitz.

Before proving the corollary , we need two lemmas.

Sublemma 1.10 ([CGM] p 35). *Let $Y_0 \subset Y$ be a pair of metric spaces and $\alpha_0 : Y_0 \mapsto \mathbb{R}$ be a Lipschitz function. Then α_0 extends to a Lipschitz function $\alpha : Y \mapsto \mathbb{R}$ such that the Lipschitz constant of α is equal to that of α_0.*

Proof of the Corollary 1.9. We want to extend

$$H_0 : \quad \tilde{P}_0 \times_\Gamma \tilde{P}_0 \mapsto \mathbb{R} \qquad H_0(p,p') \equiv H(d_0(p,p'))$$

to a function H_i on $\tilde{P}_0 \times_\Gamma \tilde{P}_i$ which is fiberwise Lipschitz . To apply the sublemma 1.10 , we need some technique. Let d be an arbitrary metric on $\tilde{P}_0 \times_\Gamma \tilde{P}_1$ such that $d|_{p \times \tilde{P}_1} = d_{\tilde{P}_1}$, $p \in \tilde{P}_0$. We recall $1 \leq d_{\tilde{P}_0}/d_{\tilde{P}_1}|_{\tilde{P}_0} \leq \nu_0$ for some ν_0 and so

$$H : \quad (\tilde{P}_0 \times_\Gamma \tilde{P}_0, d) \mapsto \mathbb{R}$$

is a fiberwise ν_0 Lipschitz map.

Let H^* be a degenerate metric on $\tilde{P}_0 \times_\Gamma \tilde{P}_0$, i.e. $H^*(z_1,z_2) = |H(z_1) - H(z_2)|$ and put $d_H = \max\{d, \nu_0^{-1} H^*\}$. It is easy to see d_H is a metric on $\tilde{P}_0 \times_\Gamma \tilde{P}_0$ and $d_H|_{p \times \tilde{P}_0} = d_{\tilde{P}_1}|_{\tilde{P}_0}$. Moreover

$$H : \quad (\tilde{P}_0 \times_\Gamma \tilde{P}_0, d_H) \mapsto \mathbb{R}$$

is itself ν_0 Lipschitz . We need to extend d_H to a metric on $\tilde{P}_0 \times_\Gamma \tilde{P}_1$.

Sublemma 1.11 ([CGM] p 42 or appendix). *Let $\{K_l\}_l$ be a Γ equivariant covering of $\tilde{P}_0$ by compact subets and put $d_l = d|_{K_l \times \tilde{P}_1}$. We assume diam K_l is sufficiently small. Moreover as a metric on $K_l \times \tilde{P}_0$, let $d_{H_l} \equiv max\,\{d_l, \nu_0^{-1} H_l^*\}$ where $H_l = H|_{K_l \times \tilde{P}_0}$. Then d_{H_l} can be extended to a metric $\bar{d}_l$ on $K_l \times \tilde{P}_1$ such that*

$$\bar{d}_l|_{y \times \tilde{P}_1} = d_{\tilde{P}_1} \qquad 2\bar{d}_l|_{K_l \times \tilde{P}_0} \geq d_{H_l}.$$

Proof of the Corollary 1.9. Using sublemmas 1.10, 1.11, we extend $H : (K_l \times \tilde{P}_0, \bar{d}_l) \mapsto \mathbb{R}$ to

$$H : \quad (K_l \times \tilde{P}_1, \bar{d}_l) \mapsto \mathbb{R}$$

$2\nu_0$ Lipschitz. Again by the same process for another l' , let $\bar{d}_{l'}$ be an extension of $d_{H_{l'}} \equiv \max\,\{d_{l'}, (2\nu_0^{-1}) H_{l'}\}$, where

$$H_{l'} : \quad (K_{l'} \times \tilde{P}_0 \cup (\Gamma(K_l) \cap K_{l'}) \times \tilde{P}_1, \bar{d}_{l'}) \mapsto \mathbb{R}$$

is $2\nu_0$ Lipschitz and we extend it to

$$H : \quad (K_{l'} \times \tilde{P}_1, \bar{d}_{l'}) \mapsto \mathbb{R}$$

$2^2\nu_0$ Lipschitz. Repeating this process , let $\bar{d}_{l''}$ be an extension of the metric $d_{H_{l''}} = \max\,\{d_{l''}, (2^2\nu_0)^{-1} H_{l''}\}$ for another l''. Then we extend $H_{l'}$ to

$$H_{l''} : \quad (K_{l''} \times \tilde{P}_1, \bar{d}_{l''}) \mapsto \mathbb{R} \qquad 2^3\nu_0 \text{ Lipschitz.}$$

Finally , we have got

$$H_1 : \quad (\tilde{P}_0 \times_\Gamma \tilde{P}_1, d_{\tilde{P}_1}) \mapsto \mathbb{R} \qquad \text{fiberwise } 2^{l_1}\nu_0 \text{ Lipschitz.}$$

By doing the same thing , we get

$$H_i : \quad \tilde{P}_0 \times_\Gamma \tilde{P}_i \mapsto \mathbb{R} \qquad \text{such that}$$

(1) H_{i+1} is an extension of H_i ,
(2) H_i is fiberwise $2^{l(i)}\nu_0 \ldots \nu_{i-1}$ Lipschitz for some $l(i)$.

Remark 1.12.
(1) Every H_i is proper automatically because $\tilde{P}_0$ is a $X_i \geq 0$ net in $\tilde{P}_i$.
(2) The above technique is used in the later sections.

Proof of the Corollary 1.9. Putting

$$\alpha_i : \quad \tilde{P}_0 \times_\Gamma \tilde{P}_i \mapsto \mathbb{R}^{N_i} \qquad \alpha_i(x,y) = H_i(x,y)F_i(x,y),$$

we will see the Lipschitz constant on $N_i(r) = \{(x,y)|d_i(x,y) \leq r\}$ is bounded by $C_i H(a_i r)$ for some C_i, a_i. Take $x_1, x_2 \in \tilde{P}_i$, $y \in \tilde{P}_0$, $(y,x_1),(y,x_2) \in N_i(r)$. Choose an arbitrary point $x_1' \in \tilde{P}_0$ such that $d_i(x_1,x_1') \leq X_i$ ($\tilde{P}_0$ is a X_i net in $\tilde{P}_i$). Then

$$\begin{aligned} d(\alpha_i(y,x_1),\alpha_i(y,x_2)) &= d(H_i(y,x_1)F_i(y,x_1), H_i(y,x_2)F_i(y,x_2)) \\ &\leq H_i(y,x_1)d_i(F_i(y,x_1),F_i(y,x_2)) + |F_i(y,x_2)||H_i(y,x_1) - H_i(y,x_2)| \\ &\leq |H(y,x_1') + H_i(x_1',x_1)|C_i' d_i(x_1,x_2) + \epsilon' d_i(x_1,x_2) 2^{l(i)} \nu_0 \dots \nu_{i-1}. \quad (*) \end{aligned}$$

Notice that C_i' is the Lipschitz constant of F_i and $|F_i(x,y)| \leq \epsilon'$. Then

$$* \leq 2\{H(\nu_0 \dots \nu_{i-1}(r + X_i))C_i' + (X_i C_i' + \epsilon')\nu_0 \dots \nu_{i-1} 2^{l(i)}\} d_i(x_1,x_2) \quad (**)$$

(Notice $d_0(x_1',y) \leq \nu_0 \dots \nu_{i-1} d_i(x_1',y)$) and

$$** \leq \qquad 2(C_i' + (C_i' X_i + \epsilon')2^{l(i)} \nu_0 \dots \nu_{i-1}) H(2\nu_0 \dots \nu_{i-1} r) d_i(x_1,x_2).$$

Thus putting $C_i = 2(C_i' + (C_i' X_i + \epsilon')2^{l(i)} \nu_0 \dots \nu_{i-1})$, $a_i = 2\nu_0 \dots \nu_{i-1}$, we have completed the proof of the Corollary 1.9.

§2 Construction of a proper Lipschitz map

In this section , we assume the existence of

$$G_i : \quad \tilde{P}_0 \times_\Gamma \tilde{P}_i \mapsto \tilde{P}_0 \times_\Gamma \tilde{P}_{i+1}$$

which is a fiberwise λ_i Lipschitz map with λ_i arbitrary small and is properly homotopic to an embedding . We construct them in the next section. Using them, we construct

$$\alpha_\infty : \quad \tilde{P}_0 \times_\Gamma \tilde{P}_0 \mapsto \mathbb{R}^\infty$$

from $\{\alpha_i : \tilde{P}_0 \times_\Gamma \tilde{P}_i \mapsto \mathbb{R}^{N_i}\}_{i=0}^\infty$. Roughly speaking, we devide $\tilde{P}_0 \times_\Gamma \tilde{P}_0$ into compact subsets $\{D_i\}$ and put $\tilde{\alpha}_\infty(x) = \mu^{-i}\alpha_i(G_{i-1} \dots G_0(x))$ for $x \in D_i$ where $0 < \mu < 1$. By modifying $\tilde{\alpha}_\infty$ slightly, we get a continuous map α_∞.

First we show on D_i , α_∞ is fiberwise Lipschitz and its constant is bounded independently of i , which implies that α_∞ is itself fiberwise Lipschitz.

Secondly we show α_∞ is properly homotopic to α_0. To show this , we use the following,

(1) G_i is properly homotopic to an embedding

(2) α_i and α_{i+1} commute with the inclusions.

Finally we show pr $\circ \alpha_\infty$ has desired properties , where pr : $\mathbb{R}^\infty \mapsto \mathbb{R}^{N_0}$ is the standard projection.

Definition 2.1. We call $\{G_i\}$ ΓDAC (Γ diagonally asymptotic contruction) if

$$G_i : \quad \tilde{P}_0 \times_\Gamma \tilde{P}_i \mapsto \tilde{P}_0 \times_\Gamma \tilde{P}_{i+1}$$

satisfies the following .

(1) $G_i : p \times \tilde{P}_i \mapsto p \times \tilde{P}_{i+1}$ is λ_i Lipschitz for any $p \in \tilde{P}_0$.

(2) There exists $\{(G_i)_t\}$ fiberwise proper homotopies between G_i and inclusion and a constant J_i such that

(a) $d((G_i)_t(z), z) \leq J_{i+1}$ for $z \in \Delta_i = \{(p,p) \in \tilde{P}_0 \times_\Gamma \tilde{P}_i, p \in \tilde{P}_0\}$,

(b) for any $p \in \tilde{P}_0, p' \in \tilde{P}_i$, $d_i(p, (G_i)_t(p,p')) \geq R_i(d_i(p,p'))$

where $R_i(t) = A_i\lambda_i t - B_i$ for some A_i, B_i.

(3) There exists C_i' such that

(a) $(G_i)_t(p, \quad) : \quad \tilde{P}_i \mapsto \tilde{P}_{i+1}$ is C_{i+1}' Lipschitz,

(b) $(G_i)_t(p,p') : \quad [0,1] \mapsto \tilde{P}_{i+1}$ is $C_{i+1}'(1 + d_i(p,p'))$ Lipschitz.

(4) For every $t \in [0,1], p \in \tilde{P}_0$, $(G_i)_t \circ G_{i-1}(p, \Gamma) \subset \tilde{P}_0$ where we regard $\Gamma \subset \tilde{P}_{i-1}$.

Remark 2.2.

(1) The conditions $3, 4$ are technical. It is needed to make pr α_∞ proper.

(2) When we construct ΓDAC in the next section , we can take λ_i arbitrarily small , while fixing the values A_i, B_i, C_i', J_i.

Proposition 2.3. *Let $\{\tilde{P}_i\}$ admit $\{G_i\}$: ΓDAC with the Lipschitz constants arbitrarily small for fixed constants A_i, B_i, C_i', J_i of the above definition and put $g_i = G_i|_{p \times \tilde{P}_i}$, $(g_i)_t = (G_i)_t|_{p \times \tilde{P}_i}$ where $p \in \tilde{P}_0$ is an arbitrary but fixed point.*

Consider the following diagrams of the family in the Proposition 1.9, (we regard $p \times \tilde{P}_i = \tilde{P}_i$)

$$\begin{array}{ccc}
\tilde{P}_0 & \xrightarrow{\alpha_0} & \mathbb{R}^{N_0} \\
{\scriptstyle incl}\downarrow & & \downarrow{\scriptstyle incl} \\
\tilde{P}_1 & \xrightarrow{\alpha_1} & \mathbb{R}^{N_1} = \mathbb{R}^{N_0} \\
{\scriptstyle incl}\downarrow & & \downarrow{\scriptstyle incl} \\
\tilde{P}_2 & \xrightarrow{\alpha_2} & \mathbb{R}^{N_2} \\
{\scriptstyle incl}\downarrow & & \downarrow{\scriptstyle incl} \\
\cdots & \longrightarrow & \cdots
\end{array}$$

(By the construction , we may assume $N_0 = N_1$) . Then

$$\alpha_0 : \quad \tilde{P}_0 \mapsto \mathbb{R}^{N_0}$$

is properly homotopic to a proper Lipschitz map

$$\alpha_\infty : \quad \tilde{P}_0 \mapsto \mathbb{R}^\infty.$$

Moreover there exists a compact set $K_0 \subset \tilde{P}_0$ such that

$$\alpha_\infty|_{K_0} = \alpha_0,$$
$$pr \circ \alpha_\infty(\tilde{P}_0 \backslash K_0) \text{ is nonzero proper.}$$

Proof. We recall every Lipschitz constant of α_i on $B_i(r) = \{p' \in \tilde{P}_i, d_i(p,p') \leq r\}$ is bounded by $C_i H(a_i r)$ where C_i, a_i are dependent only on i. We determine $\{r_i\}, \{\lambda_i\}$ later ,

$$r_0 \leq r_1 \leq \cdots \nearrow \infty, \qquad \lambda_0 \geq \lambda_1 \geq \cdots \mapsto 0.$$

Let us put

$$B_0 \equiv B_0(r_0),$$
$$B_i \equiv (g_{i-1} \ldots g_0)^{-1} B_i(r_i) \qquad i \geq 1,$$
$$D_i \equiv B_i \backslash B_{i-1}.$$

Define a map on $\tilde{P}_i$ for $i \geq 0$

$$\alpha_i' \equiv \mu_{a_i(x)} \alpha_{i+1}((g_i)_{a_i(x)}(x))$$

where $\mu_t = t\mu^{-1} + 1 - t$ and functions a_i on $\tilde{P}_i$

$$a_i(x) = \begin{cases} \{d_i(p,x) - \frac{r_{i+1}}{2}\}/\frac{r_{i+1}}{2} & r_{i+1}/2 \leq d_i(p,x) \leq r_{i+1}, \\ 1 & d_i(p,x) \geq r_{i+1}, \\ 0 & d_i(p,x) \leq r_{i+1}/2, \end{cases}$$

We assume $r_i << r_{i+1}/2$. Then define

$$\alpha_\infty(y) \equiv \begin{cases} \alpha_0(y) & y \in B_0, \\ \alpha_0'(y) & y \in D_1, \\ \mu^{-i}\alpha_i'(g_{i-1} \ldots g_0(y)) & y \in D_{i+1} \qquad \text{for } i \geq 1. \end{cases}$$

We now see $\alpha_\infty : \tilde{P}_0 \mapsto \mathbb{R}^\infty$ is continuous. Let $y_s (0 \le s \le 1)$ be a curve in $\tilde{P}_0$ and for $0 \le s < 1$ $y_s \in D_{i+1}$, $y_1 \in D_i$. Then by definition , for $0 \le s < 1$,

$$\alpha_\infty(y_s) = \mu^{-i}\alpha_i'(g_{i-1}\dots g_0(y_s)) \mapsto_{s\to 1} \mu^{-i}\alpha_i'(g_{i-1}\dots g_0(y_1)) \qquad \cdots * .$$

As $y_1 \in (g_{i-1}\dots g_0)^{-1}B_i(r_i)$ and so

$$* = \mu^{-i}\alpha_{i+1}(g_{i-1}\dots g_0(y_1)) \quad = \mu^{-i}\alpha_i(g_{i-1}\dots g_0(y_1)).$$

On the other hand,

$$\begin{aligned}\alpha_\infty(y_1) &= \mu^{-(i-1)}\alpha_{i-1}'(g_{i-2}\dots g_0(y_1)) \\ &= \mu^{-(i-1)}\mu^{-1}\alpha_i(g_{i-1}g_{i-2}\dots g_0(y_1)).\end{aligned}$$

Thus α_∞ is continuous at every boundary of D_i , which implies α_∞ itself continuous.

We remark for $y \in D_{i+1}$, $\alpha_\infty(y)$ factors through $\alpha_i'|_{g_i^{-1}(B_{i+1}(r_{i+1}))\setminus B_i(r_i)}$. We need to know Lipschitz constants of α_i' on

$$\begin{aligned}E_1 &\equiv g_i^{-1}(B_{i+1}(r_{i+1}))\setminus B_i(r_{i+1}), \\ E_2 &\equiv B_i(r_{i+1})\setminus B_i(r_{i+1}/2), \\ E_3 &\equiv B_i(r_{i+1}/2)\setminus B_i(r_i).\end{aligned}$$

For $x, y \in E_1$,

$$\begin{aligned}d(\alpha_i'(x), \alpha_i'(y)) &\le \mu^{-1}d(\alpha_{i+1}(g_i(x)), \alpha_{i+1}(g_i(y))) \\ &\le \mu^{-1}C_{i+1}H(a_{i+1}r_{i+1})\lambda_i d_i(x,y).\end{aligned}$$

We recall that if $d(p,x) \le r_{i+1}$,

$$\begin{aligned}d(p,(g_i)_t(x)) &\le d((g_i)_t(p),(g_i)_t(p)) + d(p,(g_i(p)_t) \\ &\le C_{i+1}'d(p,x) + J_{i+1} \le 2C_{i+1}'r_{i+1}.\end{aligned}$$

for sufficiently large r_{i+1}. Then for $x, y \in E_2$,

$$\begin{aligned}d(\alpha_i'(x), \alpha_i'(y)) &= d(\mu_{a_i(x)}\alpha_{i+1}((g_i)_{a_i(x)}(x)), \mu_{a_i(y)}\alpha_{i+1}((g_i)_{a_i(y)}(y))) \\ &\le |\mu_{a_i(x)} - \mu_{a_i(y)}||\alpha_{i+1}((g_i)_{a_i(x)}(x))| \\ &\quad + \mu^{-1}d(\alpha_{i+1}((g_i)_{a_i(x)}(x)), \alpha_{i+1}((g_i)_{a_i(y)}(y))).\end{aligned}$$

We estimate the first term.

$$
\begin{aligned}
&|\mu_{a_i(x)} - \mu_{a_i(y)}||\alpha_{i+1}((g_i)_{a_i(x)}(x))| \\
&= (\mu^{-1} - 1)|a_i(x) - a_i(y)|d(\alpha_{i+1}((g_i)_{a_i(x)}(x)), \alpha_{i+1}(p)) \\
&\leq (\frac{1}{\mu} - 1)\frac{d_i(x,y)}{r_{i+1}/2} C_{i+1} H(2a_{i+1}C'_{i+1}r_{i+1})(d_i((g_i)_{a_i(x)}(x), (g_i)_{a_i(x)}(p)) + J_{i+1}) \\
&\leq 2\frac{(\mu^{-1} - 1)}{r_{i+1}} d_i(x,y) C_{i+1} H(2a_{i+1}C'_{i+1}r_{i+1})(C'_{i+1}d_i(p,x) + J_{i+1}) \\
&\leq 4(\mu^{-1} - 1) C_{i+1}C'_{i+1} H(2a_{i+1}C'_{i+1}r_{i+1}) d_i(x,y),
\end{aligned}
$$

and for the second term ,

$$
\begin{aligned}
&\mu^{-1} d(\alpha_{i+1}((g_i)_{a_i(x)}(x)), \alpha_{i+1}((g_i)_{a_i(y)}(y))) \\
&\qquad \leq \mu^{-1} d(\alpha_{i+1}((g_i)_{a_i(x)}(x)), \alpha_{i+1}(g_i)_{a_i(y)}(x)) \\
&\qquad\quad + \mu^{-1} d(\alpha_{i+1}((g_i)_{a_i(y)}(x)), \alpha_{i+1}((g_i)_{a_i(y)}(y)) \\
&\qquad \leq \mu^{-1} C'_{i+1} C_{i+1}(1 + r_{i+1})|a_i(x) - a_i(y)| H(2a_{i+1}C'_{i+1}r_{i+1}) \\
&\qquad\quad + \mu^{-1} C'_{i+1} C_{i+1} d_i(x,y) H(2a_{i+1}C'_{i+1}r_{i+1}) \\
&\qquad \leq \mu^{-1} C'_{i+1} C_{i+1}(1 + r_{i+1}) d_i(x,y)/(r_{i+1}/2) H(2a_{i+1}C'_{i+1}r_{i+1}) \\
&\qquad\quad + \mu^{-1} C'_{i+1} C_{i+1} d_i(x,y) H(2a_{i+1}C'_{i+1}r_{i+1}) \\
&\qquad \leq 5\mu^{-1} C'_{i+1} C_{i+1} H(2a_{i+1}C'_{i+1}r_{i+1}) d_i(x,y).
\end{aligned}
$$

Thus we have

$$d(\alpha'_i(x), \alpha'_i(y)) \leq 9\mu^{-1} C_{i+1} C'_{i+1} H(2a_{i+1}C'_{i+1}r_{i+1}) d_i(x,y).$$

For $x, y \in E_3$,

$$d(\alpha'_i(x), \alpha'_i(y)) \leq d(\alpha_{i+1}(x), \alpha_{i+1}(y)) \leq C_{i+1} H(a_{i+1}r_{i+1}) d_i(x,y).$$

Collecting

Sublemma 2.4. *On $g_i^{-1}(B_{i+1}(r_{i+1}))\backslash B_i(r_i)$, α'_i is $\mu^{-1}C_{i+1}H(a_{i+1}r_{i+1})$ Lipschitz , where C_i, a_i are dependent only on i.*

Proof of the Proposition 2.3. Thus on every D_{i+1} , the Lipschitz constant of α_∞ is bounded by

$$(\mu^{-1}\lambda_0)\dots(\mu^{-1}\lambda_{i-1})\mu^{-1}C_{i+1}H(a_{i+1}r_{i+1}). \qquad \cdots *$$

For α_∞ to be Lipschitz on $\tilde{P}_0$, it is enough to take $\{\lambda_i\}$ and $\{r_i\}$ so that they satisfy the following,

$$\mu\lambda_i^{-1}\frac{C_{i+1}H(a_{i+1}r_{i+1})}{C_{i+2}H(a_{i+2}r_{i+2})} \geq 1. \qquad \cdots **$$

But at the same time , we require it to be properly homotopic to α_0. Thus before estimating $**$, we show α_∞ is homotopic to α_0 and in the step , we determine $\{r_i\}, \{\lambda_i\}$ so that the homotopy becomes proper and then , we finally determine H , so that they satisfy $**$.

We inductively construct a family of proper homotopies $\{(\alpha_\infty^i)_t\}_{i\geq 1}$, where $\alpha_\infty^0 \equiv \alpha_\infty$ and $(\alpha_\infty^i)_t \equiv \alpha_0$ on B_i. This implies that α_∞ and α_0 are properly homotopic. For $i \geq 1$, take a function $\delta_{i-1} : \tilde{P}_{i-1} \mapsto [0,1]$ so that

$$\delta_{i-1}(x) = \begin{cases} 0 & \text{on } g_{i-1}^{-1}(B_i(r_i)), \\ 1 & \text{on } \tilde{P}_{i-1}\backslash(g_ig_{i-1})^{-1}B_{i+1}(r_{i+1}). \end{cases}$$

and define

$$(\alpha_i')_t : \quad \tilde{P}_{i-1} \mapsto \mathbb{R}^\infty \qquad \text{by}$$

$$(\alpha_i')_t(x) = \mu_{a_i(g_{i-1}(x))}\alpha_{i+1}((g_i)_{ta_i(g_{i-1}(x))} \circ (g_{i-1})_{\delta_{i-1}(x)+t(1-\delta_{i-1}(x))}(x))$$

$$= \begin{cases} \alpha_i((g_{i-1})_t(x)) & \text{on } g_{i-1}^{-1}(B_i(r_i)), \\ \mu^{-1}\alpha_{i+1}((g_i)_t g_{i-1}(x)) & \text{on } \tilde{P}_{i-1}\backslash(g_ig_{i-1})^{-1}B_{i+1}(r_{i+1}). \end{cases}$$

By the definition of ΓDAC , there exists $A_i, B_i \geq 0$ such that

$$d_i(p,(g_{i-1})_t(x)) \geq A_i\lambda_{i-1}\lambda_{i-1}^{-1}r_i - B_i = A_ir_i - B_i$$

if $x \in (g_ig_{i-1})^{-1}B_{i+1}(r_{i+1})\backslash g_{i-1}^{-1}(B_i(r_i))$ and for $x \in \tilde{P}_i\backslash B_i(A_ir_i - B_i)$,

$$d_{i+1}(p,(g_i)_t(x)) \quad \geq \lambda_iA_{i+1}(A_ir_i - B_i) - B_{i+1}.$$

Thus there exists a constant $l(i)$ so that if $\lambda_ir_i \geq l(i)$, then ,

$$|(\alpha_i')_t(x)| >> 0$$

on $(g_ig_{i-1})^{-1}B_{i+1}(r_{i+1})\backslash g_{i-1}^{-1}(B_i(r_i))$, where $l(i)$ is dependent only on i. We define

$$(\alpha_\infty)_t(x) = \begin{cases} \alpha_0(x) & x \in B_0, \\ \mu_{ta_0(x)}\alpha_1((g_0)_{ta_0(x)}(x)) & x \in D_1, \\ \mu_t(\alpha_1')_t(x) & x \in D_2, \\ \mu_t\mu^{-(i-1)}(\alpha_i')_t(g_{i-2}\ldots g_0(x)) & x \in D_{i+1}, \quad i \geq 2. \end{cases}$$

As before it is easy to see that this is a continuous homotopy. Thus there exists a homotopy between α_∞ and α^1_∞ , where

$$\alpha^1_\infty(x) = \begin{cases} \alpha_0(x) & x \in B_1, \\ \mu^{-(i-1)}\mu_{a_i(g_{i-1}\dots g_0(x))}\alpha_i((g_{i-1})_{\delta_{i-1}(x)}g_{i-2}\dots g_0(x)) & x \in D_{i+1} \end{cases}$$

which is proper if $\lambda_i r_i \geq l(i)$. We continue this process and see that α^1_∞ is properly homotopic to some α^2_∞ . From this stage , properness is followed automatically. Take another function δ^2_{i-2} on $\tilde{P}_{i-2}$ such that

$$\delta^2_{i-2}(x) = \begin{cases} 0 & \text{on } (g_{i-1}g_{i-2})^{-1}B_i(r_i), \\ 1 & \text{on } \tilde{P}_{i-2}\backslash(g_ig_{i-1}g_{i-2})^{-1}B_{i+1}(r_{i+1}), \end{cases}$$

and define for $i \geq 2$,

$$(\alpha^2_i)_t(x) = \mu_{a_i(g_{i-1}g_{i-2}(x))}\alpha_i((g_{i-1})_{t\delta_{i-1}(g_{i-2}(x))}(g_{i-2})_{\delta^2_{i-2}(x)+t(1-\delta^2_{i-2}(x))}(x))$$

$$= \begin{cases} \alpha_{i-1}((g_{i-2})_t(x)) & x \in (g_{i-1}g_{i-2})^{-1}B_i(r_i) \\ \mu^{-1}\alpha_i((g_{i-1})_t g_{i-2}(x)) & x \in \tilde{P}_{i-2}\backslash(g_ig_{i-1}g_{i-2})^{-1}B_{i+1}(r_{i+1}). \end{cases}$$

For $x \in (g_ig_{i-1}g_{i-2})^{-1}B_{i+1}(r_{i+1})\backslash(g_{i-1}g_{i-2})^{-1}B_i(r_i)$,

$$\begin{aligned} d_i(p, (g_{i-1})_t(g_{i-2})_{t'}(x)) &\geq \lambda_{i-1}A_i(\lambda_{i-2}A_{i-1}\lambda^{-1}_{i-1}\lambda^{-1}_{i-2}r_i - B_{i-1}) - B_i \\ &\geq A_iA_{i-1}r_i - A_iB_{i-1} - B_i. \end{aligned}$$

Without loss of generality , we may assume $A_iA_{i-1}r_i - A_iB_{i-1} - B_i >> 0$. Thus we put

$$(\alpha^2_\infty)_t(x) = \begin{cases} \alpha_0(x) & x \in B_1, \\ \mu_{ta_1(g_0(x))}\alpha_1((g_0)_{t\delta_0(x)}(x)) & x \in D_2, \\ \mu_t(\alpha^2_2)_t(x) & x \in D_3, \\ \mu_t\mu^{-(i-2)}(\alpha^2_i)_t(g_{i-3}\dots g_0(x)) & x \in D_{i+1}, \quad i \geq 3. \end{cases}$$

We can see this is a continuous , proper homotopy between α^1_∞ and α^2_∞. Inductively , by constructing α^k_∞ and proper homotopies $(\alpha^k_\infty)_t$, connecting them , we finally get a proper homotopy between α_∞ and α_0.

Next we estimate $**$. We take $\lambda_i = l(i)/r_i$. Then

$$** = \mu\frac{r_i}{l(i)}\frac{C_{i+1}H(a_{i+1}r_{i+1})}{C_{i+2}H(a_{i+2}r_{i+2})}.$$

We take $r_i = \frac{C_{i+2}}{C_{i+1}} l(i) 4$ and may assume , by replacing $C_i, l(i)$ if necessarily,

$$a_i r_i << a_{i+1} r_{i+1} << \cdots \nearrow \infty.$$

Take a monotone function $H : \quad \mathbb{R}_+ \mapsto [1, \infty)$ such that

$$(1) \qquad 2H(a_i r_i) = H(a_{i+1} r_{i+1})$$
$$(2) \qquad 1 \text{ Lipschitz.}$$

Thus $** \geq 2\mu \geq 1$ (for example , we may take $\mu = 2/3$.)

So far , we have obtained α_∞, which is properly Lipschitz , properly homotopic to α_0. We must show that there exists a compact set $K_0 \subset \tilde{P}_0$ such that

$$\alpha_\infty|_{K_0} = \alpha_0$$
$$\text{pr} \circ \alpha_\infty(\tilde{P}_0 \backslash K_0) \text{ is non zero proper.}$$

We take $K_0 = B_0(r_0)$. Then by the construction , $\text{pr} \circ \alpha_\infty(x) = \alpha_\infty(x)$ for $x \in B_1 = g_0^{-1}(B_0)$. ($\dim \tilde{P}_1 = \dim \tilde{P}_0$). Notice that for $x \in D_2$, $\alpha_\infty(x)$ factors

$$D_2 \subset \tilde{P}_0 \xrightarrow{g_0} \tilde{P}_1 \xrightarrow{(g_1)_t} \tilde{P}_2 \xrightarrow{F_2} N^2_{\epsilon'}(\Delta) \backslash N^2_{\epsilon}(\Delta).$$

Let us recall that $(g_1)_t$ is Lipschitz and its constant is bounded by C_1 independently of g_0 and g_1 , Γ is X_0 net in $\tilde{P}_0$ and image$\{(g_1)_t g_0(\Gamma)\} \subset \tilde{P}_0$. So if we choose λ_0 sufficiently small, then $|\ \text{pr} \circ (g_1)_t g_0|_{D_2}| \geq \epsilon/2$ and $\text{pr} \circ \alpha_\infty|_{D_2}$ does not meet a neighbourhood of zero. By the same consideration ,by replacing $0, 1$ to i and $i+1$ respectively , it follows that $\text{pr} \circ \alpha_\infty$ has the desired properties. This completes the Proposition 2.3.

As in [CGM] , the above process is completely canonical and works for the family. Thus

Proposition 2.5. *Let $\{\tilde{P}_i\}$, $\{G_i\}$, $\{\alpha_i\}$ as before. Then α_0 is fiberwise properly homotopic to*

$$\alpha_\infty : \quad \tilde{P}_0 \times_\Gamma \tilde{P}_0 \mapsto \mathbb{R}^\infty$$

fiberwise proper Lipschitz , which has the following properties. There exists a neighbourhood of diagonal K such that

(1) $\text{pr} \circ \alpha_\infty(x) = \alpha_\infty(x) = \alpha_0(x) \quad x \in K.$

(2) $\text{pr} \circ \alpha_\infty(\tilde{P}_0 \times_\Gamma \tilde{P}_0 \backslash K)$ *is non zero and proper.*

Modification at the boundary.

Until this time , we have considered the case where $\tilde{P}_0$ is closed, (i.e. with boundary) . It is needed for α_0 to induce Poincaré duality that it is proper at the end, $\partial\tilde{P}_0 \mapsto \infty$. To do so , for example , we can multiply on α_0 some function which goes to infinity at the end. But then α_∞ may not be Lipschitz. Let us consider a simple example.

Let $\bar{I} = [0,1]$, $I = (0,1)$, $\quad \alpha : \quad I \times \bar{I} \mapsto \mathbb{R}$, $\quad \alpha(a,b) = a - b$.

Clearly α is a fiberwise Lipschitz , but not proper. If it is possible to consider

$$\alpha : \quad I \times \bar{I} \mapsto \mathbb{R} \subset \mathbb{R} \oplus \mathbb{R},$$

Then by taking a proper function

$$\delta : \quad (0,1) \mapsto \mathbb{R}_+, \qquad \text{and put } \alpha(a,b) = (a-b, \delta(a)),$$

this has become a fiberwise proper Lipschitz map on $I \times \bar{I}$.

With this in mind , we reconstruct another α_∞. We recall $\tilde{P}_0/\Gamma \subset \mathbb{R}^{N_0}$. By embedding $\mathbb{R}^\infty$ into $\mathbb{R}^\infty \oplus \mathbb{R}^2$, and identifying the latter with $\mathbb{R}^\infty$, we regard

$$\mathbb{R}^\infty = \mathbb{R}^{N_0} \oplus \mathbb{R}^2 \oplus (\mathbb{R}^{N_0})^\perp,$$

$$\alpha_\infty : \quad \tilde{P}_0 \times_\Gamma \tilde{P}_0 \mapsto \mathbb{R}^{N_0} \oplus 0 \oplus (\mathbb{R}^{N_0})^\perp.$$

We take a sufficiently small regular neighbourhood $\tilde{\mathbb{P}}_0/\Gamma$ of $\tilde{P}_0/\Gamma$ in $\mathbb{R}^{N_0+2} = \mathbb{R}^{N_0} \oplus \mathbb{R}^2$. We may assume $\partial\tilde{\mathbb{P}}_0 \cap \tilde{P}_0 = \phi$. Let

$$r : \quad \tilde{\mathbb{P}}_0 \mapsto \tilde{P}_0$$

be a retraction. We take ϵ' sufficiently small compared to ϵ. (we recall $F_0|_{N^0_\epsilon(\Delta)}$ = identity). Then for $x \in \tilde{\mathbb{P}}_0$, $N^0_{\epsilon'}(x) \subset N^0_\epsilon(r(x))$ and there exists a family of continuous maps , which is also continuous with respect to parameter $x \in \tilde{\mathbb{P}}_0$, $f_x : \quad \tilde{\mathbb{P}}_0 \mapsto \tilde{\mathbb{P}}_0$ such that

$$f_x(p) = \begin{cases} p, & d(x,p) \le \epsilon' \\ r(p), & d(x,p) \ge 2\epsilon'. \end{cases}$$

We may assume if $d(x,p) \ge \epsilon'$, then $d(f_p(x), f_x(p)) \ge \epsilon'/2$. Define

$$F(p,x) \equiv (f_x(p), f_p(x))$$

and we extend $(\alpha_\infty)_t$ to a map on $\tilde{\mathbb{P}}_0 \times_\Gamma \tilde{\mathbb{P}}_0$ by

$$(\alpha_\infty)_t(x,y) \equiv \begin{cases} exp^{-1}F(x,y), & d(x,y) \le 2\epsilon' \\ (\alpha_\infty)_t(F(x,y)), & \text{otherwise.} \end{cases}$$

Moreover we may assume $d(P_0, \partial\tilde{\mathbb{P}}_0) > 2\epsilon'$. So

$$(\alpha_\infty)_t : \quad \partial\tilde{\mathbb{P}}_0 \times_\Gamma P_0 \mapsto \mathbb{R}^{N_0} \oplus 0 \oplus (\mathbb{R}^{N_0})^\perp.$$

Let us introduce a function l on $\tilde{\mathbb{P}}_0 \times_\Gamma \tilde{\mathbb{P}}_0$ as follows ,

$$l(x,y) = \begin{cases} 0 & \frac{d(x,y)}{d(\partial\tilde{\mathbb{P}}_0,x)} \leq \epsilon'', \\ \frac{1}{d(\partial\tilde{\mathbb{P}}_0,x)} & \frac{d(x,y)}{d(\partial\tilde{\mathbb{P}}_0,x)} \geq 2\epsilon'', \end{cases}$$

for ϵ'' sufficiently small, compared to ϵ and ϵ'. Clearly

$$\tilde{\alpha_0} : \quad \tilde{\mathbb{P}}_0 \times_\Gamma \tilde{\mathbb{P}}_0 \mapsto \mathbb{R}^{N_0+2}, \qquad \tilde{\alpha_0}(x,y) = \alpha_0(x,y)exp(l(x,y))$$

induces Poincaré duality,

$$H_*^{\inf}(\tilde{\mathbb{P}}_0 : \Gamma) \mapsto H^*(\tilde{\mathbb{P}}_0 : \Gamma).$$

As P_0 is the retract of $\tilde{\mathbb{P}}_0$, we may restrict

$$\alpha_0 : \quad \tilde{\mathbb{P}}_0 \times_\Gamma P_0 \mapsto \mathbb{R}^{N_0+2}.$$

Recall for $(x,y) \in \partial\tilde{\mathbb{P}} \times_\Gamma P_0$,

$$(\alpha_\infty)_t(x,y) = (\alpha_\infty)_t(x,y)\exp(l(x,y)) \in \mathbb{R}^{N_0} \oplus 0 \oplus (\mathbb{R}^{N_0})^\perp.$$

Let $v = (0,(1,0),0)$ and define

$$(\alpha_\infty)_{t,s}(x,y) \equiv (\alpha_\infty)_t(x,y)\exp(sl(x,y)) + (1-s)l(x,y)v.$$

By this proper homotopy , we know for $(x,y) \in \tilde{\mathbb{P}}_0 \times_\Gamma P_0$,

$$\tilde{\alpha_0}_0(x,y) = \alpha_0(x,y) + l(x,y)v$$

induces Poincaré duality. Moreover if x is near $\partial\tilde{\mathbb{P}}_0$, then

$$(\tilde{\alpha}_\infty)_0(x,y) \equiv \alpha_\infty(x,y) + \frac{v}{d(\partial\tilde{\mathbb{P}}_0,x)}$$

which is Lipschitz with respect to the variable y.

This implies that $(\tilde{\alpha}_\infty)_0$ is fiberwise proper Lipschitz and all the properties stated in the above proposition remain. Thus we have the following modified proposition.

Modified Proposition 2.6. *Let us fix $P_0 = P_{d_0}(\Gamma)$ and $\tilde{P}_0$ be a Γ covering of a regular neighbourhood in $\mathbb{R}^{N_0}$ for some N_0. We regard $\frac{\tilde{P}_0}{\Gamma}$ as an open manifold without boundary. Then there exists*

$$\alpha_\infty : \quad \tilde{P}_0 \times_\Gamma P_0 \mapsto \mathbb{R}^\infty$$

such that

(a) fiberwise proper C Lipschitz for some C ,

(b) fiberwise properly homotopic to $\alpha_0 : \tilde{P}_0 \times_\Gamma P_0 \mapsto \mathbb{R}^{N_0}$ and $\alpha_0 \cap : H_*^{\inf}(\frac{\tilde{P}_0}{\Gamma}) \mapsto H^*(\frac{P_0}{\Gamma})$ induces Poincaré duality,

(c) there exists a neighbourhood of diagonal $K \subset \tilde{P}_0 \times_\Gamma P_0$ such that

$$pr \circ \alpha_\infty(x) = \alpha_\infty(x) = \alpha_0(x), \quad x \in K,$$

$$pr \circ \alpha_\infty|_{\tilde{P}_0 \times_\Gamma P_0 \setminus K} \text{ is non zero proper.}$$

In §5 , we use $(\tilde{\alpha}_\infty)_t$ instead of $(\alpha_\infty)_t$ and write $(\alpha_\infty)_t$ for $(\tilde{\alpha_\infty})_t$ and $\tilde{P}_0$ for $\tilde{\mathbb{P}}_0$.

§3 Extension of maps

In this section , we are concerned with constructing Γ DAC for our groups. But before constructing them , we need some tool to extend a map in the category of Lipschitz maps. At this and next section , we regard $\tilde{P}_0$ as a closed manifold (with boundary). The content of this section is as follows. First we have λ contruction ($\lambda < 1$ Lipschitz map)

$$\Gamma \mapsto \tilde{P}_0.$$

We extend this to $C\lambda$ contruction

$$\tilde{P}_0 \mapsto \tilde{P}_1$$

where C is independent of λ. Here we regard $\Gamma \subset \tilde{P}_0 \subset \tilde{P}_1$. Finally we do the process for famillies.

Definition 3.1. Let $\{V_i, d_i\}_{i \geq 0}$ be an increasing sequence of metric spaces and every d_i and $d_{i+1}|_{V_i}$ are Lipschitz equivalent. $\{V_i\}$ is asymptotically , uniformly and Lipschitz contructible (here after we write as-ULC) if there exists a function $C_i(\delta)$ such that for any pair of metric spaces $Y_0 \subset Y$ and $\alpha_0 : Y_0 \mapsto V_i$ λ_0 Lipschitz , diam ($\alpha_0(Y_0)$) $\leq \delta$, then there exists an extension of α_0,

$$\alpha : \quad Y \mapsto V_{i+1}$$

which is $C_i(\delta)\lambda_0$ Lipschitz map.

The important example of as-ULC is $\{\tilde{P}_i\}$. We show this fact , but before , we need Lipschitz version of the standard homotopy extension lemma.

Proposition 3.2. *Let $V_0 \subset V \subset \mathbb{R}^N$ be a pair of smooth manifolds (with boundary) where V_0 is compact and contractible in V. Then there exists*

$$p: \quad \mathbb{R}^N \mapsto V$$

which fixes V_0 and is Lipschitz.

Proof. We can assume its contructing homotopy $h_t : V_0 \mapsto V_1 \subset V$ is smooth where V_1 is compact. Let us consider $(V_0, \mathbb{R}^N)$: CW pair and its characteristic maps $\varphi_i : \quad D^k \mapsto X_i^q \subset \mathbb{R}^N$ which are into λ Lipschitz equivalent. (λ is independent of i .) On $[0,1] \times V_0$, we define $p \equiv h_t$. Then for vertices $p_i \in \mathbb{R}^N \backslash V_0$, we put $p(t, p_i) = h_0(x), x \in V_0$ and extend p on 1 simlices of $\mathbb{R}^N$ so that it becomes a Lipschitz map. Assume we have extended p to q dimensional subcomplex of $\mathbb{R}^N$ and let X^{q+1} be a $q+1$ dimensional slice , not contained in V_0. Put $Z = D^{q+1} \times [0,1]$, $B = S^q \times [0,1] \cup D^{q+1} \times 0$ and $r : Z \mapsto B$ is a smooth retraction. Then inductively , we define for $x \in X^{q+1} \backslash V_0$,

$$p(x,t) = p(r(x,t)).$$

Let us assume p is Lipschitz on $\{$ q dimensional subcomplex $\}$ $\cup V_0$. Then it follows p is Lipschitz on every $q+1$ dimensional slice, whose constant is uniformly bounded. Thus p is Lipschitz on $\{$ $q+1$ dimensional subcomplex $\}$ $\cup V_0$. We continue until p is extended to a Lipschitz map on $\mathbb{R}^N$. Then we put

$$p \equiv p(\quad, 1): \quad \mathbb{R}^N \mapsto V$$

which is Lipschitz and $p(\quad, 1)|_{V_0} \equiv$ identity.

Lemma 3.3 ([CGM] p 35). *Let $Y_0 \subset Y$ be a pair of metric spaces. Then every λ_0 Lipschitz map $\alpha_0 : Y_0 \mapsto \mathbb{R}^N$ can be extended to $\sqrt{N}\lambda_0$ Lipschitz map*

$$\alpha: \quad Y \mapsto \mathbb{R}^N.$$

Corollary 3.4. *$\{\tilde{P}_i\}$ is as-ULC .*

Proof. Take arbitrary $Y_0 \subset Y$ and diam $\alpha_0(Y_0) \leq \delta$. Because $\tilde{P}_i/\Gamma$ is compact, there exists $M > 0$ such that for some $\gamma \in \Gamma$ and a fixed $e \in \tilde{P}_i$, $\gamma\alpha_0(Y_0) \subset B_e(\delta + M) = \{x \in \tilde{P}_i, d(e,x) \leq \delta + M\}$. By the Proposition 3.2, there exists λ Lipschitz map

$$p_{M+\delta}: \quad \mathbb{R}^N \mapsto \tilde{P}_{i+1}$$

which fixes $B_e(\delta + M)$. Using the Lemma 3.3, we extend

$$Y_0 \xrightarrow{\alpha_0} \alpha_0(Y_0) \xrightarrow{\gamma} \gamma\alpha_0(Y_0) \subset \tilde{P}_{i+1} \subset \mathbb{R}^{N_{i+1}}$$

to a Lipschitz map

$$\alpha : \quad Y \mapsto \mathbb{R}^{N_{i+1}}$$

and composing with $p_{\delta+M}$ and $\gamma^{-1} : \tilde{P}_{i+1} \mapsto \tilde{P}_{i+1}$, we get

$$\alpha : \quad Y \mapsto \tilde{P}_{i+1}.$$

The Lipschitz constant of α is multiplied by some constant C , which is dependent only on diam $\alpha_0(Y_0)$.

Proposition 3.5. *Let Y be a metric spaces and $Y_0 \subset Y$ be ϵ_0 net and*

$$\alpha_i : \quad Y_0 \mapsto V_i$$

be λ_0 Lipschitz. If $\{V_i\}$ is as-ULC and Y or V_i are ULB (uniformly locally bounded) , then there exists an extension

$$\tilde{\alpha_i} : \quad Y \mapsto V_{i+k}$$

which is $C\lambda_0$ Lipschitz for some C and k.

For covenience , we first recall the definition of ULB.

Definition 3.6 ([CGM] , [G1]). Y is ULB iff there exists a covering $\{B_i\}$ of Y such that

(1) sup $_i$ diam $B_i \leq \beta < \infty$,

(2) for every $x \in Y$, the number of $\{B_x(R) \cap B_i \neq \phi\} \leq B(R)$

where the constant $B(R)$ is dependent only of R.

The equivalent definition is as follows.

Definition 3.7. For every $\Delta > 0$, there exists a covering $\{Y_i\}_{i=1}^m$ of Y such that

(1) $Y_i = \cup_j B_j^i$,

(2) sup $_{i,j}$ diam $B_j^i \leq \beta < \infty$,

(3) $d(B_j^i, B_{j'}^i) > \Delta$.

We call $\{Y_i\}$ a Δ cover .

Proof of the Proposition 3.5. First let us assume Y is ULB and take $\Delta > \epsilon_0$ cover $\{Y_i\}_{i=1}^m$ of Y . Assume we have extended α_0 to

$$\alpha_k : \quad \cup_{i=1}^k Y_i \mapsto V_{i+k}.$$

Let us write $Y_{k+1} = \cup_j B_j^{k+1}$ and $Y_{k+1}^i(\epsilon_0) \subset \cup_{j=1}^k Y_j$ be ϵ_0 neighbourhood of $B_i^{k+1} \cap (\cup_{j=1}^k Y_j)$. Extending α_k on every $Y_{k+1}^i(\epsilon_0) \cup B_i^{k+1}$, we obtain a map on $\cup_{j=1}^{k+1} Y_j$,

$$\alpha_{k+1} : \quad \cup_{j=1}^{k+1} Y_j \mapsto V_{i+k+1}$$

whose Lipschitz constant is $C_{i+k}(\lambda(\alpha_k)(\beta+2\epsilon_0))\lambda(\alpha_k)$ on $Y_{k+1}^i \cup B_i^{k+1}$. So the grobal Lipschitz constant $\lambda(\alpha_{k+1})$ of α_{k+1} is less than $5C_i(\lambda(\alpha_k)(\beta+2\epsilon_0))\lambda(\alpha_k)$, because of the following

Lemma 3.8 ([CGM] p 36). *Let $Y_0 \subset Y$ be an ϵ_0 net and $\alpha : Y \mapsto V$ be a map such that*

(1) $\alpha|_{Y_0}$ *is λ_0 Lipschitz,*

(2) *for every $x, y \in Y, d(x,y) \leq \epsilon_0, \qquad d(\alpha(x), \alpha(y)) \leq \lambda_0 d(x,y).$*

Then α itself is $5\lambda_0$ Lipschitz .

Proof of the Proposition 3.5. Next , let us assume $\{V_l\}$ as-ULC and ULB. Take a $\Delta > 3\epsilon_0\lambda(\alpha_0)$ cover $V_i = \cup_{l=1}^m V_l$, $V_l = \cup_j B_j^l$ and let $\tilde{B}_j^l$ be ϵ_0 neighbourhood of $\alpha_0^{-1}(B_j^l)$. Then $Y_l \equiv \cup_j \tilde{B_j^l}$ is ϵ_0 separated and diam $\alpha_0(Y_0 \cap \tilde{B_j^l})$ are uniformly bounded. The previous argument works for this case.

Corollary 3.9 of the Proof of the Proposition 3.5. *Let $\{V_i\}$ be as-ULC and ULB. Then for any $Y_0 \subset Y$ which is ϵ_0 net , λ_0 Lipschitz map*

$$\alpha_0 : \quad Y_0 \mapsto V_i$$

can be extended to λ Lipschitz map

$$\alpha : \quad Y \mapsto V_{i+m}$$

where $\lambda \leq C\lambda_0$, C, m is dependent only on the values $\lambda_0\epsilon_0$ and i. In particular, for sufficiently small λ_0, we can extend α_0 to a contructing map α .

This process works for families.

Corollary 3.10. *Let $Y_1' \subset Y_1$ be compact, Y_2 be a geodesic space. Moreover let $X \subset X' \subset Y_2$ be closed, ϵ_0 nets and put $Z = Y_1' \times X' \cup Y_1 \times X \subset Y_1 \times Y_2$. Then if*

$$\alpha_0 : \quad Z \mapsto V_i$$

is fiberwise λ_0 Lipschitz, then α_0 extends to a fiberwise $C\lambda_0$ Lipschitz map

$$\alpha : \quad Y_1 \times Y_2 \mapsto V_{i+N}.$$

Proof. Let d_0 be any metric on $Y_1 \times Y_2$ such that $d_0|_{y \times Y_2} = d_{Y_2}$ and put $d = \max\{d_0, \lambda_0^{-1}\alpha_0^*\}$ where α_0^* is the pull backed , degenerate metric on Z. Then it follows that

$$\alpha_0 : \quad (Z, d) \mapsto (V_i, d_i)$$

is λ_0 Lipschitz. We need the following

Lemma 3.11 ([CGM] p42 or appendix). *There exists a metric $\bar{d}$ on $Y_1 \times Y_2$ such that*

$$\bar{d}|_{y \times Y_2} = d_{Y_2}, \qquad 2\bar{d}|_Z \geq d.$$

Proof of the Corollary 3.10. By applying the Lemma 3.11 for d , we have $\bar{d}$ on $Y_1 \times Y_2$ satisfying the above conditions. Then

$$\alpha_0 : \quad (Y_1 \times X \cup Y_1' \times X_1', \bar{d}) \mapsto (V_i, d_i)$$

is $2\lambda_0$ Lipschitz (not only fiberwise). By applying the Corollary 3.9, we obtain the result.

§4 Construction of ΓDAC

We recall that our discrete groups have the following properties. There exists $f : \tilde{P}_0 \times \tilde{P}_0 \times [0,1] \mapsto \tilde{P}_i$ such that

(1) $d_i(f_t(x_1, y_1), f_t(x_2, y_2)) \leq k_i(td_0(x_1, x_2) + (1-t)d_0(y_1, y_2)) + C_i$,
(2) $d_i(f_{t_1}(x, y), f_{t_2}(x, y)) \leq k_i|t_1 - t_2|d_0(x, y) + C_i$,
(3) $d_i(x, f_t(x, y)) \geq A_i t d_0(x, y) - B_i$ for some k_i, C_i, A_i, B_i,
(4) $f(p, p', 0) = p, \qquad f(p, p', 1) = p'$,
(5) $f : \Gamma \times \Gamma \times [0,1] \mapsto G(\Gamma)$ continuous.

Because we should allow all the constructions to work for a product of torsion free quasi geodesic bicombing groups , we assume

$$f: \quad \Gamma \times \Gamma \times [0,1] \mapsto P_1(\Gamma).$$

Let us recall that we have a family of the Rips complexes $\{\tilde{P}_i\}_i$. By shifting the suffix , we may regard that there is $\tilde{P}_{i'}$ so that $\tilde{P}_i \subset \tilde{P}_{i'} \subset \tilde{P}_{i+1}$. Moreover we may assume that there is $\tilde{P}_{-1}$, $\tilde{P}_{-1} \subset \tilde{P}_0$. To construct ΓDAC , we add another $\tilde{P}_{-1} \subset \tilde{P}_0$. The contents of this section are as follows. We construct $\{G_i\}_i$ and proper homotopies between G_i and an embedding. By induction , suppose we have constructed G_{i-1}. First we construct λ contraction

$$G: \quad \Gamma \mapsto \tilde{P}_{-1}$$

and a proper homotopy

$$(G)_t: \quad \Gamma \mapsto \tilde{P}_{-1}$$

between the embedding and G. Next we extend $G : \Gamma \mapsto \tilde{P}_{-1}$ to $C\lambda$ contraction (C does not depend on λ)

$$G: \quad \Gamma \cup G_{i-1}(\Gamma) \mapsto \tilde{P}_0$$

and $(G)_t$ to a proper homotopy

$$(G)_t: \quad \Gamma \cup G_{i-1}(\Gamma) \mapsto \tilde{P}_0$$

where $(G)_0 =$ embedding , $(G)_1 = G$. Then by embedding in $\tilde{P}_i$,

$$G_i: \quad (\Gamma \cup G_{i-1}(\Gamma), d_i) \mapsto (\tilde{P}_i, d_i)$$

is $C\nu_0 \dots \nu_{i-1}\lambda$ Lipschitz and finally we extend the above map to

$$G_i: \quad (\tilde{P}_i, d_i) \mapsto (\tilde{P}_{i+1}, d_{i+1})$$

which is $C_i'\lambda$ Lipschitz and $(G)_t$ to a proper homotopy

$$(G_i)_t: \quad (\tilde{P}_i, d_i) \mapsto (\tilde{P}_{i+1}, d_{i+1})$$

between G_i and the embedding.

We need to do the construction above for families.

Construction of ΓDAC.

From now on , we are concerned with constructing ΓDAC in detail and in the following , we assume Γ is a torsion free quasi geodesic bicombing group. By moving slightly the extended bicombing map, we may assume

$$f: \quad \tilde{P_0} \times \Gamma \times [0,1] \mapsto P_1(\Gamma) \subset P_{-1} \subset P_0 \subset \ldots.$$

(we recall f is required to be continuous only on $\Gamma \times \Gamma \times [0,1]$.)

Take $\{Y_l\}$: a covering of $\tilde{P_0}$ such that $\gamma(Y_l) \cap Y_l = \phi$, $\gamma(Y_l) = Y_j$, and diam $Y_l \leq \epsilon$ for some small ϵ. Moreover associated to $\{Y_l\}$, we take $\{Z_l\}$: Δ separated nets of Γ such that $\gamma(Z_k) = Z_l$, $\gamma(Z_k) \cap Z_k = \phi$ if $\gamma \neq e$, if $Y_i \cap Y_j \neq \phi$, then $d(Z_i, Z_j) > \Delta$ and $sup_{y \in \tilde{P_0}} \epsilon(Z_y) \equiv sup_{y \in \tilde{P_0}} sup_{y' \in \tilde{P_0}} d(y', Z_y) \leq N\Delta$ where $Z_y = Z \cap y \times \tilde{P_0}$, $Z = \cup_i Y_i \times Z_i$. Δ is sufficiently large , which is determined later and N is chosen independently of Δ. For every Y_l , we choose $y_l \in Y_l$. We write $\cup_{y \in Y_l} Z_y = \tilde{Z}_{Y_l}$. For every $y \in Y_l$, $z \in Z_k$, $Y_l \cap Y_k \neq \phi$, define

$$G_l: \quad Y_l \times \tilde{Z}_{Y_l} \mapsto Y_l \times \tilde{P}_{-1} \qquad \alpha_l(y,z) = f_\lambda(y_k, z).$$

Then for $y \in Y_l$ and $Y_l \cap Y_k \neq \phi$, $z_1 \in Z_l, z_2 \in Z_k$,

$$\begin{aligned} d(G_l(y,z_1), G_k(y,z_2)) &\leq d(f_\lambda(y_l,z_1), f_\lambda(y_k,z_2)) \\ &\leq k_{-1}\{\lambda d(z_1,z_2) + (1-\lambda)d(y_l,y_k)\} + C_{-1} \\ &\leq k_{-1}\lambda d(z_1,z_2) + k_{-1}\epsilon + C_{-1} \quad \leq 2k_{-1}\lambda d(z_1,z_2). \end{aligned}$$

At this time , we need a lower bound of $\lambda\Delta$ i.e. $C_{-1} + k_{-1}\epsilon \leq k_{-1}\lambda\Delta$. Notice that the lower bound is $\frac{C_{-1}+k_{-1}\epsilon}{k_{-1}}$. Moreover for $Y_{l'} = \gamma(Y_l)$, define

$$G_{l'} = \gamma G_l \gamma^{-1}.$$

These maps are fiberwise $2k_{-1}\lambda$ Lipschitz. Now we extend G_l to

$$G: \quad Y_l \times \Gamma \mapsto Y_l \times \tilde{P}_{-1'} \qquad (\tilde{P}_{-1} \subset \tilde{P}_{-1'} \subset \tilde{P_0}).$$

To do this , let d_l be a metric on $Y_l \times \tilde{P_0}$ such that $d_{x \times \tilde{P_0}} = d_{\tilde{P_0}}$ and let $d_{G_l} = \max\{d_l, (2k_{-1}\lambda)^{-1} G_l^*\}$ be a metric on $Y_l \times \tilde{Z}_{Y_l}$. Then

$$G_l: \quad (Y_l \times \tilde{Z}_{Y_l}, d_{G_l}) \mapsto \tilde{P}_{-1}$$

is $2k_{-1}\lambda$ Lipschitz. Extending d_{G_l} to $\bar{d}_l$ on $Y_l \times \tilde{P}_0$ which satisfies $2\bar{d}_l|_{Y_l \times \tilde{Z}_{Y_l}} \geq d_{G_l}$, $\bar{d}_l|_{y \times \tilde{P}_0} = d_{\tilde{P}_0}$ and using the extension lemma of maps in the previous section , we get $4C_{-1}k_{-1}\lambda$ Lipschitz map

$$G_l : \quad Y_l \times \Gamma \mapsto \tilde{P}_{-1'}$$

where C_{-1} is independent of λ. If $Y_{l'} = \gamma(Y_l)$, we put $G_{l'} = \gamma G_l \gamma^{-1}$ as usual.

Next let us take another Y_{l_1} such that $Y_{l_1} \cap \Gamma(Y_l) = \phi$ and do the above construction for Y_{l_1}. Take Y_{l_2} such that $Y_{l_2} \cap \Gamma(Y_{l_1}) = \phi$, $Y_{l_2} \cap \Gamma(Y_l) = \phi$ and extend G_{l_2} to a map from $Y_{l_2} \times \Gamma$. We repeat this process. Namely take Y_{l_3} such that $Y_{l_3} \cap \Gamma(Y_{l_i}) = \phi$, ($0 \leq i \leq 2$), $Y_{l_0} = Y_l$ and extend G_{l_3} to a map from $Y_{l_3} \times \Gamma$. We continue this process until there is no such Y_k. Then we have a map

$$G : \quad \Gamma(\cup_{0 \leq i \leq n} Y_{l_i}) \times \Gamma \mapsto \tilde{P}_{-1'}.$$

Next take $Y_m \neq \gamma(Y_{l_i})$ ($0 \leq i \leq n$, $\gamma \in \Gamma$). For Y_m, we extend

$$G_m : Y_m \times Z_m \cup Y_m \cap \Gamma(\cup Y_{l_i}) \times \Gamma \cup_j Y_m \cap Y_{m_j} \times Z_{m_j} \mapsto \tilde{P}_{-1'}$$

to a map

$$G_m : \quad Y_m \times \Gamma \mapsto \tilde{P}_{-1''}.$$

where $Y_{m_j} \cap Y_m \neq \phi$, $Y_{m_j} \neq \gamma(Y_{l_i})$. For the remaining $Y_{m'}$, we extend $G_{m'}$ by the same method. Thus finally we get a fiberwise $\tilde{C}_{-1}k_{-1}\lambda$ Lipschitz map

$$G : \quad \tilde{P}_0 \times_\Gamma \Gamma \mapsto \tilde{P}_0 \times_\Gamma \tilde{P}_{-1''}. \qquad (\tilde{P}_{-1} \subset \tilde{P}_{-1'} \subset \tilde{P}_{-1''} \subset \tilde{P}_0)$$

Next we construct a fiberwise proper homotopy between G and an embedding such that

(1) G_t is a fiberwise C Lipschitz , where C does not depend on λ, t,

(2) $d(G_t(y,z), G_{t'}(y,z)) \leq C(1 + d(y,z))|t' - t| \qquad y \in \tilde{P}_0, z \in \Gamma.$

We put

$$G : \quad \{0,1\} \times \tilde{P}_0 \times_\Gamma \Gamma \mapsto \tilde{P}_0 \times_\Gamma \tilde{P}_{-1''}$$

where G_0 is the embedding and $G_1 = G$. Let $\{Y_l\}$ be as before and for fixed $\Delta_0 \geq 1 \geq \epsilon$, take another $\{(Z_0)_l\}$ which are Δ_0 separated $N\Delta_0$ nets satisfying properties before. First , take $y \in \tilde{P}_0, z \in \Gamma$. Then

$$\begin{aligned} &d(G(y,z), f_\lambda(y,z)) \\ &\quad \leq d(G(y,z), G(y,z')) + d(G(y,z'), f_\lambda(y,z')) + d(f_\lambda(y,z'), f_\lambda(y,z)) \\ &\quad \leq \tilde{C}_{-1}k_{-1}\lambda N\Delta + k_{-1}\lambda d(z,z') + C_{-1} \\ &\quad \leq (\tilde{C}_{-1}k_{-1}N + k_{-1}N)\lambda\Delta + C_{-1}, \end{aligned}$$

which is bounded independently of each value of λ, Δ where $z' \in \Gamma, f_\lambda(y, z') = G(y, z')$, $d(z, z') \leq N\Delta$. For every l , we fix $y_l \in Y_l$. Let

$$(Z_0)_y = \cup_{i \geq 0} D_i^y$$
$$D_i^y = \cup_l \{z \in (Z_0)_y \cap (Z_0)_l, 2^{i-1}\Delta_0 < d(y_l, z) \leq 2^i \Delta_0\}$$
$$D_0^y = (Z_0)_y \setminus \cup_{i \geq 1} D_i^y.$$

For every $i \geq 0$, We take partitions I_i on $[0, 1]$ as follows.

$$I_i = \{(t_0 = 0 < t_1 < \cdots < t_{\alpha-1} < t_{\alpha_i} = 1),$$
$$t_{a+1} - t_a = 2^{-i}\Delta_0^{-1}, \quad 0 \leq a \leq \alpha_i - 1, \quad 2^{-i}\Delta_0^{-1} \leq t_{\alpha_i} - t_{\alpha_i - 1} \leq 2^{-i+1}\Delta_0^{-1}\}.$$

On every t_a , $0 < a \leq \alpha_i - 1$, we define

$$G(t_i, y, z) = f_{\lambda t_i + 1 - t_i}(y_l, z)$$

for $y \in Y_l, z \in (Z_0)_l \cap D_i^y$ and $G(1, y, z) = G(y, z)$, $G(0, y, z) = (y, z)$. Then for every a , $0 < t_a < t_{a+1} < 1$ and $y \in Y_l, z \in (Z_0)_l \cap D_i^y$, we connect $f_{\lambda t_a + 1 - t_a}$ and $f_{\lambda t_{a+1} + 1 - t_{a+1}}(y_l, z)$ by a geodesic segment

$$h_i^a(y_l, z)(s),$$

$$0 \leq s \leq d(f_{\lambda t_a + 1 - t_a}(y_l, z), f_{\lambda t_{a+1} + 1 - t_{a+1}}(y_l, z)) \equiv s_i^a(y_l, z).$$

Then define for $z \in D_i^y \cap (Z_0)_l \quad , y \in Y_l \quad , 0 < t_a \leq t \leq t_{a+1} < 1$,

$$G(t, y, z) = h_i^a(y_l, z)(\frac{(t - t_a)s_i^a(y_l, z)}{t_{a+1} - t_a}).$$

For $t_{\alpha_i - 1} \leq t \leq 1$ ($0 \leq t \leq t_1$) , we connect $f_{\lambda t_{\alpha_i - 1} + 1 - t_{\alpha_i - 1}}(y_l, z)$ and $G(y, z)$, ((y, z) and $f_{\lambda t_1 + 1 - t_1}(y_l, z)$) by lines , respectively. But in this case , $G(y, z)$ and (y, z) vary with respect to y . So in this case , we choose ϵ' : sufficiently small and connect $f_{\lambda t_{\alpha_i - 1} + 1 - t_{\alpha_i - 1}}(y_l, z)$ and $G(y, z)$ ((y, z) and $f_{\lambda t_1 + 1 - t_1}(y_l, z)$) by continuous lines $h_i^{\alpha_i - 1}(y, z)(s)$ ($h_i^0(y, z)(s)$) where

$$0 \leq s \leq d(f_{\lambda t_{\alpha_i - 1} + 1 - t_{\alpha_i - 1}}(y_l, z), G(y, z)) + \epsilon'$$
$$(\quad 0 \leq s \leq d(f_{\lambda t_1 + 1 - t_1}(y_l, z), (y, z)) + \epsilon' \quad)$$

for some ϵ' , $y \in Y_l$. Thus , we obtain

$$G : \quad [0, 1] \times \cup_l Y_l \times (Z_0)_l \mapsto \tilde{P}_{-1''}.$$

Next , we estimate their Lipschitz constants. First , let us note that on every $z \in D_i^y \cap (Z_0)_l$ and $0 < t_a \le t \le t_{a+1} < 1$,

$$d(h_i^a(y_l,z)(\frac{s_i^a(y_l,z)(t-t_a)}{t_{a+1}-t_a}), f_{\lambda t_{a+1}+1-t_{a+1}}(y_l,z)) \le k_{-1}(1-\lambda)+C_{-1},$$

which is bounded independently of λ, i . Similarly for $t_{\alpha_i-1} \le t \le 1$,

$$\begin{aligned} &d(h_i^{\alpha_i-1}(\frac{s_i^{\alpha_i-1}(y_l,z)(t-t_{\alpha_i-1})}{1-t_{\alpha_i-1}}), G(y,z)) \\ &\qquad \le (k_{-1}(1-\lambda)+C_{-1}) + (\tilde{C}_{-1}k_{-1}N + k_{-1}N)\lambda\Delta + C_{-1} + \epsilon' \equiv M. \end{aligned}$$

As every $G(t,y,z)$ is of the form

$$h_i^a(y_l,z)(\frac{(t-t_a)s_i^a(y_l,z)}{t_{a+1}-t_a})$$

for some a , thus for $z \in (Z_0)_y \cap (Z_0)_l$, $z' \in (Z_0)_y \cap (Z_0)_k$, for $t_1 \le t \le t_{\alpha_i-1}$,

$$\begin{aligned} &d(G(t,y,z), G(t,y,z')) \\ &\quad \le 2k_{-1}(1-\lambda) + 2C_{-1} + d(f_{\lambda t_{a+1}+1-t_{a+1}}(y_l,z), f_{\lambda t_{a+1}+1-t_{a+1}}(y_k,z')) \\ &\quad \le 2(k_{-1}(1-\lambda)+C_{-1}) + d(f_{\lambda t_{a+1}+1-t_{a+1}}(y_l,z), f_{\lambda t_{a+1}+1-t_{a+1}}(y_l,z')) \\ &\qquad + d(f_{\lambda t_{a+1}+1-t_{a+1}}(y_l,z'), f_{\lambda t_{a+1}+1-t_{a+1}}(y_k,z')) \\ &\quad \le 2(k_{-1}(1-\lambda)+C_{-1}) + k_{-1}d(y_l,y_k) + C_{-1} + k_{-1}d(z,z') + C_{-1} \\ &\quad \le 2(k_{-1}(1-\lambda)+C_{-1}) + k_{-1}\epsilon + 2C_{-1} + k_{-1}d(z,z') \\ &\quad \le (k_{-1} + \Delta_0^{-1}\{2(k_{-1}(1-\lambda)+C_{-1}) + k_{-1}\epsilon + 2C_{-1}\})d(z,z') \\ &\quad \le (k_{-1} + \Delta_0^{-1}\{2M + k_{-1}\epsilon + 2C_{-1}\})d(z,z'). \end{aligned}$$

Similarly for $t_{\alpha-1} \le t \le 1$,

$$d(G(t,y,z), G(t,y,z')) \le \{\Delta_0^{-1}(2M + 2C_{-1} + k_{-1}\epsilon) + \tilde{C}_{-1}k_{-1}\lambda\}d(z,z')$$

and for $0 \le t \le t_1$,

$$\begin{aligned} &d(G(t,y,z), G(t,y,z')) \\ &\quad \le \{\Delta_0^{-1}(2(k_{-1}(1-\lambda)+C_{-1}+\epsilon+\epsilon') + 2C_{-1} + k_{-1}\epsilon) + 1\}d(z,z') \\ &\quad \le \{\Delta_0^{-1}(2M + k_{-1}\epsilon + 2C_{-1}) + 1\}d(z,z'). \end{aligned}$$

Next , for $z \in D_i^y \cap (Z_0)_l$, $0 \le t_a \le t, t' \le t_{a+1} \le 1$, $y \in Y_l$,

$$\begin{aligned} & d(G(t,y,z), G(t',y,z)) \\ & \quad = d(h_i^a(y_l,z)(\frac{(t-t_a)s_i^a(y_l,z)}{t_{a+1}-t_a}), h_i^a(y_l,z)(\frac{(t'-t_a)s_i^a(y_l,z)}{t_{a+1}-t_a})) \\ & \quad = s_i^a(y_l,z)|\frac{t-t_a}{t_{a+1}-t_a} - \frac{t'-t_a}{t_{a+1}-t_a}| \quad = s_i^a(y_l,z)\frac{|t-t'|}{t_{a+1}-t_a} \\ & \quad \le 2^i \Delta_0 s_a(y_l,z)|t-t'| \quad \le 2\Delta_0 M(1+d(y,z))|t-t'|. \end{aligned}$$

Moreover , for $z \in (Z_0)_y \cap (Z_0)_l$, $0 \le t_a \le t \le t_{a+1} \le 1$,

$$\begin{aligned} d(y, G(t,y,z)) & \ge d(y, f_{\lambda t_a + 1 - t_a}(y_l,z)) - d(f_{\lambda t_a+1-t_a}(y_l,z), G(t,y,z)) \\ & \ge A_{-1}(\lambda t_a + 1 - t_a)(d(y,z)) - B_{-1} - A_{-1}\epsilon - B_{-1} - M \\ & \ge A_{-1}\lambda d(y,z) - (A_{-1}\epsilon + 2B_{-1} + M). \end{aligned}$$

Thus we have the following lemma ,

Lemma 4.1. *$G : \quad [0,1] \times \cup_l Y_l \times (Z_0)_l \mapsto \tilde{P}_{-1''}$ has the following properties.*

(1) $G(t,y,\quad) : (Z_0)_y \mapsto \tilde{P}_{-1''}$ *is C Lipschitz.*
where C is independent of t, y and each value of λ, Δ.

(2) $G(\quad, y, z) : \quad [0,1] \mapsto \tilde{P}_{-1''}$ *is $C(1+d(y,z))$ Lipschitz.*

(3) *There exists A_{-1}, B_{-1} such that $d(y, G(t,y,z)) \ge A_{-1}\lambda d(y,z) - B_{-1}$.*

(4) $G(1,\quad,\quad) = G, \qquad G(0,\quad,\quad) = id$.

We continue the construction. Let $\tilde{d}_l$ be a metric on $Y_l \times \tilde{P}_0$ such that

(1) $G(t,\quad,\quad) : \quad (Y_l \times (\tilde{Z}_0)_{Y_l}, \tilde{d}_l) \mapsto \tilde{P}_{-1''}$ is Lipschitz ,

(2) $\tilde{d}_l|_{y \times \tilde{P}_0} = d_{\tilde{P}_0}$,

(take an extension of $\sup_t d_{G_t}$) and put $\bar{\tilde{d}}_l = \max\{d_l, \bar{d}_l, \tilde{d}_l\}$. We define a metric d_{i_l} on $[0,1] \times W_i^l$ where $W_i^l \subset Y_l \times \Gamma$, $W_i^l = \{(y,p), 2^{i-1}\Delta_0 < d(y,p) \le 2^i\Delta_0\}$, $W_0^l = Y_l \times \Gamma \setminus \cup_{i \ge 1} W_i^l$. Then for $(y,z), (y',z') \in W_i^l$, put

$$d_{i_l}((t,y,z),(t',y',z')) \equiv 2^i\Delta_0|t-t'| + \bar{\tilde{d}}_l((y,z),(y',z')).$$

Lemma 4.2. *$G_l|_{[0,1]\times\{W_i^l\cap Y_l\times(\tilde{Z}_0)_{Y_l}\}}$ is Lipschitz with respect to d_{i_l}.*

Proof. Take $(y,z),(y',z')\in W_i^l\cap Y_l\times(\tilde{Z}_0)_{Y_l}$. Then

$$\begin{aligned}
&d(G(t,y,z),G(t',y',z'))\\
&\qquad\le d(G(t,y,z),G(t,y',z'))+d(G(t,y',z'),G(t',y',z'))\\
&\qquad\le C\bar{\bar{d}}_l((y,z),(y',z'))+C(1+d(y',z'))|t-t'|\\
&\qquad\le C\{2^{i+1}\Delta_0|t-t'|+\bar{\bar{d}}_l((y,z),(y',z'))\}\\
&\qquad\le 2Cd_{i_l}((t,y,z),(t',y',z')).
\end{aligned}$$

Then using the Lipschitz extension lemma , we first extend G_l defined on every $W_i^l\cap(Y_l\times(\tilde{Z}_0)_{Y_l})$ to

$$G_l:\quad [0,1]\times W_i^l\mapsto\tilde{P}_{-1'''}\qquad(\tilde{P}_{-1'''}\subset\tilde{P}_0)$$

which is C' Lipschitz with respect to d_{i_l} for i : even. Next , by doing the same extension process for $i=$ odd , we have

$$G_l:\quad [0,1]\times Y_l\times\Gamma\mapsto\tilde{P}_{-1''''}\qquad(\tilde{P}_{-1'''}\subset\tilde{P}_{-1''''}\subset\tilde{P}_0)$$

where $G_l(\ \ ,y,z):[0,1]\mapsto\tilde{P}_{-1''''}$ is $4\Delta_0C(1+d(y,z))$ Lipschitz because for $(y,z)\in W_i^l$,

$$d(G_l(t,y,z),G_l(t',y,z))\quad\le\quad C2^{i+1}\Delta_0|t-t'|\quad\le\quad 4\Delta_0C(1+d(y,z))|t-t'|.$$

As usual , we put $G_{l'}=\gamma G_l\gamma^{-1}$ for $Y_{l'}=\gamma(Y_l)$. We extend each G_k by the same method as we have constructed a contraction $G:\tilde{P}_0\times_\Gamma\Gamma\mapsto\tilde{P}_0\times_\Gamma\tilde{P}_{-1''}$. Then we obtain a map

$$G_k:\quad [0,1]\times Y_k\times\Gamma\mapsto\tilde{P}_{-1''''}$$

which satisfies

(1) $G_k|_{Y_k\cap\gamma(Y_l)\times\Gamma}=\gamma G_l\gamma^{-1}|_{Y_k\cap\gamma(Y_l)\times\Gamma}$,
(2) $G(t,\ \ ,\ \ )$ is fiberwise C Lipschitz ,
(3) $G(\ \ ,y,z)$ is $C(1+d(y,z))$ Lipschitz.

Thus by putting $\tilde{P}_{-1''''}\equiv\tilde{P}_{-1}$, we obtain

$$G:\quad [0,1]\times\tilde{P}_0\times_\Gamma\Gamma\mapsto\tilde{P}_{-1}\qquad(\tilde{P}_{-1}\subset\tilde{P}_0).$$

Remark 4.3.

(1) Properness is followed because G is proper on $[0,1] \times \tilde{P}_0 \times_\Gamma \Gamma$ and G_t is C Lipschitz .

(2) For any $y \in \tilde{P}_0$, $z \in \Gamma$, $d(y, G_t(y,z)) \geq A_{-1}\lambda d(y,z) - B_{-1} - C\Delta_0 N$.

We do the same process. Recall $G_{i-1}(\tilde{P}_0 \times_\Gamma \Gamma) \subset \tilde{P}_0 \times_\Gamma \tilde{P}_{-1}$. First , extend $G : \tilde{P}_0 \times_\Gamma \Gamma \mapsto \tilde{P}_0 \times_\Gamma \tilde{P}_{-1}$ to a map

$$G : \quad \tilde{P}_0 \times_\Gamma \Gamma \cup G_{i-1}(\tilde{P} \times_\Gamma \Gamma) \mapsto \tilde{P}_0 \times_\Gamma \tilde{P}_{-1'}$$

which is fiberwise $C\lambda$ Lipschitz , where $\tilde{P}_{-1} \subset \tilde{P}_{-1'} \subset \tilde{P}_0$. Secondly we extend $G : \{0,1\} \times G_{i-1}(\tilde{P}_0 \times_\Gamma \Gamma) \cup [0,1] \times \tilde{P}_0 \times_\Gamma \Gamma \mapsto \tilde{P}_0 \times_\Gamma \tilde{P}_{-1'}$ to a proper homotopy

$$G : \quad [0,1] \times \{\tilde{P}_0 \times_\Gamma \Gamma \cup G_{i-1}(\tilde{P}_0 \times_\Gamma \Gamma)\} \mapsto \tilde{P}_0 \times_\Gamma \tilde{P}_0$$

where $G_0 =$ embedding , $G_1 = G$. Embedding into $\tilde{P}_0 \times_\Gamma \tilde{P}_i$, we extend G to a map

$$G : \quad (\tilde{P}_0 \times_\Gamma \tilde{P}_i, d_i) \mapsto (\tilde{P}_0 \times_\Gamma \tilde{P}_{i+1}, d_{i+1})$$

which is fiberwise $C_i\lambda$ Lipschitz where C_i does not depend on λ. Finally we extend $G_i : \{0,1\} \times \tilde{P}_0 \times_\Gamma \tilde{P}_i \mapsto \tilde{P}_0 \times_\Gamma \tilde{P}_{i+1}$ to a proper homotopy

$$G_i : \quad [0,1] \times \tilde{P}_0 \times_\Gamma \tilde{P}_i \mapsto \tilde{P}_0 \times_\Gamma \tilde{P}_{i+1'}.$$

The method is the same as above and we omit the repetition. Rewriting $\tilde{P}_{i+1}$ for $\tilde{P}_{i+1'}$, we thus get ΓDAC and complete the construction.

Appendix

For the convenience of the reader, in this section, we give a proof of the metric extension lemma ([CGM] p42). Here we prove it in a slightly weak version of [CGM] , which is enough for our purpose.

Lemma. *Let Y_1 be a compact metric space , Y_2 a (non) compact , geodesic space and $Z \subset Y_1 \times Y_2$ be Δ_0 net for some $\Delta_0 > 0$ where Z is of the form $Z = Y_1 \times X$, X is closed in Y_2. Then if d is a metric on Z such that $d|_{y_1 \times X} = d_{Y_2}|_X$, there exists a metric $\bar{d}$ on $Y_1 \times Y_2$ such that*

$$\bar{d}|_{y_1 \times Y_2} = d_{Y_2}, \qquad 2\bar{d}|_Z \geq d.$$

Proof. First we assume Y_2 is compact. Let $z_i = (y_i, y_i')$, $y_i \in Y_1, y_i' \in Y_2$ $i = 1,2$ and let us put

$$\epsilon : \quad Z \times Z \mapsto \mathbb{R}, \qquad \epsilon(z_1, z_2) \equiv d(z_1, z_2) - d_{Y_2}(y_1', y_2').$$

$$\epsilon' : \quad Y_1 \times Y_1 \mapsto \mathbb{R},$$
$$\epsilon'(y_1, y_2) = \sup \{\epsilon(z_1, z_2), z_i = (y_i, y_i'), y_1', y_2' \in Y_2, i = 1,2\}.$$

Then take a continuous function

$$\epsilon'' : \quad Y_1 \times Y_1 \mapsto \mathbb{R}_+ \qquad \text{such that}$$

$$\epsilon'' \geq \epsilon', \qquad \epsilon'' \text{ vanishes on diagonal.}$$

Let d_1 , $\bar{d}$ be metrics on Y_1 and $Y_1 \times Y_2$ respectively by

$$d_1(y_1, y_2) = sup_{y \in Y_1} |\epsilon''(y_1, y) - \epsilon''(y_2, y)|,$$
$$\bar{d}(z_1, z_2) = d_{Y_2}(y_1', y_2') + d_1(y_1, y_2).$$

Then in this case , $\bar{d}|_Z \geq d$. Next , let Y_2 be non compact and take a covering of Y_2 by compact subsets $\{B_{\frac{M}{2}}(y_l)\}_{y_l \in Y_2}$ where each $B_{\frac{M}{2}}(y_l)$ is a ball of radious $\frac{M}{2}$ with the center y_l , for a fixed M (independently of l). We take M sufficiently large so that $M >> 2\Delta_0$.

We apply the extension process on each $Y_1 \times B_{4M}(y_l)$ and write the metric $\bar{d}_l$. We recall $\bar{d}_l$ is of the form

$$\bar{d}_l = \epsilon_l + d_{Y_2}.$$

Let us write

$$\tilde{\Delta}_n \subset (Y_1 \times Y_2) \times (Y_1 \times Y_2),$$
$$\tilde{\Delta}_n = \{(z, z'), z, z' \in Y_1 \times B_{nM}(y_l) \text{ for some } l\},$$

and define

$$\tilde{d} : \quad \tilde{\Delta}_4 \mapsto \mathbb{R}_+$$

$$\tilde{d}(z, z') = \sup_l \{\bar{d}_l(z, z'), z, z' \in Y_l \times B_{4M}(y_l)\}.$$

Then we define the desired metric $\bar{d}$ by

$$\bar{d}(z, z') = \inf \{\sum_{i=1}^{m-1} \tilde{d}(z_i, z_{i+1}), z_1 = z, z_m = z', (z_i, z_{i+1}) \in \tilde{\Delta}_1\}$$

for any $z, z' \in Y_1 \times Y_2$. It is easy to see that $\bar{d}$ is a metric and $\bar{d}|_{y_1 \times Y_2} = d_{Y_2}$. We only have to show $2\bar{d}|_Z \geq d$. Let $z, z' \in Z$. It is enough to see for any

$$\{z_i\}_{i=1}^m, \quad z_1 = z, z_m = z', \quad (z_i, z_{i+1}) \in \tilde{\Delta}_1,$$

$$2\sum_{i=1}^{m-1}\tilde{d}(z_i, z_{i+1}) \geq d(z, z').$$

The result is clear if $\{z_i\} \subset Y_1 \times B_{3M}(y_l)$ for some l. In fact , in this case ,

$$\bar{d}|_{Y_1 \times B_{3M}(y_l) \cap Z} \geq d|_{Y_1 \times B_{3M}(y_l) \cap Z}.$$

Next , let

$$\begin{gathered} z_1, \ldots, z_\alpha \in Y_1 \times B_{3M}(y_l) \text{ for some } l, \\ z_\alpha \in Y_1 \times B_M(y_{l'}), \qquad z_{\alpha+1} \notin Y_1 \times B_{3M}(y_l), \\ z_1 \in Y_1 \times B_M(y_l). \end{gathered}$$

Then take

$$\begin{gathered} \omega_\alpha \in Z \cap Y_1 \times B_M(y_{l'}) \cap Y_1 \times B_{4M}(y_l), \\ z_\alpha = (y_\alpha, y'_\alpha), \qquad \omega_\alpha = (y_\alpha, y''_\alpha), \\ d_{Y_2}(y'_\alpha, y''_\alpha) \leq \Delta_0. \end{gathered}$$

By choice of M , we have

$$2\sum_{i=1}^{\alpha-1}\tilde{d}(z_i, z_{i+1}) \geq \tilde{d}(z_1, \omega_\alpha) \geq d(z_1, \omega_\alpha).$$

Beginning at z_α , we continue this process. let

$$\begin{gathered} z_\alpha, \ldots, z_{\alpha'} \in Y_1 \times B_{3M}(y_{l'}), \\ z_{\alpha'} \in Y_1 \times B_M(y_{l''}), \qquad z_{\alpha'+1} \notin Y_1 \times B_{3M}(y_{l'}). \end{gathered}$$

As before , we take

$$\begin{gathered} \omega_{\alpha'} \in Z \cap Y_1 \times B_M(y'') \cap Y_1 \times B_{4M}(y'), \\ z_{\alpha'} = (y_{\alpha'}, y'_{\alpha'}), \qquad \omega_{\alpha'} = (y_{\alpha'}, y''_{\alpha'}), \\ d_{Y_2}(y'_{\alpha'}, y''_{\alpha'}) \leq \Delta_0. \end{gathered}$$

Then we obtain as before

$$2\sum_{i=\alpha}^{\alpha'-1}\tilde{d}(z_i, z_{i+1}) \geq \tilde{d}(\omega_\alpha, \omega_{\alpha'}) \geq d(\omega_\alpha, \omega_{\alpha'}).$$

We continue this process until we have reached to z_m. Namely , let

$$\{z_i\}_{i=1}^m = \{\tilde{z}_{\alpha_1}, \ldots, \tilde{z}_{\alpha_{m'}}\},$$
$$\tilde{z}_{\alpha_i} = \{z_{\alpha_i}, \ldots, z_{\alpha_{i+1}}\}, \tilde{z_1} = \{z_1, \ldots, z_{\alpha_2}\}, \tilde{z}_{\alpha_{m'}} = \{z_{\alpha_{m'}}, \ldots, z_m\},$$
$$z_{\alpha_i} \in Y_1 \times B_M(y_{l(i)}), \tilde{z}_{\alpha_i} \subset Y_1 \times B_{3M}(y_{l(i)}), z_{\alpha_i+1} \notin Y_1 \times B_{3M}(y_{l(i)}).$$

Then , we take

$$\omega_{\alpha_i} \in Z \cap Y_1 \times B_M(y_{l(i)}) \cap Y_1 \times B_{4M}(y_{l(i-1)}),$$
$$z_{\alpha_i} = (y_{\alpha_i}, y'_{\alpha_i}), \qquad \omega_{\alpha_i} = (y_{\alpha_i}, y''_{\alpha_i}),$$
$$d_{Y_2}(y'_{\alpha_i}, y''_{\alpha_i}) \leq \Delta_0.$$

By the same reasoning as before , we have

$$2 \sum_{j=\alpha_i}^{\alpha_{i+1}-1} \tilde{d}(z_j, z_{j+1}) \geq d(\omega_{\alpha_i}, \omega_{\alpha_{i+1}})$$

for every $i \in \{1, \ldots, m'-1\}$. As we may put $\omega_{\alpha_1} = z, \omega_{\alpha_{m'+1}} = z'$, we have got

$$2 \sum_{i=1}^{m-1} \tilde{d}(z_i, z_{i+1}) \geq \sum_{i=1}^{m'} d(\omega_{\alpha_i}, \omega_{\alpha_{i+1}}) \geq d(z, z').$$

This completes the proof of the lemma.

We have shown more in the proof.

Corollary of the proof of the lemma. *Using the notations of the lemma , let $Y'_1 \subset Y_1$ be compact , $X \subset X'$ be closed, Δ_0 nets in Y_2. If d is a metric on $Z = Y_1 \times X \cup Y'_1 \times X'$ such that*

$$d_{y'_1 \times X'} = d_{Y_2}|_{X'}, \quad d_{y_1 \times X} = d_{Y_2}|_X, \quad y'_1 \in Y'_1, y_1 \in Y_1,$$

then there exists a metric $\bar{d}$ on $Y_1 \times Y_2$ such that

$$\bar{d}|_{y_1 \times Y_2} = d_{Y_2}, \qquad 2\bar{d}|_Z \geq d.$$

PART II

In the following sections , using Kasparov's KK theory , which is a generalization of K (co) homology , we show Novikov conjecture for our groups. First we review KK groups and next , we construct from α_∞ an element

$$K(\alpha) \in KK_\Gamma(C_0(\tilde{P}_0), \mathbb{C})$$

and a commutative diagram ,

$$\begin{array}{ccc} KK_\Gamma(\mathbb{C}, C_0(\tilde{P}_0)) & \xrightarrow{\phi\circ\mu} & K^*(B\Gamma) \\ {\scriptstyle ch_*}\downarrow & & \downarrow{\scriptstyle \mathrm{r}^*\circ ch^*} \\ H_*^{\inf}(\tilde{P}_0 : \Gamma) & \xrightarrow[\alpha_0\cap]{} & H^*(\frac{P_0}{\Gamma}) \end{array}$$

where

$$r^* : \quad H^*(B\Gamma) \mapsto H^*(\frac{P_0}{\Gamma})$$

is the restriction , $\alpha_0\cap$: is the Poincaré duality ,

$$\mu : \quad KK_\Gamma(\mathbb{C}, \mathbb{C}) \mapsto K(B\Gamma)$$

is a map constructed by A.S.Miscenko. (Appendix B) One of the most particular feature of KK group is the existence of the intersection product,

$$KK_\Gamma(X, Y) \times KK_\Gamma(Y, Z) \mapsto KK_\Gamma(X, Z).$$

(By definition , $KK_\Gamma(X, Y) = KK_\Gamma(C_0(X), C_0(Y))$). Then

$$\phi \equiv K(\alpha)\times : \quad KK_\Gamma(\tilde{P}_0,\ \mathrm{pt}\ ) \mapsto KK_\Gamma(\mathbb{C}, \mathbb{C}).$$

From the above diagram , we will see the following. First we fix N , but arbitrary and then we determine P_0 so that

$$r_* : \quad H_*(\frac{P_0}{\Gamma}) \mapsto H_*(B\Gamma) \qquad (0 \leq * \leq N)$$

induced from the inclusion map is surjective. For this P_0 , we do the construction of the previous sections . Then by the above diagram , for every $x \in H^*(B\Gamma)$, ($0 \leq * \leq N$) , there exists $F \in KK_\Gamma(\mathbb{C}, \mathbb{C})$ such that

$$x - ch^*(F) \in H^*(B\Gamma), \qquad * \geq N + 1.$$

This implies Novikov conjecture for our groups.

§5 Elementary definitions of Kasparov's KK group.

In the following , we mean $\hat{\otimes}$ is the graded tensor product. (See [Bl]) Let A, B be separable C^* algebras.

Definition. *$E(A,B)$ is the set of all triples (E,ϕ,F) where*

(1) E is a countably generated $\mathbb{Z}_2$ graded Hilbert module over B, (i.e. right B module with a B valued inner product $<,>$)

(2) ϕ is a degree 0 $*$ homomorphism from A to $B(E)$, where $B(E)$ is the set of B module homomorphisms with adjoint ,

(3) $F \in B(E)$ is of degree 1 satisfying

$$[F-F^*]\phi(a), \quad [\phi(a),F], \quad (F^2-1)\phi(a) \quad \in K(E),$$

where $K(E)$ = closure of linear span of $\theta_{x,y} \in B(E)$, for $x,y,z \in E$, $\theta_{x,y}(z) = x<y,z>$. We call an element of $K(E)$ a compact endomorphism.

Examples.

(1) $\hat{H}_{C_0(X)} = \hat{H}\hat{\otimes}C_0(X) = (H\oplus H)\hat{\otimes}C_0(X)$, where $C_0(X)$ is continuous functions vanishing at infinity. $\hat{H}_{C_0(X)}$ is a Hilbert $C_0(X)$ module with the obvious inner product.

(2) $K(\hat{H}_{C_0(X)}) = K\hat{\otimes}C_0(X)$ where K is the set of compact operators in $B(H\oplus H)$.

(3) $K(C_0(X)) = C_0(X)$, $B(C_0(X)) = C_b(X) = \{$ bounded continuous functions $\}$.

(4) Let X be a manifold without boundary , which admit a spin ^{c} structure S and E be a complex vector bundle over X. Then $(L^2(X, S\hat{\otimes}E), D_E(1+D_E^2)^{-\frac{1}{2}})$ is an element of $KK(X,\ pt\ )$. (X is not assumed to be compact.) D_E is the Dirac operator with coefficient E.

Homotopy equivalence. *$(E_0,\phi_0,F_0) \sim (E_1,\phi_1,F_1)$ if there exists $(E,\phi,F) \in E(A, C([0,1],B))$ such that*

$$(E\hat{\otimes}_{f_i}B, f_i\circ\phi, (f_i)_*F) \cong (E_i,\phi_i,F_i),$$

where $f_i :\quad C([0,1],B) \mapsto B$ are the evaluation maps , $i=0,1$.

Definition.

$$KK(A,B) = E(A,B)/\sim .$$

Lemma (see [Bl]).

(1) *$KK(A,B)$ is a group which is compatible with the natural semi group structure (direct sum) .*

(2) *Let* $D(A,B) = \{(E,\phi,F) \in E(A,B)$,

$$[F - F^*]\phi(a) = (F^2 - 1)\phi(a) = [\phi(a), F] \equiv 0 \text{ for all } a \in A\}.$$

Then any element of $D(A,B)$ is zero in $KK(A,B)$. Such an element is called a degenerate element.

(3) *$KK(\mathbb{C}, C_0(X)) \cong K(X)$. Under this isomorphism , the Bott element β in $K(\mathbb{R}^{2N})$ is expressed as*

$$\{C_0(\mathbb{R}^{2N}, \bigwedge \mathbb{C}^N), F = \frac{x}{1+|x|}\} \in KK(\mathbb{C}, C_0(\mathbb{R}^{2N})),$$

where the action of F on $C_0(\mathbb{R}^{2N}, \bigwedge \mathbb{C}^N)$ is the Clifford multiplication.

Functor. *If $f : X \mapsto Y$ is a proper map , then*

$$f_* : \quad KK(X,Z) \mapsto KK(Y,Z) \qquad f_*(E,\phi,F) = (E, \phi \circ f, F).$$

$$f^* : \quad KK(Z,Y) \mapsto KK(Z,X) \qquad f^*(E,\phi,F) = (E\hat{\otimes}_f C_0(X), \phi\hat{\otimes}1, F\hat{\otimes}1).$$

Γ equivariant KK group. *An element of $E_\Gamma(A,B)$ is of the form $(E,\phi,F) \in E(A,B)$ such that there exists automorphisms of Γ on A,B and an action on E so that $g(a\xi b) = (ga)(g\xi)(gb)$, $< g\xi, g\xi' >= g < \xi, \xi' >$ where $a \in A, \xi \in E, b \in B$ and $\phi(a)(gFg^{-1} - F)$ is a compact endomorphism of E. Homotopy equivalence is defined similarly and by definition ,*

$$KK_\Gamma(A,B) = E_\Gamma(A,B)/\sim .$$

§6 KK group and Poincaré duality

Clifford representation.

In the following , let $e_1, \ldots, e_{2N}$ be the standard basis of $\mathbb{R}^{2N}$ and we identify $\mathbb{R}^{2N} = \mathbb{C}^N$. As an irreducible complex representation space of the Clifford algebra $Cl(\mathbb{R}^{2N})$, we take $\Delta_N = \bigwedge \mathbb{C}^N$. Canonically , there exists embeddings

$$\Delta_N \subset \Delta_{N+1} \subset \ldots .$$

Deformation of $\alpha_0^*(\beta)$.

Let $\alpha_0 : \quad Z = \tilde{P}_0 \times_\Gamma P_0 \mapsto \mathbb{R}^{N_0}$ where $N_0 = \dim \tilde{P}_0$. Then

$$\alpha_0^*(\beta) = (C_0(Z, \bigwedge \mathbb{C}^{\frac{N_0}{2}}), F_0(x) = \frac{\alpha_0(x)}{1 + |\alpha_0(x)|})$$

is the pull back of the Bott generator.

Proposition 6.1. *$(\ pr\ \alpha_\infty)^*(\beta) \equiv (C_0(Z, \bigwedge \mathbb{C}^{\frac{N_0}{2}}), F_\infty(x) = \frac{\text{pr } \alpha_\infty(x)}{1+|\text{ pr } \alpha_\infty(x)|})$ is equal to $\alpha_0^*(\beta)$ in $KK(\mathbb{C}, C_0(Z))$.*

Proof. We have a canonical , orthogonal basis $\{e_1, \ldots, e_{2^N}\}$ on each $\bigwedge \mathbb{C}^N$ as a Hibert space. We write $[\bigwedge \mathbb{C}^\infty]$ for an infinite dimensional Hilbert space with the orthogonal basis $\{e_i\}_{i \geq 1}$. For $x \in \tilde{P}_0 \times_\Gamma P_0$, $F_\infty(x)$ and $F_0(x)$ can be considerd as endomorphisms on $\bigwedge \mathbb{C}^{\frac{N_0}{2}}$. In the following , regarding them as endomorphisms on $[\bigwedge \mathbb{C}^\infty]$, we will make a homotopy between F_∞ and F_0 through End($[\bigwedge \mathbb{C}^\infty]$). First of all , we construct $F \in E(C_0(Z, [\bigwedge \mathbb{C}^\infty]))$ such that $(C_0(Z, [\bigwedge \mathbb{C}^\infty]), F) = \alpha_0^*(\beta)$ in $KK(\mathbb{C}, C_0(Z))$. We decompose $[\bigwedge \mathbb{C}^\infty]$ as follows. $[\bigwedge \mathbb{C}^\infty] = [\oplus_{i=0}^\infty V_i]$, where

$$\begin{aligned}
V_0 &= \bigwedge \mathbb{C}^{\frac{N_0}{2}} \\
V_1 &= \mathbb{C}e_{N_0+1} \wedge V_0 \\
V_2 &= \mathbb{C}e_{N_0+3} \wedge \bigwedge \mathbb{C}^{\frac{N_0}{2}+1} \\
V_3 &= \mathbb{C}e_{N_0+5} \wedge \bigwedge \mathbb{C}^{\frac{N_0}{2}+2} \\
&\cdots
\end{aligned}$$

Remark 6.2.

(1) $\bigwedge \mathbb{C}^{\frac{N_0}{2}+i} = V_0 \oplus \cdots \oplus V_i$

(2) for $v \in \mathbb{R}^{N_0}$, $\quad v : \quad V_i \mapsto V_i$ as a Clifford multiplication .

Proof of the Proposition 6.1. As for every $x \in \tilde{P}_0 \times_\Gamma P_0$, there exists N such that $\alpha_\infty(x) \in \mathbb{R}^N$, when x is far from diagonal , we must take N sufficiently large. Thus if we regard $\alpha_\infty(x) \in \text{End}(\bigwedge \mathbb{C}^{N'})$ as the Clifford multiplication, N' will vary according to the position of x in $\tilde{P}_0 \times_\Gamma P_0$. Take functions

$$\delta, \mu : \quad Z \mapsto [0, \infty)$$

such that

(1) on a neighbourhood of diagonal K , $\delta|_K \equiv \mu|_K \equiv 0$,

(2) there exists a compact set K' such that

$$K \subset K' \qquad \alpha_\infty|_{K'} \equiv \alpha_0$$
$$\mu \leq \delta \quad , \quad 2 \leq [\mu(x)] \leq [\delta(x)] - 1 \qquad \text{for } x \in Z \backslash K',$$

(3) for $i \geq [\mu(x)] - 2$, $\alpha_t(x) : V_i \mapsto V_i$, α_t is a proper homotopy between α_0 and α_∞ , $[a]$ is the integer part of a.

We use μ later. We construct F so that

$$F(p,p')|_{\oplus_{i=0}^{[\delta(p,p')]-1} V_i} \equiv \frac{\alpha_\infty(x)}{|\alpha_\infty(x)|}$$
$$F(p,p')|_{(\oplus_{i=0}^{[\delta(p,p')]+1} U_i)^\perp} \equiv \text{ constant self adjoint unitary.}$$

First , we construct a self adjoint unitary operator of degree 1

$$G : \quad (V_0)^\perp \mapsto (V_0)^\perp$$

where $(V_0)^\perp \subset [\bigwedge \mathbb{C}^\infty]$ is the orthgonal complement of V_0. Let

$$W_1 = V_1$$
$$W_2 = V_2 \cap (e_{N_0+3} \wedge V_1)^\perp$$
$$W_3 = V_3 \cap (e_{N_0+5} W_2)^\perp$$
$$\cdots$$
$$W_i = V_i \cap (e_{N_0+2i-1} W_{i-1})^\perp$$
$$\cdots,$$

$$U_0 = V_0$$
$$U_1 = W_1 \oplus e_{N_0+3} \wedge W_1$$
$$U_2 = W_2 \oplus e_{N_0+5} \wedge W_2$$
$$U_3 = W_3 \oplus e_{N_0+7} \wedge W_3$$
$$\cdots$$
$$U_i = W_i \oplus e_{N_0+2i+1} \wedge W_i$$
$$\cdots.$$

Remark 6.3.

(1) $e_{N_0+2i+1}: \quad U_i \mapsto U_i \qquad$ for $i \geq 1$.

(2) $[\bigwedge \mathbb{C}^\infty] = [\oplus_{i=1}^{\infty} U_i] \oplus V_0$.

(3) for $v \in \mathbb{R}^{N_0}$, $\quad v: \quad W_i \mapsto W_i \qquad$ and so $\quad v: \quad U_i \mapsto U_i$.

Proof of the Proposition 6.1. Now we define

$$G|_{U_1} \equiv e_{N_0+3}$$
$$G|_{U_2} \equiv e_{N_0+5}$$
$$\cdots$$
$$G|_{U_i} \equiv e_{N_0+2i+1}$$
$$\cdots$$

We construct F so that it connects G and $\frac{\alpha_\infty(x)}{|\alpha_\infty(x)|}$ in a suitable way. Let

$$(k_i)_t: \quad \mathbb{R}^{N_0} \mapsto End(U_i)$$

$$(k_i)_t(v)(a) = ((sin\frac{t\pi}{2})v + (cos\frac{t\pi}{2})e_{N_0+2i+1})a.$$

Notice that if $|v| = 1$, then $((k_i)_t(v))^2 \equiv 1$. Define for $x \in Z \backslash K'$,

(1) $i > [\delta(x)] + 1, \qquad F|_{U_i} \equiv G|_{U_i}$.

(2) $F|_{U_{[\delta(x)]+1}} \equiv (k_{[\delta(x)]+1})_{[[\delta(x)]]}(\frac{\alpha_0(x)}{|\alpha_0(x)|})$ where $[[\delta(x)]] = \delta(x) - [\delta(x)]$.

(3) on $(W_{[\delta(x)]+1})^\perp \cap V_{[\delta(x)]+1}, \qquad F \equiv \frac{\alpha_0(x)}{|\alpha_0(x)|}$.

(4) $F|_{V_{[\delta(x)]}} \equiv \frac{\alpha_{[[\delta(x)]]}(x)}{|\alpha_{[[\delta(x)]]}(x)|}$.

(5) on $\oplus_{i=0}^{[\delta(x)]-1} V_i, \qquad F \equiv \frac{\alpha_\infty(x)}{|\alpha_\infty(x)|} = \frac{\alpha_1(x)}{|\alpha_1(x)|}$.

(We recall α_t is a proper homotopy between α_0 and α_∞.) Moreover take a cut off function

$$M: \quad Z \mapsto [0,1]$$

so that $M|_K \equiv 0$ and $M|_{Z\setminus K'} \equiv 1$. Then we put on $x \in K'$,

$$F(x) = \begin{cases} M(x)\frac{\alpha_0(x)}{|\alpha_0(x)|}, & \text{on } V_0 \\ G, & \text{on } (V_0)^{\perp}. \end{cases}$$

Thus we have

$$(C_0(Z, [\bigwedge \mathbb{C}^{\infty}]), F) \in KK(\mathbb{C}, C_0(Z)).$$

Through this element, we connect $(\text{pr } \alpha_{\infty})^*(\beta)$ and $\alpha_0^*(\beta)$ by an operator homotopy. Using μ , we deform F. We remark for $i \leq \mu(x)$, $x \in Z\setminus K'$,

$$F(x)|_{\oplus_{i=0}^{[\mu(x)]} V_i} \equiv \frac{\alpha_{\infty}(x)}{|\alpha_{\infty}(x)|} = \frac{\alpha_1(x)}{|\alpha_1(x)|}.$$

Let

$$\tilde{F}_t(x) = \frac{t \text{ pr } {}^{\perp}\alpha_{\infty}(x) + \text{ pr } \alpha_{\infty}(x)}{|t \text{ pr } {}^{\perp}\alpha_{\infty}(x) + \text{ pr } \alpha_{\infty}(x)|}$$

where pr $^{\perp}$: $\mathbb{R}^{\infty} \mapsto (\mathbb{R}^{N_0})^{\perp}$ is the standard projection. Let us define F_t by the following . For $x \in Z\setminus K'$,

(1) for $i \geq [\mu(x)]$, $F_t(x)|_{V_i} \equiv F(x)|_{V_i}$.

(2) for $i = [\mu(x)] - 1$, $F_t(x)|_{V_i} \equiv \tilde{F}_{1-(1-t)[[\mu(x)]]}$.

(3) for $i \leq [\mu(x)] - 2$, $F_t(x)|_{\oplus_{i=0}^{[\mu(x)]-2} V_i} \equiv \tilde{F}_t(x)$.

For $x \in K'$, $F_t(x) \equiv F(x)$. This defines a homotopy between

$$(C_0(Z, [\bigwedge \mathbb{C}^{\infty}]), F) \text{ and } (C_0(Z, [\bigwedge \mathbb{C}^{\infty}]), F_0)$$

and the latter splits as

$$(C_0(Z, \bigwedge \mathbb{C}^{\frac{N_0}{2}}), M(x)\frac{\text{pr } \alpha_{\infty}(x)}{|\text{ pr } \alpha_{\infty}(x)|})) \oplus (C_0(Z, \bigwedge \mathbb{C}^{\frac{N_0}{2}})^{\perp}, F')$$

where $(F')^2 \equiv 1$. So the second element is degenerate. This implies

$$(C_0(Z, [\bigwedge \mathbb{C}^{\infty}]), F) = (C_0(Z, \bigwedge \mathbb{C}^{\frac{N_0}{2}}), F_{\infty}) \in KK(\mathbb{C}, C_0(Z)).$$

(Notice $(C_0(Z, \bigwedge \mathbb{C}^{\frac{N_0}{2}}), \frac{\text{pr } \alpha_{\infty}(x)}{1+|\text{ pr } \alpha_{\infty}(x)|})$ and $(C_0(Z, \bigwedge \mathbb{C}^{\frac{N_0}{2}}), M(x)\frac{\text{pr } \alpha_{\infty}(x)}{|\text{ pr } \alpha_{\infty}(x)|})$ are operator homotopic.)

By doing the same thing (by putting $\tilde{F}_t(x) = \frac{\alpha_t}{|\alpha_t(x)|}$ for $x \in Z \backslash K'$) , we get another homotopy connecting

$$(C_0(Z, [\bigwedge \mathbb{C}^\infty]), F) \text{ and } (C_0(Z, \bigwedge \mathbb{C}^{\frac{N_0}{2}}), \frac{\alpha_0(x)}{1+|\alpha_0(x)|}).$$

This completes the proof of the Proposition 6.1.

Let us consider

$$K(\alpha) = \{C_0(\tilde{P}_0, \bigwedge \mathbb{C}^{\frac{N_0}{2}}), F_\infty|_{\tilde{P}_0}\}$$

where $F_\infty(x) = M(x)\frac{\text{pr } \alpha_\infty(x)}{|\text{ pr } \alpha_\infty(x)|}$ and we regard $\tilde{P}_0 = \tilde{P}_0 \times e, \quad e \in P_0$.

Proposition 6.4. *$K(\alpha)$ defines an element of $KK_\Gamma(\mathbb{C}, C_0(\tilde{P}_0))$.*

Proof. we only have to prove

$$\gamma F_\infty|_{\tilde{P}_0}\gamma^{-1} - F_\infty|_{\tilde{P}_0}$$

is a compact endomorphism for $\gamma \in \Gamma$. It is equivalent to

$$|F_\infty(\gamma^{-1}p, e) - F_\infty(p, e)| \mapsto 0 \qquad (p \to \infty).$$

By Γ invariance , it is

$$\begin{aligned}
&|F_\infty(p,\gamma) - F_\infty(p,e)| = |\frac{\text{pr } \alpha_\infty(p,\gamma)}{|\text{ pr } \alpha_\infty(p,\gamma)|} - \frac{\text{pr } \alpha_\infty(p,e)}{|\text{ pr } \alpha_\infty(p,e)|}| \\
&\leq |\frac{\text{pr}\alpha_\infty(p,\gamma) - \text{ pr}\alpha_\infty(p,e)}{|\text{ pr } \alpha_\infty(p,\gamma)|}| + |\text{ pr}\alpha_\infty(p,e)||\frac{1}{|\text{ pr}\alpha_\infty(p,\gamma)|} - \frac{1}{|\text{ pr}\alpha_\infty(p,e)|}| \\
&\leq \frac{2C}{|\text{ pr } \alpha_\infty(p,\gamma)|}d(p,\gamma). \text{ (pr } \alpha_\infty \text{ is a fiberwise } C \text{ Lipschitz map .)}
\end{aligned}$$

Because pr $\circ\alpha_\infty$ is proper , this goes to zero.

§ 7 Conclusion

The construction of $\mu : \quad KK_\Gamma(\mathbb{C},\mathbb{C}) \mapsto K(B\Gamma)$ is done at the appendix B. Take $y = (E,\rho,F) \in KK_\Gamma(C_0(\tilde{P}_0),\mathbb{C})$ and construct from it

$$\tilde{y} = (\mathbb{E},\rho,\mathbb{F}) \in KK(C_0(Z), C(\frac{P_0}{\Gamma}))$$

as follows. Notice that as $KK(C_0(\frac{\tilde{P}_0}{\Gamma}),\mathbb{C})$ is isomorphic to $KK_\Gamma(C_0(\tilde{P}_0),\mathbb{C})$, we may assume F commutes with the actions of Γ. (cf. Appendix A)

$$\begin{gathered}\mathbb{E} = C(P_0,E)^\Gamma \qquad (\Gamma \text{ invariant functions on } P_0 \text{ with value in } E\ ),\\ \text{for } \xi \in \mathbb{E}, \quad (\mathbb{F}\xi)(x) = F\xi(x) \quad \text{for } x \in P_0,\\ \rho : \quad C_0(Z) \mapsto End_{C_0(\tilde{P}_0)}(\mathbb{E}) \quad \text{by } (\rho(f)\xi)(x) = f(\quad ,x)\xi(x).\end{gathered}$$

By the construction , it follows

Proposition 7.1. $\mu(K(\alpha) \times_\Gamma y)|_{\frac{P_0}{\Gamma}} = \alpha_0^*(\beta) \times \tilde{y}$ *in* $KK(\mathbb{C}, C(\frac{P_0}{\Gamma}))$.

This is shown at the appendix B. As in the introduction of this part , we take P_0 so that

$$r_* : \quad H_*(\frac{P_0}{\Gamma}) \mapsto H_*(B\Gamma)$$

induced by $\frac{P_0}{\Gamma} \mapsto B\Gamma$ is surjective for $0 \le * \le N$ for N fixed. Then take $z \in K_0(\frac{P_0}{\Gamma})$,

$$\begin{aligned}< ch^*\mu(K(\alpha)\times y)|_{\frac{P_0}{\Gamma}}, ch_* z > &=< \mu(K(\alpha)\times y)|_{\frac{P_0}{\Gamma}}, z >\\ &=< \alpha_0^*(\beta)\times\tilde{y}, z > =< \alpha_0^*(\beta), \tilde{y}\times z >\\ &=< ch^*\alpha_0^*(\beta), ch_*(\tilde{y}\times z) > \qquad \cdots *\end{aligned}$$

By the construction , $ch_*(\tilde{y}\times z) = ch_*(y)\times_\Gamma ch_*(z)$, (this is proved at the appendix A) and $ch^*(\beta) = \omega$ where ω is the Bott generator in $H^{N_0}_{\text{cp}}(\mathbb{R}^{N_0})$. Thus

$$* =< \alpha_0^*(\omega), ch_*(y)\times_\Gamma ch_*(z) >= (\alpha_0 \cap ch_*(y))(ch_*(z)).$$

By varying $y \in KK_\Gamma(C_0(\tilde{P}_0),\mathbb{C})$, we see $ch^*\mu(K(\alpha)\times y)$ occupy $H^*(B\Gamma;\mathbb{Q})$ for $0 \le * \le N$.

This completes the proof of Novikov conjecture for torsion free quasi geodesic bicombing groups.

Appendix A

In this section , reviewing basic materials around KK theory , we show

$$ch_*(\tilde{y} \times z) = ch_*(y) \times_\Gamma ch_*(z)$$

where $\tilde{y} \in KK(\tilde{P}_0 \times_\Gamma P_0, \frac{P_0}{\Gamma})$ is constructed from $y \in KK_\Gamma(\tilde{P}_0,$ pt $)$ and $z \in K_0(\frac{P_0}{\Gamma})$. In PART II , we have used the fact that there exists an intersection product ,

$$KK(A,B) \times KK(B,C) \mapsto KK(A,C)$$

$$(E_1, \varphi_1, F_1) \times (E_2, \varphi_2, F_2) \mapsto (E_3, \varphi_3, F_3).$$

The product is characterized by some axioms.

Definition A.1 ([CS]). $F \in B(E_1 \hat{\otimes}_{\varphi_2} E_2)$ is called F_2 connection for E_1 if for any $x \in E_1$,

$$T_x \circ F_2 - (-1)^{\partial x \partial F_2} F \circ T_x \in K(E_2, E)$$
$$F_2 \circ T_x^* - (-1)^{\partial x \partial F_2} T_x^* \circ F \in K(E, E_2)$$

where $E = E_1 \hat{\otimes}_{\varphi_2} E_2$, $\quad T_x : E_2 \mapsto E$ by $T_x(y) = x \hat{\otimes} y$.

Example A.2. *Let S be a spin ^{c} structure on X and $\tilde{E}_1, \tilde{E}_2$ be a bundle over X and $(E_1, F_1) = (C_0(X, \tilde{E}_1), 0)$, $(E_2, F_2) = (L^2(X, S \hat{\otimes} \tilde{E}_2), \mathbb{D}_{\tilde{E}_2})$ where $\mathbb{D}_{\tilde{E}_2} = D_{\tilde{E}_2}(1 + D^2_{\tilde{E}_2})^{-\frac{1}{2}}$, $D_{\tilde{E}_2}$ is the Dirac operator with the coefficient $\tilde{E}_2$. Then $\mathbb{D}_{\tilde{E}_1 \hat{\otimes} \tilde{E}_2}$ is $\mathbb{D}_{\tilde{E}_2}$ connection , $\mathbb{D}_{\tilde{E}_1 \hat{\otimes} \tilde{E}_2} \in B(C_0(X, \tilde{E}_1) \hat{\otimes}_{C_0(X)} L^2(X, S \hat{\otimes} \tilde{E}_2)) \cong B(L^2(X, S \hat{\otimes} \tilde{E}_1 \otimes \tilde{E}_2))$.*

Definition A.3 ([CS]). Let F be F_2 connection. Then $(E_1 \hat{\otimes}_{\varphi_1} E_2, \varphi_1 \hat{\otimes} 1, F)$ is called a Kasparov product for (E_1, φ_1, F_1) and (E_2, φ_2, F_2) if

$$(E_1 \hat{\otimes}_{\varphi_2} E_2, \varphi_1 \hat{\otimes} 1, F) \in E(A, C),$$
$$\varphi_1(a) \hat{\otimes} 1 [F_1 \hat{\otimes} 1, F] \varphi_1^*(a) \hat{\otimes} 1 \geq 0 \qquad \text{for all } a \in A.$$

Proposition A.4 (see [Bl]). *Kasparov product exists and any two Kasparov products are operator homotopic.*

Thus assigning a Kasparov product , we obtain a well defined pairing ,

$$KK(A,B) \times KK(B,C) \mapsto KK(A,C).$$

It is known that $KK(X,\ \mathrm{pt}\ )\otimes\mathbb{Q}$ is isomorphic to the homology of X with infinite support $H^{\inf}_{\mathrm{ev}}(X)$. To obtain an explicit map of the isomorphism , we need to pass to K homology $K_0(X)$. An element of $K_0(X)$ is of the form (M,E,φ) where

(1) M is a C^∞ , even dimensional spin^c manifolds without boundary,

(2) E is a complex vector bundle over X ,

(3) φ is a proper map from M to X .

$K_0(X)$ is the set of the above triples devided by equivalent relations. Let D_E be an oder 0 pseudo differential operator on $S\hat{\otimes}E$ whose principal symbol is $\frac{\xi}{|\xi|}\hat{\otimes}1$ and which is bounded as an operator on $L^2(M,S\hat{\otimes}E)$. Then

$$\mu:\quad K_0(X)\cong KK(X,\ \mathrm{pt}\ )\qquad \mu((M,E,\varphi))=\varphi_*((L^2(M,S\hat{\otimes}E),D_E))$$

where for $(H,f,F)\in KK(M,\ \mathrm{pt}\ )$,

$$\varphi_*:\quad KK(M,\ \mathrm{pt}\ )\mapsto KK(X,\ \mathrm{pt}\ )\qquad \varphi_*((H,f,F))=(H,f\circ\varphi^*,F)).$$

On the other hand ,

$$ch_*:\quad K_0(X)\otimes\mathbb{Q}\cong H^{\inf}_{\mathrm{ev}}(X)$$

$$ch_*((M,\varphi,E))=\varphi_*(ch^*(E)\cup td(M)\cap[M]).$$

If X is a finite simplicial complex , embedded in $\mathbb{R}^{n-1}$, then we can take as M a boundary of a regular neighbourhood of X in $\mathbb{R}^n$ and $\varphi=r$: the retraction.([BD]) In particular if $X=\frac{P}{\Gamma}$, φ can be lifted to $\tilde{r}:\tilde{M}\mapsto P$ where $\tilde{M}$ is a Γ covering of M. On the other hand , if X is an even dimensional (non compact) spin ^{c} manifold , we can always take X itself as M.

Let M be a boundary of a regular neighbourhood of $\frac{P_0}{\Gamma}$ in $\mathbb{R}^n$ and $\tilde{M}$ its Γ covering. Take $y=(\frac{\tilde{P}_0}{\Gamma},E_1)\in K_0(\frac{\tilde{P}_0}{\Gamma})$, ($\frac{\tilde{P}_0}{\Gamma}$ is a non compact spin ^{c} manifold) , $z=(M,E_2,r)\in K_0(\frac{P_0}{\Gamma})$. Let us put $\tilde{Z}=\tilde{P}_0\times_\Gamma\tilde{M}$, $\quad Z=\tilde{P}_0\times_\Gamma P_0$ and $1\times_\Gamma\tilde{r}:\tilde{Z}\mapsto Z$ be the obvious map . Moreover put

$$y\times_\Gamma z=(\tilde{Z},\ \mathrm{pr}_1^*E_1\otimes\ \mathrm{pr}_2^*E_2,1\times_\Gamma\tilde{r})\in K_0(Z)\qquad \text{where}$$

$$\mathrm{pr}_1:\quad \tilde{Z}\mapsto\frac{\tilde{P}_0}{\Gamma}\qquad \mathrm{pr}_2:\quad \tilde{Z}\mapsto M,$$

$$S_1,S_2 \text{ are the spinors on } \frac{\tilde{P}_0}{\Gamma},M \text{ respectively.}$$

As a spin^c structure on $\tilde{Z}$, we take $S\equiv \mathrm{pr}_1S_1\hat{\otimes}\mathrm{pr}_2S_2$.

Lemma A.5. $ch_*(y \times_\Gamma z) = ch_*(y) \times_\Gamma ch_*(z) \in H_*^{\inf}(Z)$.

Proof.

$ch_*(y) \times_\Gamma ch_*(z) = (1 \times_\Gamma \tilde{r})_*\{(\ \mathrm{pr}_1^*(ch^*(E_1)Td(\frac{\tilde{P}_0}{\Gamma}))\ \mathrm{pr}_2^*(ch^*(E_2)Td(M))) \cap [\tilde{Z}]\}$. On the other hand ,

$$\begin{aligned} ch_*(y \times_\Gamma z) &= (1 \times_\Gamma \tilde{r})_*((ch^*(\ \mathrm{pr}_1^*(E_1) \hat{\otimes}\ \mathrm{pr}_2^*(E_2))Td(\tilde{Z}) \cap [\tilde{Z}]) \\ &= (1 \times_\Gamma \tilde{r})_*((\ \mathrm{pr}_1^* ch^*(E_1)\ \mathrm{pr}_2^* ch^*(E_2)Td(\tilde{Z})) \cap [\tilde{Z}]). \end{aligned}$$

Clearly $\mathrm{pr}_1^* Td(\frac{\tilde{P}_0}{\Gamma})\ \mathrm{pr}_2^* Td(M) = Td(\tilde{Z})$. Thus the result follows.

Based on [CS] , we show

$$ch_*(\tilde{y} \times z) = ch_*(y) \times_\Gamma ch_*(z).$$

Let $\mathfrak{E}$ be the completion of $C_c(\tilde{Z}, S_1)$ as $C_0(M)$ module with the inner product

$$< \xi, \eta > (m) = \int_{x \in\ \mathrm{pr}_2^{-1}(m)} < \xi(x,m), \eta(x,m) > dx \qquad \xi, \eta \in C_c(\tilde{Z}, S_1).$$

Moreover let D be a lift of an almost local pseudo differential operator on $\tilde{P}_0$ whose principal symbol is $\frac{\xi}{|\xi|}$. Then D acts on $\mathfrak{E}$ fiberwisely and $(\mathfrak{E}, D)$ is an element of $KK(\tilde{Z}, M)$. We write $(\mathfrak{E}, D) =\ \mathrm{pr}!\ (\ \mathrm{pr} :\ \tilde{Z} \mapsto M\ )$. On the other hand , if E_1 is a vector bundle on $\frac{\tilde{P}_0}{\Gamma}$, identifying $\mathrm{pr}_1^* E_1$ with E_1 , $(E_1) = (C_0(\tilde{Z}, E_1), 0)$ is an element of $KK(\tilde{Z}, \tilde{Z})$.

Lemma A.6. $(E_1) \times\ pr\,! = (C_0(\tilde{M}, L^2(\tilde{P}_0, S_1 \hat{\otimes} E_1))^\Gamma, D_{E_1})$, *where* D_{E_1} *is a lift of order* 0 *pseudo differential operator on* $\tilde{P}_0$ *whose principal symbol is* $\frac{\xi}{|\xi|} \hat{\otimes} 1$.

Proof. It is clear

$$C_0(\tilde{Z}, E_1) \otimes_{\tilde{Z}} C_0(\tilde{M}, L^2(\tilde{P}_0, S_1))^\Gamma \cong C_0(\tilde{M}, L^2(\tilde{P}_0, S_1 \hat{\otimes} E_1))^\Gamma.$$

It remains to show that for $f \in C_0(\tilde{Z}, E_1)$,

$$T_f \circ D - D_{E_1} \circ T_f, \qquad D \circ T_{f^*} - T_{f^*} \circ D_{E_1}$$

are compact endomorphisms. As D , D_{E_1} act on $C_0(\tilde{M}, L^2(\tilde{P}_0, S_1 \hat{\otimes} E_1))^\Gamma$ fiberwise , it is enough to show that for $f \in C_0(\tilde{P}_0, E_1)$,

$$T_f \circ D - D_{E_1} \circ T_f, \qquad D \circ T_{f^*} - T_{f^*} \circ D_{E_1}$$

are compact operators on $L^2(\tilde{P}_0, S_1 \hat{\otimes} E_1)$. This follows by the standard argument.

In PART II , we have constructed $\tilde{y} \in KK(\tilde{Z}, M)$ from $y \in KK_\Gamma(\tilde{P}_0,\ \mathrm{pt}\ )$.

Corollary A.7. *For* $y = (E_1, S_1) \in K_0(\frac{\tilde{P}_0}{\Gamma}) = KK_\Gamma(\tilde{P}_0,\ pt\ )$ *and* $pr : \tilde{Z} \mapsto M$,

$$\tilde{y} = (E_1) \times \ pr\ ! \in KK(\tilde{Z}, M).$$

Proof. The explicit isomorphism is as follows.

$$K_0(\frac{\tilde{P}_0}{\Gamma}) \mapsto KK_\Gamma(\tilde{P}_0,\ \text{pt}\ ) \qquad (E_1, S_1) \mapsto (L^2(\tilde{P}_0, S_1 \hat{\otimes} E_1), D_{E_1}).$$

From the Lemma $A.6$, the result follows.

For $y = (\eta, F) \in KK_\Gamma(\tilde{P}_0,\ \text{pt}\ )$, define $y' \in KK(Z, \frac{P_0}{\Gamma})$, $y'' \in KK(\tilde{Z}, M)$ by the following. Notice that by the above map , we may assume F commutes with the actions of Γ.

$$y' = (C_0(P_0, \eta)^\Gamma, F') \qquad y'' = (C_0(\tilde{M}, \eta)^\Gamma, F''),$$

where for $f \in C_0(P_0, \eta)^\Gamma$, $(F'(f))(p) = F(f(p))$ and the action of F'' is similar. Then expressing $z \in K_0(\frac{P_0}{\Gamma})$ as

$$r_*(L^2(M, S_2 \hat{\otimes} E_2), D_{E_2}) \equiv r_*(\tilde{z}) \qquad (\tilde{z} \in KK(M,\ \text{pt}\ )),$$

Lemma A.8. $y' \times r_*(\tilde{z})$ *and* $(1 \times_\Gamma r)_*(y'' \times \tilde{z})$ *coincide in* $KK(Z,\ pt\ )$.

Proof. Notice that $C(P_0, \eta)^\Gamma \hat{\otimes}_r L^2(M, E_2) \cong C(\tilde{M}, \eta)^\Gamma \hat{\otimes} L^2(M, E_2)$. It is clear that through this isomorphism , a Kasparov product of F'' and D_{E_2} is at the same time that of F' and D_{E_2}.

Proposition A.9. *Let* $\tilde{y} = (E_1) \times \ pr\ ! \in KK(\tilde{Z}, M)$ *where* $pr : \tilde{Z} \mapsto M$. *Then* $\tilde{y} \times \tilde{z} = (\ pr_1^*(E_1) \otimes \ pr_2^*(E_2)) \times \ ev_Z!$ *where* $ev_Z!$ *is defined in the proof.*

Proof. Let $\text{ev}\ ! \equiv (L^2(M, S_2), D_2) \in KK(M,\ \text{pt}\ )$. Then $\tilde{z} = (E_2) \times \ \text{ev}\ !$ follows from the Lemma $A.6$ and

$$\tilde{y} \times \tilde{z} = \ \text{pr}_1^*(E_1) \times \ \text{pr}\ ! \times (E_2) \times \ \text{ev}\ !.$$

It is easy to see $(\ \text{pr}_2^* E_2) \times \ \text{pr}\ ! = \ \text{pr}\ ! \times (E_2)$ in $KK(\tilde{Z}, M)$. Thus we only have to show

$$\text{pr}\ ! \times \ \text{ev}\ ! = \ \text{ev}_Z!$$

where $\text{ev}_Z! = (L^2(\tilde{Z}, S_1 \hat{\otimes} S_2), D)$ and we regard $S = S_1 \hat{\otimes} S_2$ as a spin^c structure on $\tilde{Z}$. (We recall S_1, S_2 are spin^c structures on $\frac{\tilde{P}_0}{\Gamma}, M$ respectively.) The difficulty lies on that $D_1 \hat{\otimes} 1$ is not a pseudo differential operator . So the direct symbolic culculus can not be applied. We follow [CS] . First notice

$$C(\tilde{M}, L^2(\tilde{P}_0, S_1))^\Gamma \hat{\otimes}_M L^2(M, S_2) \cong L^2(\tilde{Z}, S_1 \hat{\otimes} S_2)$$

and let $D \in B(L^2(\tilde{Z}, S_1 \hat{\otimes} S_2))$ be constructed as usual.

Lemma A.10 ([CS] p1146). *Let $\eta_1, \eta_1' \in C(\tilde{M}, L^2(\tilde{P}_0, S_1))^\Gamma$. Then $T^*_{\eta_1} \circ D \circ T_{\eta_1'}$ $(T^*_{\eta_1} \circ D^* \circ T_{\eta_1'})$ lies in the closure in $B(L^2(M, S_2))$ of the set of order 0 pseudo differential operators and its symbol is*

$$\sigma(T^*_{\eta_1} \circ D^{(*)} \circ T_{\eta_1'})(m, \xi) = < \eta_1, \eta_1' > (m) \frac{\xi}{|\xi|}.$$

Proof of the Proposition A.9. In particular , this shows D is D_2 connection. It remains to show for $f \in C_0(\tilde{Z})$,

$$f[D_1 \hat{\otimes} 1, D] f^* \geq 0 \qquad \text{modulo compact operators.}$$

It is enough to show this for $f \in C_c(\tilde{Z})$. Let $\frac{\xi}{|\xi|} \in Hom(S_1 \hat{\otimes} S_2)$ where $\xi \neq 0 \in T(\tilde{Z})$ and decompose $\xi = \xi_1 + \xi_2$, $\xi = \sigma_1 \hat{\otimes} 1 + 1 \hat{\otimes} \sigma_2$ where $\sigma_i = \frac{\xi_i}{|\xi|}$, $i = 1, 2$. Let

$$D = \tilde{D_1} + \tilde{D_2}$$

$$\text{where} \qquad \text{symbol } (\tilde{D_1}) = \sigma_1 \hat{\otimes} 1, \qquad \text{symbol } (\tilde{D_2}) = 1 \hat{\otimes} \sigma_2.$$

Lemma A.11 ([CS] p1147). *$\tilde{D_1}(D_1 \hat{\otimes} 1)$, $(D_1 \hat{\otimes} 1)\tilde{D_1}$ lie in the closure in $B(L^2(\tilde{Z}, S))$ of the set of order 0 pseudo differential operators , whose principal symbol is*

$$\frac{\xi_1}{|\xi_1|} \frac{\xi_1}{|\xi|} = \frac{|\xi_1|}{|\xi|}.$$

Lemma A.12 ([CS] p1147). *$f[D_1 \hat{\otimes} 1, \tilde{D_2}] f^*$ is a compact operator.*

Proof of the Proposition A.9. Let us decompose

$$f[D_1 \hat{\otimes} 1, D] f^* = f[D_1 \hat{\otimes} 1, \tilde{D_1}] f^* + f[D_1 \hat{\otimes} 1, \tilde{D_2}] f^*.$$

Then from the above lemmas , this element is positive modulo compact operators.

Corollary A.13. *$ch_*(1 \times_\Gamma \tilde{r})_*(\tilde{y} \times \tilde{z}) = ch_*(y) \times_\Gamma ch_* r_*(z)$.*

Proof. In lemma $A.5$, we have seen

$$ch_*(y \times_\Gamma z) = ch_*(y) \times_\Gamma ch_*(z)$$

$$\text{where} \qquad y \times_\Gamma z = (\tilde{Z},\ \mathrm{pr}_1^*(E_1) \otimes\ \mathrm{pr}_2^*(E_2), 1 \times_\Gamma \tilde{r}) \in K_0(Z).$$

By the Lemma $A.9$, this is equal to $(1 \times_\Gamma \tilde{r})_*(\tilde{y} \times \tilde{z})$.

Now we have

Proposition A.14. $ch_*(\tilde{y} \times \tilde{z}) = ch_*(y) \times_\Gamma ch_*(z)$.

Proof.

$$\begin{aligned} ch_*(y) \times_\Gamma ch_*(z) &= ch_*(y) \times_\Gamma ch_*(r_*(\tilde{z})) \\ &= ch_*(1 \times_\Gamma \tilde{r})_*(\tilde{y} \times \tilde{z}) \\ &= ch_*(\tilde{y} \times r_*(\tilde{z})) = ch_*(\tilde{y} \times z). \end{aligned}$$

Appendix B

In this section , we shortly explain A.S.Miscenko's construction [Mis2] and show

$$\mu(K(\alpha) \times y)|_{\frac{P_0}{\Gamma}} = \alpha_0^*(\beta) \times \tilde{y}.$$

Fredholm representation.

Any element of $KK_\Gamma(\mathbb{C}, \mathbb{C})$ is of the form (H, ρ, F) where

(1) H is a Hilbert space ,

(2) $\rho : \quad \Gamma \mapsto U(H) =$ unitary $*$ representation of degree 0 ,

(3) $F : \quad H \mapsto H$ Fredholm operator of degree 1 such that $gFg^{-1} - F$ is a compact operator for any $g \in \Gamma$.

Let $\mathbb{H} = E\Gamma \times_\Gamma H \mapsto B\Gamma$ be a Hilbert bundle over $B\Gamma$. We construct a family of Fredholm operators ,

$$\tilde{F} : \quad E\Gamma \mapsto B(H)$$

which induces an endomorphism of $\mathbb{H}$. To induce an operator on $\mathbb{H}$, it is necessary and sufficient if $\tilde{F}$ satisfy

$$\tilde{F}(\gamma x) = \gamma \tilde{F}(x) \gamma^{-1} \qquad \text{for any } \gamma \in \Gamma, x \in E\Gamma.$$

$\tilde{F}$ is constructed as follows. We take P_∞ as $E\Gamma$. Then define $\tilde{F}(e) = F$, $\tilde{F}(\gamma) = \gamma F \gamma^{-1}$ where e is the identity and $\gamma \in \Gamma \subset P_\infty$. For a 1 skelton σ , $\gamma_1, \gamma_2 \in \partial\sigma$, extend $\tilde{F}$ from $\gamma_1 \cup \gamma_2$ to $\Gamma\sigma$ such that for $x \in \sigma$,

$$\tilde{F}(\gamma_i) - \tilde{F}(x) \text{ is compact }, \qquad \tilde{F}(\gamma x) = \gamma \tilde{F}(x) \gamma^{-1}.$$

Inductively we continue this extension to all k skeltons , requiring to satisfy above. Thus we have got the desired $\tilde{F} \in End(\mathbb{H})$.

Let $y = (E, \rho, F) \in KK_\Gamma(\tilde{P}_0,\ \mathrm{pt}\ )$. By appendix A , we may assume for any $g \in \Gamma$, $gFg^{-1} \equiv F$, (lift of order 0 , almost local pseudo differential operator). We recall

$$K(\alpha) = (C_0(\tilde{P}_0, \bigwedge \mathbb{C}^{\frac{N_0}{2}}), F_\infty|_{\tilde{P}_0} = \frac{(\ \mathrm{pr}\ \alpha_\infty)|_{\tilde{P}_0}}{1 + |(\ \mathrm{pr}\ \alpha_\infty)|_{\tilde{P}_0}|})$$

where we regard $\tilde{P}_0 = \tilde{P}_0 \times e \subset Z$. First we see

$$C(P_0, C_0(\tilde{P}_0, \bigwedge \mathbb{C}^{\frac{N_0}{2}}) \hat{\otimes} E)^\Gamma \cong C_0(Z, \bigwedge \mathbb{C}^{\frac{N_0}{2}}) \hat{\otimes}_Z C(P_0, E)^\Gamma.$$

Let $\mathbb{F} = F_\infty|_{\tilde{P}_0} \times_\Gamma F$ be a Kasparov product of $F_\infty|_{\tilde{P}_0}$ and F. $\mathbb{F}$ satisfies

(1) for any $\xi \in C_0(\tilde{P}_0, \bigwedge \mathbb{C}^{\frac{N_0}{2}})$, $T_\xi \circ F - \mathbb{F} \circ T_\xi$, $F \circ T_\xi^* - T_\xi^* \circ \mathbb{F}$ are compact operators ,

(2) $[F_\infty|_{\tilde{P}_0} \hat{\otimes} 1, \mathbb{F}] \geq 0$ modulo $K(C_0(\tilde{P}_0, \bigwedge \mathbb{C}^{\frac{N_0}{2}}) \otimes_{\tilde{P}_0} E)$.

Let $H = C_0(\tilde{P}_0, \bigwedge \mathbb{C}^{\frac{N_0}{2}}) \hat{\otimes}_{\tilde{P}_0} E$ and $\tilde{\mathbb{F}}$ be constructed from the Fredholm representation $(H, \mathbb{F})$. We check $\tilde{\mathbb{F}}$ satisfies the axiom of the Kasparov product $(1,2)$ for $(\mathbb{H}, \tilde{F})$ and $\tilde{y}$. Because $\frac{P_0}{\Gamma}$ is compact , for any $\xi \in C_0(Z, \bigwedge \mathbb{C}^{\frac{N_0}{2}})$, any $p \in P_0$, it is enough to show

$(1')$ $T_{\xi(\quad, p)} \circ F - \tilde{\mathbb{F}}(p) \circ T_{\xi(\quad, p)}$, $F \circ T^*_{\xi(\quad, p)} - T^*_{\xi(\quad, p)} \circ \tilde{\mathbb{F}}(p)$ are compact operators ,

$(2')$ $[F_\infty(\quad, p) \hat{\otimes} 1, \tilde{\mathbb{F}}(p)] \geq 0$ modulo $K(C_0(\tilde{P}_0, \bigwedge \mathbb{C}^{\frac{N_0}{2}}) \hat{\otimes}_{\tilde{P}_0} E)$.

$1'$ follows because $\mathbb{F} - \tilde{\mathbb{F}}(p)$ is compact. To show $2'$, it is enough to show

$$F_\infty(\quad, e) - F_\infty(\quad, p)$$

is in $K(C_0(\tilde{P}_0, \bigwedge \mathbb{C}^{\frac{N_0}{2}}))$, which is equivalent to

$$|F_\infty(x, e) - F_\infty(x, p)| \mapsto 0 \qquad x \to \infty.$$

This follows because pr α_∞ is a fiberwise proper Lipschitz map .

References

[Al] J. Alonso, *Combings of groups, in Algorithms and Classification in Combinatorial Group theory*, vol. 23, MSRI. Publ, 1992, pp. 165-178.

[At] M.F. Atiyah, *Elliptic operators , discrete groups and von Neumann algebras*, Astérisque **32-33** (1976), 32-33.

[ABS] M.F. Atiyah, R. Bott and A. Shapiro, *Clifford modules*, Topology 3, Suppl.1 (1964), 3-38.

[AS] M.F. Atiyah and I.M. Singer, *The index of elliptic operators I , III*, Ann. of Math **87** (1968), 484-530 , 546-604.

[BD1] P. Baum, R. Douglus, *K theory and index theory*, Operator theory and applications, Proc. Simp. Pure Math. **38 (I)** (1982), 117-173.

[BD2] P. Baum, R. Douglus, *Relative K homology and C^* algebras*, K theory **5** (1991), 1-46.

[Bl] B. Blackadar, *K theory for operator algebras*, MSRI. Publ, vol. 5, 1986.

[CGM] A. Connes, M. Gromov, H. Moscovici, *Group cohomology with Lipschitz Controll and Higher Signatures*, Geometric and functional analysis **3** (1993), 1-78.

[CS] A. Connes, G. Scandalis, *The longitudinal index theorem for foliations*, Publ. Res. Inst. Math. Sci. Kyoto Univ. **20** (1984), 1139-1183.

[ECHPT] Epstein. D.B.A, Cannon. J.W, Holt. D.F, Paterson. M.S and Thurston. W.P, *Word processing and group theory*, Jones and Bartlett Publ, 1992.

[GH] W. Ballmann, E. Ghys, A. Haefiger, P. de la Harpe, E. Salem, R. Strebel, M. Troyanov, *Sur les Groupes Hyperbolique d'après Mikhael Gromov (E. Ghys, P. de la Harpe, eds.)*, Progress in Mathematics; 83, Boston, Birkhäuser, 1990.

[G1] M. Gromov, *Asymptotic invariants of infinite groups*, London Math. Soc. Lecture Note Series **182** (1991).

[G2] M. Gromov, *Hyperbolic groups, in Essays in group theory (S.M. Gerstein, ed.)*, MSRI. Publ, vol. 8, 1987, pp. 75-263.

[GL] M. Gromov, H.B. Lawson, *Positive scalar curvature and the Dirac operators on complete Riemannian manifolds*, Publ. Math IHES **58** (1983), 83-196.

[K1] G.G. Kasparov, *The operator K functor and extensions of C^* algebras*, Izv. Akad. Nauk SSSR, Ser. Mat **44** (1981), 571-636.

[K2] G.G. Kasparov, *Equivariant KK theory and the Novikov conjecture*, Inv. Math **91** (1988), 147-201.

[MS] J.W. Milnor, J.D. Stasheff, *Characteristic Classes*, Annals of Math. Studies. 176, Princeton Univ. Press.

[Mis1] A.S. Miscenko, *Homotopy invariants of non simply connected manifolds I*, Izv. Akad. Nauk SSSR, Ser. Mat. **43** (1979), 831-859.

[Mis2] A.S. Miscenko, *Infinite dimensional representations of discrete groups and higher signature*, Izv. Akad. Nauk SSSR, Ser. Mat **38** (1974), 81-106.