On Gelfand pairs associated with nilpotent Lie groups

by

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Introduction

Let G be a locally compact group and K a compact subgroup of G. The investigation of the Banach $*$-algebra $L^1(K\backslash G/K)$ of K-biinvariant integrable functions on G is an important theme in harmonic analysis on G or G/K. When $L^1(K\backslash G/K)$ is commutative, the pair (G, K) is called a Gelfand pair, and there have been many works on Gelfand pairs until now (see for example [F] and the introduction of [BJR]). In this paper, we consider the case $G = K \ltimes N$, where N is a connected, simply connected nilpotent Lie group and K acts on N as automorphisms. We shall give a necessary and sufficient condition for the pair (G, K) to be a Gelfand pair. This is equivalent to determining a condition that the Banach $*$-algebra $L^1_K(N)$ of K-invariant integrable functions on N be commutative. We call the pair $(K; N)$ a Gelfand pair associated with N if $L^1_K(N)$ is commutative.

Now $L^1_K(N)$ is commutative only if N is at most 2-step thanks to [BJR], and accordingly our object N is assumed to be 2-step. Our first theorem (Theorem A below) gives a way in which one reduces the matter to Heisenberg groups. Let us describe our method in detail.

Denote by $\mathfrak{n}$ the Lie algebra of N and by $\mathfrak{n}^*$ the dual vector space of $\mathfrak{n}$. For $l \in \mathfrak{n}^*$, let B_l be the alternative form corresponding to l : $B_l(X, Y) = l([X, Y])$ $(X, Y \in \mathfrak{n})$, and $\mathfrak{b}(l)$ the intersection of the radical of B_l with $\ker l$. Then we see that $\mathfrak{n}/\mathfrak{b}(l)$ is isomorphic to a Heisenberg algebra if $l|_{[\mathfrak{n},\mathfrak{n}]} \neq 0$. Let $\pi = \pi_l$ be the irreducible

unitary representation of N corresponding to l (see [Ki]) and K_π the stabilizer of π for the action of K on the unitary dual $\widehat{N}$ of N. We denote by $\Phi_\pi(K_\pi)$ the subgroup of $\mathrm{Aut}(\mathfrak{n}/\mathfrak{b}(l))$ formed by the K_π-actions induced on $\mathfrak{n}/\mathfrak{b}(l)$. Let $B(l)$ be the subgroup of N corresponding to $\mathfrak{b}(l)$. Considering the pair $(\Phi_\pi(K_\pi), B(l)\backslash N)$ with $\mathfrak{n}/\mathfrak{b}(l)$ regarded as the Lie algebra of $B(l)\backslash N$, we obtain the following theorem.

Theorem A. *Let N be a 2-step nilpotent Lie group and K a compact group acting on N as automorphisms. Then the pair $(K; N)$ is a Gelfand pair if and only if $(\Phi_\pi(K_\pi); B(l)\backslash N)$ is a Gelfand pair for every $l \in \mathfrak{n}^*$.*

Using Theorem A, we will show by an example a certain subtlety of 2-step nilpotent Lie groups N if the derived algebra $[\mathfrak{n}, \mathfrak{n}]$ of the Lie algebra $\mathfrak{n}$ of N is different from the center $Z(\mathfrak{n})$ of $\mathfrak{n}$. To be more precise, we decompose $\mathfrak{n}$ into K-invariant subspaces as $\mathfrak{n} = \mathfrak{n}' \oplus \mathfrak{a} \oplus [\mathfrak{n}, \mathfrak{n}]$ with $Z(\mathfrak{n}) = \mathfrak{a} \oplus [\mathfrak{n}, \mathfrak{n}]$. Put $\mathfrak{n}_1 = \mathfrak{n}' \oplus [\mathfrak{n}, \mathfrak{n}]$. The sum is a direct sum of ideals and we have $Z(\mathfrak{n}_1) = [\mathfrak{n}_1, \mathfrak{n}_1]$. Let N_1 and A be the subgroups corresponding to $\mathfrak{n}_1$ and $\mathfrak{a}$ respectively. Consider the pair $(K; N_1)$. Since $N = N_1 \times A$, we have $L^1(N) = L^1(N_1) \otimes L^1(A)$. But $L^1_K(N_1)$ and $L^1(A)$ alone do not suffice to determine the properties of $L^1_K(N)$ in general. For instance, the commutativity of $L^1_K(N_1)$ does not imply the commutativity of $L^1_K(N)$. In fact, let $\mathfrak{n}$ be the 5-dimensional Lie algebra $\mathbb{C} \times \mathbb{C} \times \mathbb{R}$ with the bracket product $[(z_1, z_2, t), (z_1', z_2', t')] = (0, 0, -\mathrm{Im} z_2 \bar{z_2}')$. Let K be the one-dimensional torus $\mathbb{T}$ acting on $\mathfrak{n}$ by $e^{\sqrt{-1}\theta} \cdot (z_1, z_2, t) = (e^{\sqrt{-1}\theta} \cdot z_1, e^{\sqrt{-1}\theta} \cdot z_2, t)$. Then $\mathfrak{n}_1 = 0 \times \mathbb{C} \times \mathbb{R}$, $\mathfrak{a} = \mathbb{C} \times 0 \times 0$. For this $\mathfrak{n}$, we show

Theorem B. *Let $N = \exp \mathfrak{n}$ and $N_1 = \exp \mathfrak{n}_1$. Then $L^1_K(N_1)$ is commutative, whereas $L^1_K(N)$ is not commutative.*

Finally, we treat the case where $K = \mathbb{T}^n$ and N is a general 2-step nilpotent

Lie group, and give a necessary and sufficient condition for $(K;N)$ to be a Gelfand pair. Such a condition was given by Leptin [L] when $[\mathfrak{n},\mathfrak{n}]=Z(\mathfrak{n})$ and the action of K is effective. In this paper, we work without these two restrictions and present a complete solution. Recall the decomposition $\mathfrak{n}=\mathfrak{n}'\oplus\mathfrak{a}\oplus[\mathfrak{n},\mathfrak{n}]$ mentioned above. Let $\widehat{K}^r$ be the family of all equivalence classes of irreducible real K-modules. We can identify $\widehat{K}^r$ with $\mathbb{Z}^n/_\sim$ where $\alpha\sim\beta$ if $\beta=\pm\alpha$ for $\alpha,\beta\in\mathbb{Z}^n$. By fixing a system of representatives, we regard $\widehat{K}^r$ as a subset of $\mathbb{Z}^n$ containing 0. Let $\mathfrak{n}'=\sum_{\alpha\in\widehat{K}^r} m_{\alpha,1}V_\alpha$, $\mathfrak{a}=\sum_{\alpha\in\widehat{K}^r} m_{\alpha,2}V_\alpha$ be the decompositions of $\mathfrak{n}'$, $\mathfrak{a}$ into irreducible real K-modules respectively. Put $S_i=\{\alpha\in\widehat{K}^r \mid m_{\alpha,i}\neq 0\}$ $(i=1,2)$. Our third theorem is the following.

Theorem C. *The pair $(K;N)$ is a Gelfand pair if and only if the following five conditions are satisfied:*

(1) $m_{0,1}=0$,

(2) S_1 *is a linearly independent system,*

(3) $m_{\alpha,1}=1$ *for all* $\alpha\in S_1$,

(4) $\mathbb{R}$-span$(S_1)\cap\mathbb{R}$-span$(S_2)=0$,

(5) K *acts on* $[\mathfrak{n},\mathfrak{n}]$ *trivially.*

1. Preliminaries

Let N be a connected, simply connected nilpotent Lie group. The Banach space $L^1(N)$ of integrable functions on N relative to the Haar measure has a structure of Banach $*$-algebra with convolution and involution defined respectively by

$$f*g(x)=\int_N f(y)g(y^{-1}x)dy,\quad f^*(x)=\overline{f(x^{-1})}.$$

Let K be a compact Lie group acting on N through a homomorphism $\phi : K \to \mathrm{Aut}(N)$, where $\mathrm{Aut}(N)$ denotes the automorphism group of N. Replacing K by $K/\ker\phi$ if necessary, we assume throughout this paper that K is a subgroup of $\mathrm{Aut}(N)$. Let $\mathfrak{n}$ be the Lie algebra of N. Since N is connected and simply connected, we shall identify $\mathrm{Aut}(N)$ with the automorphism group $\mathrm{Aut}(\mathfrak{n})$ of $\mathfrak{n}$. The group K acts also on $L^1(N)$ as automorphisms of $*$-algebra by $(k \cdot f)(x) = f(k^{-1} \cdot x)$. We denote by $L^1_K(N)$ the closed $*$-subalgebra of K-invariant functions in $L^1(N)$.

Definition 1.1. We call the pair $(K; N)$ a Gelfand pair if $L^1_K(N)$ is commutative.

We remark that by forming the semidirect product $K \ltimes N$, the pair $(K; N)$ is a Gelfand pair if and only if the algebra $L^1(K\backslash K \ltimes N/K)$ of K-biinvariant functions is commutative.

By Theorem 2.4 in [BJR], we assume from now on that N is a 2-step nilpotent Lie group. Let $\widehat{N}$ be the unitary dual of N, that is, the space of all equivalence classes of irreducible unitary representations of N with Fell topology. For $k \in K$ and $\pi \in \widehat{N}$, we define a representation π_k of N by $\pi_k(x) = \pi(k \cdot x)$ $(x \in N)$. Then, K acts continuously on $\widehat{N}$ from the right. Let K_π be the stabilizer of π in K: $K_\pi = \{k \in K \mid \pi_k \simeq \pi\}$. For each $k \in K_\pi$, there exists a unitary operator $W_\pi(k)$ on the representation space H_π of π such that $\pi_k(x) = W_\pi(k)\pi(x)W_\pi(k)^{-1}$ for all $x \in N$. By Schur's lemma, the operator $W_\pi(k)$ is determined up to a scalar multiple of absolute value 1. On the analogy of the theory of unitary representations of compact groups, we can decompose W_π as a direct sum of irreducible projective representations of K_π:

$$W_\pi = \sum c(T, W_\pi)T,$$

where $c(T, W_\pi)$ is the multiplicity of T in W_π. We state here the following theorem due to Carcano [C, p. 1094] for later references.

Theorem 1.2 [C]. *The following three conditions are equivalent:*

(1) $(K; N)$ *is a Gelfand pair.*

(2) *One has* $c(T, W_\pi) \leq 1$ *for each* $\pi \in \widehat{N}$.

(3) *There is a dense subset* S *of* $\widehat{N}$ *such that* $c(T, W_\pi) \leq 1$ *for each* $\pi \in S$.

For a subset S of $\widehat{N}$, we denote by $S \cdot K$ the union of all K-orbits of elements of S: $S \cdot K = \{\pi_k \mid \pi \in S, k \in K\}$. Then,

Corollary 1.3. *The conditions in Theorem 1.2 are also equivalent to*
(4) *there is a subset* S *of* $\widehat{N}$ *with dense* $S \cdot K$ *in* $\widehat{N}$ *such that one has* $c(T, W_\pi) \leq 1$ *for each* $\pi \in \widehat{N}$.

Now, Kirillov's theory [Ki] tells us that there is a bijection between the coadjoint orbit space $\mathfrak{n}^*/N$ and the unitary dual $\widehat{N}$. By [Br], this bijection is a homeomorphism when $\mathfrak{n}^*/N$ is equipped with the quotient topology. For $l \in \mathfrak{n}^*$, we denote by π_l the irreducible unitary representation of N corresponding to l. Define the right action of K on $\mathfrak{n}^*$ by

$$(l \cdot k)(X) = l(k \cdot X) \quad (l \in \mathfrak{n}^*,\ k \in K,\ X \in \mathfrak{n}).$$

Then we see that $(\pi_l)_k \simeq \pi_{l \cdot k}$. Moreover, we have for $x \in N$

$$\begin{aligned}((\mathrm{Ad}^*(x)l) \cdot k)(X) &= l(\mathrm{Ad}(x^{-1})(k \cdot X)) = l(k \cdot (\mathrm{Ad}(k^{-1} \cdot x^{-1})X)) \\ &= (l \cdot k)(\mathrm{Ad}(k^{-1} \cdot x^{-1})X) = (\mathrm{Ad}^*(k^{-1} \cdot x)(l \cdot k))(X).\end{aligned}$$

Denoting by O_l the coadjoint orbit through $l \in \mathfrak{n}^*$, we get

$$O_l \cdot k = \{(\mathrm{Ad}^*(x)l) \cdot k \mid x \in N\} = \{\mathrm{Ad}^*(x)(l \cdot k) \mid x \in N\} = O_{l \cdot k}.$$

Therefore, $(\mathrm{Ad}^*(N)l) \cdot K = \mathrm{Ad}^*(N)(l \cdot K)$. This says that in Corollary 1.3, we can take the set $\{\pi_{l_\alpha}\}_{\alpha \in A}$ as S, where $\{l_\alpha\}_{\alpha \in A}$ is a complete system of representatives of (N, K)-orbits in $\mathfrak{n}^*$ such that the union $\bigcup_{\alpha \in A} (\mathrm{Ad}^*(N)l_\alpha \cdot K)$ is dense in $\mathfrak{n}^*$.

2. Reduction to Heisenberg Groups

We consider first the case of the $(2n+1)$-dimensional Heisenberg group H_n. Let $\mathfrak{h}_n$ be the Lie algebra of H_n and K a compact subgroup of $\mathrm{Aut}(\mathfrak{h}_n)$. Since the one-dimensional center $Z(\mathfrak{h}_n)$ of $\mathfrak{h}_n$ is invariant under K, and since K is compact, there is a character χ of K with image $\{1\}$ or $\{\pm 1\}$ such that

$$k \cdot X = \chi(k)X \quad (k \in K, X \in Z(\mathfrak{h}_n)).$$

We suppose that K acts on $Z(\mathfrak{h}_n)$ trivially. Let T be a generator of $Z(\mathfrak{h}_n)$. Let V be a K-invariant subspace of $\mathfrak{h}_n$ complementary to $Z(\mathfrak{h}_n)$. Then we can define a symplectic form ω on V such that

$$[X, Y] = \omega(X, Y)T \qquad (X, Y \in V).$$

Since K acts on $Z(\mathfrak{h}_n)$ trivially, we have $\omega(k \cdot X, k \cdot Y) = \omega(X, Y)$ for all $k \in K$, $X, Y \in V$. Hence K can be regarded as a compact subgroup of the symplectic group $\mathrm{Sp}(\omega)$ of ω. Since $[V, V] = [\mathfrak{h}_n, \mathfrak{h}_n] = Z(\mathfrak{h}_n)$, the form ω is non-degenerate.

Take a basis $\{X'_1, Y'_1, \ldots, X'_n, Y'_n\}$ of V such that

$$\omega(X'_i, Y'_j) = \delta_{ij}, \quad \omega(X'_i, X'_j) = \omega(Y'_i, Y'_j) = 0.$$

Defining a complex structure I_0 on V by $I_0X'_i = Y'_i$, $I_0Y'_i = -X'_i$, we let

$$\omega_0(X, Y) = \omega(X, I_0Y) - \sqrt{-1}\,\omega(X, Y) \qquad (X, Y \in V).$$

It is easy to see that ω_0 is a hermitian inner product on the complex vector space (V, I_0). The unitary group $\mathrm{U}(\omega_0)$ of ω_0 is a maximal compact subgroup of $\mathrm{Sp}(\omega)$. Hence there is $\psi \in \mathrm{Sp}(\omega)$ such that $K \subset \psi\mathrm{U}(\omega_0)\psi^{-1}$. Set $I = \psi I_0 \psi^{-1}$ and

$$\tilde{\omega}(X, Y) = \omega(X, IY) - \sqrt{-1}\,\omega(X, Y) \qquad (X, Y \in V).$$

Then, since $\psi^{-1}k\psi \in \mathrm{U}(\omega_0)$ commutes with I_0, we have

$$I(k \cdot X) = \psi I_0(\psi^{-1}k\psi) \cdot (\psi^{-1}X) = \psi(\psi^{-1}k\psi) \cdot I_0(\psi^{-1}X) = k \cdot IX$$

for $k \in K$, $X \in V$. Hence we get

$$\begin{aligned}
\tilde{\omega}(k \cdot X, k \cdot Y) &= \omega(k \cdot X, I(k \cdot Y)) - \sqrt{-1}\,\omega(k \cdot X, k \cdot Y) \\
&= \omega(k \cdot X, k \cdot (IX)) - \sqrt{-1}\,\omega(k \cdot X, k \cdot Y) \\
&= \omega(X, IY) - \sqrt{-1}\,\omega(X, Y) = \tilde{\omega}(X, Y)
\end{aligned}$$

for $k \in K$, $X, Y \in V$. Consequently we see that $K \subset \mathrm{U}(\tilde{\omega})$. In what follows we regard V as an n-dimensional complex vector space (V, I) with the inner product $\tilde{\omega}$.

As is well-known, the irreducible unitary representations of H_n which are non-trivial on the center are determined by their central characters. We denote by R_λ ($\lambda \neq 0$) the irreducible unitary representation with central character $\chi_\lambda(\exp tT) = e^{\sqrt{-1}\lambda t}$ ($t \in \mathbb{R}$). Moreover, $\{R_\lambda\}_{\lambda\neq 0}$ is dense in $\widehat{H}_n$ with respect to Fell topology. Then, by Theorem 1.2, it is sufficient to consider the representations $\{R_\lambda\}_{\lambda\neq 0}$. We will realize R_λ by means of the Fock models (see for example [Ba]). For $\lambda > 0$, let $\mathfrak{H}_\lambda$ (resp. $\mathfrak{H}_{-\lambda}$) be the Hilbert space of entire holomorphic (resp. antiholomorphic) functions f on V such that

$$\int_V |f(w)|^2 e^{-\lambda|w|^2/2} dw < +\infty.$$

The representation operators are given as follows:

$$(R_\lambda(z,t)f)(w) = e^{\sqrt{-1}\lambda t - \lambda w\bar{z}/2 - \lambda|z|^2/4} f(w+z) \quad (\lambda > 0), \tag{2.1}$$

$$(R_{-\lambda}(z,t)f)(w) = e^{-\sqrt{-1}\lambda t - \lambda \bar{w}z/2 - \lambda|z|^2/4} f(w+z) \quad (\lambda > 0). \tag{2.2}$$

If $\lambda > 0$, then $\mathfrak{H}_\lambda$ contains the algebra $\mathbb{C}[V]$ of holomorphic polynomials densely, and $\mathfrak{H}_{-\lambda}$ contains the algebra $\overline{\mathbb{C}[V]}$ of antiholomorphic polynomials densely.

Recalling that K is contained in the unitary group of V, we define the unitary operator $W_\lambda(k)$ ($\lambda > 0, k \in K$) on $\mathfrak{H}_\lambda$ by

$$(W_\lambda(k)f)(w) = f(k^{-1} \cdot w).$$

Then, an easy computation shows $R_\lambda(k \cdot (z,t))W_\lambda(k) = W_\lambda(k)R_\lambda(z,t)$. Hence $W_\lambda(k)$ is an intertwining operator between R_λ and $(R_\lambda)_k$. It is obvious that

$W_\lambda(k_1k_2) = W_\lambda(k_1)W_\lambda(k_2)$. Moreover, $\mathbb{C}[V]$ is invariant under W_λ, and the representation operators of W_λ are the same on $\mathbb{C}[V]$ for all $\lambda > 0$. The case $\lambda < 0$ being treated analogously, we have only to consider the particular case $\lambda = \lambda_0$, say.

Proposition 2.1 [BJR]. *The pair $(K; H_n)$ is a Gelfand pair if and only if W_{λ_0} decomposes into irreducibles with multiplicity one.*

We return to the case where N is a 2-step nilpotent Lie group. We will see that every infinite-dimensional irreducible unitary representation of N factors through a Heisenberg group. For $l \in \mathfrak{n}^*$, let B_l be the alternative form on $\mathfrak{n}$ defined by

$$B_l(X, Y) = l([X, Y]).$$

Define subspaces $\mathfrak{n}(l)$, $\mathfrak{b}(l)$ of $\mathfrak{n}$ as follows:

$$\mathfrak{n}(l) = \{X \in \mathfrak{n} \mid B_l(X, Y) = 0 \text{ for all } Y \in \mathfrak{n}\}, \qquad \mathfrak{b}(l) = \mathfrak{n}(l) \cap (\ker l). \tag{2.3}$$

Proposition 2.2. *Let l be a non-zero element of $\mathfrak{n}^*$. Then,*

(1) $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}(l)$. *In particular $\mathfrak{n}(l)$ is an ideal of $\mathfrak{n}$.*

(2) $[\mathfrak{n}(l), \mathfrak{n}] \subset \mathfrak{b}(l)$. *In particular $\mathfrak{b}(l)$ is an ideal of $\mathfrak{n}$.*

(3) $\dim(\mathfrak{n}(l)/\mathfrak{b}(l)) = 1$.

(4) $l|_{[\mathfrak{n},\mathfrak{n}]} \neq 0$ *if and only if* $\mathfrak{n}(l) \neq \mathfrak{n}$. *In this case,* $\dim(\mathfrak{n}/\mathfrak{b}(l)) > 1$.

Proof. (1) Since $\mathfrak{n}$ is 2-step, $[\mathfrak{n}, \mathfrak{n}]$ is included in the center $Z(\mathfrak{n})$. Therefore $B_l([\mathfrak{n}, \mathfrak{n}], \mathfrak{n}) \subset B_l(Z(\mathfrak{n}), \mathfrak{n}) = 0$. Hence $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}(l)$.

(2) This follows from (1) and the definition of $\mathfrak{n}(l)$.

(3) Clearly $\dim(\mathfrak{n}(l)/\mathfrak{b}(l)) \leq 1$. Suppose first $l|_{[\mathfrak{n},\mathfrak{n}]} \neq 0$. Then we have $\mathfrak{n}(l) \neq \mathfrak{b}(l)$ by (1). Suppose next $l|_{[\mathfrak{n},\mathfrak{n}]} = 0$. Then $B_l(\mathfrak{n},\mathfrak{n}) = 0$, so that we get $\mathfrak{n}(l) = \mathfrak{n}$. On the other hand we have $\mathfrak{b}(l) \neq \mathfrak{n}$, because $l \neq 0$.

(4) By the proof of (3), if $\mathfrak{n}(l) \neq \mathfrak{n}$, then $l|_{[\mathfrak{n},\mathfrak{n}]} \neq 0$. Suppose conversely that $l|_{[\mathfrak{n},\mathfrak{n}]} \neq 0$. Then there are $X, Y \in \mathfrak{n}$ such that $l([X,Y]) \neq 0$, so $X, Y \notin \mathfrak{n}(l)$. ∎

Let π_l be the irreducible unitary representation of N corresponding to $l \in \mathfrak{n}^*$.

Lemma 2.3. *Let $l, l' \in \mathfrak{n}^*$, Then one has $\pi_l \simeq \pi_{l'}$ if and only if*
(1) $\mathfrak{n}(l) = \mathfrak{n}(l')$, (2) $l|_{\mathfrak{n}(l)} = l'|_{\mathfrak{n}(l)}$.

Proof. See [M], Theorem 2.3 (3). ∎

Now, let K be a compact subgroup of $\mathrm{Aut}(\mathfrak{n})$.

Lemma 2.4. *Let $k \in K$. Then $k \in K_{\pi_l}$ if and only if*
(1) $k^{-1} \cdot \mathfrak{n}(l) = \mathfrak{n}(l)$, (2) $l \cdot k|_{\mathfrak{n}(l)} = l|_{\mathfrak{n}(l)}$.

Proof. Since $(\pi_l)_k \simeq \pi_{l \cdot k}$ for $k \in K$, we have $k \in K_{\pi_l} \Leftrightarrow \pi_{l \cdot k} \simeq \pi_l$. Now Lemma 2.4 follows from Lemma 2.3 by noting $\mathfrak{n}(l \cdot k) = k^{-1} \cdot \mathfrak{n}(l)$. ∎

We suppose from now on that $l|_{[\mathfrak{n},\mathfrak{n}]} \neq 0$. Then, Proposition 2.2 (4) says that $\dim(\mathfrak{n}/\mathfrak{b}(l)) > 1$. Moreover,

$$Z(\mathfrak{n}/\mathfrak{b}(l)) = \mathfrak{n}(l)/\mathfrak{b}(l), \quad (\mathfrak{n}/\mathfrak{b}(l))/(\mathfrak{n}(l)/\mathfrak{b}(l)) \simeq \mathfrak{n}/\mathfrak{n}(l). \tag{2.4}$$

Since the second algebra in (2.4) is abelian by Proposition 2.2 (1), $\mathfrak{n}/\mathfrak{b}(l)$ is isomorphic to a Heisenberg algebra $\mathfrak{h}_n$ where $n = \frac{1}{2}\dim(\mathfrak{n}/\mathfrak{n}(l))$. Put $B(l) = \exp \mathfrak{b}(l)$. Denote by p_l the canonical projection of N onto $B(l)\backslash N$ and by l_0 the element of $(\mathfrak{n}/\mathfrak{b}(l))^*$ such that $l = l_0 \circ p_l$.

Lemma 2.5. *Denote by σ_{l_0} the unitary representation of $B(l)\backslash N$ corresponding to l_0. Then, one has $\pi_l \simeq \sigma_{l_0} \circ p_l$.*

Proof. Let $\overline{\mathfrak{m}}$ be a polarization for l_0. Put $\mathfrak{m} = p_l^{-1}(\overline{\mathfrak{m}})$. Then $\mathfrak{m}$ is a polarization for l. Put $\overline{M} = \exp \overline{\mathfrak{m}}$ and $M = \exp \mathfrak{m}$ respectively. Obviously $\overline{M} = B(l)\backslash M$. Let χ_l and χ_{l_0} be the characters of M and $\overline{M}$ respectively such that

$$\chi_l(\exp X) = \exp \sqrt{-1}\, l(X) \quad (X \in \mathfrak{m}),$$
$$\chi_{l_0}(\exp \overline{X}) = \exp \sqrt{-1}\, l_0(\overline{X}) \quad (\overline{X} \in \overline{\mathfrak{m}}).$$

Then $\chi_{l_0}(p_l(x)) = \chi_l(x)$ for all $x \in M$. In particular $\chi_l(b) = 1$ for all $b \in B(l)$.

The representations π_l and σ_{l_0} can be considered as the induced representations $\mathrm{Ind}\chi_l$ and $\mathrm{Ind}\chi_{l_0}$ respectively. Let $d\mu$, $d\nu$ and $d\rho$ be Haar measures on N, M and $B(l)$ respectively. Let $d\dot{\mu}$, $d\bar{\mu}$ and $d\bar{\nu}$ be the invariant measures on $M\backslash N$, $B(l)\backslash N$ and $B(l)\backslash M$ respectively such that $d\mu = d\nu d\dot{\mu} = d\rho d\bar{\mu}$, $d\nu = d\rho d\bar{\nu}$. Identifying $M\backslash N$ with $(B(l)\backslash M)\backslash(B(l)\backslash N)$, we regard $d\dot{\mu}$ as the invariant measure on $(B(l)\backslash M)\backslash(B(l)\backslash N)$ such that $d\bar{\mu} = d\bar{\nu}d\dot{\mu}$. The representation space $\mathfrak{H}_l$ of π_l is the Hilbert space of functions f satisfying

$$f(mx) = \chi_l(m)f(x) \quad (m \in M, x \in N), \quad \int |f|^2 d\dot{\mu} < +\infty.$$

We have a similar description for the representation space $\mathfrak{H}_{l_0}$ of σ_{l_0}. Now for $b \in B(l)$ and $x \in N$, we have $f(bx) = \chi_l(b)f(x) = f(x)$ for all $f \in \mathfrak{H}_l$. Hence the map $f \mapsto \bar{f}$, where $\bar{f}(p_l(x)) = f(x)$ $(x \in N)$, gives rise to an identification of $\mathfrak{H}_l$ with $\mathfrak{H}_{l_0}$. Then, for $x, n \in N$, we get

$$\sigma_{l_0}(p_l(x))\bar{f}(p_l(n)) = \bar{f}(p_l(nx)) = f(nx) = \pi_l(x)f(n).$$

Therefore $\pi_l \simeq \sigma_{l_0} \circ p_l$. ∎

Put $\pi = \pi_l$ for brevity. Consider the subgroup $\mathrm{Aut}(N)_\pi$ of $\mathrm{Aut}(N)$:

$$\mathrm{Aut}(N)_\pi = \{\varphi \in \mathrm{Aut}(N) \mid (\pi)_\varphi \simeq \pi\},$$

where $(\pi)_\varphi(x) = \pi(\varphi(x))$. Then K_π is a subgroup of $\mathrm{Aut}(N)_\pi$. Recalling the projection $p_l : N \to B(l)\backslash N$ we define a map Φ_π as follows:

$$\Phi_\pi : \mathrm{Aut}(N)_\pi \ni \varphi \mapsto \bar{\varphi} \in \mathrm{Aut}(B(l)\backslash N), \quad \bar{\varphi}(p_l(x)) = p_l(\varphi(x)). \tag{2.5}$$

Then Φ_π is well-defined thanks to Lemma 2.3, and $\Phi_\pi(K_\pi)$ stabilizes the elements in the center of $B(l)\backslash N$.

Theorem 2.6. *For $l \in \mathfrak{n}^*$, denote by $\pi = \pi_l$ the irreducible unitary representation of N corresponding to $l \in \mathfrak{n}^*$. Then, $(K; N)$ is a Gelfand pair if and only if $(\Phi_\pi(K_\pi); B(l)\backslash N)$ is a Gelfand pair for every $l \in \mathfrak{n}^*$.*

Proof. By Theorem 1.2, it suffices to treat the case $l|_{[\mathfrak{n},\mathfrak{n}]} \neq 0$. For such an l, we have $\pi_l \simeq \sigma_{l_0} \circ p_l$ for some $l_0 \in (\mathfrak{n}/\mathfrak{b}(l))^*$. For $k \in K_\pi$,

$$\pi_l(k \cdot x) = \sigma_{l_0}(p_l(k \cdot x)) = \sigma_{l_0}(\Phi_\pi(k)p_l(x)),$$

by (2.5). Since σ_{l_0} is unitarily equivalent to some R_{λ_0} in (2.1) or (2.2), there is an intertwining representation W of $\Phi_\pi(K_\pi)$ such that

$$\sigma_{l_0}(\Phi_\pi(k)h) = W(\Phi_\pi(k))\sigma_{l_0}(h)W(\Phi_\pi(k))^{-1},$$

for all $k \in K_\pi$, $h \in H_n$. Therefore we get

$$\pi_l(k \cdot x) = W(\Phi_\pi(k))\sigma_{l_0}(p_l(x))W(\Phi_\pi(k))^{-1}.$$

Thus we can take $W \circ \Phi_\pi$ as an intertwining representation W_π of K_π. Then it is evident that

$$c(T, W_\pi) \leq 1 \text{ for } T \in \widehat{K}_\pi \iff c(T', W) \leq 1 \text{ for } T' \in \Phi_\pi(K_\pi)\widehat{\ }.$$

This together with Theorem 1.2 and Proposition 2.1 completes the proof. ∎

3. A Counterexample

In this section we give some applications of Theorem 2.6. Let N be a 2-step nilpotent Lie group, $\mathfrak{n}$ the Lie algebra of N. We denote by $Z(\mathfrak{n})$ the center of $\mathfrak{n}$ and by $[\mathfrak{n}, \mathfrak{n}]$ the derived algebra of $\mathfrak{n}$. Since $\mathfrak{n}$ is 2-step, we have $[\mathfrak{n}, \mathfrak{n}] \subset Z(\mathfrak{n})$. We suppose that $[\mathfrak{n}, \mathfrak{n}] \neq Z(\mathfrak{n})$. Let K be a compact group acting on $\mathfrak{n}$ as automorphisms. Then there is a K-invariant real inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$. Let $\mathfrak{n}'$ (resp. $\mathfrak{a}$) be the K-invariant orthogonal complement of $Z(\mathfrak{n})$ (resp. of $[\mathfrak{n}, \mathfrak{n}]$) in $\mathfrak{n}$ (resp. in $Z(\mathfrak{n})$) relative to this inner product, so that we have the following K-invariant orthogonal decompositions:

$$\mathfrak{n} = \mathfrak{n}' \oplus \mathfrak{a} \oplus [\mathfrak{n}, \mathfrak{n}], \tag{3.1}$$

$$Z(\mathfrak{n}) = \mathfrak{a} \oplus [\mathfrak{n}, \mathfrak{n}]. \tag{3.2}$$

Put $\mathfrak{n}_1 = \mathfrak{n}' \oplus [\mathfrak{n}, \mathfrak{n}]$. Then $\mathfrak{n}_1$ is a Lie subalgebra and we have $Z(\mathfrak{n}_1) = [\mathfrak{n}_1, \mathfrak{n}_1] = [\mathfrak{n}, \mathfrak{n}]$. Moreover, $\mathfrak{a}$ and $\mathfrak{n}_1$ are ideals of $\mathfrak{n}$ and

$$\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{a} \qquad \text{(direct sum of ideals)}. \tag{3.3}$$

Let N_1, A be the normal subgroups of N corresponding to the ideals $\mathfrak{n}_1$, $\mathfrak{a}$ respectively.

Proposition 3.1. *If $(K;N)$ is a Gelfand pair, so is $(K;N_1)$.*

Proof. Given $l_1 \in \mathfrak{n}_1^*$, we denote by l the linear form on $\mathfrak{n}$ defined by $l(X_1+Z) = l_1(X_1)$ $(X_1 \in \mathfrak{n}_1,\ Z \in \mathfrak{a})$. By (2.3) we have easily

$$\mathfrak{n}(l) = \mathfrak{n}_1(l_1) \oplus \mathfrak{a}, \quad \mathfrak{b}(l) = \mathfrak{b}_1(l_1) \oplus \mathfrak{a}.$$

By the latter equality we can identify $\mathfrak{n}_1/\mathfrak{b}(l_1)$ with $\mathfrak{n}/\mathfrak{b}(l)$. Put $\pi_1 = \pi_{l_1} \in \widehat{N}_1$ for simplicity. Then π_1 is equivalent to $\pi_l|_{N_1}$. Furthermore the stabilizer K_{π_1} coincides with K_{π_l}. In fact,

$$\begin{aligned} K_{\pi_1} &= \{k \in K \mid k^{-1} \cdot \mathfrak{n}_1(l_1) = \mathfrak{n}_1(l_1),\ l_1 \cdot k|_{\mathfrak{n}_1(l_1)} = l_1|_{\mathfrak{n}_1(l_1)}\} \quad \text{(by Lemma 2.4)} \\ &= \{k \in K \mid k^{-1} \cdot \mathfrak{n}(l) = \mathfrak{n}(l),\ l \cdot k|_{\mathfrak{n}(l)} = l|_{\mathfrak{n}(l)}\} = K_{\pi_l}. \end{aligned}$$

Let $B_1(l_1) = \exp \mathfrak{b}_1(l_1)$. Let Φ_{π_l} be the map in (2.5) and $\Phi^1_{\pi_1}$ the map $\mathrm{Aut}(N_1)_{\pi_1} \to \mathrm{Aut}(B_1(l_1)\backslash N_1)$ defined similarly through the data N_1, l_1, π_1. Then we can identify $(\Phi_{\pi_l}(K_{\pi_l}); B(l)\backslash N)$ with $(\Phi^1_{\pi_1}(K_{\pi_1}); B_1(l_1)\backslash N_1)$. Since $(K;N)$ is a Gelfand pair, so is $(\Phi_{\pi_l}(K_{\pi_l}); B(l)\backslash N)$ by Theorem 2.6. Hence $(\Phi^1_{\pi_1}(K_{\pi_1}); B_1(l_1)\backslash N_1)$ is a Gelfand pair, so that $(K;N_1)$ is also a Gelfand pair by Theorem 2.6 again. ∎

Now, we consider the converse of Proposition 3.1. It is stated in [L, p. 59] and [BJR, p. 105] that the commutativities of $L^1_K(N)$ and $L^1_K(N_1)$ are equivalent. However it turns out that the commutativity of $L^1_K(N_1)$ does not imply the commutativity of $L^1_K(N)$ as will be shown by the following counterexample.

Let N be a nilpotent Lie group homeomorphic to $\mathbb{C} \times \mathbb{C} \times \mathbb{R}$, $\mathfrak{n}$ the Lie algebra of N with the bracket product given by

$$[(z_1, z_2, t), (z_1', z_2', t')] = (0, 0, -\mathrm{Im} z_2 \bar{z_2}').$$

Let $K = \mathbb{T}$ act on $\mathfrak{n}$ as follows:

$$e^{\sqrt{-1}\theta} \cdot (z_1, z_2, t) = (e^{\sqrt{-1}\theta} z_1, e^{\sqrt{-1}\theta} z_2, t).$$

Using the same notations as the beginning of this section we have

$$Z(\mathfrak{n}) = \mathbb{C} \times 0 \times \mathbb{R}, \qquad [\mathfrak{n}, \mathfrak{n}] = 0 \times 0 \times \mathbb{R},$$
$$\mathfrak{n}_1 = 0 \times \mathbb{C} \times \mathbb{R}, \qquad \mathfrak{a} = \mathbb{C} \times 0 \times 0.$$

It is clear that $\mathfrak{n}_1$ is isomorphic to the 3-dimensional Heisenberg algebra. Denote by N_1 the subgroup of N corresponding to $\mathfrak{n}_1$.

Theorem 3.2. *$(K; N_1)$ is a Gelfand pair, whereas $(K; N)$ is not a Gelfand pair.*

Proof. Since N_1 is isomorphic to the 3-dimensional Heisenberg group H_1 and $K = \mathbb{T}$, the pair $(K; N_1)$ is a Gelfand pair as is well-known (see [BJR]). To show that $(K; N)$ fails to be a Gelfand pair we take a basis of $\mathfrak{n}$ as follows:

$$E_1^R = (1, 0, 0), \quad E_1^I = (\sqrt{-1}, 0, 0),$$
$$E_2^R = (0, 1, 0), \quad E_2^I = (0, \sqrt{-1}, 0), \quad T = (0, 0, 1).$$

Let l be an element of $\mathfrak{n}^*$ such that $l(z_1, z_2, t) = \mathrm{Re} z_1 + t$. Then we have $\mathfrak{n}(l) = Z(\mathfrak{n})$, $\mathfrak{b}(l) = \mathbb{R}E_1^I + \mathbb{R}(T - E_1^R)$ and $\mathfrak{n}/\mathfrak{b}(l)$ is isomorphic to the 3-dimensional Heisenberg algebra. Let $k = e^{\sqrt{-1}\theta} \in K$. Then

$$l(e^{\sqrt{-1}\theta} \cdot (z_1, 0, t)) = x_1 \cos\theta - y_1 \sin\theta + t,$$

where $z_1 = x_1 + \sqrt{-1}\, y_1$ $(x_1, y_1 \in \mathbb{R})$. Obviously $k \cdot \mathfrak{n}(l) = \mathfrak{n}(l)$. Letting $\pi \in \widehat{N}$ be corresponding to l, we have $K_\pi = \{1\}$ by Lemma 2.4. Hence $\Phi_\pi(K_\pi) = \{1\}$,

so that $(\Phi_\pi(K_\pi); B(l)\backslash N)$ is not a Gelfand pair. This together with Theorem 2.6 completes the proof. ∎

By (3.3), we have $N = N_1 \times A$. Hence we have $L^1(N) = L^1(N_1) \otimes L^1(A)$. But Theorem 3.2 says that $L^1_K(N)$ is not isomorphic to $L^1_K(N_1) \otimes B$ for any subalgebra B of $L^1(A)$.

4. Leptin's Problem

Let N be a 2-step nilpotent Lie group and $K = \mathbb{T}^n$ an n-dimensional torus acting on N as automorphisms. We consider the following problem posed by Leptin [L].

Problem 4.1. *When is $(K; N)$ a Gelfand pair?*

For example, $(\mathbb{T}^n; H_n)$ is a Gelfand pair, [HR]. When $[\mathfrak{n}, \mathfrak{n}] = Z(\mathfrak{n})$ and $\mathbb{T}^n$ acts on N effectively, Leptin gave an answer as follows [L]:

$(\mathbb{T}^n; N)$ is a Gelfand pair if and only if N is a quotient group of $(H_1)^n$ by a central subgroup and $\mathbb{T}^n$ acts on $(H_1)^n$ naturally. In this case, $\mathbb{T}^n$ acts on $Z(N)$ trivially.

We investigate now the case $[\mathfrak{n}, \mathfrak{n}] \neq Z(\mathfrak{n})$ and give a complete solution to Problem 4.1. We have the decompositions (3.1), (3.2) and (3.3) in the previous section. We write $\widehat{K}^r$ for the family of all equivalence classes of irreducible real K-modules. Then $\widehat{K}^r$ is identified with $\mathbb{Z}^n/\sim$ where $\alpha \sim \beta$ if $\beta = \pm\alpha$ for $\alpha, \beta \in \mathbb{Z}^n$. By fixing a system of representatives, we may regard $\widehat{K}^r$ as a subset of $\mathbb{Z}^n$. We

decompose further $\mathfrak{n}'$, $\mathfrak{a}$ into isotypic real K-modules:

$$\mathfrak{n}' = \sum_{\alpha \in \widehat{K}^r} V'_\alpha, \qquad \mathfrak{a} = \sum_{\alpha \in \widehat{K}^r} V''_\alpha. \tag{4.1}$$

For non-zero $\alpha \in \mathbb{Z}^n$, let V_α be the 2-dimensional real irreducible K-module $\mathbb{R}X_\alpha + \mathbb{R}Y_\alpha$ such that

$$(\exp U \cdot X_\alpha,\ \exp U \cdot Y_\alpha) = (X_\alpha, Y_\alpha) \begin{pmatrix} \cos\alpha(U) & -\sin\alpha(U) \\ \sin\alpha(U) & \cos\alpha(U) \end{pmatrix}, \tag{4.2}$$

where $U \in \mathfrak{k}$, the Lie algebra of K. If $\alpha = 0$, V_α denote the 1-dimensional trivial real K-module. Let

$$V'_\alpha = m_{\alpha,1} V_\alpha, \qquad V''_\alpha = m_{\alpha,2} V_\alpha, \tag{4.3}$$

where $m_{\alpha,1}$, $m_{\alpha,2}$ are the multiplicities of V_α in $\mathfrak{n}'$ and in $\mathfrak{a}$ respectively. We also write

$$V'_\alpha = \sum_{i=1}^{m_{\alpha,1}} V'_{\alpha,i}, \qquad V''_\alpha = \sum_{i=1}^{m_{\alpha,2}} V''_{\alpha,i}, \tag{4.4}$$

where $V'_{\alpha,i}$, $V''_{\alpha,i} \simeq V_\alpha$ for all i, and $V'_{\alpha,i} \perp V'_{\alpha,j}$, $V''_{\alpha,i} \perp V''_{\alpha,j}$ if $i \neq j$. Let S_1, S_2 be the subsets of $\widehat{K}^r$ such that

$$S_1 = \{\alpha \in \widehat{K}^r \mid m_{\alpha,1} \neq 0\}, \quad S_2 = \{\alpha \in \widehat{K}^r \mid m_{\alpha,2} \neq 0\}. \tag{4.5}$$

Theorem 4.2. *$(K; N)$ is a Gelfand pair if and only if the following five conditions are satisfied:*

(1) $m_{0,1} = 0$,

(2) S_1 *is a linearly independent system,*

(3) $m_{\alpha,1} = 1$ *for all* $\alpha \in S_1$,

(4) $\mathbb{R}$-span$(S_1) \cap \mathbb{R}$-span$(S_2) = 0$,

(5) K *acts on* $[\mathfrak{n}, \mathfrak{n}]$ *trivially.*

In order to prove this theorem, we need the following lemma.

Lemma 4.3. *Let K be an n-dimensional compact abelian Lie group (not necessarily connected), and suppose K acts on H_m effectively as automorphisms. Then $(K; H_m)$ is a Gelfand pair if and only if $n = m$.*

Proof. See [BJR, p. 103], the proof of Theorem 5.17. ∎

Proof of Theorem 4.2. Suppose first that $(K; N)$ is a Gelfand pair.

Step 1: Let V_1, V_2 be mutually orthogonal K-invariant subspaces of $\mathfrak{n}'$. Then we have $[V_1, V_2] = 0$ (see the proof of Leptin's theorem in [BJR, p. 107]). In particular, $[V_0', V_\alpha'] = 0$ for all $\alpha \neq 0$. Clearly $[V_0', V_0'] = 0$ and $[V_0', Z(\mathfrak{n})] = 0$. Hence $[V_0', \mathfrak{n}] = 0$ by (3.1) and (3.2). This means $V_0' \subset Z(\mathfrak{n}) \bigcap \mathfrak{n}' = 0$, whence (1). We next show (5). Indeed, for non-zero $\alpha \in \widehat{K}^r$, and for $U \in \mathfrak{k}$, we have by (4.2),

$$\begin{aligned}
&[\exp U \cdot X_\alpha,\ \exp U \cdot Y_\alpha] \\
&\quad = [\cos\alpha(U)X_\alpha + \sin\alpha(U)Y_\alpha,\ -\sin\alpha(U)X_\alpha + \cos\alpha(U)Y_\alpha] \\
&\quad = \cos^2\alpha(U)[X_\alpha, Y_\alpha] - \sin^2\alpha(U)[Y_\alpha, X_\alpha] = [X_\alpha, Y_\alpha].
\end{aligned}$$

Since $[V_{\alpha,i}', V_{\alpha,j}'] = 0$ for $i \neq j$, we see that K acts on $[\mathfrak{n}, \mathfrak{n}]$ trivially.

Step 2: We prove (2), (3) and (4). Take bases $\{X_{\alpha,i}', Y_{\alpha,i}'\}$, $\{X_{\beta,j}'', Y_{\beta,j}''\}$ of $V_{\alpha,i}'$, $V_{\beta,j}''$ respectively similarly to (4.2). Let l be an element of $\mathfrak{n}^*$ with $l|_{\mathfrak{n}'} = 0$ such that for each $\alpha \in S_1$, $l([X_{\alpha,i}', Y_{\alpha,i}']) \neq 0$ for any i with $1 \leq i \leq m_{\alpha,1}$, and

for each $\beta \in S_2$, $l(X''_{\beta,j}) = 1$, $l(Y''_{\beta,j}) = 0$ for any j with $1 \leq j \leq m_{\beta,2}$. Then $\mathfrak{n}(l) = \mathfrak{a} + [\mathfrak{n}, \mathfrak{n}] = Z(\mathfrak{n})$ and $\mathfrak{n}/\mathfrak{b}(l) \simeq \mathfrak{h}_m$ with $m = \sum_{\alpha \in S_1} m_{\alpha,1}$. Write $\pi = \pi_l$ for simplicity. Then, we have by Lemma 2.4,

$$\begin{aligned} K_\pi &= \{k \in K \mid l \cdot k = l \text{ on } Z(\mathfrak{n})\} \\ &= \{k \in K \mid k \cdot X = X \text{ for all } X \in Z(\mathfrak{n})\}. \end{aligned}$$

Let $\mathfrak{k}_\pi$ be the Lie algebra of K_π. Then we have by the above,

$$\begin{aligned} \mathfrak{k}_\pi &= \{U \in \mathfrak{k} \mid U \cdot X = 0 \text{ for all } X \in Z(\mathfrak{n})\} \\ &= \{U \in \mathfrak{k} \mid \beta(U) = 0 \text{ for all } \beta \in S_2\} = \bigcap_{\beta \in S_2} \ker \beta \end{aligned}$$

Let Φ_π be the map $K_\pi \to \mathrm{Aut}(\mathfrak{n}/\mathfrak{b}(l))$ obtained by making the K_π-action factor through $\mathfrak{n}/\mathfrak{b}(l)$. Then for the differential $d\Phi_\pi$ which maps $\mathfrak{k}_\pi$ to the derivation algebra $\mathrm{Der}(\mathfrak{n}/\mathfrak{b}(l))$, we have

$$\ker d\Phi_\pi = \{U \in \mathfrak{k}_\pi \mid \alpha(U) = 0 \text{ for all } \alpha \in S_1\} = (\bigcap_{\alpha \in S_1} \ker \alpha) \bigcap (\bigcap_{\beta \in S_2} \ker \beta),$$

so that $\dim \Phi_\pi(K_\pi) = \dim d\Phi_\pi(\mathfrak{k}_\pi)$. By Lemma 4.3, we obtain $\dim \Phi_\pi(K_\pi) = m$. Moreover, we have

$$\begin{aligned} \dim \Phi_\pi(K_\pi) &= \dim \mathfrak{k}_\pi - \dim \ker d\Phi_\pi \\ &= \dim(\bigcap_{\beta \in S_2} \ker \beta) - \dim((\bigcap_{\alpha \in S_1} \ker \alpha) \bigcap (\bigcap_{\beta \in S_2} \ker \beta)) \\ &\leq \dim \mathfrak{k} - \dim(\bigcap_{\alpha \in S_1} \ker \alpha) \leq \sharp S_1 \leq \sum_{\alpha \in S_1} m_{\alpha,1} = m. \end{aligned}$$

Hence (4), (2) and (3) are proved by the first, second and third inequality.

Suppose conversely that the conditions (1) $\sim$ (5) are satisfied. We will show that $(K; N)$ is a Gelfand pair. In order to prove this, it is sufficient to deal with the elements l in the maximal dimensional coadjoint orbits in $\mathfrak{n}^*$. We note here that (1) says every $\alpha \in S_1$ is non-zero and that (3) implies the K-module V'_α $(\alpha \in S_1)$ is irreducible. Hence for each $\alpha \in S_1$ we have $V'_\alpha = \mathbb{R}X'_\alpha + \mathbb{R}Y'_\alpha$ for some X'_α, Y'_α with the K-action (4.2). First, we show that

$$[V'_\alpha, V'_\beta] = 0 \quad \text{if } \alpha, \beta \in S_1, \alpha \neq \beta. \tag{4.6}$$

Suppose that (4.6) is not true, then there are elements $Z_\alpha \in V'_\alpha$, $Z_\beta \in V'_\beta$ such that $[Z_\alpha, Z_\beta] \neq 0$. Transforming Z_α by an element of K if necessary, we may assume $Z_\alpha = X'_\alpha$. If $U \in \ker\beta$, then $\exp tU \cdot Z_\beta = Z_\beta$ for all $t \in \mathbb{R}$. We get

$$\begin{aligned}
[X'_\alpha, Z_\beta] &= [\exp tU \cdot X'_\alpha, \quad \exp tU \cdot Z_\beta] \qquad (\text{by } (5)) \\
&= [(\cos t\alpha(U))X'_\alpha + (\sin t\alpha(U))Y'_\alpha, \quad Z_\beta] \\
&= \cos t\alpha(U)[X'_\alpha, Z_\beta] + \sin t\alpha(U)[Y'_\alpha, Z_\beta].
\end{aligned}$$

If $[X'_\alpha, Z_\beta]$ and $[Y'_\alpha, Z_\beta]$ are linearly independent, then $\cos t\alpha(U) = 1, \sin t\alpha(U) = 0$ for all $t \in \mathbb{R}$. If $[X'_\alpha, Z_\beta]$ and $[Y'_\alpha, Z_\beta]$ are linearly dependent, then there is $c \in \mathbb{R}$ such that $[Y'_\alpha, Z_\beta] = c[X'_\alpha, Z_\beta]$, $\cos t\alpha(U) + c \sin t\alpha(U) = 1$ for all $t \in \mathbb{R}$. Therefore $\alpha(U) = 0$, so that $\ker\beta \subset \ker\alpha$. This contradicts (2).

By (4.6), the coadjoint orbit O_l through $l \in \mathfrak{n}^*$ is of maximal dimension if and only if $l([V'_\alpha, V'_\alpha]) \neq 0$ for any $\alpha \in S_1$. Hence we may assume that $l|_{[V'_\alpha, V'_\alpha]} \neq 0$ for any $\alpha \in S_1$ and that $l|_{\mathfrak{n}'} = 0$. Then $\mathfrak{n}(l) = \mathfrak{a} + [\mathfrak{n}, \mathfrak{n}] = Z(\mathfrak{n})$ and $\mathfrak{n}/\mathfrak{b}(l) \simeq \mathfrak{h}_m$ with $m = \sharp S_1$. We denote by $\pi = \pi_l$ the irreducible unitary representation of N

corresponding to l. Then we have

$$K_\pi = \{k \in K \mid l \cdot k = l \text{ on } Z(\mathfrak{n})\}.$$

Using (5), we get

$$\mathfrak{k}_\pi = \{U \in \mathfrak{k} \mid l(U \cdot X) = 0 \text{ for all } X \in \mathfrak{a}\}.$$

Put $a_{\beta,j} = l(X''_{\beta,j})$, $b_{\beta,j} = l(Y''_{\beta,j})$ for simplicity. Let $S_{2,l} = \{\beta \in S_2 \mid a_{\beta,j} = b_{\beta,j} = 0 \text{ for all } j \text{ with } 1 \leq j \leq m_{\beta,2}\}$. Then $\mathfrak{k}_\pi = \bigcap_{\beta \in S_2 \setminus S_{2,l}} \ker \beta$. Consider the map $\Phi_\pi : K_\pi \to \mathrm{Aut}(\mathfrak{n}/\mathfrak{b}(l))$ and its differential $d\Phi_\pi$ as in Step 2, then we have

$$\ker d\Phi_\pi = (\bigcap_{\alpha \in S_1} \ker \alpha) \bigcap (\bigcap_{\beta \in S_2 \setminus S_{2,l}} \ker \beta).$$

Hence we get

$$\begin{aligned}
\dim \Phi_\pi(K_\pi) &= \dim d\Phi_\pi(\mathfrak{k}_\pi) = \dim \mathfrak{k}_\pi - \dim \ker d\Phi_\pi \\
&= \dim(\bigcap_{\beta \in S_2 \setminus S_{2,l}} \ker \beta) - \dim((\bigcap_{\alpha \in S_1} \ker \alpha) \bigcap (\bigcap_{\beta \in S_2 \setminus S_{2,l}} \ker \beta)) \\
&= \dim \mathfrak{k} - \dim(\bigcap_{\alpha \in S_1} \ker \alpha) \quad \text{(by (4))} \\
&= \sharp S_1 = m \quad \text{(by (2) and (3))}.
\end{aligned}$$

Then $(\Phi_\pi(K_\pi); B(l)\backslash N)$ is a Gelfand pair by virtue of Lemma 4.3. Theorem 2.6 now shows that $(K; N)$ is a Gelfand pair. ∎

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