On the Multiplicity of the Discrete Series Representations of $Sp_2(\mathbb{R})$

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§0. Introduction

Let Γ be a torsion free cocompact discrete subgroup of $G = SL_2(\mathbb{R})$. Then the dimension of the space $S_k(\Gamma)$ of modular forms of weight k is given by

$$\begin{aligned} \dim S_k(\Gamma) &= \mathrm{Vol}(\Gamma\backslash G)\,\frac{k-1}{\sqrt{2}\,\pi} \qquad (k \geqq 3) \\ \dim S_2(\Gamma) &= \mathrm{Vol}(\Gamma\backslash G)\,\frac{1}{\sqrt{2}\,\pi} + 1 \quad (k = 2). \end{aligned} \tag{0.1}$$

For our normalization of invariant measures see Section 1. (The bilinear form is given by $B(X,Y) := \mathrm{tr}(XY)/2$). When $k = 2$, there exists an error term "+1". This kind of error terms for lower weight are of our interest.

We can give a representation theoretic interpretation to the formula (0.1). For a locally compact topological group G and a cocompact discrete subgroup Γ of G, it is proved by Gel'fand–Graev–Pyatetskii-Shapiro [6] that the representation r on $L^2(\Gamma\backslash G)$ by right translation splits into a discrete sum of a countable number of irreducible unitary representations, each of finite multiplicity m_π. Let D_k^+ be the holomorphic discrete series representation of $SL_2(\mathbb{R})$. Then

$$\begin{aligned} m_{D_k^+} &= \mathrm{Vol}(\Gamma\backslash G)\,\frac{k-1}{\sqrt{2}\,\pi} \qquad (k \geqq 3) \\ m_{D_2^+} &= m_{\mathrm{Trivial}} + \mathrm{Vol}(\Gamma\backslash G)\,\frac{1}{\sqrt{2}\,\pi} \quad (k = 2). \end{aligned} \tag{0.2}$$

It is easy to see that the multiplicity m_{Trivial} of the trivial representation is "+1". It turns out that the contribution of the trivial representation gives the error term "+1" in (0.1).

In this paper we will investigate the error terms of the discrete series representations of $Sp_2(\mathbb{R})$. Let G be the symplectic group $Sp_2(\mathbb{R})$ and let Γ be a cocompact discrete subgroup of G. We define $L^2(\Gamma\backslash G)$, r and m_π as in the case of $SL_2(\mathbb{R})$. There are 4 conjugacy classes of Cartan subgroups in G. We fix a representative of each conjugacy class; H_0, H_1, H_2 and H_3. The Cartan subgroup H_0 is split and H_3 is contained in the maximal compact subgroup K of G. We denote by $(x,y)_i$ the element of $(\mathfrak{h}_i^{\mathbb{C}})'$ (see Section 1). Let $\pi_\lambda = D(n_1,n_2)$ be the discrete series representation of G with Harish-Chandra parameter $\lambda = (n_1 i, n_2 i)_3$ (see Section 2). If $n_i \in \mathbb{Z}$ and $n_1 > n_2 > 0$ then $D(n_1,n_2)$ is a holomorphic discrete series representation and $D(k-1,k-2)$ corresponds to the Siegel modular form of weight k. For simplicity we assume that Γ has no torsion element.

THEOREM A.

$$m_{D(k-1,k-2)} = \mathrm{Vol}\,(\Gamma\backslash G)\,\frac{(k-1)(k-2)(2k-3)}{8\pi^3} \qquad (k \geqq 4)$$

$$m_{D(2,1)} = m_{\mathrm{Trivial}} + m_{\overline{[-1]\otimes D_3^+(1)}} + \mathrm{Vol}\,(\Gamma\backslash G)\,\frac{3}{4\pi^3} \qquad (k = 3).$$

The representation $\overline{[-1]\otimes D_3^+(1)}$ *is the unique irreducible unitary representation such that* $\mathrm{H}^{2,0}(\mathfrak{g}, K; \pi) \neq 0$.

When $k = 3$, two nontempered unitary representations contribute to the multiplicity. (See Section 4 and Corollary 6.8). For the multiplicity formula of general discrete series representations of G, see Theorem 4.11 and Section 5.

It is proved by Clozel–Delorme [4] that for each discrete series representation π_λ, there exists a pseudo-coefficient $\phi_{\pi_\lambda} = \phi_\lambda$. Let $\mathfrak{h}_i$ be the Lie algebra of H_i. Put

$$\Delta_3 := \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}_3^{\mathbb{C}}),$$

the set of roots of $\mathfrak{g}^{\mathbb{C}}$. For each discrete series representation π_λ, put

$$\Delta_\lambda^+ := \{\alpha \in \Delta_3 | \langle \lambda, \check{\alpha}\rangle > 0\},$$

$$\rho_\lambda := \frac{1}{2}\sum_{\alpha\in\Delta_\lambda^+} \alpha,$$

$$\mathrm{Wall}(\pi_\lambda) := \left\{ \alpha \in \Delta_\lambda^+ \,\middle|\, \begin{array}{ll} \mathrm{i}) & \alpha \text{ is noncompact} \\ \mathrm{ii}) & \langle \lambda - \rho_\lambda, \check{\alpha}\rangle \leqq 0 \end{array} \right\}.$$

Let τ_{π_λ} be the minimal K-type of π_λ and Λ_λ be the highest weight of τ_{π_λ}. Then put

$$T_\lambda := \left\{ \Lambda = \Lambda_\lambda - \sum_{\alpha\in\mathrm{Wall}(\pi_\lambda)} \epsilon_\alpha \alpha \,\middle|\, \begin{array}{ll} \mathrm{i}) & \epsilon_\alpha = 0, 1 \\ \mathrm{ii}) & \Lambda \text{ is dominant for} \\ & \text{the positive compact root.} \end{array} \right\}$$

Let $\lambda = \rho_\lambda + \delta$. We denote by F_δ the finite dimensional irreducible representation of G with highest weight δ and denote by F_δ' its contragredient representation. Let $\mathfrak{p}$ be the -1 eigenspace of the Cartan involution of $\mathfrak{g}^{\mathbb{C}}$. Then it was proved by Vogan–Zuckerman [15] that for any K-type τ there exists at most one irreducible unitary representation π_τ with infinitesimal character χ_λ such that

$$\mathrm{H}^{p,q}(\mathfrak{g}, K; \pi_\tau \otimes F_\delta') \cong \mathrm{Hom}_K(\wedge^{p,q}\mathfrak{p} \otimes F_\delta, \pi_\tau) \neq 0$$

for some integers p, q and

$$\mathrm{Hom}_K(\tau, \wedge^{p,q}\mathfrak{p} \otimes F_\delta) \neq 0, \quad \mathrm{Hom}_K(\tau, \pi_\tau) \neq 0.$$

For the notation $\wedge^{p,q}\mathfrak{p}$ and $\mathrm{H}^{p,q}(\mathfrak{g}, K; \pi \otimes F'_\delta)$, see Section 6. We say that K-type τ *contributes to the* $(\mathfrak{g}, K)$*-cohomology with infinitesimal character* χ_λ if and only if there exists a representation π_τ which satisfies the conditions above. We denote by τ_Λ the irreducible representation of K whose highest weight is Λ. Put

$$T_\lambda^{coh} := \left\{ \tau_\Lambda \in \widehat{K} \;\middle|\; \begin{array}{ll} \text{i)} & \Lambda \in T_\lambda \\ \text{ii)} & \tau_\Lambda \text{ contributes to the } (\mathfrak{g}, K)\text{-cohomology} \\ & \text{with infinitesimal character } \chi_\lambda \end{array} \right\}.$$

For each $\tau \in T_\lambda^{coh}$, let $d_\tau = \prod_{\alpha \in \mathrm{Wall}(\pi_\lambda)} (-1)^{\epsilon_\alpha}$ be the number defined in Theorem 6.17. Put

$$J_\gamma(\phi_\lambda) = \int_{G_\gamma \backslash G} \phi_\lambda(g^{-1}\gamma\, g)\, dg.$$

By comparing Theorem 4.11 with the results of Section 6, we get

THEOREM B.

$$\sum_{\tau \in T_\lambda^{coh}} d_\tau m_{\pi_\tau} = \sum_{\{\gamma\}_\Gamma} \mathrm{Vol}\,(\Gamma_\gamma \backslash G_\gamma) J_\gamma(\phi_\lambda).$$

It is expected that a similar formula holds also for each real semisimple Lie group which has discrete series representations.

Suppose $\mathrm{Wall}(\pi_\lambda) = \emptyset$. Then $T_\lambda^{coh} = \{\tau = \tau_{\pi_\lambda}\}$, $\pi_\tau = \pi_\lambda$ and $d_\tau = 1$.

COROLLARY C. *If* $\mathrm{Wall}(\pi_\lambda) = \emptyset$, *then*

$$m_{\pi_\lambda} = \sum_{\{\gamma\}_\Gamma} \mathrm{Vol}\,(\Gamma_\gamma \backslash G_\gamma) J_\gamma(\phi_\lambda).$$

For each $\pi_\lambda = D(k-1, k-2)$ $(k \geqq 4)$, we have $\mathrm{Wall}(\pi_\lambda) = \emptyset$. If $\pi_\lambda = D(2,1)$, then $\mathrm{Wall}(\pi_\lambda) = \{\ (0,2i)_3, (2i,0)_3, (i,i)_3\ \}$ and $T_\lambda^{coh} = \{\ \tau_{(3i,3i)_3}, \tau_{(3i,i)_3}, \tau_{(0,0)_3}\ \}$. Moreover

$$\begin{array}{lll} \pi_{\tau_{(3i,3i)_3}} = D(2,1), & & d_{\tau_{(3i,3i)_3}} = \ \ 1 \\ \pi_{\tau_{(3i,i)_3}} = \overline{[-1] \otimes D_3^+(1)}, & & d_{\tau_{(3i,i)_3}} = -1 \\ \pi_{\tau_{(0,0)_3}} = \text{Trivial}, & & d_{\tau_{(0,0)_3}} = -1. \end{array}$$

(See Section 2). Hence we get the following result.

COROLLARY D.

$$m_{D(k-1,k-2)} = \sum_{\{\gamma\}_\Gamma} \mathrm{Vol}\,(\Gamma_\gamma \backslash G_\gamma) J_\gamma(\phi_{D(k-1,k-2)}) \quad (k \geqq 4)$$

$$m_{D(2,1)} - m_{\mathrm{Trivial}} - m_{\overline{[-1]\otimes D_3^+(1)}} = \sum_{\{\gamma\}_\Gamma} \mathrm{Vol}\,(\Gamma_\gamma \backslash G_\gamma) J_\gamma(\phi_{D(2,1)}) \qquad (k = 3).$$

Theorem A follows from this corollary. We can calculate the orbital integral $J_\gamma(\phi_\lambda)$. Put

$$h_{\theta_1,\theta_2} := \begin{pmatrix} \cos\theta_1 & & \sin\theta_1 & \\ & \cos\theta_2 & & \sin\theta_2 \\ -\sin\theta_1 & & \cos\theta_1 & \\ & -\sin\theta_2 & & \cos\theta_2 \end{pmatrix}.$$

The trace formula for ϕ_λ is given by the Theorem E below.

THEOREM E. *Let π_λ be a discrete series representation of Harish-Chandra parameter $\lambda = (n_1 i, n_2 i)_3$ $(n_1 - n_2 > 0)$. Then*

$$\begin{aligned}
\sum_{\tau \in T_\lambda^{coh}} d_\tau m_{\pi_\tau} &= (-1)^\epsilon \operatorname{Vol}(\Gamma\backslash G)\, \frac{n_1^3 n_2 - n_1 n_2^3}{8\pi^3} \\
&+ (-1)^\epsilon \operatorname{Vol}(\Gamma\backslash G)\, \frac{(-1)^{n_1+n_2}(n_1 n_2^3 - n_1^3 n_2)}{8\pi^3} \\
&+ (-1)^\epsilon \sum_{\{\gamma\}_\Gamma \sim h_{0,\pi}} \operatorname{Vol}(\Gamma_\gamma\backslash G_\gamma)\, \frac{((-1)^{n_2} - (-1)^{n_1})\, n_1 n_2}{8\pi^2} \\
&+ (-1)^\epsilon \sum_{\{\gamma\}_\Gamma \sim h_{\theta,\theta}} \operatorname{Vol}(\Gamma_\gamma\backslash G_\gamma)\, \frac{(n_2 - n_1)\, e^{-(n_1+n_2)i\theta}}{(e^{-i\theta} - e^{i\theta})^3} \\
&+ (-1)^\epsilon \sum_{\{\gamma\}_\Gamma \sim h_{\theta,-\theta}} \operatorname{Vol}(\Gamma_\gamma\backslash G_\gamma)\, \frac{(n_1 + n_2)(e^{i(n_1-n_2)\theta} - e^{i(-n_1+n_2)\theta})}{2\sqrt{2}\,\pi\,(e^{-i\theta} - e^{i\theta})^3} \\
&+ (-1)^\epsilon \sum_{\{\gamma\}_\Gamma \sim h_{0,\theta}} \operatorname{Vol}(\Gamma_\gamma\backslash G_\gamma)\, \frac{n_2\, e^{-in_1\theta} - n_1\, e^{-in_2\theta}}{\sqrt{2}\,\pi\,(e^{-i\theta/2} - e^{i\theta/2})^2\,(e^{-i\theta} - e^{i\theta})} \\
&+ (-1)^\epsilon \sum_{\{\gamma\}_\Gamma \sim h_{\pi,\theta}} \operatorname{Vol}(\Gamma_\gamma\backslash G_\gamma)\, \frac{n_1\,(-1)^{n_1} e^{-in_2\theta} - n_2\,(-1)^{n_2} e^{-in_1\theta}}{\sqrt{2}\,\pi\,(e^{-i\theta/2} + e^{i\theta/2})^2\,(e^{-i\theta} - e^{i\theta})} \\
&+ (-1)^\epsilon \sum_{\{\gamma\}_\Gamma \sim h_{\theta_1,\theta_2}} \operatorname{Vol}(\Gamma_\gamma\backslash G_\gamma)\, \frac{-e^{-in_1\theta_1 - in_2\theta_2} + e^{-in_2\theta_1 - in_1\theta_2}}{D(h_{\theta_1,\theta_2})},
\end{aligned}$$

where $\epsilon = 0$(resp. 1) if $\pi_\lambda \in D_{\mathrm{I}}$ or D_{III} (resp. if $\pi_\lambda \in D_{\mathrm{II}}$ or D_{IV}), $e^{\theta i} \neq \pm 1$, $(\theta_1 i, \theta_2 i)_3$ is regular and $D(h_{\theta_1,\theta_2})$ is the Weyl denominator. (See Section 1 and Section 2).

The Kazhdan–Lusztig conjecture for χ_ρ was proved by Vogan [13]. The unitary dual of $Sp_2(\mathbb{R})$ was determined by Nzoukoudi [12]. Using these results, we will compute the spectral part of the trace formula for ϕ_λ in Section 4. We will compute $J_\gamma(\phi_\lambda)$ in Section 5 by using the results of Harish-Chandra [7]. In Section 6, we will compute the $(\mathfrak{g}, K)$-cohomology and will get Theorem E.

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§1. Notation

In this paper, the Lie algebra of each Lie group will be denoted by the corresponding German letter. For the Lie algebra $\mathfrak{l}$, we denote its complexification by

$\mathfrak{l}^{\mathbb{C}}$. If V is a finite dimensional (real or complex) vector space, we denote its dual space by V'.

Put

$$G = Sp_2(\mathbb{R}) = \left\{ g \in SL_4(\mathbb{R}) \,\middle|\, {}^t g \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \right\}.$$

We fix a Cartan involution θ of $\mathfrak{g}$ as minus transpose, and we denote by Θ the corresponding automorphism of G.

Put

$$K = \left\{ \begin{pmatrix} \mathrm{Re}(U) & \mathrm{Im}(U) \\ -\mathrm{Im}(U) & \mathrm{Re}(U) \end{pmatrix} \,\middle|\, U \in U(2, \mathbb{C}) \right\}.$$

Then K is the Θ-invariant maximal compact subgroup. We denote by $\mathbf{C}_c^{\infty}(G, K)$ (resp. $\mathbf{C}_c^{n}(G, K)$) the space of smooth (resp. n-times continuously differentiable) and compactly supported functions on G that are left and right K-finite.

It is easy to see that G has 4 conjugacy classes of Cartan subgroups and following H_i $(i = 0, 1, 2, 3)$ are Θ-stable representatives;

$$H_0 = \left\{ \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_1^{-1} & \\ & & & a_2^{-1} \end{pmatrix} \right\},$$

$$H_1 = \left\{ \begin{pmatrix} a\cos(\theta) & a\sin(\theta) & & \\ -a\sin(\theta) & a\cos(\theta) & & \\ & & a^{-1}\cos(\theta) & a^{-1}\sin(\theta) \\ & & -a^{-1}\sin(\theta) & a^{-1}\cos(\theta) \end{pmatrix} \right\},$$

$$H_2 = \left\{ \begin{pmatrix} a & & & \\ & \cos(\theta) & & \sin(\theta) \\ & & a^{-1} & \\ & -\sin(\theta) & & \cos(\theta) \end{pmatrix} \right\},$$

$$H_3 = \left\{ \begin{pmatrix} \cos(\theta_1) & & \sin(\theta_1) & \\ & \cos(\theta_2) & & \sin(\theta_2) \\ -\sin(\theta_1) & & \cos(\theta_1) & \\ & -\sin(\theta_2) & & \cos(\theta_2) \end{pmatrix} \right\}.$$

Then the corresponding Cartan subalgebras are

$$\mathfrak{h}_0 = \left\{ \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & -a_1 & \\ & & & -a_2 \end{pmatrix} \right\},$$

$$\mathfrak{h}_1 = \left\{ \begin{pmatrix} a & \theta & & \\ -\theta & a & & \\ & & -a & \theta \\ & & -\theta & -a \end{pmatrix} \right\},$$

$$\mathfrak{h}_2 = \left\{ \begin{pmatrix} a & & & \\ & 0 & & \theta \\ & & -a & \\ & -\theta & & 0 \end{pmatrix} \right\},$$

$$\mathfrak{h}_3 = \left\{ \begin{pmatrix} & & \theta_1 & 0 \\ & & 0 & \theta_2 \\ -\theta_1 & 0 & & \\ 0 & -\theta_2 & & \end{pmatrix} \right\}.$$

Define $(x,y)_i \in (\mathfrak{h}_i^{\mathbb{C}})'$ by

$$(x,y)_0 : \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & -a_1 & \\ & & & -a_2 \end{pmatrix} \to xa_1 + ya_2,$$

$$(x,y)_1 : \begin{pmatrix} a & \theta & & \\ -\theta & a & & \\ & & -a & \theta \\ & & -\theta & -a \end{pmatrix} \to xa + y\theta,$$

$$(x,y)_2 : \begin{pmatrix} a & & & \\ & 0 & \theta & \\ & -\theta & 0 & \\ & & & -a \end{pmatrix} \to xa + y\theta,$$

$$(x,y)_3 : \begin{pmatrix} & & \theta_1 & 0 \\ & & 0 & \theta_2 \\ -\theta_1 & 0 & & \\ 0 & -\theta_2 & & \end{pmatrix} \to x\theta_1 + y\theta_2.$$

Let $\Delta_i = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}_i^{\mathbb{C}})$ be the set of roots. We fix positive systems Δ_i^+;

$$\begin{aligned} \Delta_0^+ =& \{\ (1,-1)_0,\ (2,0)_0,\ (0,2)_0,\ (1,1)_0\ \}, \\ \Delta_1^+ =& \{\ (0,2i)_1,\ (2,2i)_1,\ (2,-2i)_1,\ (2,0)_1\ \}, \\ \Delta_2^+ =& \{\ (1,-i)_2,\ (2,0)_2,\ (0,2i)_2,\ (1,i)_2\ \}, \\ \Delta_3^+ =& \{\ (i,-i)_3,\ (2i,0)_3,\ (0,2i)_3,\ (i,i)_3\ \}. \end{aligned}$$

Let $\Delta_{i,r}$ be the subset of the real roots of Δ_i. We denote by $\Delta_{i,r}^+$ the set of the positive real roots; $\Delta_{i,r} \cap \Delta_i^+$. Define the Weyl denominator $D_i(h)$ by

$$D_i(h) := e^{\rho_i(\log h)} \prod_{\alpha \in \Delta_{i,r}^+} |1 - e^{-\alpha(\log h)}| \prod_{\alpha \in \Delta_i^+ - \Delta_{i,r}^+} (1 - e^{-\alpha(\log h)}),$$

where $\rho_i = \frac{1}{2}\sum_{\alpha \in \Delta_i^+} \alpha$.

The Borel subalgebra corresponding to $\mathfrak{h}_0$ and Δ_0^+ is

$$\mathfrak{p}_0 := \left\{ \begin{pmatrix} a_1 & b & c_1 & d \\ 0 & a_2 & d & c_2 \\ 0 & 0 & -a_1 & 0 \\ 0 & 0 & -b & -a_2 \end{pmatrix} \right\}.$$

It is easy to see that G has 4 parabolic subalgebras containing $\mathfrak{p}_0$; $\mathfrak{p}_0$ itself, $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3 = \mathfrak{g}$, where

$$\mathfrak{p}_1 = \left\{ \begin{pmatrix} a_1 & b_1 & c_1 & d \\ b_2 & a_2 & d & c_2 \\ 0 & 0 & -a_1 & -b_2 \\ 0 & 0 & -b_1 & -a_2 \end{pmatrix} \right\}, \quad \mathfrak{p}_2 = \left\{ \begin{pmatrix} a_1 & b & c_1 & d \\ 0 & a_2 & d & c_2 \\ 0 & 0 & -a_1 & 0 \\ 0 & c_3 & -b & -a_2 \end{pmatrix} \right\}.$$

We denote the corresponding parabolic subgroups by P_0, P_1, P_2 and P_3, and we denote the corresponding Langlands decompositions by $N_0A_0M_0, \ldots, N_3A_3M_3$. It is easy to see that all of them are cuspidal parabolic subgroups. Let $U(\mathfrak{g}^{\mathbb{C}})$ be

the universal enveloping algebra and let $Z(\mathfrak{g}^{\mathbb{C}})$ be the center of $U(\mathfrak{g}^{\mathbb{C}})$. Using the Harish-Chandra homomorphism $\gamma_{\mathfrak{h}_i}$ from $Z(\mathfrak{g}^{\mathbb{C}})$ to $U(\mathfrak{h}_i^{\mathbb{C}})^W \subset U(\mathfrak{h}_i^{\mathbb{C}})$, we can represent its homomorphisms to $\mathbb{C}$ by the elements of $(\mathfrak{h}_i^{\mathbb{C}})'$. For each $\lambda \in (\mathfrak{h}_i^{\mathbb{C}})'$ let χ_λ be the infinitesimal character represented by λ. Put

$$B(X,Y) := \frac{1}{2}\mathrm{tr}(XY) \quad (X,Y \in \mathfrak{g}).$$

Then $B(X,Y)$ is a bilinear form on $\mathfrak{g}$ and the quadratic form

$$\|X\|^2 := -B(X,\theta(X)) \quad (X \in \mathfrak{g})$$

is positive definite. As $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, $B(X,Y)$ can be regarded as a bilinear form on $\mathfrak{g}^{\mathbb{C}}$.

We normalize Haar measures following [7]. Suppose L is a Lie subgroup of G. Restriction of $\| \quad \|$ on $\mathfrak{l}$ defines a Euclidean norm. Let $d\mathfrak{l}$ be the Euclidean measure on $\mathfrak{l}$ corresponding to $\| \quad \|$. We can choose an open neighborhood $\mathfrak{l}_0$ of 0 in $\mathfrak{l}$ such that the exponential mapping defines a diffeomorphism of $\mathfrak{l}_0$ onto $L_0 = \exp \mathfrak{l}_0$. We define left invariant Haar measure dL on L such that

$$\int_{L_0} f dL = \int_{\mathfrak{l}_0} F d\mathfrak{l} \quad (f \in \mathbf{C}_c^\infty(L_0)).$$

where

$$F(X) = f(\exp X)|\det((1-e^{-\mathrm{ad}X})/\mathrm{ad}X)_{\mathfrak{l}}| \quad (X \in \mathfrak{l}_0).$$

If L is compact, we put

$$v(L) = \int_L dL,$$

and $v(L)^{-1}dL$ is called the normalized Haar measure of L.

Let M be a Θ-stable reductive subgroup of G, and $N_M A_M K_M$ be its Iwasawa decomposition corresponding to $\Theta|_M$. Put $da = dA_M$ and $dn = dN_M$. Let dk be the normalized Haar measure of K_M. We define the Haar measure dx of M by

$$\begin{aligned} dx &= e^{2\rho_M(\log a)} dk da dn \quad (x = kan) \\ &= e^{-2\rho_M(\log a)} dn da dk \quad (x = nak). \end{aligned}$$

We call dx the standard Haar measure of M.

For each $\alpha \in (\mathfrak{h}_i^{\mathbb{C}})'$, define $H_\alpha \in \mathfrak{h}_i^{\mathbb{C}}$ by

$$B(X, H_\alpha) = \alpha(X), \text{ for all } X \in \mathfrak{h}_i^{\mathbb{C}}.$$

§2. The Standard Representations

Since

$$M_0 = \left\{ M = \begin{pmatrix} \epsilon_1 & & & \\ & \epsilon_2 & & \\ & & \epsilon_1 & \\ & & & \epsilon_2 \end{pmatrix} \middle| \, \epsilon_i = \pm 1 \right\},$$

$$A_0 = \left\{ A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_1^{-1} & \\ & & & a_2^{-1} \end{pmatrix} \middle| \, a_i \in \mathbb{R}_+^\times \right\};$$

each quasi-character of M_0, A_0 is of the form

$$\sigma_{+,+}(M) = 1, \quad \sigma_{+,-}(M) = \epsilon_2,$$
$$\sigma_{-,+}(M) = \epsilon_1, \quad \sigma_{-,-}(M) = \epsilon_1\epsilon_2,$$

$$\nu(A) = (\nu_1, \nu_2)(A) = a_1^{\nu_1} a_2^{\nu_2} \quad (\nu_1, \nu_2 \in \mathbb{C}).$$

Put

$$P^{+,+}(\nu_1, \nu_2) := \mathrm{Ind}_{N_0 A_0 M_0}^G (1 \otimes \nu \otimes \sigma_{+,+})$$
$$P^{+,-}(\nu_1, \nu_2) := \mathrm{Ind}_{N_0 A_0 M_0}^G (1 \otimes \nu \otimes \sigma_{+,-})$$
$$P^{-,+}(\nu_1, \nu_2) := \mathrm{Ind}_{N_0 A_0 M_0}^G (1 \otimes \nu \otimes \sigma_{-,+})$$
$$P^{-,-}(\nu_1, \nu_2) := \mathrm{Ind}_{N_0 A_0 M_0}^G (1 \otimes \nu \otimes \sigma_{-,-}).$$

The Langlands quotient of $P^{+,+}(2, 1)$ is the trivial representation.

LEMMA 2.1. *The standard representations corresponding to P_0 are*

$$P^{\pm,\pm}(\nu_1, \nu_2),$$

and $P^{\pm,\pm}(\nu_1, \nu_2)$ has the infinitesimal character $\chi_{(\nu_1,\nu_2)_0}$.

PROOF: See the book of Knapp [9, Proposition 8.22]. □

Since

$$M_1 = \left\{ \begin{pmatrix} M & 0 \\ 0 & {}^tM^{-1} \end{pmatrix} \middle| \, M \in GL_2(\mathbb{R}),\ \det M = \pm 1 \right\},$$

$$A_1 = \left\{ A = \begin{pmatrix} a & & & \\ & a & & \\ & & a^{-1} & \\ & & & a^{-1} \end{pmatrix} \middle| \, a \in \mathbb{R}_+^\times \right\};$$

the discrete series representations of M_1 are

$$D_n \quad (n = 2, 3, 4, \dots),$$

and the limit of discrete series representation of M_1 is D_1. Each quasi-character of A_1 is of the form

$$\nu(A) = a^\nu \quad (\nu \in \mathbb{C}).$$

Put

$$D_n(\nu) := \mathrm{Ind}_{N_1 A_1 M_1}^G (1 \otimes \nu \otimes D_n).$$

LEMMA 2.2. *The standard representations corresponding to P_1 are*

$$D_n(\nu) \quad (n = 2, 3, \dots),$$

and $D_n(\nu)$ has the infinitesimal character

$$\chi_{(\nu,(n-1)i)_1} = \chi_{((\nu+n-1)/2,(\nu-n+1)/2)_0}.$$

PROOF: Since D_1 is not nondegenerate (see [9, Chap.XIV §17]) and $D_1(\nu)$ is infinitesimally equivalent to $P^{+,-}(\nu/2, \nu/2)$, the standard representations corresponding to P_1 are as above. Since we know that the infinitesimal character of D_n is represented by the element

$$\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \to (n-1)\, i\theta,$$

we get the result by [9, Proposition 8.22]. □

Since

$$M_2 = \left\{ M = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & \epsilon & 0 \\ 0 & c & 0 & d \end{pmatrix} \middle| \, ad - bc = 1,\ \epsilon = \pm 1 \right\},$$

$$A_2 = \left\{ A = \begin{pmatrix} a & & & \\ & 1 & & \\ & & a^{-1} & \\ & & & 1 \end{pmatrix} \middle| \, a \in \mathbb{R}^\times_+ \right\};$$

the discrete series representations of M_2 are

$$[\pm 1] \otimes D_n^+, \quad [\pm 1] \otimes D_n^- \quad (n = 2, 3, \dots),$$

where

$$[+1]\,(\epsilon) = 1, \quad [-1]\,(\epsilon) = \epsilon.$$

The limit of discrete series representations of M_2 are

$$[\pm 1] \otimes D_1^+, \quad [\pm 1] \otimes D_1^-.$$

Each quasi-character of A_2 is of the form

$$\nu(A) = a^\nu \quad (\nu \in \mathbb{C}).$$

Put

$$[\pm 1] \otimes D_n^\pm(\nu) := \mathrm{Ind}_{N_2 A_2 M_2}^G (1 \otimes \nu \otimes ([\pm 1] \otimes D_n^\pm)).$$

LEMMA 2.3. *The standard representations corresponding to P_2 are*

$$[\pm 1] \otimes D_n^\pm(\nu) \quad (n = 1, 2, 3, \dots),$$

and $[\pm 1] \otimes D_n^\pm(\nu)$ has the infinitesimal character $\chi_{(\nu,(n-1)i)_2} = \chi_{(\nu,n-1)_0}$.

PROOF: The limit of discrete series representations $[\pm 1] \otimes D_1^\pm$ are nondegenerate. □

Let $\pi_\lambda = D(n_1, n_2)$ be the discrete series representation of Harish–Chandra parameter $\lambda = (n_1 i,\ n_2 i)_3$.

LEMMA 2.4. *The group G has 4 series of discrete series representations;*

$$\begin{aligned}
D_{\mathrm{I}} &:= \{\ D(n_1,n_2) \mid n_1 > n_2 > 0,\ n_i \in \mathbb{Z}\ \},\\
D_{\mathrm{II}} &:= \{\ D(n_1,n_2) \mid n_1 > -n_2 > 0,\ n_i \in \mathbb{Z}\ \},\\
D_{\mathrm{III}} &:= \{\ D(n_1,n_2) \mid -n_2 > n_1 > 0,\ n_i \in \mathbb{Z}\ \},\\
D_{\mathrm{IV}} &:= \{\ D(n_1,n_2) \mid -n_2 > -n_1 > 0,\ n_i \in \mathbb{Z}\ \}.
\end{aligned}$$

The series D_{I} is the holomorphic discrete series and the series D_{IV} is the anti–holomorphic discrete series. The representation $D(n_1,n_2)$ has the infinitesimal character $\chi_{(n_1 i, n_2 i)_3} = \chi_{(n_1,n_2)_0}$.

PROOF: See [9, Theorem 9.20 and Theorem 12.21]. □

Put

$$\begin{aligned}
\Delta^+_{3,\mathrm{I}} &:= \{\ (i,-i)_3,\ (2i,0)_3,\ (0,2i)_3,\ (i,i)_3\ \},\\
\Delta^+_{3,\mathrm{II}} &:= \{\ (i,-i)_3,\ (2i,0)_3,\ (0,-2i)_3,\ (i,i)_3\},\\
\Delta^+_{3,\mathrm{III}} &:= \{\ (i,-i)_3,\ (2i,0)_3,\ (0,-2i)_3,\ (-i,-i)_3\},\\
\Delta^+_{3,\mathrm{IV}} &:= \{\ (i,-i)_3,\ (-2i,0)_3,\ (0,-2i)_3,\ (-i,-i)_3\}.
\end{aligned}$$

Then each $\Delta^+_{3,\mathrm{i}}$ ($\mathrm{i} = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$) is one of the positive systems corresponding to the discrete series D_{i}. Let $D(\lambda, \Delta^+_{3,\mathrm{i}})$ be the limit of discrete series representation with Harish-Chandra parameter λ and positive system $\Delta^+_{3,\mathrm{i}}$.

LEMMA 2.5. *The limit of discrete series representations of G are*

$$\begin{aligned}
&\{\ D(\lambda,\Delta^+_{3,\mathrm{I}}) \mid \lambda = (n_1 i, n_2 i)_3,\ \ n_1 \geqq 1,\ n_2 = 0,\ n_i \in \mathbb{Z}\ \},\\
&\{\ D(\lambda,\Delta^+_{3,\mathrm{II}}) \mid \lambda = (n_1 i, n_2 i)_3,\ \ n_1 \geqq 1,\ n_2 = 0,\ n_i \in \mathbb{Z}\ \},\\
&\{\ D(\lambda,\Delta^+_{3,\mathrm{II}}) \mid \lambda = (n_1 i, n_2 i)_3,\ \ n_1 \geqq 1,\ n_1 + n_2 = 0,\ n_i \in \mathbb{Z}\ \},\\
&\{\ D(\lambda,\Delta^+_{3,\mathrm{III}}) \mid \lambda = (n_1 i, n_2 i)_3,\ \ n_1 \geqq 1,\ n_1 + n_2 = 0,\ n_i \in \mathbb{Z}\ \},\\
&\{\ D(\lambda,\Delta^+_{3,\mathrm{III}}) \mid \lambda = (n_1 i, n_2 i)_3,\ \ n_1 = 0,\ -n_2 \geqq 1,\ n_i \in \mathbb{Z}\ \},\\
&\{\ D(\lambda,\Delta^+_{3,\mathrm{IV}}) \mid \lambda = (n_1 i, n_2 i)_3\ \ n_1 = 0,\ -n_2 \geqq 1,\ n_i \in \mathbb{Z}\ \};
\end{aligned}$$

and each representation has infinitesimal character $\chi_{(n_1 i,\, n_2 i)_3} = \chi_{(n_1,n_2)_0}$.

PROOF: See [9, Theorem 12.26 and Corollary 12.27]. □

Put

$$h_{\theta_1,\theta_2} := \begin{pmatrix} \cos\theta_1 & & \sin\theta_1 & \\ & \cos\theta_2 & & \sin\theta_2 \\ -\sin\theta_1 & & \cos\theta_1 & \\ & -\sin\theta_2 & & \cos\theta_2 \end{pmatrix}$$

and

$$D(h_{\theta_1,\theta_2}) = e^{2i\theta_1 + i\theta_2}(1 - e^{-i\theta_1 + i\theta_2})(1 - e^{-2i\theta_1})(1 - e^{-2i\theta_2})(1 - e^{-i\theta_1 - i\theta_2}),$$

the Weyl denominator corresponding to Δ^+_3.

LEMMA 2.6. *On H_3, the character $\Theta_{D(n_1,n_2)}$ of the discrete series representation $D(n_1,n_2)$ is given by*

$$\Theta_{D(n_1,n_2)}(h_{\theta_1,\theta_2}) = (-1)^{\epsilon}\frac{-e^{n_1 i\theta_1+n_2 i\theta_2}+e^{n_2 i\theta_1+n_1 i\theta_2}}{D(h_{\theta_1,\theta_2})},$$

where $\epsilon = 0$ (resp. 1) if $D(n_1,n_2) \in D_{\mathrm{I}}$ or D_{III} (resp. $D(n_1,n_2) \in D_{\mathrm{II}}$ or D_{IV}).

PROOF: See [9, Theorem 12.7]. □

We recall the realization of holomorphic discrete series on Siegel upper half space of degree 2 (see [9, Chap. VI]);

$$\mathfrak{H} := \{Z \in M_2(\mathbb{C}) | {}^tZ = Z \text{ and } \mathrm{Im}Z \gg 0\}.$$

Let W_N be a $N+1$ dimensional vector space over $\mathbb{C}$. Then W_N can be regarded as the space of polynomials of t whose degree is at most N. For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{C})$, we define

$$j(A,t) := -bt+d,$$
$$A\langle t\rangle := (at-c)(-bt+d)^{-1}.$$

Let $\rho_{M,N}$ be the representation of $GL_2(\mathbb{C})$ on W_N whose action is given by

$$\rho_{M,N}(A)f(t) := \det(A)^M j(A,t)^N f(A\langle t\rangle).$$

Since $W_N \equiv \mathbb{C}^{N+1}$, we can define a norm $\| \ \|$ on W_N by $\|w\|^2 = \sum_{i=1}^{N+1}|w_i|^2$. It is easy to see that Lemma 2.7 below is independent of the choice of the isomorphism $W_N \to \mathbb{C}^{N+1}$.

LEMMA 2.7. *Suppose $n_1 > n_2 > 0$ ($n_i \in \mathbb{Z}$). Let $V(n_1,n_2)$ be the space of functions $\varphi : \mathfrak{H} \to W_{n_1-n_2-1}$ satisfying the following two conditions.*

$$\begin{cases} \text{i) } \varphi(z) \text{ is a holomorphic function.} \\ \text{ii) } \displaystyle\int_G \left\| D(n_1,n_2)[g]\varphi\left[\left(\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right)\right]\right\|^2 dg < +\infty. \end{cases}$$

Then the holomorphic discrete series $D(n_1,n_2)$ is realized on $V(n_1,n_2)$ and the action of G is given by

$$D(n_1,n_2)[g]\varphi(Z) = \rho_{n_2+2,n_1-n_2-1}(-{}^tBZ+{}^tD)^{-1}\varphi[({}^tAZ-{}^tC)(-{}^tBZ+{}^tD)^{-1}],$$

where $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp_2(\mathbb{R})$, $A,B,C,D \in M_2(\mathbb{R})$.

§3. The Trace Formula

We define the action of G on the Siegel upper half space $\mathfrak{H}$ by

$$g\langle Z\rangle := ({}^tAZ - {}^tC)(-{}^tBZ + {}^tD)^{-1} \quad (g \in G,\ Z \in \mathfrak{H}).$$

This definition is a right action of G but we express it as above. Let $\Gamma \subset G$ be a discrete subgroup. We assume that $\Gamma\backslash\mathfrak{H}$ is compact. Then $\Gamma\backslash G$ is also compact as $\mathfrak{H} \cong G/K$. Since G is a semisimple Lie group of rank 2 and Γ is an irreducible lattice, Γ is arithmetic (see the book of Margulis [11]). It is known that semisimple algebraic group H defined over $\mathbb{Q}$ is anisotropic if and only if $H_{\mathbb{Q}}\backslash H_{\mathbb{A}}$ is compact, where $\mathbb{A}$ is the adele ring of $\mathbb{Q}$. Moreover if H is anisotropic, each element of $H_{\mathbb{Q}}$ is semisimple. Hence we get the following lemma.

LEMMA 3.1. *Each element of Γ is semisimple.*

Let r be the representation of G on $L^2(\Gamma\backslash G)$ given by the right translation.

LEMMA 3.2 ([6]). *The representation r splits into a discrete sum of a countable number of irreducible unitary representations, each of finite multiplicity.*

Next lemma is due to Duflo–Labesse [5] and Arthur [1].

LEMMA 3.3 ([1, COROLLARY 4.2]). *For any number n and $\varphi \in \mathbf{C}_c^\infty(G, K)$, there exist functions $\varphi_i \in \mathbf{C}_c^n(G, K)$ $(i = 1, 2, 3, 4)$ such that*

$$\varphi = \varphi_1 * \varphi_2 + \varphi_3 * \varphi_4.$$

COROLLARY 3.4. *The operator $r(\varphi) = \int_G r(g)\varphi(g)dg$ on $L^2(\Gamma\backslash G)$ is of trace class.*

For irreducible admissible representation π, put $\Theta_\pi(\varphi) := \mathrm{Trace}[\pi(\varphi)]$. The trace formula for $r(\varphi)$ is

$$\sum_\pi m_\pi \Theta_\pi(\varphi) = \sum_{\{\gamma\}_\Gamma} \mathrm{Vol}\,(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} \varphi(g^{-1}\gamma\, g)\, dg,$$

where m_π is the multiplicity of π in r, G_γ and Γ_γ are the centralizer of γ in G and in Γ respectively, dg is the quotient measure and the summation on the right-hand side is extended over the conjugacy classes of Γ. In general G_γ is not a Θ-stable subgroup. But by Lemma 3.1 there exists an element $x \in G$ such that $x^{-1}\gamma\, x$ is contained in at least one of the H_i. We know that G_γ is reductive. Hence we can define the standard Haar measure dg' on $x^{-1}G_\gamma\, x$; and pulling back the measure dg' on G_γ, we get the Haar measure dg on G_γ. It is easy to see that dg does not depend on the choice of x. We call it the standard Haar measure on G_γ. From now on we use the standard measures on G_γ and G.

§4. The Spectral Part

Let $\mathcal{D}_\lambda$ be the Grothendieck group of the category of the Harish-Chandra modules with integral nonsingular infinitesimal character χ_λ. We denote by $\mathcal{D}_\lambda^{st}$ the

set of the elements of $\mathcal{D}_\lambda$ which correspond to the standard representations. We know that $\mathcal{D}_\lambda^{st}$ gives a base of $\mathcal{D}_\lambda$. For $\xi \in \mathcal{D}_\lambda^{st}$, let $\overline{\xi}$ be the Langlands quotient of a suitable representative of ξ. Then $\overline{\xi}$ can be expressed as

$$\overline{\xi} = \sum_{\eta \in \mathcal{D}_\lambda^{st}} M(\eta, \xi)\, \eta.$$

Using the results of Vogan [13] and [14], we can calculate $M(\eta, \xi)$ in this case.

If $\chi_\lambda = \chi_{(2,1)_0}$,then $\mathcal{D}_\lambda^{st}$ has 18 elements. We denote these representations by $A, B, \ldots, R$.

$$\begin{array}{ll}
A = D(2,1), & B = D(2,-1), \\
C = D(1,-2), & D = D(-1,-2), \\
E = [-1] \otimes D_3^+(1), & F = D_4(1), \\
G = [-1] \otimes D_3^-(1), & H = [+1] \otimes D_2^+(2), \\
I = D_2(3), & J = [+1] \otimes D_2^-(2), \\
K = P^{-,-}(2,1), & L = P^{+,+}(2,1), \\
M = [+1] \otimes D_3^+(1), & N = [+1] \otimes D_3^-(1), \\
O = [-1] \otimes D_2^+(2), & P = [-1] \otimes D_2^-(2), \\
Q = P^{+,-}(2,1), & R = P^{-,+}(2,1).
\end{array}$$

Put $\mathcal{M}_\lambda := \mathcal{D}_\lambda \otimes_{\mathbb{Z}} \mathbb{Z}[q^{1/2}, q^{-1/2}]$.

LEMMA 4.1 ([13, THEOREM 7.1]). *There exists a unique $\mathbb{Z}$-linear map*

$$\mathbf{D} : \mathcal{M}_\lambda \to \mathcal{M}_\lambda$$

with the properties of [13, Lemma 6.8].

LEMMA 4.2.

$$\begin{aligned}
\mathbf{D}(A) &= A, \qquad \mathbf{D}(B) = B, \qquad \mathbf{D}(C) = C, \qquad \mathbf{D}(D) = D, \\
\mathbf{D}(E) &= q^{-1}[(1-q)(A+B) + E], \\
\mathbf{D}(F) &= q^{-1}[(1-q)(B+C) + F], \\
\mathbf{D}(G) &= q^{-1}[(1-q)(C+D) + G], \\
\mathbf{D}(H) &= q^{-2}[(1-q)(A+C+E+F) + (1-q)^2 B + H], \\
\mathbf{D}(I) &= q^{-2}[(1-q)(A+D+E+F+G) + (1-q)^2(B+C) + I], \\
\mathbf{D}(J) &= q^{-2}[(1-q)(B+D+F+G) + (1-q)^2 C + J], \\
\mathbf{D}(K) &= q^{-3}[-q(1-q)^2(B+C) + q(q-1)(E+F+G) + (1-q)I + K], \\
\mathbf{D}(L) &= q^{-3}[(1-q)^3(B+C) + (1-q)^2(A+D+E+G) \\
&\quad - (1-q)(2q-1)F + (1-q)(H+I+J) + L], \\
\mathbf{D}(M) &= q^{-1}M, \qquad \mathbf{D}(N) = q^{-1}N, \\
\mathbf{D}(O) &= q^{-2}[(1-q)M + O], \quad \mathbf{D}(P) = q^{-2}[(1-q)N + P], \\
\mathbf{D}(Q) &= q^{-3}Q, \\
\mathbf{D}(R) &= q^{-3}[-q(1-q)(M+N) + (1-q)(O+P) + R].
\end{aligned}$$

PROOF: This is a consequence of Kazhdan-Lusztig algorithm in [14]. Like as the case of $PSp_2(\mathbb{R})$ in [14], we can calculate our $Sp_2(\mathbb{R})$ case in the straight forward manner. $\square$

LEMMA 4.3 ([13, THEOREM 7.1]). *For each $\xi \in \mathcal{D}_\lambda^{st}$, there exists a unique element*

$$\mathcal{C}_\xi = \sum_{\eta \in \mathcal{D}_\lambda^{st}} P_{\eta,\xi}(q)\,\eta \quad (\ P_{\eta,\xi} \in \mathbb{Z}[q^{1/2}, q^{-1/2}]\ ),$$

with the properties of [13, Corollary 6.12].

LEMMA 4.4.

$$\begin{aligned}
&\mathcal{C}_A = A, \qquad \mathcal{C}_B = B, \qquad \mathcal{C}_C = C, \qquad \mathcal{C}_D = D,\\
&\mathcal{C}_E = A + B + E, \qquad \mathcal{C}_F = B + C + F, \qquad \mathcal{C}_G = C + D + G,\\
&\mathcal{C}_H = A + B + C + E + F + H,\\
&\mathcal{C}_I = A + B + C + D + E + F + G + I,\\
&\mathcal{C}_J = B + C + D + F + G + J,\\
&\mathcal{C}_K = qA + qD + I + K,\\
&\mathcal{C}_L = A + B + C + D + E + F + G + H + I + J + L,\\
&\mathcal{C}_M = M, \qquad \mathcal{C}_N = N,\\
&\mathcal{C}_O = M + O, \qquad \mathcal{C}_P = N + P,\\
&\mathcal{C}_Q = Q, \qquad \mathcal{C}_R = O + P + R.
\end{aligned}$$

PROOF: We get the result by the Kazhdan-Lusztig algorithm. □

For each $\xi \in \mathcal{D}_\lambda^{st}$, we can define the length of ξ as in [14, Definition 5.3]. Let $l(\xi)$ be the length of $\xi \in \mathcal{D}_\lambda^{st}$.

LEMMA 4.5.

$$\begin{aligned}
&l(A) = 0, \quad l(B) = 0, \quad l(C) = 0, \quad l(D) = 0, \quad l(E) = 1, \quad l(F) = 1,\\
&l(G) = 1, \quad l(H) = 2, \quad l(I) = 2, \quad l(J) = 2, \quad l(K) = 3, \quad l(L) = 3,\\
&l(M) = 1, \quad l(N) = 1, \quad l(O) = 2, \quad l(P) = 2, \quad l(Q) = 3, \quad l(R) = 3.
\end{aligned}$$

The Kazhdan-Lusztig conjecture asserts that

$$M(\eta, \xi) = (-1)^{l(\xi) - l(\eta)} P_{\eta,\xi}(1).$$

PROPOSITION 4.6. *Irreducible representations $\overline{A}, \overline{B}, \ldots, \overline{R}$ can be expressed as*

$$\begin{aligned}
&\overline{A} = A, \qquad \overline{B} = B, \qquad \overline{C} = C, \qquad \overline{D} = D,\\
&\overline{E} = -A - B + E, \qquad \overline{F} = -B - C + F, \qquad \overline{G} = -C - D + G,\\
&\overline{H} = A + B + C - E - F + H,\\
&\overline{I} = A + B + C + D - E - F - G + I,\\
&\overline{J} = B + C + D - F - G + J,\\
&\overline{K} = -A - D - I + K,\\
&\overline{L} = -A - B - C - D + E + F + G - H - I - J + L,\\
&\overline{M} = M, \qquad \overline{N} = N,\\
&\overline{O} = -M + O, \qquad \overline{P} = -N + P,\\
&\overline{Q} = Q, \qquad \overline{R} = -O - P + R.
\end{aligned}$$

PROOF: The Kazhdan-Lusztig conjecture was proved for the representations whose infinitesimal character is same as some finite dimensional representation of G [13]. Especially it is proved for $\mathcal{D}_{(2,1)_0}$, for it has the trivial representation. Hence by Lemma 4.4 and Lemma 4.5, we get the equations above. $\square$

Let χ_λ, $\chi_{\lambda+\delta}$ be nonsingular infinitesimal characters such that λ is dominant and δ is dominant integral. Then the Zuckerman translation functor induces the isomorphism between $\mathcal{D}_\lambda$ and $\mathcal{D}_{\lambda+\delta}$, since Sp_2 is simply connected. It maps the standard representations to the standard representations and the Langlands quotients to the Langlands quotients. Especially if λ is integral, the Zuckerman translation functor ${}^G\psi^\lambda_{(2,1)_0}$ maps $\mathcal{D}_\lambda$ to $\mathcal{D}_{(2,1)_0}$. (See [9, Chap. X §9 and Chap. XIV §12] and the paper of Zuckerman [16]).

LEMMA 4.7. *Let $\lambda_1 > \lambda_2 > 0$, $\lambda_i \in \mathbb{Z}$. If $\lambda_1 \equiv 0$, $\lambda_2 \equiv 0$ then the Zuckerman translation functor ${}^G\psi^\lambda_{(2,1)_0}$ is given by*

$$
\begin{array}{llll}
D(\lambda_1,\lambda_2) & \to A & D(\lambda_1,-\lambda_2) & \to B \\
D(\lambda_2,-\lambda_1) & \to C & D(-\lambda_2,-\lambda_1) & \to D \\
[+1]\otimes D^+_{\lambda_1+1}(\lambda_2) & \to E & D_{\lambda_1+\lambda_2+1}(\lambda_1-\lambda_2) & \to F \\
[+1]\otimes D^-_{\lambda_1+1}(\lambda_2) & \to G & [+1]\otimes D^+_{\lambda_2+1}(\lambda_1) & \to H \\
D_{\lambda_1-\lambda_2+1}(\lambda_1+\lambda_2) & \to I & [+1]\otimes D^-_{\lambda_2+1}(\lambda_1) & \to J \\
P^{-,+}(\lambda_1,\lambda_2) & \to K & P^{+,-}(\lambda_1,\lambda_2) & \to L \\
[-1]\otimes D^+_{\lambda_1+1}(\lambda_2) & \to M & [-1]\otimes D^-_{\lambda_1+1}(\lambda_2) & \to N \\
[-1]\otimes D^+_{\lambda_2+1}(\lambda_1) & \to O & [-1]\otimes D^-_{\lambda_2+1}(\lambda_1) & \to P \\
P^{+,+}(\lambda_1,\lambda_2) & \to Q & P^{-,-}(\lambda_1,\lambda_2) & \to R.
\end{array}
$$

If $\lambda_1 \equiv 1$, $\lambda_2 \equiv 0$ then the Zuckerman translation functor ${}^G\psi^\lambda_{(2,1)_0}$ is given by

$$
\begin{array}{llll}
D(\lambda_1,\lambda_2) & \to A & D(\lambda_1,-\lambda_2) & \to B \\
D(\lambda_2,-\lambda_1) & \to C & D(-\lambda_2,-\lambda_1) & \to D \\
[+1]\otimes D^+_{\lambda_1+1}(\lambda_2) & \to E & D_{\lambda_1+\lambda_2+1}(\lambda_1-\lambda_2) & \to F \\
[+1]\otimes D^-_{\lambda_1+1}(\lambda_2) & \to G & [-1]\otimes D^+_{\lambda_2+1}(\lambda_1) & \to H \\
D_{\lambda_1-\lambda_2+1}(\lambda_1+\lambda_2) & \to I & [-1]\otimes D^-_{\lambda_2+1}(\lambda_1) & \to J \\
P^{+,+}(\lambda_1,\lambda_2) & \to K & P^{-,-}(\lambda_1,\lambda_2) & \to L \\
[-1]\otimes D^+_{\lambda_1+1}(\lambda_2) & \to M & [-1]\otimes D^-_{\lambda_1+1}(\lambda_2) & \to N \\
[+1]\otimes D^+_{\lambda_2+1}(\lambda_1) & \to O & [+1]\otimes D^-_{\lambda_2+1}(\lambda_1) & \to P \\
P^{-,+}(\lambda_1,\lambda_2) & \to Q & P^{+,-}(\lambda_1,\lambda_2) & \to R.
\end{array}
$$

If $\lambda_1 \equiv 0$, $\lambda_2 \equiv 1$ then the Zuckerman translation functor ${}^G\psi^{\lambda}_{(2,1)_0}$ is given by

$$\begin{array}{llll}
D(\lambda_1,\lambda_2) & \to A & D(\lambda_1,-\lambda_2) & \to B \\
D(\lambda_2,-\lambda_1) & \to C & D(-\lambda_2,-\lambda_1) & \to D \\
[-1]\otimes D^+_{\lambda_1+1}(\lambda_2) & \to E & D_{\lambda_1+\lambda_2+1}(\lambda_1-\lambda_2) & \to F \\
[-1]\otimes D^-_{\lambda_1+1}(\lambda_2) & \to G & [+1]\otimes D^+_{\lambda_2+1}(\lambda_1) & \to H \\
D_{\lambda_1-\lambda_2+1}(\lambda_1+\lambda_2) & \to I & [+1]\otimes D^-_{\lambda_2+1}(\lambda_1) & \to J \\
P^{-,-}(\lambda_1,\lambda_2) & \to K & P^{+,+}(\lambda_1,\lambda_2) & \to L \\
[+1]\otimes D^+_{\lambda_1+1}(\lambda_2) & \to M & [+1]\otimes D^-_{\lambda_1+1}(\lambda_2) & \to N \\
[-1]\otimes D^+_{\lambda_2+1}(\lambda_1) & \to O & [-1]\otimes D^-_{\lambda_2+1}(\lambda_1) & \to P \\
P^{+,-}(\lambda_1,\lambda_2) & \to Q & P^{-,+}(\lambda_1,\lambda_2) & \to R.
\end{array}$$

If $\lambda_1 \equiv 1$, $\lambda_2 \equiv 1$ then the Zuckerman translation functor ${}^G\psi^{\lambda}_{(2,1)_0}$ is given by

$$\begin{array}{llll}
D(\lambda_1,\lambda_2) & \to A & D(\lambda_1,-\lambda_2) & \to B \\
D(\lambda_2,-\lambda_1) & \to C & D(-\lambda_2,-\lambda_1) & \to D \\
[-1]\otimes D^+_{\lambda_1+1}(\lambda_2) & \to E & D_{\lambda_1+\lambda_2+1}(\lambda_1-\lambda_2) & \to F \\
[-1]\otimes D^-_{\lambda_1+1}(\lambda_2) & \to G & [-1]\otimes D^+_{\lambda_2+1}(\lambda_1) & \to H \\
D_{\lambda_1-\lambda_2+1}(\lambda_1+\lambda_2) & \to I & [-1]\otimes D^-_{\lambda_2+1}(\lambda_1) & \to J \\
P^{+,-}(\lambda_1,\lambda_2) & \to K & P^{-,+}(\lambda_1,\lambda_2) & \to L \\
[+1]\otimes D^+_{\lambda_1+1}(\lambda_2) & \to M & [+1]\otimes D^-_{\lambda_1+1}(\lambda_2) & \to N \\
[+1]\otimes D^+_{\lambda_2+1}(\lambda_1) & \to O & [+1]\otimes D^-_{\lambda_2+1}(\lambda_1) & \to P \\
P^{-,-}(\lambda_1,\lambda_2) & \to Q & P^{+,+}(\lambda_1,\lambda_2) & \to R.
\end{array}$$

PROOF: Suppose that $\lambda, \delta \in (\mathfrak{h}_i^{\mathbb{C}})'$ are integral nonsingular for $\Delta(\mathfrak{a}_i^{\mathbb{C}} \oplus \mathfrak{m}_i^{\mathbb{C}}, \mathfrak{h}_i^{\mathbb{C}})$ and dominant for some positive root system of $\Delta(\mathfrak{a}_i^{\mathbb{C}} \oplus \mathfrak{m}_i^{\mathbb{C}}, \mathfrak{h}_i^{\mathbb{C}})$. Let ${}^{A_iM_i}\psi^{\lambda+\delta}_{\lambda}$ be the Zuckerman translation functor of A_iM_i. Then by [9, Theorem 14.67] we get the relation

$${}^G\psi^{\lambda+\delta}_{\lambda}(\mathrm{Ind}^G_{P_i})(1\otimes V) \cong \mathrm{Ind}^G_{P_i}(1\otimes {}^{A_iM_i}\psi^{\lambda+\delta}_{\lambda}(V))$$

for any Harish-Chandra module V of A_iM_i with infinitesimal character $\chi_{\lambda+\delta}$. It is easy to see that

$$\begin{aligned}
{}^{A_0M_0}\psi^{(\lambda_1+\delta_1,\lambda_2+\delta_2)_0}_{(\lambda_1,\lambda_2)_0}&[(\lambda_1+\delta_1,\lambda_2+\delta_2)\otimes\sigma_{\epsilon_1,\epsilon_2}] \\
&= (\lambda_1,\lambda_2)\otimes\sigma_{\epsilon_1',\epsilon_2'} \quad (\epsilon_i=\pm,\ \epsilon_i'=\epsilon_i(-)^{\delta_i}), \\
{}^{A_1M_1}\psi^{(\lambda_1+\delta_1,(\lambda_2+\delta_2)i)_1}_{(\lambda_1,\lambda_2 i)_1}&[(\lambda_1+\delta_1)\otimes D_{\lambda_2+\delta_2+1}] \\
&= (\lambda_1)\otimes D_{\lambda_2+1},
\end{aligned}$$

$$
\begin{aligned}
&{}_{A_2M_2}\psi^{(\lambda_1+\delta_1,(\lambda_2+\delta_2)i)_2}_{(\lambda_1,\lambda_2 i)_2}[\,(\lambda_1+\delta_1)\otimes([\epsilon_1]\otimes D^{\epsilon_2}_{\lambda_2+\delta_2+1})\,]\\
&\qquad = (\lambda_1)\otimes([\epsilon_1']\otimes D^{\epsilon_2}_{\lambda_2+1}) \quad (\epsilon_1=\pm 1,\ \epsilon_2=\pm,\ \epsilon_1'=\epsilon_1(-1)^{\delta_1}),\\
&{}_{G}\psi^{((\lambda_1+\delta_1)i,(\lambda_2+\delta_2)i\,)_3}_{(\lambda_1 i,\lambda_2 i\,)_3}[\,D(\lambda_1+\delta_1,\lambda_2+\delta_2)\,]\\
&\qquad = D(\lambda_1,\lambda_2)\,.
\end{aligned}
$$

Hence we get the result. □

B. Nzoukoudi determined the unitary dual of $Sp_2(\mathbb{R})$ in [12]. We summarize the result.

PROPOSITION 4.8. *If π is infinitesimally equivalent to a unitary representation then π is infinitesimally equivalent to one of the following representations.*

1) *The (limit of) discrete series representations*
2) $\overline{D_{2m+1}(\nu)}$ $(m\geqq 1)$ $\nu\in i\mathbb{R}-\{0\}$
   $\overline{D_{2m}(\nu)}$ $(m\geqq 1)$ (i) $\nu\in i\mathbb{R}$ *or*
   (ii) $\nu\in\mathbb{R},\ 0<\nu\leqq 1$
3) $\overline{[+1]\otimes D^{\pm}_n(\nu)}$ $(n\geqq 2)$ $\nu\in i\mathbb{R}-\{0\}$
   $\overline{[-1]\otimes D^{\pm}_n(\nu)}$ $(n\geqq 2)$ (i) $\nu\in i\mathbb{R}$ *or*
   (ii) $\nu\in\mathbb{R},\ 0<\nu\leqq 1$
   $\overline{[\pm 1]\otimes D^{\pm}_1(\nu)}$ (i) $\nu\in i\mathbb{R}$ *or*
   (ii) $\nu\in\mathbb{R},\ 0<\nu\leqq 1$
4) $\overline{P^{+,+}(\nu_1,\nu_2)}$ (i) $\nu\in i(\mathbb{R}\oplus\mathbb{R})$ *or*
   (ii) $\nu_1=\overline{\nu_2},\ 0<\mathrm{Re}(\nu_1)\leqq 1/2$ *or*
   (iii) $\nu_1\in\mathbb{R},\ 0<\nu_1\leqq 1,\ \nu_2\in i\mathbb{R}$ *or*
   (iv) $\nu_1,\nu_2\in\mathbb{R},\ \nu_1\geqq\nu_2\geqq 0,\ \nu_1+\nu_2\leqq 1$ *or*
   (v) $\nu=(2,1)$ (*the trivial representation*)
   $\overline{P^{+,-}(\nu_1,\nu_2)}$ (i) $\nu\in i(\mathbb{R}\oplus\mathbb{R})-\{0\}$ *or*
   (ii) $\nu_1\in\mathbb{R},\ 0<\nu_1\leqq 1,\nu_2\in i\mathbb{R}$
   $\overline{P^{-,+}(\nu_1,\nu_2)}$ $\nu\in i(\mathbb{R}\oplus\mathbb{R})-\{0\}$
   $\overline{P^{-,-}(\nu_1,\nu_2)}$ (i) $\nu\in i(\mathbb{R}\oplus\mathbb{R})-\{0\}$ *or*
   (ii) $\nu_1=\overline{\nu_2},\ 0<\mathrm{Re}(\nu_1)\leqq 1/2$

The overline means the Langlands quotients of the standard representations.

We use the same symbol for the unitary representations and their infinitesimally equivalent representations in the list of Proposition 4.8.

LEMMA 4.9. *Let χ_λ be an integral nonsingular infinitesimal character. Then $\mathcal{D}_\lambda$ has irreducible nontempered representations that can be unitarizable, if and only if*

$$\chi_\lambda=\chi_{(n+2,n+1)_0}\quad (n\in\mathbb{Z},n\geqq 0)$$

or

$$\chi_\lambda = \chi_{(n+2,1)_0} \quad (n \in \mathbb{Z}, n \geqq 0),$$

and the nontempered unitary representations with infinitesimal character χ_λ are given by the list below.

Case $\lambda = (2,1)_0$

$$\begin{aligned}
\overline{[-1] \otimes D_3^+(1)} &= [-1] \otimes D_3^+(1) - D(2,1) - D(2,-1) \\
\overline{[-1] \otimes D_3^-(1)} &= [-1] \otimes D_3^-(1) - D(1,-2) - D(-1,-2) \\
\overline{P^{+,+}(2,1)} &= P^{+,+}(2,1) - D_2(3) + D_4(1) \\
&\quad - [+1] \otimes D_2^+(2) - [+1] \otimes D_2^-(2) \\
&\quad + [-1] \otimes D_3^+(1) + [-1] \otimes D_3^-(1) \\
&\quad - D(2,1) - D(2,-1) - D(1,-2) - D(-1,-2) \\
\overline{D_4(1)} &= D_4(1) - D(2,-1) - D(1,-2)
\end{aligned}$$

Case $\lambda = (n+2, n+1)_0$ $(n \in \mathbb{Z}, n \geqq 1)$

$$\overline{D_{2n+4}(1)} = D_{2n+4}(1) - D(n+2,-n-1) - D(n+1,-n-2)$$

Case $\lambda = (n+2,1)_0$ $(n \in \mathbb{Z}, n \geqq 1)$

$$\begin{aligned}
\overline{[-1] \otimes D_{n+3}^+(1)} &= [-1] \otimes D_{n+3}^+(1) - D(n+2,1) - D(n+2,-1) \\
\overline{[-1] \otimes D_{n+3}^-(1)} &= [-1] \otimes D_{n+3}^-(1) - D(1,-n-2) - D(-1,-n-2)
\end{aligned}$$

Proof: By Proposition 4.8, we get the first statement. Their equations are consequences of Proposition 4.6 and Lemma 4.7. □

Let π_λ be a discrete series representation of G. We call that $\phi_\lambda \in \mathbf{C}_c^\infty(G,K)$ is a pseudo-coefficient of π_λ, if ϕ_λ satisfies

$$\Theta_\pi(\phi_\lambda) = \begin{cases} 1 & \text{if } \pi = \pi_\lambda, \\ 0 & \text{if } \pi \neq \pi_\lambda, \pi \in \mathcal{D}^{st}. \end{cases}$$

Lemma 4.10 ([4]). *Any discrete series representation π_λ of G has a pseudo-coefficient ϕ_λ.*

Theorem 4.11. *Let π_λ be a discrete series representation of G. Then the spectral*

part of the trace formula for ϕ_λ is given by the following formula. ($n \in \mathbb{Z}\ n \geqq 1$)

$$\sum_\pi m_\pi\, \Theta_\pi(\phi_\lambda) = \begin{cases} m_{D(2,1)} - m_{\overline{[-1]\otimes D_3^+(1)}} - m_{\text{Trivial}} & \text{if } \pi_\lambda = D(2,1) \\ m_{D(2,-1)} - m_{\overline{[-1]\otimes D_3^+(1)}} - m_{\overline{D_4(1)}} - m_{\text{Trivial}} & \text{if } \pi_\lambda = D(2,-1) \\ m_{D(1,-2)} - m_{\overline{[-1]\otimes D_3^-(1)}} - m_{\overline{D_4(1)}} - m_{\text{Trivial}} & \text{if } \pi_\lambda = D(1,-2) \\ m_{D(-1,-2)} - m_{\overline{[-1]\otimes D_3^-(1)}} - m_{\text{Trivial}} & \text{if } \pi_\lambda = D(-1,-2) \\ m_{D(n+2,-n-1)} - m_{\overline{D_{2n+4}(1)}} & \text{if } \pi_\lambda = D(n+2,-n-1) \\ m_{D(n+1,-n-2)} - m_{\overline{D_{2n+4}(1)}} & \text{if } \pi_\lambda = D(n+1,-n-2) \\ m_{D(n+2,1)} - m_{\overline{[-1]\otimes D_{n+3}^+(1)}} & \text{if } \pi_\lambda = D(n+2,1) \\ m_{D(n+2,-1)} - m_{\overline{[-1]\otimes D_{n+3}^+(1)}} & \text{if } \pi_\lambda = D(n+2,-1) \\ m_{D(1,-n-2)} - m_{\overline{[-1]\otimes D_{n+3}^-(1)}} & \text{if } \pi_\lambda = D(1,-n-2) \\ m_{D(-1,-n-2)} - m_{\overline{[-1]\otimes D_{n+3}^-(1)}} & \text{if } \pi_\lambda = D(-1,-n-2) \\ m_{\pi_\lambda} & \text{for other } \pi_\lambda. \end{cases}$$

PROOF: Suppose $\chi_\lambda \neq \chi_{\lambda'}$. Since $\Theta_\pi(\phi_\lambda) = 0$ for all $\pi \in \mathcal{D}_{\lambda'}^{st}$ and $\mathcal{D}_{\lambda'}^{st}$ gives the base of $\mathcal{D}_{\lambda'}$, we get $\Theta_\pi(\phi_\lambda) = 0$ for all $\pi \in \mathcal{D}_{\lambda'}$. If π is not unitary, then $m_\pi = 0$. Hence $m_\pi\, \Theta_\pi(\phi_\lambda) \neq 0$ only if π is a unitary representation with infinitesimal character χ_λ. We know that the tempered representations with infinitesimal character χ_λ are discrete series representations. Hence we get the result by Lemma 4.9. □

§5. The Geometric Part

Let γ be a regular semisimple element of H_i ($i = 0, 1, 2, 3$) and let $D_i(\gamma)$ be the Weyl denominator corresponding to Δ_i^+. For $\varphi \in \mathbf{C}_c^\infty(G, K)$, put

$$F_\varphi(\gamma) := D_i(\gamma) \int_{H_i \backslash G} \varphi(g^{-1}\gamma\, g)\, dg.$$

Then $F_\varphi(\gamma)$ is a smooth function on the regular subset of H_i.

LEMMA 5.1. *F_{ϕ_λ} can be extended to a smooth function on each H_i. Moreover we have*

$$F_{\phi_\lambda} \equiv 0$$

on H_0, H_1 and H_2.

PROOF: Since H_0 is a split torus, $F_{\phi_\lambda}(\gamma)$ can be extended to a smooth function on H_0. For any $\pi = P^{\pm,\pm}(\nu_1, \nu_2)$, we get $\Theta_\pi(\phi_\lambda) = 0$. This means that the Fourier transform of F_{ϕ_λ} is 0 as a function on H_0. Hence $F_{\phi_\lambda} \equiv 0$ on H_0. The Cartan

subalgebra $\mathfrak{h}_1$ has one positive noncompact imaginary root $(0,2i)_1$, and we have $\mathbf{c}_{(0,2i)_1}(\mathfrak{h}_1) = \mathfrak{h}_0$, where $\mathbf{c}_{(0,2i)_1}$ is a Cayley transform. Since $F_{\phi_\lambda} \equiv 0$ on H_0, F_{ϕ_λ} can be extended to a smooth function on H_1 (see [7, Theorem 9.1]). For any $\pi = D_n(\nu)$, we get $\Theta_\pi(\phi_\lambda) = 0$. This means that the Fourier transform of F_{ϕ_λ} is 0 as a function on H_1. Hence $F_{\phi_\lambda} \equiv 0$ on H_1. The Cartan subalgebra $\mathfrak{h}_2$ has one positive noncompact imaginary root $(0,2i)_2$, and we have $\mathbf{c}_{(0,2i)_2}(\mathfrak{h}_2) = \mathfrak{h}_0$. Hence we can prove that F_{ϕ_λ} is 0 on H_2 as in the case of H_1. The Cartan subalgebra $\mathfrak{h}_3$ has 3 positive noncompact imaginary root; $(2i,0)_3$, $(0,2i)_3$ and $(i,i)_3$. Their Cayley transforms $\mathbf{c}_\alpha(\mathfrak{h}_3)$ are conjugate to either $\mathfrak{h}_1$ or $\mathfrak{h}_2$. Hence F_{ϕ_λ} can be extended to a smooth function on H_3. $\square$

A semisimple element γ (not supposed to be regular) is called elliptic if one of its conjugation is contained in H_3.

LEMMA 5.2 (SELBERG PRINCIPLE). *For any semisimple element γ which is not elliptic, we get*

$$\int_{G_\gamma \backslash G} \phi_\lambda(g^{-1}\gamma\, g)\, dg = 0.$$

PROOF: If γ is regular, it is the statement of Lemma 5.1. For singular γ, we use [8, Lemma 23]. Since γ is semisimple, $\mathfrak{g}_\gamma$ is reductive. Let $\mathfrak{b}$ be a fundamental Cartan subalgebra of $\mathfrak{g}_\gamma$, which has no real roots. We fix a positive root system $\Delta^+_{\mathfrak{g}_\gamma}$ of $\Delta(\mathfrak{g}_\gamma^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$. Put

$$\varpi_\gamma := \prod_{\alpha \in \Delta^+_{\mathfrak{g}_\gamma}} H_\alpha.$$

With these notations [8, Lemma 23] states that there exists a non-zero constant c such that

$$\varpi_\gamma \cdot F_{\phi_\lambda}(\gamma) = c \int_{G_\gamma \backslash G} \phi_\lambda(g^{-1}\gamma\, g)\, dg.$$

Since γ cannot be contained in any conjugate of H_3, B is not conjugate to H_3. By Lemma 5.1, we get $F_{\phi_\lambda} \equiv 0$ on B. $\square$

Define

$$h_{\theta_1,\theta_2} = \begin{pmatrix} \cos\theta_1 & & \sin\theta_1 & \\ & \cos\theta_2 & & \sin\theta_2 \\ -\sin\theta_1 & & \cos\theta_1 & \\ & -\sin\theta_2 & & \cos\theta_2 \end{pmatrix},$$

and

$$D(h_{\theta_1,\theta_2}) = e^{2i\theta_1 + i\theta_2}(1 - e^{-i\theta_1 + i\theta_2})(1 - e^{-2i\theta_1})(1 - e^{-2i\theta_2})(1 - e^{-i\theta_1 - i\theta_2}),$$

as in Section 2.

PROPOSITION 5.3. *Let $\gamma = h_{\theta_1,\theta_2} \in H_3$ be a regular element and let π_λ be the discrete series representation with Harish–Chandra parameter $\lambda = (n_1 i, n_2 i)_3$. Then*

$$\int_{H_3 \backslash G} \phi_\lambda(g^{-1}\gamma\, g)\, dg = (-1)^\epsilon \frac{-e^{-in_1\theta_1 - in_2\theta_2} + e^{-in_2\theta_1 - in_1\theta_2}}{D(h_{\theta_1,\theta_2})},$$

where $\epsilon = 0$ *(resp.* 1*) if* $D(n_1, n_2) \in D_{\mathrm{I}}$ *or* D_{III} *(resp.* $D(n_1, n_2) \in D_{\mathrm{II}}$ *or* D_{IV}*).*

PROOF: By Lemma 5.1, we know that F_{ϕ_λ} can be extended to a smooth function on H_3. Hence F_{ϕ_λ} has the Fourier series expansion

$$\frac{1}{2} F_{\phi_\lambda}(h_{\theta_1,\theta_2}) = \sum_{n_1,n_2} c_{n_1,n_2} e^{in_1\theta_1 + in_2\theta_2}.$$

Since $h_{\theta_1,\theta_2} \simeq h_{\theta_2,\theta_1}$ and $D(h_{\theta_1,\theta_2}) = -D(h_{\theta_2,\theta_1})$, we get $F_{\phi_\lambda}(h_{\theta_1,\theta_2}) = -F_{\phi_\lambda}(h_{\theta_2,\theta_1})$. This means that $c_{n_1,n_2} = -c_{n_2,n_1}$. We get

$$\Theta_{D(n_1,n_2)}(\phi_\lambda) = \frac{1}{2} \int_{H_3} F_{\phi_\lambda}(h)\, \Theta_{D(n_1,n_2)}(h)\, D(h)\, dh.$$

By the character formula of discrete series representation, we have

$$\Theta_{D(n_1,n_2)}(\phi_\lambda) = \begin{cases} -c_{-n_1,-n_2} + c_{-n_2,-n_1} = -2c_{-n_1,-n_2} & \text{if } D(n_1,n_2) \in D_{\mathrm{I}} \text{ or } D_{\mathrm{III}} \\ c_{-n_1,-n_2} - c_{-n_2,-n_1} = 2c_{-n_1,-n_2} & \text{if } D(n_1,n_2) \in D_{\mathrm{II}} \text{ or } D_{\mathrm{VI}}. \end{cases}$$

If π is the limit of discrete series representations, then $\Theta_\pi(\phi_\lambda) = 0$. Since ϕ_λ is a pseudo-coefficient of π_λ, we get the proposition. □

LEMMA 5.4. *Suppose* γ *is a singular elliptic element, then* γ *is conjugate to one of the following elements.*

$$\begin{array}{cc} h_{0,\tau} \simeq h_{\tau,0} & (e^{i\tau} \neq \pm 1) \\ h_{\pi,\tau} \simeq h_{\tau,\pi} & (e^{i\tau} \neq \pm 1) \\ h_{\tau,\tau} & (e^{i\tau} \neq \pm 1) \\ h_{\tau,-\tau} \simeq h_{-\tau,\tau} & (e^{i\tau} \neq \pm 1) \\ h_{0,\pi} \simeq h_{\pi,0} & \\ h_{0,0} & \\ h_{\pi,\pi} & \end{array}$$

PROOF: Since γ is an elliptic element, γ is conjugate to some $h_{\theta_1,\theta_2} \in H_3$. The positive roots of H_3 are $e^{i\theta_1 - i\theta_2}$, $e^{2i\theta_1}$, $e^{2i\theta_2}$ and $e^{i\theta_1 + i\theta_2}$. Hence the singular elements are as above. It is easy to see that if h_{θ_1,θ_2} and $h_{\theta_1',\theta_2'}$ are conjugate, then $\theta_1 = \theta_1',\ \theta_2 = \theta_2'$ or $\theta_1 = \theta_2',\ \theta_2 = \theta_1'$. □

Let γ be an element of H_3. Then $\mathfrak{h}_3$ is a fundamental Cartan subalgebra of $\mathfrak{g}_\gamma$.

Put

$$\Delta_\gamma := \Delta(\mathfrak{g}_\gamma^{\mathbb{C}}, \mathfrak{h}_3^{\mathbb{C}}),$$
$$\Delta_\gamma^+ := \Delta_\gamma \cap \Delta_3^+,$$
$$\rho_\gamma := \frac{1}{2} \sum_{\alpha \in \Delta_\gamma^+} \alpha,$$
$$'D_\gamma(h) := \prod_{\alpha \in \Delta_\gamma^+} (1 - \alpha(h^{-1})) \quad (h \in H_3),$$
$$\varpi'_\gamma := \prod_{\alpha \in \Delta_\gamma^+} (H_\alpha + \rho_\gamma(H_\alpha)).$$

Let $\Delta_{\gamma,c}^+$ be the set of the positive compact roots. Put $\rho_{\gamma,c} := \frac{1}{2} \sum_{\alpha \in \Delta_{\gamma,c}^+} \alpha$ and $\varpi_{\gamma,c} := \prod_{\alpha \in \Delta_{\gamma,c}^+} H_\alpha$. We define N_γ, A_γ, K_γ and $\mathbf{C}_c^\infty(G_\gamma, K_\gamma)$ as in the case of G. Let G_γ^0 be the connected component of G_γ. For $\varphi \in \mathbf{C}_c^\infty(G_\gamma, K_\gamma)$ and regular element $h \in H_3$, we define

$$'F_\varphi^{G_\gamma}(h) = 'F_\varphi^\gamma(h) := 'D_\gamma(h) \int_{H_3 \backslash G_\gamma} \varphi(g^{-1} h\, g)\, dg.$$

LEMMA 5.5. *There exists a constant c_γ such that for any $\varphi \in \mathbf{C}_c^\infty(G_\gamma, K_\gamma)$,*

$$c_\gamma \varphi(1) = \varpi'_\gamma \cdot 'F_\varphi^\gamma(1).$$

Moreover

$$c_\gamma = (-1)^q\, (2\pi)^q\, 2^{\nu/2}\, \varpi_{\gamma,c}(\rho_{\gamma,c})\, [W(G_\gamma^0, H_3)],$$

where $[W(G_\gamma^0, H_3)] = [\mathrm{Norm}_{G_\gamma^0}(H_3)/H_3]$ is the order of the Weyl group,

$$q = \frac{1}{2} \mathrm{dim} G_\gamma / K_\gamma,$$

and

$$\nu = \dim(\mathfrak{n}_\gamma).$$

PROOF: This is a special case of [7, Theorem 37.1 and Lemma 17.5]. □

COROLLARY 5.6. *Let γ be one of the elements of Lemma 5.4. Then*

$$c_\gamma \int_{G_\gamma \backslash G} \varphi(x^{-1} \gamma\, x)\, dx = \varpi'_\gamma \cdot 'D_\gamma(h) \int_{H_3 \backslash G} \varphi(g^{-1} \gamma\, h\, g)\, dg \Big|_{h=1}.$$

PROOF: We have

$$\int_{H_3 \backslash G} \varphi(g^{-1} \gamma\, h\, g)\, dg = \int_{G_\gamma \backslash G} dg_1 \int_{H_3 \backslash G_\gamma} \varphi(g_1^{-1} g_2^{-1} \gamma\, h\, g_2\, g_1)\, dg_2.$$

By the argument of the proof of [8, Lemma 23], we can interchange the integral $\int_{G_\gamma \backslash G} dg_1$ and the operator ϖ'_γ, then it is a direct consequence of Lemma 5.5. □

PROPOSITION 5.7. *Suppose* $\gamma = h_{0,\tau}$ $(e^{i\tau} \neq \pm 1)$. *Then*

$$G_\gamma = \left\{ \begin{pmatrix} a & & b & \\ & \cos\theta & & \sin\theta \\ c & & d & \\ & -\sin\theta & & \cos\theta \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \right\},$$
$$c_\gamma = -2\sqrt{2}\pi,$$
$$\varpi'_\gamma = -2i\frac{\partial}{\partial\theta_1} + 2.$$

If $\pi_\lambda = D(n_1, n_2) \in D_{\mathrm{I}}$ *or* D_{III}, *then*

$$\int_{G_\gamma\backslash G} \phi_\lambda(g^{-1}\gamma\, g)\, dg = \frac{n_2\, e^{-in_1\tau} - n_1\, e^{-in_2\tau}}{\sqrt{2}\,\pi\,(e^{-i\tau/2} - e^{i\tau/2})^2\,(e^{-i\tau} - e^{i\tau})};$$

and if $\pi_\lambda = D(n_1, n_2) \in D_{\mathrm{II}}$ *or* D_{IV}, *then*

$$\int_{G_\gamma\backslash G} \phi_\lambda(g^{-1}\gamma\, g)\, dg = \frac{-n_2\, e^{-in_1\tau} + n_1\, e^{-in_2\tau}}{\sqrt{2}\,\pi\,(e^{-i\tau/2} - e^{i\tau/2})^2\,(e^{-i\tau} - e^{i\tau})}.$$

PROPOSITION 5.8. *Suppose* $\gamma = h_{\pi,\tau}$ $(e^{i\tau} \neq \pm 1)$. *Then*

$$G_\gamma = G_{h_{0,\tau}},$$
$$c_\gamma = -2\sqrt{2}\pi,$$
$$\varpi'_\gamma = -2i\frac{\partial}{\partial\theta_1} + 2.$$

If $\pi_\lambda = D(n_1, n_2) \in D_{\mathrm{I}}$ *or* D_{III}, *then*

$$\int_{G_\gamma\backslash G} \phi_\lambda(g^{-1}\gamma\, g)\, dg = \frac{n_1\,(-1)^{n_1} e^{-in_2\tau} - n_2\,(-1)^{n_2} e^{-in_1\tau}}{\sqrt{2}\,\pi\,(e^{-i\tau/2} + e^{i\tau/2})^2\,(e^{-i\tau} - e^{i\tau})};$$

and if $\pi_\lambda = D(n_1, n_2) \in D_{\mathrm{II}}$ *or* D_{IV}, *then*

$$\int_{G_\gamma\backslash G} \phi_\lambda(g^{-1}\gamma\, g)\, dg = \frac{-n_1\,(-1)^{n_1} e^{-in_2\tau} + n_2\,(-1)^{n_2} e^{-in_1\tau}}{\sqrt{2}\,\pi\,(e^{-i\tau/2} + e^{i\tau/2})^2\,(e^{-i\tau} - e^{i\tau})}.$$

PROPOSITION 5.9. *Suppose* $\gamma = h_{\tau,\tau}$ $(e^{i\tau} \neq \pm 1)$. *Then*

$$G_\gamma = K,$$
$$c_\gamma = 2,$$
$$\varpi'_\gamma = -i\frac{\partial}{\partial\theta_1} + i\frac{\partial}{\partial\theta_2} + 1.$$

If $\pi_\lambda = D(n_1, n_2) \in D_{\mathrm{I}}$ *or* D_{III}, *then*

$$\int_{G_\gamma \backslash G} \phi_\lambda(g^{-1}\gamma\, g)\, dg = \frac{-(n_1 - n_2)e^{-(n_1+n_2)i\tau}}{(e^{-i\tau} - e^{i\tau})^3};$$

and if $\pi_\lambda = D(n_1, n_2) \in D_{\mathrm{II}}$ *or* D_{IV}, *then*

$$\int_{G_\gamma \backslash G} \phi_\lambda(g^{-1}\gamma\, g)\, dg = \frac{(n_1 - n_2)e^{-(n_1+n_2)i\tau}}{(e^{-i\tau} - e^{i\tau})^3}.$$

PROPOSITION 5.10. *Suppose* $\gamma = h_{\tau,-\tau}$ $(e^{i\tau} \neq \pm 1)$. *Then*

$$G_\gamma = \left\{ \frac{1}{2} \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \\ i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & & & a_{12} \\ & b_{11} & b_{12} & \\ & b_{21} & b_{22} & \\ a_{21} & & & a_{22} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & -i \\ -i & 0 & 1 & 0 \\ 0 & -i & 0 & 1 \end{pmatrix} \right\},$$

$$c_\gamma = -2\sqrt{2}\pi,$$

$$\varpi'_\gamma = -i(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}) + 1,$$

where

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{pmatrix}^{-1} \in U(1,1).$$

If $\pi_\lambda = D(n_1, n_2) \in D_{\mathrm{I}}$ *or* D_{III}, *then*

$$\int_{G_\gamma \backslash G} \phi_\lambda(g^{-1}\gamma\, g)\, dg = \frac{(n_1 + n_2)(e^{i(n_1-n_2)\tau} - e^{i(-n_1+n_2)\tau})}{2\sqrt{2}\,\pi\,(e^{-i\tau} - e^{i\tau})^3};$$

and if $\pi_\lambda = D(n_1, n_2) \in D_{\mathrm{II}}$ *or* D_{IV}, *then*

$$\int_{G_\gamma \backslash G} \phi_\lambda(g^{-1}\gamma\, g)\, dg = \frac{-(n_1 + n_2)(e^{i(n_1-n_2)\tau} - e^{i(-n_1+n_2)\tau})}{2\sqrt{2}\,\pi\,(e^{-i\tau} - e^{i\tau})^3}.$$

PROPOSITION 5.11. *Suppose* $\gamma = h_{0,\pi}$. *Then*

$$G_\gamma = \left\{ \left. \begin{pmatrix} a_{11} & & a_{12} & \\ & b_{11} & & b_{12} \\ a_{21} & & a_{22} & \\ & b_{21} & & b_{22} \end{pmatrix} \right| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in SL(2, \mathbb{R}) \right\},$$

$$c_\gamma = 8\pi^2,$$

$$\varpi'_\gamma = (-2i\frac{\partial}{\partial\theta_1} + 2)(-2i\frac{\partial}{\partial\theta_2} + 2).$$

If $\pi_\lambda = D(n_1, n_2) \in D_{\mathrm{I}}$ *or* D_{III}, *then*

$$\int_{G_\gamma \backslash G} \phi_\lambda(g^{-1}\gamma\, g)\, dg = \frac{((-1)^{n_2} - (-1)^{n_1})n_1 n_2}{8\pi^2};$$

and if $\pi_\lambda = D(n_1, n_2) \in D_{\mathrm{II}}$ *or* D_{IV}, *then*

$$\int_{G_\gamma \backslash G} \phi_\lambda(g^{-1}\gamma\, g)\, dg = \frac{-((-1)^{n_2} - (-1)^{n_1})n_1 n_2}{8\pi^2}.$$

PROPOSITION 5.12. *Suppose* $\gamma = h_{0,0}$. *Then*

$$G_\gamma = G,$$
$$c_\gamma = -64\pi^3,$$
$$\varpi'_\gamma = (-i\frac{\partial}{\partial\theta_1} + i\frac{\partial}{\partial\theta_2} + 1)(-2i\frac{\partial}{\partial\theta_1} + 4)(-2i\frac{\partial}{\partial\theta_2} + 2)(-i\frac{\partial}{\partial\theta_1} - i\frac{\partial}{\partial\theta_2} + 3).$$

If $\pi_\lambda = D(n_1, n_2) \in D_{\mathrm{I}}$ *or* D_{III}, *then*

$$\int_{G_\gamma \backslash G} \phi_\lambda(g^{-1}\gamma\, g)\, dg = \frac{-(n_1 n_2^3 - n_1^3 n_2)}{8\pi^3};$$

and if $\pi_\lambda = D(n_1, n_2) \in D_{\mathrm{II}}$ *or* D_{IV}, *then*

$$\int_{G_\gamma \backslash G} \phi_\lambda(g^{-1}\gamma\, g)\, dg = \frac{(n_1 n_2^3 - n_1^3 n_2)}{8\pi^3}.$$

PROPOSITION 5.13. *Suppose* $\gamma = h_{\pi,\pi}$. *Then*

$$G_\gamma = G,$$
$$c_\gamma = -64\pi^3,$$
$$\varpi'_\gamma = (-i\frac{\partial}{\partial\theta_1} + i\frac{\partial}{\partial\theta_2} + 1)(-2i\frac{\partial}{\partial\theta_1} + 4)(-2i\frac{\partial}{\partial\theta_2} + 2)(-i\frac{\partial}{\partial\theta_1} - i\frac{\partial}{\partial\theta_2} + 3).$$

If $\pi_\lambda = D(n_1, n_2) \in D_{\mathrm{I}}$ *or* D_{III}, *then*

$$\int_{G_\gamma \backslash G} \phi_\lambda(g^{-1}\gamma\, g)\, dg = \frac{(-1)^{n_1+n_2}(n_1 n_2^3 - n_1^3 n_2)}{8\pi^3};$$

and if $\pi_\lambda = D(n_1, n_2) \in D_{\mathrm{II}}$ *or* D_{IV}, *then*

$$\int_{G_\gamma \backslash G} \phi_\lambda(g^{-1}\gamma\, g)\, dg = \frac{-(-1)^{n_1+n_2}(n_1 n_2^3 - n_1^3 n_2)}{8\pi^3}.$$

§6. $(\mathfrak{g}, K)$-cohomology

As in Section 1 we fix a positive root system Δ_3^+. Let $\Delta_{3,c}^+ := \{\,(i,-i)_3\,\}$ be the set of positive compact root, and let $\Delta_{3,n}^+ := \{\,(2i,0)_3,\ (0,2i)_3,\ (i,i)_3\,\}$ be the set of positive noncompact roots. Put $\mathfrak{p}^+ := \sum_{\alpha\in\Delta_{3,n}^+} \mathfrak{g}_\alpha^{\mathbb{C}}$, $\mathfrak{p}^- := \sum_{\alpha\in\Delta_{3,n}^+} \mathfrak{g}_{-\alpha}^{\mathbb{C}}$ and $\mathfrak{p} := \mathfrak{p}^+ \oplus \mathfrak{p}^-$. We define $\wedge^n\mathfrak{p}$ and $\wedge^{p,q}\mathfrak{p} = (\wedge^p\mathfrak{p}^+) \wedge (\wedge^q\mathfrak{p}^-)$ as usual (see [2]). The maximal compact subgroup K acts on $\wedge^n\mathfrak{p}$, $\wedge^{p,q}\mathfrak{p}$ by the adjoint action. Let F_δ be the finite dimensional irreducible representation of G with highest weight δ. Let F'_δ be the contragredient representation of F_δ.

LEMMA 6.1 ([2, CHAP.II §3]). *If π is an irreducible unitary representation with infinitesimal character $\chi_{\rho+\delta}$, then*

$$\mathrm{H}^{p,q}(\mathfrak{g}, K; \pi \otimes F'_\delta) = \mathrm{Hom}_K((\wedge^{p,q}\mathfrak{p} \otimes F_\delta), \pi).$$

In Section 2 we have defined the irreducible finite dimensional representation $\rho_{M,N}$ of $GL_2(\mathbb{C})$. Restriction of each $\rho_{M,N}$ to $U(2,\mathbb{C})$ defines an irreducible representation of $U(2,\mathbb{C})$. We use the same symbol $\rho_{M,N}$ to denote the representation $\rho_{M,N}|_K$.

LEMMA 6.2. *The highest weight of $\rho_{M,N}$ is $((M+N)i, Mi)_3$. Hence for any irreducible representation τ of K, there exists a pair of integers M, N such that τ is equivalent to $\rho_{M,N}$.*

LEMMA 6.3. *As a K-module, $\wedge^{p,q}\mathfrak{p}$ decomposes as follows.*

$$\begin{aligned}
\wedge^{0,0}\mathfrak{p} &\cong \rho_{0,0}, & \wedge^{1,2}\mathfrak{p} &\cong \rho_{-3,4} \oplus \rho_{-2,2} \oplus \rho_{-1,0},\\
\wedge^{1,0}\mathfrak{p} &\cong \rho_{0,2}, & \wedge^{0,3}\mathfrak{p} &\cong \rho_{-3,0},\\
\wedge^{0,1}\mathfrak{p} &\cong \rho_{-2,2}, & \wedge^{3,1}\mathfrak{p} &\cong \rho_{1,2},\\
\wedge^{2,0}\mathfrak{p} &\cong \rho_{1,2}, & \wedge^{2,2}\mathfrak{p} &\cong \rho_{-2,4} \oplus \rho_{-1,2} \oplus \rho_{0,0},\\
\wedge^{1,1}\mathfrak{p} &\cong \rho_{-2,4} \oplus \rho_{-1,2} \oplus \rho_{0,0}, & \wedge^{1,3}\mathfrak{p} &\cong \rho_{-3,2},\\
\wedge^{0,2}\mathfrak{p} &\cong \rho_{-3,2}, & \wedge^{3,2}\mathfrak{p} &\cong \rho_{0,2},\\
\wedge^{3,0}\mathfrak{p} &\cong \rho_{3,0}, & \wedge^{2,3}\mathfrak{p} &\cong \rho_{-2,2},\\
\wedge^{2,1}\mathfrak{p} &\cong \rho_{-1,4} \oplus \rho_{0,2} \oplus \rho_{1,0}, & \wedge^{3,3}\mathfrak{p} &\cong \rho_{0,0}.
\end{aligned}$$

LEMMA 6.4. *Let $\rho_{M,N}$ be a K-type of $[-1] \otimes D_n^+(1)$. Then $N + n \equiv 1\ (2)$ and there exists a pair of integers; $k, m \in \mathbb{Z},\ 0 \leqq k \leqq N,\ m \geqq 0$ such that*

$$M + N - k = n + 2m\,.$$

Moreover the multiplicity of $\rho_{M,N}$ in $[-1] \otimes D_n^+(1)$ is equal to the number of the pairs k, m satisfying the conditions above.

PROOF: Suppose that n is an even number. Let $P^{+,\nu}$ be the principal series representation of $SL_2(\mathbb{R})$. Then

$$\begin{aligned}
P^{-,+}(\nu, n-1) &= \mathrm{Ind}_{P_0}^G(1 \otimes (\nu, n-1) \otimes \sigma_{-,+})\\
&= \mathrm{Ind}_{P_2}^G(1 \otimes \nu \otimes ([-1] \otimes P^{+,n-1})).
\end{aligned}$$

Hence $[-1] \otimes D_n^+(1)$ is a subrepresentation of $P^{-,+}(1, n-1)$. The matrix coefficients $V_{M,N,k,l}$ of $\rho_{M,N}$ are given by

$$V_{M,N,k,l}(u) = \det u^M \sum_{j=0}^{N} \binom{l}{j} \binom{N-l}{k-j} u_{11}^j\, u_{12}^{k-j}\, u_{21}^{l-j}\, u_{22}^{N-k-l+j},$$

where

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in U(2,\mathbb{C}), \quad \begin{pmatrix} \mathrm{Re}(u) & \mathrm{Im}(u) \\ -\mathrm{Im}(u) & \mathrm{Re}(u) \end{pmatrix} \in K$$

and $0 \leqq k, l \leqq N$. Hence $\{\ V_{M,N,k,l} \,|\, M+k \equiv 1,\ N \equiv 1\ (2)\ \}$ gives a base of $P^{-,+}(\nu, n-1)$. Let $h_{0,\theta}$ be the element defined in Section 4. Then

$$V_{M,N,k,l}(h_{0,\theta}\, u) = e^{(M+N-k)i\theta}\, V_{M,N,k,l}(u) \quad (u \in K).$$

Hence $\{\ V_{M,N,k,l} \,|\, M+N-k = n+2m,\ N \equiv 1\ (2),\ m \geqq 0\ \}$ gives a base of $[-1] \otimes D_n^+(1)$. We can prove the case when n is an odd number by the same argument. □

LEMMA 6.5. *Let $\rho_{M,N}$ be a K-type of $[-1] \otimes D_n^-(1)$. Then $N+n \equiv 1\ (2)$ and there exists a pair of integers; $k, m \in \mathbb{Z}$, $0 \leqq k \leqq N$, $m \geqq 0$ such that*

$$M+N-k = -n-2m.$$

Moreover the multiplicity of $\rho_{M,N}$ in $[-1] \otimes D_n^-(1)$ is equal to the number of the pairs k, m satisfying the conditions above.

PROOF: By the same argument as in the case of Lemma 6.4, we can prove the lemma. □

LEMMA 6.6. *Let $\rho_{M,N}$ be a K-type of $D_n(1)$. Then $N+n \equiv 0\ (2)$ and there exists a pair of integers; $k, m \in \mathbb{Z}$, $0 \leqq k \leqq N$, $m \geqq 0$ such that*

$$N-2k = n+2m.$$

Moreover the multiplicity of $\rho_{M,N}$ in $D_n(1)$ is equal to the number of the pairs k, m satisfying the conditions above.

PROOF: We can choose another base of $L^2(K)$,

$$\{\ \tilde{V}_{M,N,k,l} = V'_{M,N,k,l} - (-1)^{M+k}\, i^N\, V'_{M,N,N-k,l}\ \},$$

where $0 \leqq k \leqq [N/2]$ and

$$V'_{M,N,k,l}(u) = V_{M,N,k,l}\left(\mathrm{Ad}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}\right](u)\right).$$

We can prove that $\{\ \tilde{V}_{M,N,k,l} \mid N-2k = n+2m,\ m \geqq 0\ \}$ gives a base of $D_n(1)$ as in the case of Lemma 6.4. □

LEMMA 6.7. *Put* $\lambda = (2,1)_0$. *Then the list of* $\wedge^{p,q}\mathfrak{p}$ *and the irreducible unitary representations* π *with infinitesimal character* χ_λ *such that*

$$\mathrm{Hom}_K(\wedge^{p,q}\mathfrak{p}, \pi) \neq 0$$

is given below.

$\wedge^{p,q}\mathfrak{p}$	π	*common* K*-type*
$\wedge^{0,0}\mathfrak{p}$	Trivial	$\rho_{0,0}$
$\wedge^{2,0}\mathfrak{p}$	$\overline{[-1] \otimes D_3^+(1)}$	$\rho_{1,2}$
$\wedge^{1,1}\mathfrak{p}$	$\overline{D_4(1)}$	$\rho_{-2,4}$
$\wedge^{1,1}\mathfrak{p}$	Trivial	$\rho_{0,0}$
$\wedge^{0,2}\mathfrak{p}$	$\overline{[-1] \otimes D_3^-(1)}$	$\rho_{-3,2}$
$\wedge^{3,0}\mathfrak{p}$	$D(2,1)$	$\rho_{3,0}$
$\wedge^{2,1}\mathfrak{p}$	$D(2,-1)$	$\rho_{-1,4}$
$\wedge^{1,2}\mathfrak{p}$	$D(1,-2)$	$\rho_{-3,4}$
$\wedge^{0,3}\mathfrak{p}$	$D(-1,-2)$	$\rho_{-3,0}$
$\wedge^{3,1}\mathfrak{p}$	$\overline{[-1] \otimes D_3^+(1)}$	$\rho_{1,2}$
$\wedge^{2,2}\mathfrak{p}$	$\overline{D_4(1)}$	$\rho_{-2,4}$
$\wedge^{2,2}\mathfrak{p}$	Trivial	$\rho_{0,0}$
$\wedge^{1,3}\mathfrak{p}$	$\overline{[-1] \otimes D_3^-(1)}$	$\rho_{-3,2}$
$\wedge^{3,3}\mathfrak{p}$	Trivial	$\rho_{0,0}$

Moreover we get $\mathrm{Hom}_K(\wedge^{p,q}\mathfrak{p}, \pi) \cong \mathbb{C}$, *for any pair* ($\wedge^{p,q}\mathfrak{p}$, π) *listed above.*

PROOF: By Proposition 4.8 it is easy to see that all the unitary representations with infinitesimal character $\chi_{(2,1)_0}$ are in the list. If τ is a K-type of $D(2,1)$, then the highest weight of τ can be written as

$$(3i, 3i)_3 + \sum_{\alpha \in \Delta_{3,n}^+} n_\alpha \alpha \quad (n_\alpha \geqq 0).$$

Hence by Lemma 6.3 it is easy to see that $\mathrm{Hom}_K(\wedge^{p,q}\mathfrak{p}, D(2,1)) \neq 0$ only if $p = 3$, $q = 0$ and the common K-type is $\rho_{3,0}$. Moreover the multiplicity of $\rho_{3,0}$ in $D(2,1)$ is 1 and the multiplicity of $\rho_{3,0}$ in $\wedge^{3,0}\mathfrak{p}$ is 1. Likewise for the other discrete series representations $D(2,-1)$, $D(1,-2)$ and $D(-1,-2)$, we get the result by Lemma 6.3. By Lemma 3.4 we get

$$\overline{[-1] \otimes D_3^+(1)} = [-1] \otimes D_3^+(1) - D(2,1) - D(2,-1).$$

By Lemma 6.4 it is easy to see that the common K-types of $\oplus_{p,q} \wedge^{p,q} \mathfrak{p}$ and $[-1] \otimes D_3^+(1)$ are $\rho_{1,2}$, $\rho_{3,0}$ and $\rho_{-1,4}$. Moreover their multiplicities in $[-1] \otimes D_3^+(1)$ are 1. Since $\rho_{3,0}$ is a K-type of $D(2,1)$ and $\rho_{-1,4}$ is a K-type of $D(2,-1)$, we get the result for $\overline{[-1] \otimes D_3^+(1)}$. By the same argument we get the result for $\overline{[-1] \otimes D_3^+(1)}$ and $\overline{D_4(1)}$. It is easy to calculate the results for the trivial representation $\overline{P^{+,+}(2,1)}$. □

COROLLARY 6.8. *Let λ be as in Lemma 6.7. Assume that π is an irreducible unitary representation with infinitesimal character χ_λ, then we get*

$$\mathrm{H}^{0,0}(\mathfrak{g}, K; \pi) \cong \begin{cases} \mathbb{C} & \text{if } \pi = \text{Trivial} \\ 0 & \text{otherwise} \end{cases}$$
$$\mathrm{H}^{1,0}(\mathfrak{g}, K; \pi) \cong \mathrm{H}^{0,1}(\mathfrak{g}, K; \pi) = 0$$
$$\mathrm{H}^{2,0}(\mathfrak{g}, K; \pi) \cong \begin{cases} \mathbb{C} & \text{if } \pi = \overline{[-1] \otimes D_3^+(1)} \\ 0 & \text{otherwise} \end{cases}$$
$$\mathrm{H}^{1,1}(\mathfrak{g}, K; \pi) \cong \begin{cases} \mathbb{C} & \text{if } \pi = \overline{D_4(1)} \text{ or } \pi = \text{Trivial} \\ 0 & \text{otherwise} \end{cases}$$
$$\mathrm{H}^{0,2}(\mathfrak{g}, K; \pi) \cong \begin{cases} \mathbb{C} & \text{if } \pi = \overline{[-1] \otimes D_3^-(1)} \\ 0 & \text{otherwise} \end{cases}$$
$$\mathrm{H}^{3,0}(\mathfrak{g}, K; \pi) \cong \begin{cases} \mathbb{C} & \text{if } \pi = D(2,1) \\ 0 & \text{otherwise} \end{cases}$$
$$\mathrm{H}^{2,1}(\mathfrak{g}, K; \pi) \cong \begin{cases} \mathbb{C} & \text{if } \pi = D(2,-1) \\ 0 & \text{otherwise} \end{cases}$$
$$\mathrm{H}^{1,2}(\mathfrak{g}, K; \pi) \cong \begin{cases} \mathbb{C} & \text{if } \pi = D(1,-2) \\ 0 & \text{otherwise} \end{cases}$$
$$\mathrm{H}^{0,3}(\mathfrak{g}, K; \pi) \cong \begin{cases} \mathbb{C} & \text{if } \pi = D(-1,-2) \\ 0 & \text{otherwise} \end{cases}$$
$$\mathrm{H}^{3,1}(\mathfrak{g}, K; \pi) \cong \begin{cases} \mathbb{C} & \text{if } \pi = \overline{[-1] \otimes D_3^+(1)} \\ 0 & \text{otherwise} \end{cases}$$
$$\mathrm{H}^{2,2}(\mathfrak{g}, K; \pi) \cong \begin{cases} \mathbb{C} & \text{if } \pi = \overline{D_4(1)} \text{ or } \pi = \text{Trivial} \\ 0 & \text{otherwise} \end{cases}$$
$$\mathrm{H}^{1,3}(\mathfrak{g}, K; \pi) \cong \begin{cases} \mathbb{C} & \text{if } \pi = \overline{[-1] \otimes D_3^-(1)} \\ 0 & \text{otherwise .} \end{cases}$$
$$\mathrm{H}^{3,2}(\mathfrak{g}, K; \pi) \cong \mathrm{H}^{2,3}(\mathfrak{g}, K; \pi) = 0$$
$$\mathrm{H}^{3,3}(\mathfrak{g}, K; \pi) \cong \begin{cases} \mathbb{C} & \text{if } \pi = \text{Trivial} \\ 0 & \text{otherwise .} \end{cases}$$

PROOF: This is an immediate consequence of Lemma 6.1 and Lemma 6.7 .□

LEMMA 6.9. *Put $\lambda = (n+2, 1)_0$ and $\delta = (ni, 0)_3$ $(n \geqq 1)$. Then the list of $\wedge^{p,q}\mathfrak{p} \otimes F_\delta$ and the irreducible unitary representations π with infinitesimal character χ_λ such that*

$$\mathrm{Hom}_K(\wedge^{p,q}\mathfrak{p} \otimes F_\delta, \pi) \neq 0$$

is given below.

$\wedge^{p,q}\mathfrak{p}\otimes F_\delta$	π	*common K-type*
$\wedge^{2,0}\mathfrak{p}\otimes F_\delta$	$\overline{[-1]\otimes D^+_{n+3}(1)}$	$\rho_{1,n+2}$
$\wedge^{0,2}\mathfrak{p}\otimes F_\delta$	$\overline{[-1]\otimes D^-_{n+3}(1)}$	$\rho_{-n-3,n+2}$
$\wedge^{3,0}\mathfrak{p}\otimes F_\delta$	$D(n+2,1)$	$\rho_{3,n}$
$\wedge^{2,1}\mathfrak{p}\otimes F_\delta$	$D(n+2,-1)$	$\rho_{-1,n+4}$
$\wedge^{1,2}\mathfrak{p}\otimes F_\delta$	$D(1,-n-2)$	$\rho_{-n-3,n+4}$
$\wedge^{0,3}\mathfrak{p}\otimes F_\delta$	$\overline{D(-1,-n-2)}$	$\rho_{-n-3,n}$
$\wedge^{3,1}\mathfrak{p}\otimes F_\delta$	$\overline{[-1]\otimes D^+_{n+3}(1)}$	$\rho_{1,n+2}$
$\wedge^{1,3}\mathfrak{p}\otimes F_\delta$	$\overline{[-1]\otimes D^-_{n+3}(1)}$	$\rho_{-n-3,n+2}$

Moreover we get $\mathrm{Hom}_K(\wedge^{p,q}\mathfrak{p}\otimes F_\delta,\pi)\cong\mathbb{C}$ *for any pair* $(\wedge^{p,q}\mathfrak{p}\otimes F_\delta,\,\pi)$ *listed above.*

Proof: Let $\rho_{M,N}\otimes F_\delta\cong\oplus\rho_{M_j,N_j}$ be a decomposition as a K-module. As F_δ is a highest weight module, one and only one of the ρ_{M_j,N_j} has the highest weight $\delta+((M+N)i,Mi)_3$ and the highest weight of each ρ_{M_j,N_j} is of the form

$$\delta+((M+N)i,Mi)_3-\sum_{\alpha\in\Delta^+_{3,n}}n_{\alpha,j}\alpha\quad(n_{\alpha,j}\geqq 0).$$

It is easy to see that the highest weight of each K-type of $D(n+2,1)$ is of the form

$$((n+3)i,3i)_3+\sum_{\alpha\in\Delta^+_{3,n}}n_\alpha\alpha\quad(n_\alpha\geqq 0).$$

Hence we get the result for $D(n+2,1)$. If we change Δ^+_3 for a suitable positive root system Δ^+_{i} ($\mathrm{i}=\mathrm{I},\,\mathrm{II},\,\mathrm{III},\,\mathrm{IV}$) which corresponds to the Harish-Chandra parameter, we get the result for the other discrete series representations. For $\overline{[-1]\otimes D^+_{n+3}(1)}$, we get the result by the same argument as in the case of Lemma 6.7.□

Corollary 6.10. *Let λ and F_δ be as in Lemma 6.9. Assume that π is an irreducible unitary representation with infinitesimal character χ_λ, then we get*

$$\mathrm{H}^{0,0}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\mathrm{H}^{1,0}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\mathrm{H}^{0,1}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong 0$$

$$\mathrm{H}^{2,0}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\begin{cases}\mathbb{C} & \text{if } \pi=\overline{[-1]\otimes D^+_{n+3}(1)}\\ 0 & \text{otherwise}\end{cases}$$

$$\mathrm{H}^{1,1}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong 0$$

$$\mathrm{H}^{0,2}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\begin{cases}\mathbb{C} & \text{if } \pi=\overline{[-1]\otimes D^-_{n+3}(1)}\\ 0 & \text{otherwise}\end{cases}$$

$$\mathrm{H}^{3,0}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\begin{cases}\mathbb{C} & \text{if } \pi=D(n+2,1)\\ 0 & \text{otherwise}\end{cases}$$

$$\mathrm{H}^{2,1}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\begin{cases}\mathbb{C} & \text{if } \pi=D(n+2,-1)\\ 0 & \text{otherwise}\end{cases}$$
$$\mathrm{H}^{1,2}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\begin{cases}\mathbb{C} & \text{if } \pi=D(1,-n-2)\\ 0 & \text{otherwise}\end{cases}$$
$$\mathrm{H}^{0,3}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\begin{cases}\mathbb{C} & \text{if } \pi=D(-1,-n-2)\\ 0 & \text{otherwise}\end{cases}$$
$$\mathrm{H}^{3,1}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\begin{cases}\mathbb{C} & \text{if } \pi=\overline{[-1]\otimes D^+_{n+3}(1)}\\ 0 & \text{otherwise}\end{cases}$$
$$\mathrm{H}^{2,2}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong 0$$
$$\mathrm{H}^{1,3}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\begin{cases}\mathbb{C} & \text{if } \pi=\overline{[-1]\otimes D^-_{n+3}(1)}\\ 0 & \text{otherwise}\end{cases}$$
$$\mathrm{H}^{3,2}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\mathrm{H}^{2,3}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\mathrm{H}^{3,3}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong 0.$$

LEMMA 6.11. *Put $\lambda=(n+2,n+1)_0$ and $\delta=(ni,ni)_3$ $(n\geqq 1)$. Then the list of $\wedge^{p,q}\mathfrak{p}\otimes F_\delta$ and the irreducible unitary representations π with infinitesimal character χ_λ such that*

$$\mathrm{Hom}_K(\wedge^{p,q}\mathfrak{p}\otimes F_\delta,\pi)\neq 0$$

is given below.

$\wedge^{p,q}\mathfrak{p}\otimes F_\delta$	π	*common K-type*
$\wedge^{1,1}\mathfrak{p}\otimes F_\delta$	$\overline{D_{2n+4}(1)}$	$\rho_{n-2,4}$
$\wedge^{3,0}\mathfrak{p}\otimes F_\delta$	$D(n+2,n+1)$	$\rho_{n+3,0}$
$\wedge^{2,1}\mathfrak{p}\otimes F_\delta$	$D(n+2,-n-1)$	$\rho_{-n-1,2n+4}$
$\wedge^{1,2}\mathfrak{p}\otimes F_\delta$	$D(n+1,-n-2)$	$\rho_{-n-3,2n+4}$
$\wedge^{0,3}\mathfrak{p}\otimes F_\delta$	$D(-n-1,-n-2)$	$\rho_{-n-3,0}$
$\wedge^{2,2}\mathfrak{p}\otimes F_\delta$	$\overline{D_{2n+4}(1)}$	$\rho_{n-2,4}$

Moreover we get $\mathrm{Hom}_K(\wedge^{p,q}\mathfrak{p}\otimes F_\delta,\pi)\cong\mathbb{C}$ for any pair $(\wedge^{p,q}\mathfrak{p}\otimes F_\delta,\,\pi)$ listed above.

PROOF: By the same argument as in the case of Lemma 6.9, we get the result.□

COROLLARY 6.12. *Let λ and F_δ be as in Lemma 6.11. For irreducible unitary representation π with infinitesimal character χ_λ we get*

$$\mathrm{H}^{0,0}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\mathrm{H}^{1,0}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\mathrm{H}^{0,1}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong 0$$
$$\mathrm{H}^{1,1}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\begin{cases}\mathbb{C} & \text{if } \pi=\overline{D_{2n+4}(1)}\\ 0 & \text{otherwise}\end{cases}$$
$$\mathrm{H}^{2,0}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\mathrm{H}^{0,2}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong 0$$
$$\mathrm{H}^{3,0}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\begin{cases}\mathbb{C} & \text{if } \pi=D(n+2,n+1)\\ 0 & \text{otherwise}\end{cases}$$

$$\mathrm{H}^{2,1}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\begin{cases}\mathbb{C} & \text{if } \pi=D(n+2,-n-1)\\ 0 & \text{otherwise}\end{cases}$$
$$\mathrm{H}^{1,2}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\begin{cases}\mathbb{C} & \text{if } \pi=D(n+1,-n-2)\\ 0 & \text{otherwise}\end{cases}$$
$$\mathrm{H}^{0,3}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\begin{cases}\mathbb{C} & \text{if } \pi=D(-n-1,-n-2)\\ 0 & \text{otherwise}\end{cases}$$
$$\mathrm{H}^{3,1}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\mathrm{H}^{1,3}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong 0$$
$$\mathrm{H}^{2,2}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\begin{cases}\mathbb{C} & \text{if } \pi=\overline{D_{2n+4}(1)}\\ 0 & \text{otherwise}\end{cases}$$
$$\mathrm{H}^{3,2}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\mathrm{H}^{2,3}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\mathrm{H}^{3,3}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong 0.$$

LEMMA 6.13. *Put $\lambda=(n_1+2,n_2+1)_0$ and $\delta=(n_1i,n_2i)_3$ $(n_1>n_2>0)$. Then the list of $\wedge^{p,q}\mathfrak{p}\otimes F_\delta$ and the irreducible unitary representations π with infinitesimal character χ_λ such that*

$$\mathrm{Hom}_K(\wedge^{p,q}\mathfrak{p}\otimes F_\delta,\pi)\neq 0$$

is given below.

$\wedge^{p,q}\mathfrak{p}\otimes F_\delta$	π	*common K-type*
$\wedge^{3,0}\mathfrak{p}\otimes F_\delta$	$D(n_1+2,n_2+1)$	ρ_{n_2+3,n_1-n_2}
$\wedge^{2,1}\mathfrak{p}\otimes F_\delta$	$D(n_1+2,-n_2-1)$	ρ_{-n_2-1,n_1+n_2+4}
$\wedge^{1,2}\mathfrak{p}\otimes F_\delta$	$D(n_2+1,-n_1-2)$	ρ_{-n_1-3,n_1+n_2+4}
$\wedge^{0,3}\mathfrak{p}\otimes F_\delta$	$D(-n_2-1,-n_1-2)$	ρ_{-n_1-3,n_1-n_2}

Moreover we get $\mathrm{Hom}_K(\wedge^{p,q}\mathfrak{p}\otimes F_\delta,\pi)\cong\mathbb{C}$ for any pair $(\wedge^{p,q}\mathfrak{p}\otimes F_\delta,\,\pi)$ listed above.

PROOF: By the same argument as in the case of Lemma 6.9, we get the result.□

COROLLARY 6.14. *Let λ and F_δ be as in Lemma 6.13. Assume that π is an irreducible unitary representation with infinitesimal character χ_λ, then we get*

$$\mathrm{H}^{3,0}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\begin{cases}\mathbb{C} & \text{if } \pi=D(n_1+2,n_2+1)\\ 0 & \text{otherwise}\end{cases}$$
$$\mathrm{H}^{2,1}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\begin{cases}\mathbb{C} & \text{if } \pi=D(n_1+2,-n_2-1)\\ 0 & \text{otherwise}\end{cases}$$
$$\mathrm{H}^{1,2}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\begin{cases}\mathbb{C} & \text{if } \pi=D(n_2+1,-n_1-2)\\ 0 & \text{otherwise}\end{cases}$$
$$\mathrm{H}^{0,3}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong\begin{cases}\mathbb{C} & \text{if } \pi=D(-n_2-1,-n_1-2)\\ 0 & \text{otherwise}\end{cases}$$
$$\mathrm{H}^{p,q}(\mathfrak{g},K;\pi\otimes F'_\delta)\cong 0 \text{ if } p+q\neq 3.$$

We will formulate the result of Theorem 4.11 more suggestively. Let τ_{π_λ} be the minimal K-type of π_λ and Λ_λ be the highest weight of τ_{π_λ} corresponding to $\Delta_{3,c}^+$. For each discrete series representation π_λ, put

$$\Delta_\lambda^+ := \{\, \alpha \in \Delta_3 \mid \langle \lambda, \check{\alpha} \rangle > 0 \,\},$$
$$\rho_\lambda := \frac{1}{2} \sum_{\alpha \in \Delta_\lambda^+} \alpha,$$

$$\mathrm{Wall}(\pi_\lambda) = \left\{ \alpha \in \Delta_\lambda^+ \;\middle|\; \begin{array}{ll} \text{i)} & \alpha \text{ is noncompact} \\ \text{ii)} & \langle \lambda - \rho_\lambda, \check{\alpha} \rangle \leqq 0 \end{array} \right\}.$$

Since we have $\pi_\lambda \cong \pi_{s_\alpha(\lambda)}$, where s_α is the reflection corresponding to the compact root α, we may assume that Δ_λ^+ contains $\Delta_{3,c}^+$. Then put

$$T_\lambda := \left\{ \Lambda = \Lambda_\lambda - \sum_{\alpha \in \mathrm{Wall}(\pi_\lambda)} \epsilon_\alpha \alpha \;\middle|\; \begin{array}{ll} \text{i)} & \epsilon_\alpha = 0, 1 \\ \text{ii)} & \Lambda \text{ is dominant for } \Delta_{3,c}^+ \end{array} \right\}.$$

Definition 6.15. *We say that K-type τ contributes to the $(\mathfrak{g}, K)$-cohomology with infinitesimal character $\chi_{\rho+\delta}$, if and only if there exist an irreducible unitary representation π with infinitesimal character $\chi_{\rho+\delta}$ and a pair of numbers p, q such that*

$$\mathrm{H}^{p,q}(\mathfrak{g}, K; \pi \otimes F_\delta') \neq 0$$

and τ is the common K-type of π and $\wedge^{p,q}\mathfrak{p} \otimes F_\delta$.

The following lemma is due to Vogan–Zuckerman.

Lemma 6.16 ([15, Theorem 1.4]). *If τ contributes to the $(\mathfrak{g}, K)$-cohomology with infinitesimal character $\chi_{\rho+\delta}$, then only one π can satisfy the conditions of Definition 6.15.*

For each τ which contributes to the $(\mathfrak{g}, K)$-cohomology with infinitesimal character $\chi_{\rho+\delta}$, we denote by π_τ the unique representation which satisfies the conditions of Definition 6.15. We denote by τ_Λ the irreducible representation of K whose highest weight is Λ.

Theorem 6.17. *Let π_λ be a discrete series representation. Suppose $\chi_\lambda = \chi_{\rho+\delta}$ and δ is dominant. Put*

$$T_\lambda^{coh} := \left\{ \tau_\Lambda \in \widehat{K} \;\middle|\; \begin{array}{ll} \text{i)} & \Lambda \in T_\lambda \\ \text{ii)} & \tau_\Lambda \textit{ contributes to the } (\mathfrak{g}, K)\textit{-cohomology} \\ & \textit{with infinitesimal character } \chi_{\rho+\delta} \end{array} \right\}.$$

For each element $\tau \in T_\lambda^{coh}$ we associate the number d_τ given by

$$d_\tau := \prod_{\alpha \in \mathrm{Wall}(\pi_\lambda)} (-1)^{\epsilon_\alpha},$$

where $\tau = \tau_\Lambda$, $\Lambda = \Lambda_\lambda - \sum_{\alpha \in \mathrm{Wall}(\pi_\lambda)} \epsilon_\alpha \alpha$. *Then the spectral part of the trace formula for* ϕ_λ *can be expressed as*

$$\sum_\pi m_\pi \, \Theta_\pi(\phi_\lambda) = \sum_{\tau \in T_\lambda^{coh}} d_\tau \, m_{\pi_\tau}.$$

The geometric part of the trace formula for ϕ_λ has been calculated in Section 5. Thus we have proved Theorem E.

§7. Remark

By Theorem 4.11 we have

$$m_{D(2,1)} = m_{\overline{[-1]\otimes D_3^+(1)}} + 1 + \sum_{\{\gamma\}_\Gamma} \mathrm{Vol}\,(\Gamma_\gamma \backslash G_\gamma) J_\gamma(\phi_{D(2,1)}).$$

Hence to calculate the multiplicity of $D(2,1)$, we have to know the multiplicity of $\overline{[-1]\otimes D_3^+(1)}$, but the author could not calculate it. By Matsushima–Murakami's Theorem we get

$$\dim \mathrm{H}^{2,0}(\Gamma \backslash \mathfrak{H}) = m_{\overline{[-1]\otimes D_3^+(1)}}.$$

LEMMA 7.1. *The cohomology group* $\mathrm{H}^{2,0}(\Gamma\backslash\mathfrak{H})$ *is isomorphic to the space of the* W_2 *valued holomorphic forms on* $\mathfrak{H}$ *which satisfy*

$$\rho_{1,2}[(-{}^tBZ + {}^tD)^{-1}]\varphi(({}^tAZ - {}^tC)(-{}^tBZ + {}^tD)^{-1}) = \varphi(Z)$$

for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$.

PROOF: In terms of the standard coordinates z_1, z_2 and z_3 such that $Z = \left(\begin{smallmatrix} z_1 & z_2 \\ z_2 & z_3 \end{smallmatrix}\right)$, the cochain complex $\mathcal{C}^{2,0}$ is given by

$$\mathcal{C}^{2,0} = \{\ \varphi(Z) = \sum_{i<j} \varphi_{ij}(Z) dz_i \wedge dz_j\ \}.$$

and the operator $\overline{\partial} : \mathcal{C}^{2,0} \to \mathcal{C}^{2,1}$ is given by

$$\overline{\partial}\varphi(Z) = \sum_k \sum_{i<j} \frac{\partial \varphi_{ij}(Z)}{\partial \overline{z_k}} d\overline{z_k} \wedge dz_i \wedge dz_j.$$

Hence

$$\overline{\partial}\varphi(Z) = 0 \Longleftrightarrow \varphi_{ij}(Z) \text{ are holomorphic.}$$

Put

$$M = (-{}^tBZ + {}^tD)^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Since the action of γ on $\mathfrak{H}$ is

$$\mu_\gamma(Z) = \gamma\langle Z\rangle = ({}^tAZ - {}^tC)(-{}^tBZ + {}^tD)^{-1},$$

the map μ_γ^* of cotangent spaces induced by μ_γ is given by

$$\begin{aligned}
(\mu_\gamma^*)_Z(dz_1)_{\gamma<Z>} =& a_{11}^2(dz_1)_Z + 2a_{11}a_{21}(dz_2)_Z + a_{21}^2(dz_3)_Z,\\
(\mu_\gamma^*)_Z(dz_2)_{\gamma<Z>} =& a_{11}a_{12}(dz_1)_Z + (a_{11}a_{22} + a_{12}a_{21})(dz_2)_Z + a_{21}a_{22}(dz_3)_Z,\\
(\mu_\gamma^*)_Z(dz_3)_{\gamma<Z>} =& a_{12}^2(dz_1)_Z + 2a_{12}a_{22}(dz_2)_Z + a_{22}^2(dz_3)_Z.
\end{aligned}$$

Hence if we regard $\varphi(Z)$ as a vector valued function

$$\varphi(Z) = (\varphi_{12}(Z), \varphi_{13}(Z), \varphi_{23}(Z)),$$

φ must satisfy the relation

$$\rho_{1,2}(M)\varphi(\gamma\langle Z\rangle) = \varphi(Z).$$

□

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