Solutions to Knizhnik-Zamolodchikov equations with coefficients in non-bounded modules

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Abstract

We explicitly write dowm integral formulas for solutions to Knizhnik-Zamolodchikov equations with coefficients in non-bounded – neither highest nor lowest weight – $\mathfrak{sl}_{n+1}$-modules. The formulas are closely related to WZNW model at a rational level.

1 Introduction

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, $\hat{\mathfrak{g}}$ the corresponding non-twisted affine Lie algebra. Let λ be a weight of $\mathfrak{g}$, $M(\lambda,\, k)$ ($M(\lambda,\, k)^c$) be the Verma (contragredient Verma) module over $\hat{\mathfrak{g}}$ with the central charge k; for a $\mathfrak{g}-$module V be denote by $V((z))$ the module of formal Laurent series in z with coefficients in V, regarded as a $\hat{\mathfrak{g}}-$module with the central charge equal to 0.

Vertex operator is a $\hat{\mathfrak{g}}-$linear map

$$\Phi(z):\ M(\lambda_1,\, k) \to M(\lambda_2,\, k)^c \otimes V((z)). \tag{1}$$

If highest weights $(\lambda_1, k), \cdots (\lambda_{N+1}, k)$ are generic then $M(\lambda_i, k) \approx M(\lambda_i, k)^c,\ 1 \leq i \leq N+1$ and one may consider a product of vertex operators $\Phi_N(z_N)\circ\cdots\circ\Phi_1(z_1)$. Matrix element $\langle v^*_{\lambda_{N+1}},\, \Phi_N(z_N)\circ\cdots\circ\Phi_1(z_1)v_{\lambda_1}\rangle$

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related to vacuum vectors is called a correlation function. One of the central results of conformal field theory (see [1]) is that a correlation function satisifies a remarkable system of Knizhnik-Zamolodchikov equations. We prepare notations in order to write down the trigonometric form of Knizhnik-Zamolodchikov equations.

Let

$$\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

be a root space decomposition. Fix an invariant inner product on $\mathfrak{g}$ and a basis $\{h_i \in \mathfrak{h},\ g_\alpha \in \mathfrak{g}_\alpha :\ 1 \leq i \leq n,\ \alpha \in \Delta\}$ of $\mathfrak{g}$ so that $(h_i, h_j) = \delta_{i,j},\ (g_\alpha, g_\beta) = \delta_{\alpha,-\beta}$. For each $\mu \in \mathfrak{h}^*$ denote by h_μ an element of $\mathfrak{h}$ satisfying (and uniquely determined) by the condition $(h_\mu, h) = \mu(h)$.

Set

$$r = \frac{1}{2}\sum_{i=1}^{n} h_i \otimes h_i + \sum_{\alpha \in \Delta_+} g_\alpha \otimes g_{-\alpha}.$$

Being an element of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ r naturally acts on a tensor product of 2 $\mathfrak{g}-$modules. There are N^2 different ways to make it act on a tensor product of N $\mathfrak{g}-$modules via the following N^2 embeddings of $U(\mathfrak{g})^{\otimes 2}$ in $U(\mathfrak{g})^{\otimes N}$: each of them is associated to a pair of numbers $1 \leq i, j \leq N$ and sends

$$U(\mathfrak{g})^{\otimes 2} \ni \omega \mapsto \omega_{ij} \in U(\mathfrak{g})^{\otimes N},$$

so that

$$\text{if } \omega = \sum_s a_s \otimes b_s \text{ then } \omega_{ij} = \sum_s \underbrace{1 \otimes \cdots 1 \otimes a_s}_{i} \overbrace{\otimes 1 \otimes \cdots \otimes 1 \otimes b_s}^{j} \otimes 1 \otimes \cdots \otimes 1.$$

For a pair $1 \leq i, j \leq N$ introduce the following function in 2 complex variables with values in $U(\mathfrak{g})^{\otimes N}$:

$$r(z_i, z_j) = \frac{r_{ij} z_i + r_{ji} z_j}{z_i - z_j}.$$

Theorem 1.1 (Knizhnik, Zamolodchikov) *The correlation function*

$$\Psi(z) = \langle v^*_{\lambda_{N+1}} \Phi_N(z_N) \circ \cdots \circ \Phi_1(z_1) v_{\lambda_1} \rangle$$

satisfies the following system of differential equations

$$\begin{aligned} (k + h^\vee) z_i \frac{\partial \Psi}{\partial z_i} &= \{\sum_{j \neq i} r_{ij}(z_i, z_j) - \frac{1}{2}(\lambda_1 + \lambda_{N+1} + 2\rho)^{(i)}\}\Psi, \qquad (2) \\ 1 \leq \quad i \quad &\leq N, \end{aligned}$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ and for each $\mu \in \mathfrak{h}^*$ $\mu^{(i)}$ stands for the operator acting on $V_1 \otimes \cdots \otimes V_N$ as h_μ applied to the $i-$th factor of $V_1 \otimes \cdots \otimes V_N$.

To keep track of the parameters we will be referring to (2) as $KZ(\lambda_{N+1}, \lambda_1)$.

The deep theory of KZ equations has been developed by several authors (see e.g. [2, 3, 4, 5]) in the case when V_i are highest weight modules. It has also been realized that this theory is relevant to physics applications in the case when (λ_i, k) is either integral or generic.Indeed, if conflicting with the above assumptions, some of $M(\lambda_i, k)$ are reducible then the product $\Phi_N(z_N) \circ \cdots \circ \Phi_1(z_1)$ does not exist unless each of the operators $\Phi_i(z_i)$ can be pushed down to a map

$$\Phi_i(z_i) : \ L(\lambda_i, \ k) \to L(\lambda_{i+1}, \ k) \otimes V_i((z_i)), \tag{3}$$

where $L(\lambda, k)$ stands for an irreducible highest weight module with the highest weight (λ, k). In the case when (λ, k) is an admissible weight (for example, dominant integral weight) [6] the last condition reduces to the *singular vector decoupling condition*: matrix elements of $\Phi_i(z_i)$ related to singular vectors of $M(\lambda_i)$ vanish. It is known [7] that if each (λ_i, k) is dominant integral then everything goes through nicely, in particular, the Schechtman-Varchenko integral solutions to (2) come from products of vertex operators (3). However if the central charge is not integral it has been realized ([10], see also [8, 9]) that the singular vector decoupling condition implies that V_i is neither highest nor lowest weight module. Though some results for such models were obtained in [8, 9], where in particular the connection to quantum hamiltonian reduction was revealed, not much is known about KZ equations in this case.

In [11] a new method of constructing solutions to (2) was proposed which seems to be relevant to the problem. Let G be a complex Lie group related to $\mathfrak{g}$, $F = G/B$ be a flag manifold and $F^0 \subset F$ be the big cell. There is a family of embeddings of $\mathfrak{g}$ into the algebra of order 1 differential operators on F^0

$$\pi_\mu : \mathfrak{g} \to Diff^1(F^0), \ \mu \in \mathfrak{h}^*.$$

This makes the space of analytic functions on F^0 into a huge $\mathfrak{g}-$module. Different $\mathfrak{g}-$closed subspaces give realization of different $\mathfrak{g}-$modules. For example, contragredient Verma modules are realized in the space of polynomials on F^0, μ being the highest weight and a constant function being a highest weight vector; this observation has been extensively used recently with regards to Wakimoto modules [12, 13, 14]. The spaces of multi-valued functions give modules with quite different properties, the simplest example being that of $\mathfrak{sl}_2$: in this case the big cell is $\mathbf{C}$, contragredient Verma modules are realized in $\mathbf{C}[x]$; the space $x^\nu \mathbf{C}[x, x^{-1}]$, $\nu \in \mathbf{C}$ is also closed under the action of $\mathfrak{sl}_2$ and the embedding π_μ, $\mu \in \mathbf{C}$ makes it into generically irreducible $\mathfrak{sl}_2-$module. This module is transparently neither highest nor lowest weight one.

Regarding V in (1) as a $\mathfrak{g}-$ module realized in functions on the big cell one identifies elements of $V((z))$ with functions of 2 groups of variables: x and z, where x stands for a (vector) coordinate on the big cell and z is a coordinate on $\mathbf{C}$. Likewise, the correlation function

$$\Psi(z) = \langle v^*_{\lambda_{N+1}}, \ \Phi_N(z_N) \circ \cdots \circ \Phi_1(z_1) v_{\lambda_1} \rangle$$

is identified with a function of $x^{(1)},\dots,x^{(N)};z_1,\dots,z_N$ where $x^{(i)}$ is a coordinate on the $i-$th copy of the big cell, $z_i\in\mathbf{C},\ 1\le i\le N$. One of the advantages of this functional realization is that the emebedding $\pi_\lambda:\mathfrak{g}\to Diff^1(F^0)$ lifts to the mapping of the group G: for $g\in\mathfrak{g}$ the exponent $\exp(-tg)$ is a well-defined operator.

Let W be the Weyl group of $\widehat{\mathfrak{g}}$, $w=r_{m_1}r_{m_2}\cdots r_{m_l}\in W$ be a decomposition(not necessarily reduced),where r_m denotes the reflection at the corresponding simple root. Set,

$$\beta_j=\frac{2(r_{m_{l+2-j}}\cdots r_{m_l}\cdot\lambda_1,\alpha_{m_{l+1-j}})}{(\alpha_{m_{l+1-j}},\alpha_{m_{l+1-j}})}+1,\ 1\le j\le l.$$

Given

$$\Psi_{old}(z)=\langle v^*_{\lambda_{N+1}},\,\Phi_N(z_N)\circ\cdots\circ\Phi_1(z_1)v_{\lambda_1}\rangle,$$

set

$$\Psi_{new}=\prod_{j=1}^{l}\Gamma(-\beta_j)^{-1}\int\{\exp(-t_lF_{m_l})\cdots\exp(-t_1F_{m_1})\Psi_{old}\}\prod_{j=1}^{l}t_j^{-\beta_j-1}\,dt_1\cdots dt_l, \tag{4}$$

where the integration is carried out over an arbitrary cycle of the highest homology group related to the multi-valued integrand. In (4) it is set that $E_i,F_i,H_i,\ 0\le i\le rk\,\mathfrak{g}$ are canonical Cartan generators of $\hat{\mathfrak{g}}$ and $E_i,F_i,H_i,\ 1\le i\le rk\,\mathfrak{g}$ are the ones coming from the inclusion $\mathfrak{g}\subset\hat{\mathfrak{g}}$.

Theorem 1.2 *[11] Ψ_{new} is a solution to $KZ(\lambda_{N+1},w\cdot\lambda_1)$.*

Theorem 1.2 works as follows: given a solution to KZ it generates new ones labelled by elements of the affine Weyl group. In our notations the simplest solution to $KZ(\lambda_{N+1},\lambda_1)$ is given by

$$\overset{\circ}{\Psi}_{old}=\prod_{i<j}(z_i-z_j)^{2(\mu_i,\mu_j)/(k+h^\vee)}(z_iz_j)^{-(\mu_i,\mu_j)/(k+h^\vee)}\times\prod_i z_i^{(\lambda_1+\lambda_{N+1}+2\rho,\mu_i)/2(k+h^\vee)},$$

where μ_i is a highest weight of V_i, $1\le i\le N$. In particular, $\overset{\circ}{\Psi}_{old}$ is independent of x's. The purpose of this paper is to explicitly write down the integral

$$\Psi_{new}=\overset{\circ}{\Psi}_{old}\times\prod_{j=1}^{l}\Gamma(-\beta_j)^{-1}\int\{\exp(-t_lF_{m_l})\cdots\exp(-t_1F_{m_1})1\}\prod_{j=1}^{l}t_j^{-\beta_j-1}\,dt_1\cdots dt_l, \tag{5}$$

for $\mathfrak{g}=\mathfrak{sl}_{n+1}$, generalizing the calculation carried out in [11] for $\mathfrak{sl}_2$.

Remark. The integral representation of Ψ_{new} in Theorem 1.2 is nothing but the conventional definition of $F^{\beta_l}_{m_l} \cdots F^{\beta_1}_{m_1} \cdot \Psi_{old}$. The latter comes from looking at the "matrix element"

$$\langle v^*_{\lambda_{N+1}} \Phi_N(z_N) \circ \cdots \circ \Phi_1(z_1) F^{\beta_1}_{m_1} \cdots F^{\beta_l}_{m_l} v_{\lambda_1} \rangle.$$

Though the expression $F^{\beta_1}_{m_1} \cdots F^{\beta_l}_{m_l} v_{\lambda_1}$ is not understood as an element of $M(\lambda_1, k)$, the powers are chosen in such a way that it formally satisfies the singular vector conditions [11, 15], which makes the statement of Theorem 1.2 almost obvious. One can similarly consider an expression

$$\langle E^{\beta'_1}_{m_1} \cdots E^{\beta'_l}_{m_l} \cdot v^*_{\lambda_{N+1}} \Phi_N(z_N) \circ \cdots \circ \Phi_1(z_1) v_{\lambda_1} \rangle,$$

for appropriate $\beta'_1, \ldots \beta'_l$ and write down another solution in the form close to (4) but with F's replaced with E's or combine both methods or, finally, apply them to other solutions obtained in [4, 8, 9].

As to relation of our solution (4) to correlation functions, we have been able to verify in simplest cases that (4) indeed gives a matrix element of a product of vertex operators and hope that (4) will prove useful for investigation of other rational level models.

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2 Integral formulas for soilutions of $KZ(\lambda_{N+1}, \lambda_1)$

2.1 Main result

Here we are going to write down the integral (5) in the case of $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $\hat{\mathfrak{g}} = \widehat{\mathfrak{sl}_{n+1}}$. In this case there are $3(n+1)$ Cartan generators E_i, F_i, H_i, $0 \leq i \leq n$, where E_i, F_i, H_i, $1 \leq i \leq n$ are the ones coming from the inclusion $\mathfrak{g} \subset \hat{\mathfrak{g}}$. Explicitly the generators are described as follows. If $e_{ij} = (a_{st})$ is an $(n+1) \times (n+1)$ matrix then $E_i = e_{ii+1}$, $F_i = e_{i+1i}$, $H_i = e_{ii} - e_{i+1i+1}$, $1 \leq i \leq n$ and $E_0 = e_{n+11} \otimes z$, $F_0 = e_{1n+1} \otimes z^{-1}$ (see [16] for details). The $\mathfrak{g}-$weight μ is considered as a vector $(\mu_1, \ldots, \mu_n)$, $\mu_i = \mu(H_i)$. The embedding

$$\pi_\mu : \mathfrak{sl}_{n+1} \to Diff^1(F^0), \ \mu = (\mu_1, \ldots, \mu_n)$$

is calculated in [12] (see also [13]). To recall this result we choose coordinates of the big cell F^0 to be $\{x_{ij} : 1 \leq i < j \leq n\}$ identifying as usual the big cell

with the subgroup of matrices

$$\begin{pmatrix} 1 & x_{11} & \cdots & \cdots & x_{1n} \\ & 1 & \ddots & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & x_{nn} \\ 0 & & & & 1 \end{pmatrix}.$$

For $1 \leq i \leq n$ set

$$\partial_{x_{ij}} := \frac{\partial}{\partial x_{ij}}.$$

Then π_μ acts on Cartan generators by

$$E_i \mapsto -\partial_{x_{ii}} - \sum_{j=i+1}^{n} x_{i+1j}\partial_{x_{ij}},$$

$$F_i \mapsto x_{ii}\left(\sum_{j=1}^{i} x_{ji}\partial_{x_{ji}} - \sum_{j=1}^{i-1} x_{ji-1}\partial_{x_{ji-1}}\right) - \sum_{j=i+1}^{n} x_{ij}\partial_{x_{i+1,j}} + \sum_{j=1}^{i-1} x_{ji}\partial_{x_{ji-1}} + \mu_i x_{ii}.$$

Here $x_{ij} = 0$ unless $1 \leq i \leq j \leq n$.

The matrix e_{1n+1} may be written as

$$e_{1n+1} := [\cdots[E_1, E_2], \cdots], E_n].$$

Using the above formulas one proves the following

Lemma 2.1

$$\pi_\mu(e_{1n+1}) = -\partial_{x_{1n}}$$

¿From now on till the end of this section we omit writing π_μ identifying Lie algebra elements with their images under π_μ.

The action of the Lie algebra $\hat{\mathfrak{g}}$ on a function on $F^0 \times \mathbf{C}^*$ is determined by the evaluation map $g \otimes z^k \mapsto z^k g$. In particular

$$F_0 = e_{1n+1} \otimes z^{-1} \mapsto -z^{-1}\partial_{x_{1n}}.$$

The result of exponentiation of these formulas is given by

Lemma 2.2 .

(i) If $\mu = 0$ then

$$1)\ \exp(-tF_0) : x_{kl} \longmapsto \begin{cases} z^{-1}t + x_{1n} & (k,l)=(1,n) \\ x_{kl} & otherwise \end{cases}$$

$$2)\ \exp(-tF_i) : x_{kl} \longmapsto \begin{cases} \frac{x_{ki}}{1+x_{ii}t} & for\ l = i \\ -\left(x_{ki} - x_{ki-1}x_{ii}\right)t + x_{ki-1} & for\ l = i-1 \\ x_{il}t + x_{i+1l} & for\ k = i+1 \\ x_{kl} & otherwise \end{cases}$$

(ii) Generically

$$\exp(-tF_i).\psi(x) = (1 + x_{ii}t)^{\mu_i}\psi(x'),\ 1 \leq i \leq n$$

where x' is given by the substitution of the item (i) while action of F_0 is independent of μ.

Proof

If $\mu = 0$ then all F's are vector fields. The problem of evaluating an exponent of a vector field is, actually, a problem of the theory of ordinary differential equations: the exponent of a vector field is an element of a 1-parametric family of diffeomorphisms generated by the vector field and, therefore, is given by a general solution to the corresponding system of o.d.e.'s. In our case the system and the solution are (resp.):

1):

$$\dot{x}_{kl} = \delta_{k1}\delta_{ln}z^{-1} \Longrightarrow x_{1n} = z^{-1}t + x_{1n};$$

2):

$$\begin{array}{lcl} \dot{x}_{ii} = x_{ii}^2 & \Longrightarrow & x_{ii} = \frac{x_{ii}(0)}{1-x_{ii}(0)t} \\ \dot{x}_{ji} = x_{ii}x_{ji} & \Longrightarrow & x_{ji} = \frac{x_{ji}(0)}{1-x_{ii}(0)t} \\ \dot{x}_{ji-1} = x_{ji} - x_{ji-1}x_{ii} & \Longrightarrow & x_{ji-1} = \left(x_{ji}(0) - x_{ji-1}(0)x_{ii}(0)\right)t + x_{ji-1}(0) \\ \dot{x}_{i+1j} = -x_{ij}\ \dot{x_{ij}} = 0 \text{ for } j \geq i+1 & \Longrightarrow & x_{ij} = x_{ij}(0) \\ & \Longrightarrow & x_{i+1j} = -x_{ij}(0)t + x_{i+1j}(0) \end{array},$$

which completes proof of the item (i). As to the item (ii), one shows that any order 1 diffrential operator is conjugated to a vector field by the multiplication by a function it annihilates. This implies (ii) since $F_i \cdot (x_{ii}^{-\mu_i}) = 0$. **Q.E.D.**

Now by using all this one can calculate the integrand of (5).But to formulate the result it is convenient to give some more notations.

Set

$$T = (t_{ij}) := \begin{pmatrix} x_{11} & \cdots & \cdots & x_{1n} \\ 1 & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & x_{nn} \end{pmatrix};$$

for $1 \le i_k \le i_{k-1} \le \cdots \le i_1 \le j$, $j+k \le n+1$,

$$I^j_{i_1,i_2,\cdots,i_k} := \{j+1-i_1, j+2-i_2, \cdots, j+k-i_k\}$$

$$J^j_k := \{j, j+1, \cdots, j+k-1\}$$

$$T^j_{i_1,i_2,\cdots i_k} := (t_{ij})_{i \in I^j_{i_1,i_2,\cdots,i_k}, j \in J^j_k}$$

$$Q^j_{i_1,i_2,\cdots i_k} := \begin{cases} \det(T^j_{i_1,i_2,\cdots i_k}) & \text{for } k>0 \\ 1 & \text{for } k=0 \end{cases}$$

Introduce a collection of functions on the big cell along with an ordering on it.

Definition We write

$$Q^j_{i_1,i_2,\cdots,i_k} \xrightarrow{F_l} Q'$$
$$\overset{\text{def}}{\Longleftrightarrow} \exp(-tF_l) Q^j_{i_1,i_2,\cdots,i_k} = \begin{cases} \frac{1}{1+x_{jj}t}\left\{Q't + Q^j_{i_1,i_2,\cdots,i_k}\right\} & \text{for } l=j \\ Q't + Q^j_{i_1,i_2,\cdots,i_k} & \text{for } l \ne j \end{cases}$$

In the definition it is assumed that $\mu = 0$.

Lemma 2.3

1)	$Q^j_{i_1,i_2,\cdots,i_k}$	$\xrightarrow{F_{j+r-1-i_r}}$	$Q^j_{i_1,i_2,\cdots,i_r+1,\cdots,i_k}$	*for* $1 \le r \le k$
2)	$Q^j_{i_1,i_2,\cdots,i_k}$	$\xrightarrow{F_{j+k}}$	$Q^j_{i_1,i_2,\cdots,i_k,1}$	*for* $k \ge 0$
3)	$Q^j_{j,i_2,\cdots,i_{n+1-j}}$	$\xrightarrow{F_0}$	$(-1)^{n-j} Q^j_{i_2-1,\cdots,i_{k'}-1} z^{-1}$	*where* $k'-1 := \sharp\{r:\ r \ge 2, i_r > 1\}$

$$Otherwise, Q \xrightarrow{F_l} 0$$

The proof of this lemma is a standard calculation of linear algebra using Lemma 2.2; in particular we use Laplace expansion of a certain determinant to prove 2).

The above definition suggests to introduce the following $n+1-$colored graph Γ. The set of vertices of Γ is the set of all $Q \ne 0$ such that

$$1 \xrightarrow{F_{j_1}} Q_1 \xrightarrow{F_{j_2}} Q_2 \longrightarrow \cdots \longrightarrow Q_{r-1} \xrightarrow{F_{j_r}}, Q$$

for some $j_1, \ldots, j_r$. It follows from Lemma 2.3 that each vertex is of the form $(-1)^{(n-j)r} Q^j_{i_1,i_2,\cdots i_k} z^{-r}$. Define a function on the set of vertices by

$$l((-1)^{(n-j)r} Q^j_{i_1,i_2,\cdots i_k} z^{-r}) = (n+1)r + \sum_{p=1}^{k} i_p.$$

Two vertices P, Q are connected by an edge of the color i if and only if

$$P \xrightarrow{F_i} Q.$$

With any vertex $Q \in \Gamma$ associate a set $\mathcal{P}(Q)$ of all oriented paths connecting 1 and Q.

Lemma 2.4 *All* $\gamma \in \mathcal{P}((-1)^{(n-j)r} Q^j_{i_1,i_2,\cdots i_k} z^{-r})$ *are of the same length* $l((-1)^{(n-j)r} Q^j_{i_1,i_2,\cdots i_k} z^{-r})$.

Proof

Lemma 2.3 shows that if there is an edge going from P to Q then $l(Q) = l(P) + 1$. The lemma now follows from the obvious remark that $l(1) = 0$.

Q.E.D

We are in a position to write down the integral (5). Recall that W is the Weyl group of $\widehat{\mathfrak{g}}$ and $w = r_{m_1} r_{m_2} \cdots r_{m_l} \in W$ is a decomposition (not necessarily reduced),where r_m denotes the reflection at the corresponding simple root (α_m). m can be viewed as a map from $I_1 (= \{1, 2, \cdots, l\})$ to $I_2 (= \{0, 1, \cdots, n\})$.Therefore,$m^{-1}(j),\ j \in I_2$, is a subset of I_1. Set,

$$\beta_j = \frac{2(r_{m_{l+2-j}} \cdots r_{m_l} \cdot \lambda_1, \alpha_{m_{l+1-j}})}{(\alpha_{m_{l+1-j}}, \alpha_{m_{l+1-j}})} + 1,\ 1 \le j \le l$$

and

$$K_w(t_1, t_2, \cdots, t_l) = \prod_{j=1}^{l} \Gamma(-\beta_j)^{-1} \times \{\exp(-t_l F_{m_l}) \cdots \exp(-t_1 F_{m_1}) \prod_{j=1}^{l} t_j^{-\beta_j - 1} 1\},$$

where 1 is viewed as an element of $V_1 \otimes \cdots \otimes V_N$ equal to the unit function on the product of N copies of the flag manifold. With any path

$$\gamma :\ 1 \xrightarrow{F_{j_1}} Q_1 \xrightarrow{F_{j_2}} Q_2 \longrightarrow \cdots \longrightarrow Q_{r-1} \xrightarrow{F_{j_r}} Q,\ r = l(Q)$$

associate a polynomial in t's:

$$f_\gamma(t) = \sum_{p_1 < \ldots < p_r, p_i \in m^{-1}(j_i)} t_{p_1} t_{p_2} \cdots t_{p_r}.$$

(This is the only point where the decomposition $w = r_{m_1} r_{m_2} \cdots r_{m_l}$ enters the calculation.) Denote by Γ^j the subgraph of Γ consisting of all vertices connected with Q_1^j by an oriented path. It is equivalently defined as a subgraph generated by all vertices $(-1)^{(n-j)r} Q^j_{i_1,i_2,\cdots i_k} z^{-r}$ with the fixed superscript j. Set

$$P^j_w(x, z; t_1, t_2, \cdots, t_l) = \sum_{l'=0}^{l} \sum_{Q \in \Gamma^j : l(Q) = l'} Q \sum_{\gamma \in \mathcal{P}(Q)} f_\gamma(t). \qquad (6)$$

Theorem 2.5

$$K_w(t_1,t_2,\cdots t_l)=\prod_{j=1}^{l}\Gamma(-\beta_j)^{-1}\prod_{p=1}^{N}\prod_{j=1}^{n}\{P_w^j(x^{(p)},z_p;t_1,t_2,\cdots,t_l)\}^{\mu_j^{(p)}}\prod_{j=1}^{l}t_j^{-\beta_j-1}, \tag{7}$$

where $\mu^{(p)}=(\mu_1^{(p)},\ldots,\mu_n^{(p)})$, $1\le p\le N$, *is a highest weight of* V_p *and* $x^{(p)}$, $1\le p\le N$, *is a coordinate in the* $p-th$ *copy of the flag manifold.*

This theorem can be proved by induction on l using Lemma2.3 and Lemma2.4.

Let $\mathcal{M}$ be the local system of continuous branches of $K_w(t_1,\cdots,t_l)$ over the Domain of $K_w(t_1,\cdots,t_l)$(say $\mathcal{D}$).Then finally we obtain

Theorem 2.6 *For any* $\sigma\in H_l(\mathcal{M},\mathcal{D})$,*the integral*

$$\overset{\circ}{\Psi}\int_\sigma K_w(t_1,t_2,\cdots,t_l)dt_1dt_2\cdots dt_l$$

satisfies the system $KZ(\lambda_{N+1},w\cdot\lambda_1)$

Remark. Theorem 2.6 gives solutions as an integral over a certain cycle depending on parameters (x,z). These cycles belong to a homology group of a complement to a collection of hypersurfaces $K_w(t_1,t_2,\cdots,t_l)=0$ with coefficients in a local system defined over this complement. Note that generically $(l>2)$, and much unlike the case of Schechtman-Varchenko integral formulas, $K_w(t_1,t_2,\cdots,t_l)=0$ is a union of hypersurfaces not isomorphic to hyperplanes and, therefore, investigation of the integral cannot be carried out by usual methods. We have already encountered with the same phenomenon in a different but related framework. As we argued in the Introduction, our integral formulas are intimately related to $\hat{\mathfrak{g}}$ or $\mathfrak{g}-$modules extended by complex powers of a Lie algebra generators. Rigorous treatment of such modules requires consideration of a Lie algebra action on sections of a local system defined over a complement to – highly non-linear – set of "shifted" Schubert cells on a flag manifold; for details see [11].

Note also that if $l=1,2$ then $K_w(t_1,t_2,\cdots,t_l)=0$ is isomorphic to a union of affine hyperplanes and the number of cycles can be calculated using results of [5].

2.2 Some examples

Theorem 2.5 produces rather an algorithm to write down the kernel of the integral (5) than a completely explicit formula for it: (7) relies on (6), while the latter is a linear combination of explicitly given polynomials $Q^j_{i_1\ldots i_k}z^{-r}$ with coefficients in the form $\sum_p t_{p_1}\cdots t_{p_r}$ determined by the combinatorial data. We have been able to "resolve" the combinatorial part of the formula in the cases

$\mathfrak{g} = \mathfrak{sl}_2,\ \mathfrak{sl}_3$. Although the $\mathfrak{sl}_2-$case was treated in [15], we discuss here both in a unified way for completeness.

The $\mathfrak{sl}_2-$case. In this case the flag manifold is $\mathbf{CP}^1$, the big cell is $\mathbf{C} \subset \mathbf{CP}^1$. Fix a coordinate x on $\mathbf{C}$. Then the matrix T (via which the polynomials $Q^j_{i_1,\dots,i_k}$ are defined) is given by $T = (x)$. The set of all $Q^j_{i_1,\dots,i_k}$ consists of 2 elements: $Q^1 = 1,\ Q^1_1 = x$. The vertices of the graph Γ are all of the form: $A^\epsilon_i(x,z) = z^{-i}x^\epsilon,\ \epsilon = 0,1,\ i = 0,1,2,3,\dots$. Further, Γ coincides with Γ^1 and is given by

$$1 \xrightarrow{F_1} x \xrightarrow{F_0} z^{-1} \xrightarrow{F_1} z^{-1}x \xrightarrow{F_0} z^{-2} \xrightarrow{F_1} z^{-2}x \cdots.$$

Observe that the Weyl group of $\widehat{\mathfrak{sl}}_2$ is a free group generated by 2 reflections $r_0,\ r_1$ and, therefore, each element is uniquely expanded as either

$$r_0 r_1 \cdots$$

or

$$r_1 r_0 \cdots$$

the second one being relevant to our calculation. Setting

$$w = \underbrace{r_1 r_0 \cdots r_\epsilon}_{l},$$

one obtains

$$P_w(x,z;t_1,\dots,t_l)\ (= P^1_w(x,z;t_1,\dots,t_l)) \ = \sum_{\epsilon=0}^{1}\sum_{i=0}^{l-\epsilon} z^{-i}x^\epsilon \sigma_{i+\epsilon}(t_1,\dots,t_l),$$

$$\sigma_j(t_1,\dots,t_l) = \sum_{0\le i_1<i_2<\cdots<i_j<l/2} t_{2i_1+1}t_{2i_2+1}\cdots t_{2i_j+1},$$

completing the $\mathfrak{sl}_2-$case.

The $\mathfrak{sl}_3-$case. The big cell is $\mathbf{C}^3$ with coordinates x_{11}, x_{12}, x_{22}. The matrix T is given by $T = \begin{pmatrix} x_{11} & x_{12} \\ 1 & x_{22} \end{pmatrix}$. The set of all $Q^j_{i_1,\cdots,i_k}$ consists of 5 elements:

$$Q^1 = 1,\ Q^j_1 = x_{jj}(j = 1,2)\ Q^1_{11} = x_{11}x_{22} - x_{12}, Q^2_2 = x_{12}$$

.The graph Γ and its subgraphs $\Gamma^1,\ \Gamma^2$ are given by

$$\begin{array}{lccccccccccc}
\Gamma_1: & & Q^1_1 & \xrightarrow{F_2} & Q^1_{11} & \xrightarrow{F_0} & -z^{-1} & \xrightarrow{F_1} & -z^{-1}Q^1_1 & \xrightarrow{F_2} & -z^{-1}Q^1_{11} & \cdots \\
 & \overset{F_1}{\nearrow} & & & & & & & & & & \\
1 & & & & & & & & & & & \\
 & \underset{F_2}{\searrow} & & & & & & & & & & \\
\Gamma_2: & & Q^2_1 & \xrightarrow{F_1} & Q^2_2 & \xrightarrow{F_0} & z^{-1} & \xrightarrow{F_2} & z^{-1}Q^2_1 & \xrightarrow{F_1} & z^{-1}Q^2_2 & \cdots
\end{array}$$

The Weyl group W of $\widehat{\mathfrak{sl}_3}$ is realized as a group generated by reflections at a certain collection of affine lines on the plane (see [16]). These lines produce a covering of the plane by triangles, called *alcoves*, which W acts on effectively. Looking at this action one obtains a collection of elements of W so that a reduced decomposition of any element of W is contained in it:

Put $c := r_0 r_1 r_2$(c is called a Coxeter element), then any $w \in W$ can be written as $w = sc^k tc^{-l}u$ where $s = e, r_2, r_1 r_2$, $u = e, r_2, r_2 r_1$,$t = r_0, r_0 r_1, r_0 r_1 r_0$ if $kl \neq 0$ and if $kl = 0$, then t can also be equal to e.

Further one obtains

$$P_w^j(x, z; t_1, t_2, \cdots, t_l) = \sum_{l'=0}^{l} A_{l'}^j f_{l'}^j(t),\ j = 1, 2,$$

where

$$f_{l'}^j(t) = \sum_{p_1 < p_2 < \cdots < p_{l'}, p_i \in m^{-1}((ji)_3)} t_{p_1} t_{p_2} \cdots t_{p_{l'}},$$

$(k)_3 \in \{0, 1, 2\}$ signifies the residue of k modulo 3 and $m : \{1, \ldots l\} \to \{0, 1, 2\}$ is a function determining a reduced decomposition of w,

$$A_{l'}^1 = \begin{cases} (-z)^{-q} & \text{if} \quad l' = 3q \\ (-z)^{-q} x_{11} & \text{if} \quad l' = 3q+1 \\ (-z)^{-q}(x_{11}x_{22} - x_{12}) & \text{if} \quad l' = 3q+2, \end{cases}$$

$$A_{l'}^2 = \begin{cases} z^{-q} & \text{if} \quad l' = 3q \\ z^{-q} x_{22} & \text{if} \quad l' = 3q+1 \\ z^{-q} x_{12} & \text{if} \quad l' = 3q+2. \end{cases}$$

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