SUBADDITIVITY OF THE LOGARITHMIC KODAIRA DIMENSION FOR MORPHISMS OF RELATIVE DIMENSION ONE REVISITED

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Abstract. The main purpose of this paper is to make the subadditivity theorem of the logarithmic Kodaira dimension for morphisms of relative dimension one, which is Kawamata's theorem, more accessible. We give a proof without depending on Kawamata's original paper. For this purpose, we discuss algebraic fiber spaces whose general fibers are of general type in detail. We also discuss elliptic fibrations. One of the main new ingredients of our proof is the effective freeness due to Popa and Schnell, which is a clever application of Kollár's vanishing theorem. We note that our approach to the subadditivity conjecture of the Kodaira dimension is slightly simpler and clearer than the classical approaches thanks to the weak semistable reduction theorem by Abramovich and Karu. Obviously, this paper is heavily indebted to Viehweg's ideas.

Contents

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1. Introduction

This paper is a completely revised and extremely expanded version of the author's unpublished short note [F1]:

- *Osamu Fujino, $\overline{C}_{n,n-1}$ revisited, preprint (2003).*

Roughly speaking, the final section (see Section 9) of this paper is a slightly expanded and revised version of the above short note and the other sections are new. If the reader is familiar with $Q_{n,n-1}$ and $C^{+}_{n,n-1}$ and is only interested in $\overline{C}_{n,n-1}$ (see Theorem 1.1), then we recommend him to go directly to Section 9.

Let us recall $\overline{C}_{n,n-1}$, that is, the subadditivity theorem of the logarithmic Kodaira dimension for morphisms of relative dimension one, which is the main result of [Kaw1]. Note that [Kaw1] is one of Kawamata's master theses to the Faculty of Science, University of Tokyo.

Theorem 1.1 ([Kaw1, Theorem 1])**.** *Let $f : X \to Y$ be a dominant morphism of algebraic varieties defined over the complex number field $\mathbb{C}$. We assume that the general fiber $X_y = f^{-1}(y)$ is an irreducible curve. Then we have the following inequality for logarithmic Kodaira dimensions:*

$$\overline{\kappa}(X) \geq \overline{\kappa}(Y) + \overline{\kappa}(X_y).$$

Note that Theorem 1.1 plays very important roles in [F13]. The main purpose of this paper is to make Theorem 1.1 more accessible. Since the author is not sure if some technical arguments in [Kaw1] are correct, we give a proof of Theorem 1.1 without depending on Kawamata's original paper [Kaw1]. In general, we have the following conjecture.

Conjecture 1.2 (Subadditivity of logarithmic Kodaira dimension)**.** *Let $f : X \to Y$ be a dominant morphism between algebraic varieties whose general fibers are irreducible. Then we have the following inequality*

$$\overline{\kappa}(X) \geq \overline{\kappa}(Y) + \overline{\kappa}(X_y),$$

where X_y is a sufficiently general fiber of $f : X \to Y$.

Therefore, Theorem 1.1 says that Conjecture 1.2 holds true when $\dim X - \dim Y = 1$. Conjecture 1.2 is usually called Conjecture $\overline{C}_{n,m}$ when $\dim X = n$ and $\dim Y = m$. Thus, Theorem 1.1 means that $\overline{C}_{n,n-1}$ is true. We note the following theorem, which is one of the main consequences of [F11].

Theorem 1.3. *Conjecture 1.2 follows from the generalized abundance conjecture for projective divisorial log terminal pairs.*

The generalized abundance conjecture is one of the most important and difficult problems in the minimal model program and is still open. For the details, see [F11] and [F10].

Before we go further, let us quote the introduction of [F1] for the reader's convenience.

> In spite of its importance, the proof of $\overline{C}_{n,n-1}$ is not so easy to access for the younger generation, including myself. After [Kaw1] was published, the birational geometry has drastically developed. When Kawamata wrote [Kaw1], the following techniques and results are not known nor fully matured.
>
> - Kawamata's covering trick,
> - moduli theory of curves, especially, the notion of level structures and the existence of tautological families,
> - various notions of singularities such as rational singularities, canonical singularities, and so on.
>
> See [Kaw2, §2], [AbK, Section 5], [AbO, Part II], [vaGO], [Vi2], and [KoM]. In the mid 1990s, de Jong gave us fantastic results: [dJ1] and [dJ2]. The alteration paradigm generated the weak semistable reduction theorem [AbK]. This paper shows how to simplify the proof of the main theorem of [Kaw1] by using the weak semistable reduction. The proof may look much simpler than Kawamata's original proof (note that we have to read [Vi1] and [Vi2] to understand [Kaw1]). However, the alteration theorem grew out from the deep investigation of the moduli space of stable pointed curves (see [dJ1] and [dJ2]). So, don't misunderstand the real value of this paper. We note that we do not enforce Kawamata's arguments. We only recover his main result. Of course, this paper is not self-contained.

Anyway, it is much easier to give a rigorous proof of Theorem 1.1 without depending on Kawamata's paper [Kaw1] than to check all the details of [Kaw1] and correct some mistakes in [Kaw1]. We note that [Kaw1, Lemma 2] does not take Viehweg's correction [Vi2] into account.

1.4 (Background and motivation)**.** In the proof of [Kaw1, Lemma 4], Kawamata considered the following commutative diagram:

$$\begin{tikzcd} & X_0 \arrow[dl, "h"'] \arrow[dd, "g"] \\ X_1 \arrow[dr, "f"'] & \\ & X_2 \end{tikzcd}$$

in order to prove $R^i f_*\mathcal{O}_{X_1}(-D_1) = 0$ for every $i > 0$. In the first half of the proof of [Kaw1, Lemma 4], he proved $R^i g_*\mathcal{O}_{X_0}(-D_0) = 0$ for every $i > 0$ by direct easy calculations. The author is not sure if Kawamata's argument in the proof of [Kaw1, Lemma 4] is sufficient for proving $R^i f_*\mathcal{O}_{X_1}(-D_1) = 0$ for every $i > 0$ from $R^i g_*\mathcal{O}_{X_0}(-D_0) = 0$ for every $i > 0$. Of course, we can check $R^i f_*\mathcal{O}_{X_1}(-D_1) = 0$ for $i > 0$ as follows.

Let us consider the usual spectral sequence:

$$E_2^{p,q} = R^p f_* R^q h_*\mathcal{O}_{X_0}(-D_0) \Rightarrow R^{p+q} g_*\mathcal{O}_{X_0}(-D_0).$$

Note that $h_*\mathcal{O}_{X_0}(-D_0) \simeq \mathcal{O}_{X_1}(-D_1)$ by the definitions of D_0 and D_1. Since

$$E_2^{1,0} \simeq R^1 f_*\mathcal{O}_{X_1}(-D_1) \hookrightarrow R^1 g_*\mathcal{O}_{X_0}(-D_0) = 0,$$

we obtain $R^1 f_*\mathcal{O}_{X_1}(-D_1) = 0$. By applying this argument to $h : X_0 \to X_1$, we can prove $R^1 h_*\mathcal{O}_{X_0}(-D_0) = 0$. This is a crucial step. This implies that $E_2^{p,1} = 0$ for every p. Thus we obtain the inclusion

$$E_2^{2,0} \simeq E_\infty^{2,0} \hookrightarrow R^2 g_*\mathcal{O}_{X_0}(-D_0) = 0.$$

Therefore, we have $E_2^{2,0} \simeq R^2 f_*\mathcal{O}_{X_1}(-D_1) = 0$. As above, we obtain $R^2 h_*\mathcal{O}_{X_0}(-D_0) = 0$. This implies that $E_2^{p,1} = E_2^{p,2} = 0$ for every p. Then we get the inclusion

$$E_2^{3,0} \simeq E_\infty^{3,0} \hookrightarrow R^3 g_*\mathcal{O}_{X_0}(-D_0) = 0$$

and $E_2^{3,0} \simeq R^3 f_*\mathcal{O}_{X_1}(-D_1) = 0$. By repeating this process, we finally obtain $R^i f_*\mathcal{O}_{X_1}(-D_1) = 0$ for every $i > 0$.

The author does not know whether the above understanding of [Kaw1, Lemma 4] is the same as what Kawamata wanted to say in the proof of [Kaw1, Lemma 4] or not. It seemed to the author that Kawamata only proves that the composition

$$\begin{aligned} \mathbf{R}f_*(\varphi_{01}) \circ \varphi_{12} : \mathcal{O}_{X_2}(-D_2) &\longrightarrow \mathbf{R}f_*\mathcal{O}_{X_1}(-D_1) \\ &\longrightarrow \mathbf{R}f_*\mathbf{R}h_*\mathcal{O}_{X_0}(-D_0) \end{aligned}$$

is a quasi-isomorphism in the derived category of coherent sheaves on X_2. Of course, we think that we can easily check the statement of [Kaw1, Lemma 4] by using the weak factorization theorem in [AKMW], which was obtained much later than [Kaw1].

As we have already pointed it out above, [Kaw1] does not take Viehweg's correction [Vi2] into account. Note that the statement of [Kaw1, Lemma 2] is obviously wrong. This mistake comes from an error in [Vi1]. Therefore, we have to correct the statement of [Kaw1, Lemma 2] and modify some related statements in [Kaw1] in order to complete the proof of Theorem 1.1 in [Kaw1].

Anyway, the author gave up checking the technical details of [Kaw1] and correcting mistakes in [Kaw1], and decided to give a proof of Theorem 1.1 without depending on [Kaw1]. We will not use [Kaw1, Lemma 2] nor [Kaw4, Lemma 4]. We will adopt a slightly different approach to Theorem 1.1 in this paper. The author believes that his decision is much more constructive. We also note that the reader does not have to refer to [Vi1] in order to understand the proof of Theorem 1.1 in this paper. Therefore, the author thinks that the proof of Theorem 1.1 in this paper is much more accessible than the original proof in [Kaw1].

Let us recall various conjectures related to Conjecture 1.2. Obviously, Conjecture 1.2 is a generalization of the famous Iitaka conjecture C.

Conjecture 1.5 (Iitaka Conjecture C)**.** *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Then the inequality*

$$\kappa(X) \geq \kappa(X_y) + \kappa(Y)$$

holds, where X_y is a sufficiently general fiber of $f : X \to Y$.

The following more precise conjecture is due to Viehweg.

Conjecture 1.6 (Generalized Iitaka Conjecture C^+)**.** *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Assume that $\kappa(Y) \geq 0$. Then the inequality*

$$\kappa(X) \geq \kappa(X_y) + \max\{\mathrm{Var}(f), \kappa(Y)\}$$

holds, where X_y is a sufficiently general fiber of $f : X \to Y$.

In Section 6, we describe that Conjecture 1.6 follows from Viehweg's conjecture Q (see Conjecture 1.7 below). For this purpose, we treat the basic properties of weakly positive sheaves and big sheaves, and Viehweg's base change trick in Section 3. Almost everything in Section 3 is contained in Viehweg's papers [Vi3] and [Vi4]. Moreover, we discuss very important Viehweg's arguments for direct images of pluricanonical bundles and adjoint bundles in Section 5, which are also contained in Viehweg's papers [Vi3] and [Vi4]. Our treatment in Section 6 is essentially the same as Viehweg's original one (see [Vi3, §7]). However, it is slightly simplified and refined by the use of the weak semistable reduction theorem due to Abramovich–Karu (see [AbK]).

We note that Viehweg's conjecture Q is as follows:

Conjecture 1.7 (Viehweg Conjecture Q)**.** *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Assume that* $\mathrm{Var}(f) = \dim Y$*. Then $f_*\omega_{X/Y}^{\otimes k}$ is big for some positive integer k.*

If $\dim X = n$ and $\dim Y = m$ in the above conjectures, then Conjectures C, C^+, and Q are usually called Conjectures $C_{n,m}$, $C^+_{n,m}$, and $Q_{n,m}$ respectively.

In [Kaw4], Kawamata proves Conjecture 1.7 under the assumption that the geometric generic fiber of $f : X \to Y$ has a good minimal model (see [Kaw4, Theorem 1.1]). Note that [Kaw4], which is a generalization of Viehweg's paper [Vi4], treats infinitesimal Torelli problems for the proof of Conjecture 1.7. In this paper, we do not discuss infinitesimal Torelli problems nor the results in [Kaw4].

In Section 7, we give a relatively simple proof of Viehweg's conjecture Q (see Conjecture 1.7) under the assumption that the geometric generic fiber of $f : X \to Y$ is of general type. The main theorem of Section 7, that is, Theorem 7.1, is slightly better than the well-known results by Kollár [Ko2] and Viehweg [Vi6] for algebraic fiber spaces whose general fibers are of general type.

Theorem 1.8 (Theorem 7.1 and Remark 7.3)**.** *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Assume that the geometric generic fiber $X_{\overline{\eta}}$ of $f : X \to Y$ is of general type and that* $\mathrm{Var}(f) = \dim Y$*. Then there exists a generically finite surjective morphism $\tau : Y' \to Y$ from a smooth projective variety Y' such that $f'_*\omega_{X'/Y'}^{\otimes k}$ is a semipositive and big locally free sheaf on Y' for some positive integer k, where X' is a resolution of the main component of $X \times_Y Y'$ and $f' : X' \to Y'$ is the induced morphism.*

We do not need the theory of variations of (mixed) Hodge structure for the proof of Theorem 1.8. One of the main new ingredients of Theorem 1.8 (see Theorem 7.1) is the effective freeness due to Popa–Schnell (see [PopS]).

Theorem 1.9 (Theorem 4.1)**.** *Let $f : X \to Y$ be a surjective morphism from a smooth projective variety X to a projective variety Y*

with $\dim Y = n$*. Let* k *be a positive integer and let* $\mathcal{L}$ *be an ample invertible sheaf on* Y *such that* $|\mathcal{L}|$ *is free. Then we have*

$$H^i(Y, f_*\omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes l}) = 0$$

for every $i > 0$ *and every* $l \geq nk + k - n$*. By Castelnuovo–Mumford regularity,* $f_*\omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes l}$ *is generated by global sections for every* $l \geq k(n+1)$*.*

We prove this effective freeness in Section 4 for the reader's convenience (see Theorem 4.1). The proof of Theorem 4.1 is a clever application of a generalization of Kollár's vanishing theorem and is very simple. Anyway, we have:

Theorem 1.10 (..., Kollár, Viehweg, ...)**.** *Let* $f : X \to Y$ *be a surjective morphism between smooth projective varieties with connected fibers whose general fibers are of general type. Then we have*

$$\begin{aligned}\kappa(X) &\geq \kappa(X_y) + \max\{\mathrm{Var}(f), \kappa(Y)\}\\ &= \dim X - \dim Y + \max\{\mathrm{Var}(f), \kappa(Y)\}\end{aligned}$$

where X_y *is a sufficiently general fiber of* $f : X \to Y$*.*

In Section 8, we quickly review elliptic fibrations and see that Conjecture 1.7 holds for elliptic fibrations. Therefore, we have:

Theorem 1.11 (Viehweg, ...)**.** *Let* $f : X \to Y$ *be a surjective morphism between smooth projective varieties with connected fibers whose general fibers are elliptic curves. Then we have*

$$\begin{aligned}\kappa(X) &\geq \kappa(X_y) + \max\{\mathrm{Var}(f), \kappa(Y)\}\\ &= \max\{\mathrm{Var}(f), \kappa(Y)\}\end{aligned}$$

where X_y *is a sufficiently general fiber of* $f : X \to Y$*.*

By combining Theorem 1.10 with Theorem 1.11, we have:

Corollary 1.12 (Viehweg [Vi1])**.** *Let* $f : X \to Y$ *be a surjective morphism between smooth projective varieties whose general fibers are irreducible curves. Then we have*

$$\kappa(X) \geq \kappa(X_y) + \kappa(Y)$$

where X_y *is a general fiber of* $f : X \to Y$*.*

Note that the proof of Theorem 1.1 in Section 9 uses Theorem 1.10 and Theorem 1.11. More precisely, we use the solution of Conjecture 1.7 for morphisms of relative dimension one. We also note that Kawamata's original proof of Theorem 1.1 heavily depends on Viehweg's paper [Vi1]. We do not directly use [Vi1] in this paper. Therefore, the reader can understand the proof of Theorem 1.1 in this paper without referring to [Vi1].

Finally, this paper is also an introduction to Viehweg's theory of weakly positive sheaves and big sheaves. Some of Viehweg's arguments in [Vi3] and [Vi4] are simplified by the use of the weak semistable reduction theorem due to Abramovich and Karu. We hope that this paper will make Viehweg's ideas in [Vi3] and [Vi4] more accessible.

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We will work over $\mathbb{C}$, the complex number field, throughout this paper.

2. Preliminaries

In this section, we collect some basic notations and results for the reader's convenience. For the details, see [U], [KoM], [Mo], [F6], [F10], and so on.

2.1 (Generically generation)**.** Let $\mathcal{F}$ be a coherent sheaf on a smooth quasi-projective variety X. We say that $\mathcal{F}$ is *generated by global sections over* U, where U is a Zariski open set of X, if the natural map

$$H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \to \mathcal{F}$$

is surjective over U. We say that $\mathcal{F}$ is *generically generated by global sections* if $\mathcal{F}$ is generated by global sections over some nonempty Zariski open set of X.

2.2. Let $\mathcal{F}$ be a coherent sheaf on a normal variety X. We put

$$\mathcal{F}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$$

and

$$\mathcal{F}^{**} = (\mathcal{F}^*)^*.$$

We also put

$$\widehat{S}^\alpha(\mathcal{F}) = (S^\alpha(\mathcal{F}))^{**}$$

for every positive integer α, where $S^\alpha(\mathcal{F})$ is the α-th symmetric product of $\mathcal{F}$, and

$$\widehat{\det}(\mathcal{F}) = (\wedge^r \mathcal{F})^{**}$$

where $r = \mathrm{rank}\mathcal{F}$. When X is smooth, $\widehat{\det}(\mathcal{F})$ is invertible since it is a reflexive sheaf of rank one.

We note the following definition of exceptional divisors.

2.3 (Exceptional divisors)**.** Let $f : X \to Y$ be a proper surjective morphism between normal varieties. Let E be a Weil divisor on X. We say that E is *f-exceptional* if $\mathrm{codim}_Y f(\mathrm{Supp}E) \geq 2$. Note that f is not always assumed to be birational. When $f : X \to Y$ is a birational morphism, $\mathrm{Exc}(f)$ denotes the exceptional locus of f.

We sometimes use $\mathbb{Q}$-divisors in this paper.

2.4 (Operations for $\mathbb{Q}$-divisors)**.** Let $D = \sum_i a_i D_i$ be a $\mathbb{Q}$-divisor on a normal variety X, where D_i is a prime divisor on X for every i, $D_i \neq D_j$ for $i \neq j$, and $a_i \in \mathbb{Q}$ for every i. Then we put $\lfloor D \rfloor = \sum_i \lfloor a_i \rfloor D_i$, $\{D\} = D - \lfloor D \rfloor$, and $\lceil D \rceil = -\lfloor -D \rfloor$. Note that $\lfloor a_i \rfloor$ is the integer which satisfies $a_i - 1 < \lfloor a_i \rfloor \leq a_i$. We also note that $\lfloor D \rfloor$, $\lceil D \rceil$, and $\{D\}$ are called the *round-down*, *round-up*, and *fractional part* of D respectively.

2.5 (Dualizing sheaves and canonical divisors)**.** Let X be a normal quasi-projective variety. Then we put $\omega_X = \mathcal{H}^{-\dim X}(\omega_X^\bullet)$, where $\omega_X^\bullet$ is the dualizing complex of X, and call ω_X the *dualizing sheaf* of X. We put $\omega_X \simeq \mathcal{O}_X(K_X)$ and call K_X the *canonical divisor* of X. Note that K_X is a well-defined Weil divisor on X up to the linear equivalence. Let $f : X \to Y$ be a morphism between Gorenstein varieties. Then we put $\omega_{X/Y} = \omega_X \otimes f^*\omega_Y^{\otimes -1}$.

2.6 (Singularities of pairs)**.** Let X be a normal variety and let Δ be an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a resolution of singularities. We write

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i$$

and $a(E_i, X, \Delta) = a_i$. Note that the *discrepancy* $a(E, X, \Delta) \in \mathbb{Q}$ can be defined for every prime divisor E *over* X. If $a(E, X, \Delta) > -1$ for every exceptional divisor E over X, then (X, Δ) is called a *plt* pair. If $a(E, X, \Delta) > -1$ for every divisor E over X, then (X, Δ) is called a *klt* pair. In this paper, if $\Delta = 0$ and $a(E, X, 0) \geq 0$ for every divisor E over X, then we say that X has only *canonical singularities*.

For the details of singularities of pairs, see [F6] and [F10].

2.7 (Iitaka dimension and Kodaira dimension)**.** Let D be a Cartier divisor on a normal projective variety X. The *Iitaka dimension* $\kappa(X, D)$ is defined as follows:

$$\kappa(X, D) = \begin{cases} \max\limits_{m>0}\{\dim \Phi_{|mD|}(X)\} & \text{if } |mD| \neq \emptyset \text{ for some } m > 0 \\ -\infty & \text{otherwise} \end{cases}$$

where $\Phi_{|mD|} : X \dashrightarrow \mathbb{P}^{\dim |mD|}$ and $\Phi_{|mD|}(X)$ denotes the closure of the image of the rational map $\Phi_{|mD|}$. Let D be a $\mathbb{Q}$-Cartier divisor on X. Then we put

$$\kappa(X, D) = \kappa(X, m_0 D)$$

where m_0 is a positive integer such that $m_0 D$ is Cartier.

Let X be a smooth projective variety. Then we put $\kappa(X) = \kappa(X, K_X)$. Note that $\kappa(X)$ is usually called the *Kodaira dimension* of X. If X is an arbitrary projective variety. Then we put $\kappa(X) = \kappa(\widetilde{X}, K_{\widetilde{X}})$, where $\widetilde{X} \to X$ is a projective birational morphism from a smooth projective variety $\widetilde{X}$.

The following inequality is well known and is easy to check.

Lemma 2.8 (Easy addition)**.** *Let $f : X \to Y$ be a surjective morphism between normal projective varieties with connected fibers and let D be a $\mathbb{Q}$-Cartier divisor on X. Then we have*

$$\kappa(X, D) \leq \dim Y + \kappa(X_y, D_y)$$

where X_y is a general fiber of $f : X \to Y$ and $D_y = D|_{X_y}$.

Proof. We take a large and divisible positive integer m such that $\Phi_{|mD|} : X \dashrightarrow \mathbb{P}^N$ gives an Iitaka fibration. We consider the following diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\varphi"] \arrow[d, "f"'] & \mathbb{P}^N \times Y \arrow[r, "p_1"] \arrow[dl, "p_2"] & \mathbb{P}^N \\ Y & & \end{tikzcd}$$

where $\varphi = \Phi_{|mD|} \times f$ and p_1 and p_2 are natural projections. Let Z be the image of φ in $\mathbb{P}^N \times Y$. Then we obtain that

$$\begin{aligned} \kappa(X, D) &= \dim p_1(Z) \\ &\leq \dim Z \\ &= \dim Y + \dim Z_y \\ &\leq \dim Y + \kappa(X_y, D|_{X_y}) \end{aligned}$$

where y is a general point of Y. This is the desired inequality. $\square$

2.9 (Logarithmic Kodaira dimension)**.** Let V be an irreducible algebraic variety. By Nagata, we have a complete algebraic variety $\overline{V}$ which contains V as a dense Zariski open subset. By Hironaka, we have a smooth projective variety $\overline{W}$ and a projective birational morphism $\mu : \overline{W} \to \overline{V}$ such that if $W = \mu^{-1}(V)$, then $\overline{D} = \overline{W} - W = \mu^{-1}(\overline{V} - V)$ is a simple normal crossing divisor on $\overline{W}$. The *logarithmic Kodaira dimension* $\overline{\kappa}(V)$ of V is defined as

$$\overline{\kappa}(V) = \kappa(\overline{W}, K_{\overline{W}} + \overline{D})$$

where κ denotes Iitaka dimension in 2.7. Note that $\overline{\kappa}(V)$ is well defined, that is, $\overline{\kappa}(V)$ is independent of the choice of $(\overline{W}, \overline{D})$.

We note the following easy but important example.

Example 2.10. Let C be a (not necessarily complete) smooth curve. Then we can easily see that

$$\overline{\kappa}(C) = \begin{cases} -\infty & C = \mathbb{P}^1 \text{ or } \mathbb{A}^1, \\ 0 & C \text{ is an elliptic curve or } \mathbb{G}_m, \\ 1 & \text{otherwise.} \end{cases}$$

2.11 (Sufficiently general fibers)**.** Let $f : X \to Y$ be a morphism between algebraic varieties. Then a *sufficiently general fiber* F of $f : X \to Y$ means that $F = f^{-1}(y)$ where y is any point contained in a countable intersection of nonempty Zariski open subsets of Y. A sufficiently general fiber is sometimes called a *very general fiber* in the literature.

2.12 (Horizontal and vertical divisors)**.** Let $f : X \to Y$ be a dominant morphism between normal varieties and let D be a $\mathbb{Q}$-divisor on X. We can write $D = D_{\mathrm{hor}} + D_{\mathrm{ver}}$ such that every irreducible component of D_{hor} (resp. D_{ver}) is mapped (resp. not mapped) onto Y. If $D = D_{\mathrm{hor}}$ (resp. $D = D_{\mathrm{ver}}$), D is said to be *horizontal* (resp. *vertical*).

In this paper, we will repeatedly use the notion of weakly semistable morphisms due to Abramovich–Karu (see [AbK] and [Kar]).

2.13 (Weakly semistable morphisms)**.** Let $f : X \to Y$ be a projective surjective morphism between quasi-projective varieties. Then $f : X \to Y$ is called *weakly semistable* if

(i) the varieties X and Y admit toroidal structures $(U_X \subset X)$ and $(U_Y \subset Y)$ with $U_X = f^{-1}(U_Y)$,
(ii) with this structure, the morphism f is toroidal,
(iii) the morphism f is equidimensional,
(iv) all the fibers of the morphism f are reduced, and

(v) Y is smooth.

Note that $(U_X \subset X)$ and $(U_Y \subset Y)$ are toroidal embeddings without self-intersection in the sense of [KKMS, Chapter II, §1]. For the details, see [AbK] and [Kar].

The following lemma is easy but very useful.

Lemma 2.14. *Let $f : X \to Y$ and $g : Z \to Y$ be weakly semistable. Then $V = X \times_Y Z$ has only rational Gorenstein singularities. We consider the following commutative diagram.*

$$\begin{tikzcd} X \arrow[d, "f"'] & V \arrow[l, "g'"'] \arrow[d, "f'"] \\ Y & Z \arrow[l, "g"] \end{tikzcd}$$

Then we have that

$$g'^*\omega_{X/Y} = \omega_{V/Z}$$

and

$$g^*f_*\omega_{X/Y}^{\otimes n} = f'_*g'^*\omega_{X/Y}^{\otimes n} = f'_*\omega_{V/Z}^{\otimes n}$$

for every integer n.

Proof. By the flat base change theorem [Ve, Theorem 2] (see also [H1], [C], and so on), we see that V is Gorenstein and $g'^*\omega_{X/Y} = \omega_{V/Z}$. Since f and g are weakly semistable, we see that V is smooth in codimension one. Therefore, V is a normal variety. Since V is local analytically isomorphic to a toric variety, V has only rational singularities. By the flat base change theorem (see [H2, Chapter III, Proposition 9.3]), we obtain $g^*f_*\omega_{X/Y}^{\otimes n} = f'_*g'^*\omega_{X/Y}^{\otimes n}$ for every integer n. □

The following lemma is an easy consequence of Kawamata's covering trick and Abhyankar's lemma (see [Kaw2, Corollary 19]).

Lemma 2.15. *Let $f : Y \to X$ be a finite surjective morphism from a normal projective variety Y to a smooth projective variety X. Assume that f is étale over $X \setminus \Sigma_Y$, where Σ_Y is a simple normal crossing divisor on X. Then we can take a finite surjective morphism $g : Z \to Y$ from a smooth projective variety Z such that $f \circ g : Z \to X$ is étale over $X \setminus \Sigma_Z$, where Σ_Z is a simple normal crossing divisor on X such that $\Sigma_Y \leq \Sigma_Z$ and that* $\mathrm{Supp}(f \circ g)^*\Sigma_Z$ *is a simple normal crossing divisor on Z.*

Proof. Without loss of generality, we may assume that $f : Y \to X$ is Galois. We put $\Sigma_Y = \sum D_i$, where D_i is a prime divisor for every i and $D_i \neq D_j$ for $i \neq j$. We write $f^*D_i = m_i(f^*D_i)_{\mathrm{red}}$ for every i. By taking a Kawamata cover $\tau : \widetilde{X} \to X$ from a smooth projective variety $\widetilde{X}$, where $\widetilde{X}$ is étale over $X \setminus \Sigma_{\widetilde{X}}$, $\Sigma_{\widetilde{X}}$ is a simple normal crossing divisor on X with $\Sigma_Y \leq \Sigma_{\widetilde{X}}$, and $\tau^*D_i = \sum_j m_{ij}D_{ij}$ such that m_i divides m_{ij} for every i, j, where $\tau^*D_i = \sum_j m_{ij}D_{ij}$ is the irreducible decomposition of τ^*D_i. Let Z be the normalization of an irreducible component of the fiber product $Y \times_X \widetilde{X}$.

$$\begin{tikzcd} Y \arrow[d, "f"'] & Z \arrow[l, "g"'] \arrow[d] \\ X & \widetilde{X} \arrow[l, "\tau"] \end{tikzcd}$$

Then Z is étale over $\widetilde{X}$. Therefore, Z is a smooth projective variety. Moreover, $Z \to X$ is étale over $X \setminus \Sigma_Z$ with $\Sigma_Z = \Sigma_{\widetilde{X}}$ and $\mathrm{Supp}(f \circ g)^*\Sigma_Z$ is a simple normal crossing divisor on Z. □

Finally, we give some supplementary results on abelian varieties for the reader's convenience (see [F2, §5. Some remaks on Abelian varieties]). We will use Corollary 2.19 in the proof of Theorem 1.1 in Section 9.

2.16 (On Abelian varieties)**.** Let Y be a not necessarily complete variety and let A be an Abelian variety. We put $Z = Y \times A$. Let $\mu : A \times A \to A$ be the multiplication. Then A acts on A naturally by the group law of A. This action induces a natural action on Z. We denote it by $m : Z \times A \to Z$, that is,

$$m : ((y, a), b) \mapsto (y, a + b),$$

where $(y, a) \in Y \times A = Z$ and $b \in A$. Let $p_{1i} : Z \times A \times A \to Z \times A$ be the projection onto the $(1, i)$-factor for $i = 2, 3$, and let $p_{23} : Z \times A \times A \to A \times A$ be the projection onto $(2, 3)$-factor. Let $p : Z \times A \times A \to Z$ be the first projection and let $p_i : Z \times A \times A \to A$ be the i-th projection for $i = 2, 3$. We define the projection $\rho : Z = Y \times A \to A$. We fix a section $s : A \to Z$ such that $s(A) = \{y_0\} \times A$ for a point $y_0 \in Y$. We define morphisms as follows:

$$\begin{aligned} \pi_i &= p_i \circ (s \times id_A \times id_A) \quad \text{for } i = 2, 3 \\ \pi_{23} &= p_{23} \circ (s \times id_A \times id_A), \quad \text{and} \\ \pi &= \rho \times id_A \times id_A. \end{aligned}$$

Let L be an invertible sheaf on Z. We define an invertible sheaf $\mathcal{L}$ on $Z \times A \times A$ as follows:

$$\begin{aligned} \mathcal{L} =& p^*L \otimes (id_Z \times \mu)^*m^*L \otimes (p_{12}^*m^*L)^{\otimes -1} \otimes (p_{13}^*m^*L)^{\otimes -1} \\ & \otimes \pi^*((\pi_{23}^*\mu^*s^*L)^{\otimes -1} \otimes \pi_2^*s^*L \otimes \pi_3^*s^*L). \end{aligned}$$

Lemma 2.17. *Under the above notation, we have that*

$$\mathcal{L} \simeq \mathcal{O}_{Z\times A\times A}.$$

Proof. It is easy to see that the restrictions $\mathcal{L}$ to $Z \times \{0\} \times A$ and $Z \times A \times \{0\}$ are trivial by the definition of $\mathcal{L}$, where 0 is the origin of A. We can also check that the restriction of $\mathcal{L}$ to $s(A) \times A \times A$ is trivial (see [Mu, Section 6, Corollary 2]). In particular, $\mathcal{L}|_{\{z_0\}\times A\times A}$ is trivial for any point $z_0 \in s(A) \subset Z$. Therefore, by the theorem of cube (see [Mu, Section 6, Theorem]), we obtain that $\mathcal{L}$ is trivial. □

We write $T_a = m|_{Z\times\{a\}} : Z \simeq Z \times \{a\} \to Z$, that is,

$$T_a : (y, b) \mapsto (y, b + a),$$

for $(y, b) \in Y \times A = Z$.

Corollary 2.18. *By restricting $\mathcal{L}$ to $Z \times \{a\} \times \{b\}$, we obtain*

$$L \otimes T^*_{a+b}L \simeq T^*_a L \otimes T^*_b L,$$

where $a, b \in A$.

As an application of Corollary 2.18, we have:

Corollary 2.19. *Let D be a Cartier divisor on Z. Then we have*

$$2D \sim T^*_a D + T^*_{-a}D$$

for every $a \in A$. In particular, if Y is complete and D is effective and is not vertical with respect to $Y \times A \to A$, then $\kappa(Z, D) > 0$.

Proof. We put $L = \mathcal{O}_X(D)$ and $b = -a$. Then we have $2D \sim T^*_a D + T^*_{-a}D$ by Corollary 2.18. We assume that D is not vertical. Then we have $\mathrm{Supp}D \neq \mathrm{Supp}T^*_a D$ if we choose $a \in A$ suitably. Therefore, $\kappa(X, D) > 0$ if D is effective and is not vertical. □

3. Weakly positive sheaves and big sheaves

In this section, we discuss the basic properties of weakly positive sheaves and big sheaves. We also discuss Viehweg's base change trick. Almost everything is contained in Viehweg's papers [Vi3] and [Vi4].

Definition 3.1 (Weak positivity and bigness)**.** Let $\mathcal{F}$ be a torsion-free coherent sheaf on a smooth quasi-projective variety W. We say that $\mathcal{F}$ is *weakly positive* if, for every positive integer α and every ample invertible sheaf $\mathcal{H}$, there exists a positive integer β such that $\widehat{S}^{\alpha\beta}(\mathcal{F})\otimes\mathcal{H}^{\otimes\beta}$ is generically generated by global sections. We say that a nonzero torsion-free coherent sheaf $\mathcal{F}$ is *big* if, for every ample invertible sheaf $\mathcal{H}$, there exists a positive integer a such that $\widehat{S}^a(\mathcal{F}) \otimes \mathcal{H}^{\otimes -1}$ is weakly positive.

Note that there are several different definitions of weak positivity (see [Mo, (5.1) Definition]).

Remark 3.2. If $\widehat{S}^{\alpha\beta}(\mathcal{F}) \otimes \mathcal{H}^{\otimes\beta}$ is generically generated by global sections, then $\widehat{S}^{\alpha\beta\gamma}(\mathcal{F}) \otimes \mathcal{H}^{\otimes\beta\gamma}$ is also generically generated by global sections for every positive integer γ.

Remark 3.3. Let $\mathcal{L}$ be an invertible sheaf on a smooth projective variety X. Then $\mathcal{L}$ is weakly positive if and only if $\mathcal{L}$ is pseudo-effective. We also note that $\mathcal{L}$ is big in the sense of Definition 3.1 if and only if $\mathcal{L}$ is big in the usual sense, that is, $\kappa(X, \mathcal{L}) = \dim X$.

We will use the notion of semipositive locally free sheaves in Section 7.

Definition 3.4 (Semipositivity)**.** Let $\mathcal{E}$ be a locally free sheaf of finite rank on a smooth projective variety X. If $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ is nef, then $\mathcal{E}$ is said to be *semipositive* or *nef*.

Remark 3.5. Let $\mathcal{E}$ be a semipositive locally free sheaf on a smooth projective variety X. Let $\mathcal{H}$ be an ample invertible sheaf on X and let α be a positive integer. Then there exists a positive integer β_0 such that $S^{\alpha\beta}(\mathcal{E}) \otimes \mathcal{H}^{\otimes\beta}$ is generated by global sections for every integer $\beta \geq \beta_0$. Note that $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(\alpha) \otimes \pi^*\mathcal{H}$ is an ample invertible sheaf on $\mathbb{P}(\mathcal{E})$, where $\pi : \mathbb{P}_X(\mathcal{E}) \to X$. Therefore, $\mathcal{E}$ is weakly positive.

We can easily check the following properties of weakly positive sheaves.

Lemma 3.6 ([Vi3, (1.3) Remark and Lemma 1.4])**.** *Let $\mathcal{F}$ and $\mathcal{G}$ be torsion-free coherent sheaves on a smooth quasi-projective variety W. Then we have the following properties.*

(i) *In order to check whether $\mathcal{F}$ is weakly positive, we may replace W with $W \setminus \Sigma$ for some closed subset Σ of codimension ≥ 2.*
(ii) *Let $\mathcal{F} \to \mathcal{G}$ be a generically surjective morphism. If $\mathcal{F}$ is weakly positive, then $\mathcal{G}$ is also weakly positive.*
(iii) *If $\widehat{S}^{a}(\mathcal{F})$ is weakly positive for some positive integer a, then $\mathcal{F}$ is weakly positive.*
(iv) *Let $\delta : W \to W''$ be a projective birational morphism to a smooth quasi-projective variety W'' and let E be a δ-exceptional Cartier divisor on W. If $\mathcal{F} \otimes \mathcal{O}_W(E)$ is weakly positive, then $\delta_*\mathcal{F}$ is weakly positive.*
(v) *Let $\tau : W' \to W$ be a finite morphism from a smooth quasi-projective variety W'. If $\tau^*\mathcal{F}$ is weakly positive, then $\mathcal{F}$ is weakly positive.*
(vi) *If $\mathcal{F}$ is weakly positive, then $\widehat{\det}(\mathcal{F})$ is weakly positive.*

(vii) *If $\mathcal{F}$ and $\mathcal{G}$ are weakly positive, then $\mathcal{F}\otimes\mathcal{G}/\mathrm{torsion}$ is also weakly positive.*

Proof. (i) and (ii) are obvious by the definition of weakly positive sheaves. By the natural map

$$\widehat{S}^{\alpha}\widehat{S}^{\beta}(\mathcal{F}) \to \widehat{S}^{\alpha\beta}(\mathcal{F}),$$

which is generically surjective, we obtain (iii). (iv) is obvious by (i). Let $\mathcal{H}$ be an ample invertible sheaf on W. In order to prove (v), we may shrink W and may assume that $\mathcal{F}$ is locally free by (i). Since $\tau^*\mathcal{F}$ is weakly positive, we see that $S^{2\alpha\beta}(\tau^*\mathcal{F})\otimes\tau^*\mathcal{H}^{\otimes\beta}$ is generically generated by global sections for every positive integer α and some large positive integer β. We note that we have a surjection

$$\tau_*\tau^*S^{2\alpha\beta}(\mathcal{F})\otimes\mathcal{H}^{\otimes\beta} \to S^{2\alpha\beta}(\mathcal{F})\otimes\mathcal{H}^{\otimes\beta}.$$

Hence we obtain a generically surjective morphism

$$\bigoplus_{\text{finite}}\tau_*\mathcal{O}_{W'}\otimes\mathcal{H}^{\otimes\beta} \to S^{2\alpha\beta}(\mathcal{F})\otimes\mathcal{H}^{2\beta}.$$

We may assume that $\tau_*\mathcal{O}_{W'}\otimes\mathcal{H}^{\otimes\beta}$ is generated by global sections since we may assume that β is sufficiently large (see Remark 3.2). Thus $S^{2\alpha\beta}(\mathcal{F})\otimes\mathcal{H}^{\otimes 2\beta}$ is generically generated by global sections. This means that $\mathcal{F}$ is weakly positive. So we obtain (v). We put $r=\mathrm{rank}(\mathcal{F})$. Let α be a positive integer and let $\mathcal{H}$ be an ample invertible sheaf. Then there exists a positive integer β such that $\widehat{S}^{\alpha\beta r}(\mathcal{F})\otimes\mathcal{H}^{\otimes\beta}$ is generically generated. Hence $\widehat{\det}(\mathcal{F})^{\otimes\alpha b}\otimes\mathcal{H}^{\otimes b}$ is generically generated for $b=\mathrm{rank}(\widehat{S}^{\alpha\beta r}(\mathcal{F}))\beta$. Thus, we obtain (vi). Since we do not use (vii) in this paper, we omit the proof of (vii) here. For the proof, see [Vi4, Lemma 3.2 iii)]. Note that the proof of (vii) is much harder than the proof of the other properties. ∎

For bigness, we have the following lemma.

Lemma 3.7 ([Vi4, Lemma 3.6])**.** *Let $\mathcal{F}$ be a nonzero torsion-free coherent sheaf on a smooth quasi-projective variety W. Then the following three conditions are equivalent.*

(i) *There exist an ample invertible sheaf $\mathcal{H}$ on W, some positive integer ν, and an inclusion $\bigoplus\mathcal{H}\hookrightarrow\widehat{S}^{\nu}(\mathcal{F})$, which is an isomorphism over a nonempty Zariski open set of W.*

(ii) *For every invertible sheaf $\mathcal{M}$ on W, there exists some positive integer γ such that $\widehat{S}^{\gamma}(\mathcal{F})\otimes\mathcal{M}^{\otimes -1}$ is weakly positive. In particular, $\mathcal{F}$ is a big sheaf.*

(iii) *There exist some positive integer* γ *and an ample invertible sheaf* $\mathcal{M}$ *such that* $\widehat{S}^{\gamma}(\mathcal{F})\otimes\mathcal{M}^{\otimes -1}$ *is weakly positive.*

Proof. First, we assume (i). For every positive integer β, there exists a map $\bigoplus\mathcal{H}^{\otimes\beta}\to\widehat{S}^{\beta\nu}(\mathcal{F})$, which is generically surjective. If we choose β large enough, we may assume that $\mathcal{H}^{\otimes\beta}\otimes\mathcal{M}^{\otimes -1}$ is very ample. Therefore, $\widehat{S}^{\beta\nu}(\mathcal{F})\otimes\mathcal{M}^{\otimes -1}$ is weakly positive by the generically surjective map $\bigoplus\mathcal{H}^{\otimes\beta}\otimes\mathcal{M}^{\otimes -1}\to\widehat{S}^{\beta\nu}(\mathcal{F})\otimes\mathcal{M}^{\otimes -1}$ by Lemma 3.6 (ii). Thus we obtain (ii). Since (iii) is a special case of (ii), (iii) follows from (i).

Next, we assume (iii). If $\widehat{S}^{\gamma}(\mathcal{F})\otimes\mathcal{M}^{\otimes -1}$ is weakly positive for some ample invertible sheaf $\mathcal{M}$ on W, then $\widehat{S}^{2\beta\gamma}(\mathcal{F})\otimes\mathcal{M}^{\otimes -2\beta}\otimes\mathcal{M}^{\otimes\beta}$ is generically generated by global sections for some positive integer β. Thus we get a map

$$\bigoplus_{\text{finite}}\mathcal{M}^{\otimes\beta}\to\widehat{S}^{2\beta\gamma}(\mathcal{F}),$$

which is surjective over a nonempty Zariski open set of W. By choosing $\mathrm{rank}(\widehat{S}^{2\beta\gamma}(\mathcal{F}))$ copies of $\mathcal{M}^{\otimes\beta}$ such that the corresponding sections generates the sheaf $\widehat{S}^{2\beta\gamma}(\mathcal{F})\otimes\mathcal{M}^{\otimes -\beta}$ in the general point of W, we obtain (i) with $\mathcal{H}=\mathcal{M}^{\otimes\beta}$ and $\nu=2\beta\gamma$. ☐

3.8 (Viehweg's base change trick)**.** Let us discuss Viehweg's clever base change arguments. They are very useful and important. The following results are contained in [Vi3, §3]. We closely follow [Mo, §4].

Lemma 3.9 ([Mo, (4.9) Lemma])**.** *Let* V *be an irreducible reduced Gorenstein variety and let* $\rho: V'\to V$ *be a resolution. Then, for every positive integer* n*, we have* $\rho_*\omega_{V'}^{\otimes n}\subset\omega_V^{\otimes n}$*. Furthermore, if* V *has only rational singularities, then we have* $\omega_V^{\otimes n}=\rho_*\omega_{V'}^{\otimes n}$ *for every positive integer* n*.*

Proof. Since V is Cohen–Macaulay, we may assume that ρ is finite by shrinking V in order to check $\rho_*\omega_{V'}^{\otimes n}\subset\omega_V^{\otimes n}$. Since ρ is birational, the trace map $\rho_*\omega_{V'}\to\omega_V$ gives $\rho_*\omega_{V'}\subset\omega_V$. Since ρ is finite, we obtain $\omega_{V'}\subset\rho^*\omega_V$ by $\rho_*\omega_{V'}\subset\omega_V$. Therefore, we have

$$\rho_*\omega_{V'}^{\otimes n}\subset\rho_*(\omega_{V'}\otimes\rho^*\omega_V^{\otimes n-1})=\rho_*\omega_{V'}\otimes\omega_V^{\otimes n-1}\subset\omega_V^{\otimes n}$$

by induction on n. We further assume that V has only rational singularities. Then it is well known that V has only canonical Gorenstein singularities. Therefore, we have $\omega_V^{\otimes n}=\rho_*\omega_{V'}^{\otimes n}$ for every positive integer n. ☐

Lemma 3.10 (Base Change Theorem, see [Mo, (4.10)])**.** *Let* $f: V\to W$ *be a projective surjective morphism between smooth quasi-projective varieties. Let* $\tau: W'\to W$ *be a flat projective surjective morphism* *from a smooth quasi-projective variety W'. We consider the following commutative diagram:*

$$\begin{tikzcd} V \arrow[d, "f"'] & \widetilde{V} \arrow[l, "\widetilde{\rho}"'] \arrow[d, "\widetilde{f}"'] & V' \arrow[l, "\rho"'] \arrow[dl, "f'"] \\ W & W' \arrow[l, "\tau"] & \end{tikzcd}$$

where $\widetilde{V} = V \times_W W'$ and $\rho : V' \to \widetilde{V}$ is a resolution. Then we have the following properties.

(i) *There is an inclusion*

$$f'_* \omega_{V'/W'}^{\otimes n} \subset \tau^*(f_* \omega_{V/W}^{\otimes n})$$

for every positive integer n.

Let P be a codimension one point of W'. Assume that $\widetilde{V}$ has only rational singularities over a neighborhood of P. Then we have

$$f'_* \omega_{V'/W'}^{\otimes n} = \widetilde{f}_* \omega_{\widetilde{V}/W'}^{\otimes n} = \tau^*(f_* \omega_{V/W}^{\otimes n})$$

at P.

(ii) *Let P be a codimension one point of W'. If $\tau(P)$ is a codimension one point of W and f is semistable in a neighborhood of $\tau(P)$, then $\widetilde{V}$ has only rational Gorenstein singularities over a neighborhood of P.*

(iii) *There is an inclusion*

$$\tau_* f'_* \omega_{V'/W}^{\otimes n} \subset (f_* \omega_{V/W}^{\otimes n} \otimes \tau_* \omega_{W'/W}^{\otimes n})^{**},$$

which is an isomorphism at codimension one point P of W if f or τ is semistable in a neighborhood of P.

Proof. Since τ is flat, $\widetilde{V}$ is an irreducible reduced Gorenstein variety and $\omega_{\widetilde{V}/W'} = \widetilde{\rho}^* \omega_{V/W}$ by the flat base change theorem [Ve, Theorem 2] (see also [H1], [C], and so on). Then we have that $\tau^* f_* \omega_{V/W}^{\otimes n} = \widetilde{f}_* \omega_{\widetilde{V}/W}^{\otimes n}$ for every positive integer n by the flat base change theorem (see [H2, Chapter III, Proposition 9.3]). By Lemma 3.9, we have $\rho_* \omega_{V'/W'}^{\otimes n} \subset \omega_{\widetilde{V}/W'}^{\otimes n}$. Therefore, we obtain $f'_* \omega_{V'/W'}^{\otimes n} \subset \tau^* f_* \omega_{V/W}^{\otimes n}$ for every positive integer n. The latter statement in (i) is obvious by the above argument and Lemma 3.9.

For (ii), it is sufficient to prove that $\widetilde{V}$ has only rational singularities over a neighborhood of P. By shrinking W around $\tau(P)$, we may assume that $\tau(P)$ is a smooth divisor on W and that $f : V \to W$ is (weakly) semistable. By shrinking W' around P, we may assume that

$P = \tau^{-1}(\tau(P))$ and that P is a smooth divisor on W'. Then we obtain that $\widetilde{f} : \widetilde{V} \to W'$ is weakly semistable by [AbK, Lemma 6.2]. Thus, $\widetilde{V}$ has only rational Gorenstein singularities.

By (i), we have

$$\begin{aligned}\tau_* f'_* \omega^{\otimes n}_{V'/W} &= \tau_* f'_* (\omega^{\otimes n}_{V'/W'} \otimes f'^* \omega^{\otimes n}_{W'/W}) \\ &= \tau_* ((f'_* \omega^{\otimes n}_{V'/W'}) \otimes \omega^{\otimes n}_{W'/W}) \\ &\subset \tau_* ((\tau^* f_* \omega^{\otimes n}_{V/W}) \otimes \omega^{\otimes n}_{W'/W}) \\ &\subset (f_* \omega^{\otimes n}_{V/W} \otimes \tau_* \omega^{\otimes n}_{W'/W})^{**}\end{aligned}$$

by projection formula. This is nothing but the inclusion in (iii). Without loss of generality, we may shrink W and assume that f is also flat for (iii). Since (iii) is symmetric with respect to f and τ, it is enough to check that the inclusion is an equality at P when f is semistable in a neighborhood of P. Then, by (i) and (ii), we have the equality at P. $\square$

3.11 (Viehweg's fiber product trick)**.** Let $f : V \to W$ be a projective surjective morphism between smooth quasi-projective varieties and let $V^s = V \times_W V \times_W \cdots \times_W V$ be the s-fold fiber product. Let $V^{(s)}$ be an arbitrary resolution of the component of V^s dominating W and let $f^{(s)} : V^{(s)} \to W$ be the induced morphism. Note that $f^{(s)}_* \omega^{\otimes n}_{V^{(s)}/W}$ is independent of the choice of resolution $V^{(s)}$ for every positive integer n.

Corollary 3.12 ([Vi3, Lemma 3.5] and [Mo, (4.11) Corollary])**.** *Let $f : V \to W$ be a projective surjective morphism between smooth quasi-projective varieties. Let s and n be arbitrary positive integers. Then there exists a generically isomorphic injection*

$$a : (f^{(s)}_* \omega^{\otimes n}_{V^{(s)}/W})^{**} \hookrightarrow \left(\bigotimes^s f_* \omega^{\otimes n}_{V/W} \right)^{**}.$$

Let P be a codimension one point of W such that f is semistable in a neighborhood of P. Then a is an isomorphism at P.

Proof. Since deleting closed subsets of W of codimension ≥ 2 does not change the double dual of torsion-free sheaves, we may assume that $f^{(i)}$ are flat for $i = 1, 2, \cdots, s$ and that $f_* \omega^{\otimes n}_{V/W}$ is locally free on W. By Lemma 3.10 (iii), we obtain an injection

$$f^{(s)}_* \omega^{\otimes n}_{V^{(s)}/W} \hookrightarrow f_* \omega^{\otimes n}_{V/W} \otimes f^{(s-1)}_* \omega^{\otimes n}_{V^{(s-1)}/W}$$

such that the above injection is an isomorphism at P if f is semistable in a neighborhood of P. This proves the assertion by induction on s. $\square$

3.13. Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Then we can always take a generically finite surjective morphism $\tau : Y' \to Y$ from a smooth projective variety Y' such that $f' : X' \to Y'$ is semistable in codimension one (see [KKMS] and [Vi3, Proposition 6.1]) or $f' : X' \to Y'$ factors through a weak semistable reduction $f^\dagger : X^\dagger \to Y'$ (see [AbK, Theorem 0.3]), where X' is a resolution of the main component of $X \times_Y Y'$.

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d, "f'"] \\ Y & Y' \arrow[l, "\tau"] \end{tikzcd}$$

Lemma 3.14. *In the notation in 3.13, if $f'_*\omega^{\otimes n}_{X'/Y'}$ is big for some positive integer n, then $f_*\omega^{\otimes n}_{X/Y}$ is also big.*

Proof. Let $\mathcal{H}$ be an ample invertible sheaf on Y. Then there exists a positive integer a such that $\widehat{S}^a(f'_*\omega^{\otimes n}_{X'/Y'}) \otimes \tau^*\mathcal{H}^{\otimes -1}$ is weakly positive by Lemma 3.7. By removing a suitable closed subset Σ of codimension ≥ 2 from Y, we assume that $f_*\omega^{\otimes n}_{X/Y}$ is locally free and that τ is finite and flat. Then, by Lemma 3.10, we obtain a generically isomorphic injection

$$\widehat{S}^a(f'_*\omega^{\otimes n}_{X'/Y'}) \otimes \tau^*\mathcal{H}^{\otimes -1} \subset \tau^*(S^a(f_*\omega^{\otimes n}_{X/Y}) \otimes \mathcal{H}^{\otimes -1}).$$

By Lemma 3.6 (i), (ii), and (v), we see that $\widehat{S}^a(f_*\omega^{\otimes n}_{X/Y}) \otimes \mathcal{H}^{\otimes -1}$ is weakly positive. This means that $f_*\omega^{\otimes n}_{X/Y}$ is big. $\square$

We close this section with a useful observation.

3.15. By Lemma 3.14, we may assume that $f : X \to Y$ is semistable in codimension one or

$$f : X \xrightarrow{\delta} X^\dagger \xrightarrow{f^\dagger} Y$$

such that $f^\dagger : X^\dagger \to Y$ is weakly semistable and that δ is a resolution of singularities when we prove Viehweg's conjecture Q (see Conjecture 1.7).

4. Effective freeness due to Popa–Schnell

In this section, we discuss the effective freeness due to Popa–Schnell (see [PopS]). The following statement is a special case of [PopS, Theorem 1.7].

Theorem 4.1 ([PopS, Theorem 1.4])**.** *Let $f : X \to Y$ be a surjective morphism from a smooth projective variety X to a projective variety Y with $\dim Y = n$. Let k be a positive integer and let $\mathcal{L}$ be an ample invertible sheaf on Y such that $|\mathcal{L}|$ is free. Then we have*

$$H^i(Y, f_*\omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes l}) = 0$$

for every $i > 0$ and every $l \geq nk + k - n$. By Castelnuovo–Mumford regularity, $f_*\omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes l}$ is generated by global sections for every $l \geq k(n+1)$.

The proof of Theorem 4.1 is surprisingly easy. Before we prove Theorem 4.1, we note the following remark.

Remark 4.2. In Theorem 4.1, by Kollár's vanishing theorem (see [Ko1, Theorem 2.1 (iii)]), we have

$$H^i(Y, f_*\omega_X \otimes \mathcal{A}^{\otimes l}) = 0$$

for every $i > 0$ and every $l > 0$, where $\mathcal{A}$ is any ample invertible sheaf on Y. Therefore, by Castelnuovo–Mumford regularity, we obtain that $f_*\omega_X \otimes \mathcal{L}^{\otimes l}$ is generated by global sections for every $l \geq n+1$.

Let us prove Theorem 4.1.

Proof of Theorem 4.1. Let us consider

$$\mathcal{M} = \mathrm{Im}\left(f^*f_*\omega_X^{\otimes k} \to \omega_X^{\otimes k}\right).$$

By taking blow-ups, we may assume that $\mathcal{M}$ is an invertible sheaf such that $\omega_X^{\otimes k} = \mathcal{M} \otimes \mathcal{O}_X(E)$ for some effective divisor E on X. We may further assume that Supp$E$ is a simple normal crossing divisor. We can take the smallest integer $m \geq 0$ such that $f_*\omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes m}$ is generated by global sections because $\mathcal{L}$ is ample. Then $\omega_X^{\otimes k} \otimes \mathcal{O}_X(-E) \otimes f^*\mathcal{L}^{\otimes m}$ is also generated by global sections. Note that $\omega_X^{\otimes k} \otimes \mathcal{O}_X(-E) = \mathcal{M}$, $f_*\mathcal{M} = f_*\omega_X^{\otimes k}$, and $f^*f_*\mathcal{M} \to \mathcal{M}$ is surjective. Therefore, we can take a smooth general effective divisor D such that $\mathrm{Supp}(D+E)$ is a simple normal crossing divisor on X such that

$$kK_X + mf^*L \sim D + E,$$

where $\mathcal{L} = \mathcal{O}_X(L)$. Thus we have

$$(k-1)K_X \sim_{\mathbb{Q}} \frac{k-1}{k}D + \frac{k-1}{k}E - \frac{k-1}{k}mf^*L.$$

So we obtain

$$\begin{aligned} &kK_X - \left\lfloor \frac{k-1}{k}E \right\rfloor + lf^*L \\ &\sim_{\mathbb{Q}} K_X + \frac{k-1}{k}D + \left\{ \frac{k-1}{k}E \right\} + \left(l - \frac{k-1}{k}m \right) f^*L. \end{aligned}$$

By the vanishing theorem (see, for example, [F6, Theorem 6.3 (ii)]), which is a generalization of Kollár's vanishing theorem, we obtain

$$H^i(Y, f_*\omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes l}) = 0$$

for every $i > 0$ if $l - \frac{k-1}{k}m > 0$. Note that

$$f_*\mathcal{O}_X \left(kK_X - \left\lfloor \frac{k-1}{k}E \right\rfloor \right) \simeq f_*\omega_X^{\otimes k}$$

by the definition of E. Therefore, if $l > \frac{k-1}{k}m + n$, then $f_*\omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes l}$ is generated by global sections by Castelnuovo–Mumford regularity. By the choice of m, we obtain

$$m \leq \frac{k-1}{k}m + n + 1.$$

This implies $m \leq k(n+1)$. Therefore, we obtain that if

$$l > \frac{k-1}{k} \cdot k(n+1) = kn + k - n - 1$$

then

$$H^i(Y, f_*\omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes l}) = 0$$

for every $i > 0$. □

By combining Theorem 4.1 with Viehweg's fiber product trick (see Corollary 3.12), we can easily recover Viehweg's weak positivity theorem.

Theorem 4.3 (Viehweg's weak positivity theorem (see [Vi3, Theorem III]))**.** *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties. Then $f_*\omega_{X/Y}^{\otimes k}$ is weakly positive for every positive integer k.*

The following proof is due to Popa–Schnell (see [PopS]).

Proof. Let $f^s : X^s = X \times_Y X \times_Y \cdots \times_Y X \to Y$ be the s-fold fiber product. Then we obtain a generically isomorphic injection

$$a : f_*^{(s)}\omega_{X^{(s)}/Y}^{\otimes k} \hookrightarrow \left(\bigotimes^s f_*\omega_{X/Y}^{\otimes k} \right)^{**}$$

for every $k \geq 1$ and every $s \geq 1$ by Corollary 3.12, where $X^{(s)} \to X^s$ is a resolution of the component of X^s dominating Y and $f^{(s)} : X^{(s)} \to Y$ is the induced morphism. Let $\mathcal{H}$ be any ample invertible sheaf on Y. We take a positive integer p such that $|\mathcal{H}^{\otimes p}|$ is base point free. Then, by Theorem 4.1, we obtain that

$$f^{(s)}_* \omega^{\otimes k}_{X^{(s)}/Y} \otimes \omega^{\otimes k}_Y \otimes \mathcal{H}^{\otimes pk(n+1)}$$

is generated by global sections for every $s \geq 1$ and every $k \geq 1$, where $n = \dim Y$. From now on, we fix a positive integer k. We take a positive integer q such that $|\mathcal{H}^{\otimes r} \otimes \omega^{\otimes -k}_Y|$ is base point free for every integer $r \geq q$. Then

$$\left(\bigotimes^{s} f_* \omega^{\otimes k}_{X/Y} \right)^{**} \otimes \mathcal{H}^{\otimes \beta}$$

is generically generated by global sections for every $\beta \geq q + pk(n+1)$ by the generically isomorphic injection a. Therefore, for every positive integer α,

$$\widehat{S}^{\alpha\beta}(f_* \omega^{\otimes k}_{X/Y}) \otimes \mathcal{H}^{\otimes \beta}$$

is generically generated by global sections for $\beta \geq q + pk(n+1)$. This implies that $f_* \omega^{\otimes k}_{X/Y}$ is weakly positive. $\square$

Remark 4.4. The proof of Theorem 4.3 says that

$$\left(\bigotimes^{s} f_* \omega^{\otimes k}_{X/Y} \right)^{**} \otimes \omega^{\otimes k}_Y \otimes \mathcal{A}^{\otimes k(n+1)}$$

is generated by global sections over U, where $\mathcal{A}$ is an ample invertible sheaf on Y such that $|\mathcal{A}|$ is free and U is a nonempty Zariski open set of Y such that f is smooth over U. Note that the inclusion a in the proof of Theorem 4.3 is an isomorphism over U.

We close this section with an obvious corollary of Theorem 4.1.

Corollary 4.5. *Let $f : X \to Y$ be a surjective morphism from a projective variety X to a smooth projective variety Y. Assume that X has only rational Gorenstein singularities. Let $\mathcal{L}$ be an ample invertible sheaf on Y such that $|\mathcal{L}|$ is free and let k be a positive integer. Then*

$$f_* \omega^{\otimes k}_X \otimes \mathcal{L}^{\otimes l} \simeq f_* \omega^{\otimes k}_{X/Y} \otimes \omega^{\otimes k}_Y \otimes \mathcal{L}^{\otimes l}$$

is generated by global sections for $l \geq k(\dim Y + 1)$.

Proof. Since X has only rational Gorenstein singularities, X has only canonical Gorenstein singularities. Therefore, by replacing X with its resolution, we may assume that X is a smooth projective variety. Then this corollary follows from Theorem 4.1. $\square$

We will use this corollary in the proof of Theorem 7.1.

5. Weak positivity of direct images of pluricanonical bundles

Let us discuss weak positivity of direct images of (pluri-)canonical divisors and adjoint divisors, and some related topics. We closely follow [Vi3, §5] and [Vi4, §3].

Lemma 5.1 ([Vi3, Theorem 4.1])**.** *Let $f : V \to W$ be a surjective morphism between smooth projective varieties. Then $f_*\omega_{V/W}$ is weakly positive.*

This result is well known. We have already proved a more general result (see Theorem 4.3) by using the effective freeness due to Popa–Schnell (see Theorem 4.1). So we omit the detailed proof here. Note that this lemma can be proved without using the theory of variations of Hodge structure (see, for example, [Ko1] and [Vi5, 5. Weak positivity]). We can prove it as an application of Kollár's vanishing theorem (see also the proof of Theorem 4.3 and [F5, Section 5]).

For the reader's convenience, we give a sketch of the original proof of Lemma 5.1.

Sketch of the proof of Lemma 5.1. Let Σ be a closed subset of W such that f is smooth over $W_0 = W \setminus \Sigma$. Let $\tau : W' \to W$ be a projective birational morphism from a smooth projective variety W' such that $f^{-1}(\Sigma)$ is a simple normal crossing divisor on W'. By Lemma 3.6 (iv), we can replace W with W'. In this situation, $f_*\omega_{V/W}$ is locally free and can be characterized as the upper canonical extension of a suitable Hodge bundle. By Lemma 3.6 (v), (ii), and the unipotent reduction theorem, we may further assume that all the local monodromies on $R^d f_{0*}\mathbb{C}_{V_0}$ around Σ are unipotent, where $d = \dim X - \dim Y$ and $f_0 = f|_{V_0} : V_0 = f^{-1}(W_0) \to W_0$. In this case, we know that $f_*\omega_{V/W}$ is a semipositive locally free sheaf by the theory of variations of Hodge structure (see [Kaw2, Theorem 5]). Therefore, we obtain that $f_*\omega_{V/W}$ is weakly positive (see Remark 3.5). $\square$

Remark 5.2. The Hodge theoretic part of [Kaw2] seems to be insufficient. So we recommend the reader to see [FF] and [FFS] for the Hodge theoretic aspect of the semipositivity of $f_*\omega_{V/W}$ and some generalizations.

The following lemma may look technical and artificial but is a very important lemma.

Lemma 5.3 ([Vi3, Lemma 5.1])**.** *Let $f : V \to W$ be a projective surjective morphism between smooth quasi-projective varieties. Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves on V and let E be an effective divisor on V such that* Supp$E$ *is a simple normal crossing divisor. Assume that*

$$\mathcal{L}^{\otimes N} = \mathcal{M} \otimes \mathcal{O}_V(E)$$

for some positive integer N. We further assume that there exists a nonempty Zariski open set U of W such that some power of $\mathcal{M}$ is generated by global sections over $f^{-1}(U)$. Then we obtain that

$$f_*(\omega_{V/W} \otimes \mathcal{L}^{(i)})$$

is weakly positive for $0 \leq i \leq N-1$, where

$$\mathcal{L}^{(i)} = \mathcal{L}^{\otimes i} \otimes \mathcal{O}_V \left(- \left\lfloor \frac{iE}{N} \right\rfloor \right).$$

Proof. Since the statement is compatible with replacing N by NN', E by $N'E$, and $\mathcal{M}$ by $\mathcal{M}^{\otimes N'}$ for some positive integer N', we may assume that $\mathcal{M}$ itself is generated by global sections over $f^{-1}(U)$. Without loss of generality, we may shrink U if necessary. Let $B + F$ be the zero set of a general section of $\mathcal{M}$ such that every irreducible component of B is dominant onto W and that Supp$F \subset V \setminus f^{-1}(U)$. By Bertini, we may assume that B is smooth and Supp$(B + E)$ is a simple normal crossing divisor on $f^{-1}(U)$. We note that $\mathcal{M} = \mathcal{O}_V(B+F)$. By taking a suitable birational modification outside $f^{-1}(U)$, we may assume that B is smooth and that Supp$(B + E + F)$ is a simple normal crossing divisor. In fact, if $\rho : V' \to V$ is a birational modification which is an isomorphism over $f^{-1}(U)$ and if $\mathcal{L}' = \rho^*\mathcal{L}$, $\mathcal{M}' = \rho^*\mathcal{M}$, and $E' = \rho^*E$, then we can easily check that $\rho_*(\omega_{V'} \otimes \mathcal{L}'^{(i)})$ is contained in $\omega_V \otimes \mathcal{L}^{(i)}$. By construction, $\rho_*(\omega_{V'} \otimes \mathcal{L}'^{(i)})$ coincides with $\omega_V \otimes \mathcal{L}^{(i)}$ on $f^{-1}(U)$. When we prove the weak positivity of $f_*(\omega_{V/W} \otimes \mathcal{L}^{(i)})$, by replacing E with $E + F$, we may assume that $F = 0$ (see Lemma 3.6 (ii)). Note that every irreducible component of F is vertical with respect to $f : V \to W$. By taking a cyclic cover $p : Z' \to X$ associated to $\mathcal{L}^{\otimes N} = B + E$, that is, Z' is the normalization of Spec$_X \bigoplus_{i=1}^{N-1} \mathcal{L}^{\otimes -i}$. Let Z be a resolution of the cyclic cover Z' and let $g : Z \to W$ be the corresponding morphism.

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[drr, "g"'] & Z' \arrow[r, "p"] & V \arrow[d, "f"] \\ & & W \end{tikzcd}$$

It is well known that Z' has only quotient singularities and

$$p_*q_*\omega_Z \simeq p_*\omega_{Z'} \simeq \bigoplus_{i=0}^{N-1} \omega_V \otimes \mathcal{L}^{(i)}.$$

Thus, we obtain

$$g_*\omega_{Z/W} \simeq \bigoplus_{i=0}^{N-1} f_*(\omega_{V/W} \otimes \mathcal{L}^{(i)}).$$

Therefore, by Lemma 5.1 and Lemma 3.6 (ii), $f_*(\omega_{V/W} \otimes \mathcal{L}^{(i)})$ is weakly positive for every $0 \le i \le N-1$. $\square$

As an application of Lemma 5.4, we have:

Lemma 5.4 ([Vi3, Corollary 5.2])**.** *Let $f : V \to W$ be a projective surjective morphism between smooth quasi-projective varieties and let $\mathcal{H}$ be an ample invertible sheaf on W such that $\widehat{S}^{\nu}(f_*\omega_{V/W}^{\otimes k} \otimes \mathcal{H}^{\otimes k})$ is generically generated by global sections for a given positive integer k and some positive integer ν. Then $f_*\omega_{V/W}^{\otimes k} \otimes \mathcal{H}^{\otimes k-1}$ is weakly positive.*

Proof. By replacing W with $W \setminus \Sigma$, where Σ is a suitable closed subset of codimension ≥ 2, we may assume that f is flat and that $f_*\omega_{V/W}^{\otimes k}$ is locally free. We put $\mathcal{L} = \omega_{V/W} \otimes f^*\mathcal{H}$ and

$$\mathcal{M} = \mathrm{Im}\left(f^*(f_*\omega_{V/W}^{\otimes k} \otimes \mathcal{H}^{\otimes k}) \to \omega_{V/W}^{\otimes k} \otimes f^*\mathcal{H}^{\otimes k}\right).$$

By taking blow-ups, we may assume that $\mathcal{M}$ is invertible and that $\mathcal{L}^{\otimes k} = \mathcal{M} \otimes \mathcal{O}_V(E)$ for some effective divisor E on V such that $\mathrm{Supp} E$ is a simple normal crossing divisor. By assumption, we see that $\mathcal{M}^{\otimes \nu}$ is generated by global sections over $f^{-1}(U)$, where U is a nonempty Zariski open set of W. By Lemma 5.3, we obtain that $f_*(\omega_{V/W} \otimes \mathcal{L}^{(k-1)})$ is weakly positive, where

$$\mathcal{L}^{(k-1)} = \mathcal{L}^{\otimes k-1} \otimes \mathcal{O}_V\left(-\left\lfloor \frac{k-1}{k} E \right\rfloor\right).$$

Note that

$$\mathcal{M} \otimes f^*\mathcal{H}^{\otimes -1} \subset \omega_{V/W} \otimes \mathcal{L}^{(k-1)}$$

and that
$$f_*(\omega_{V/W}\otimes \mathcal{L}^{(k-1)})\subset f_*\omega_{V/W}^{\otimes k}\otimes \mathcal{H}^{\otimes k-1}.$$
By the definition of $\mathcal{M}$, we have
$$f_*\mathcal{M}\otimes \mathcal{H}^{\otimes -1}=f_*(\omega_{V/W}\otimes \mathcal{L}^{(k-1)})=f_*\omega_{V/W}^{\otimes k}\otimes \mathcal{H}^{\otimes k-1}.$$
Thus we obtain that
$$f_*\omega_{V/W}^{\otimes k}\otimes \mathcal{H}^{\otimes k-1}$$
is weakly positive. □

By using Lemma 5.4, Viehweg cleverly obtained:

Theorem 5.5 (Viehweg's weak positivity theorem (see [Vi3, Theorem III])**.** *Let $f: V\to W$ be a surjective morphism between smooth projective varieties. Then $f_*\omega_{V/W}^{\otimes k}$ is weakly positive for every positive integer k.*

Note that we have already proved Theorem 5.5 by using the effective freeness due to Popa–Schnell (see Theorem 4.3). However, we give Viehweg's original proof here since it is interesting and useful for some other applications (see, for example, [F8]).

Proof. We divide the proof into two steps.

Step 1. Let $\mathcal{H}$ be any ample invertible sheaf on W. We put
$$r=\min\{s\in \mathbb{Z}_{>0}\ ;\ f_*\omega_{V/W}^{\otimes k}\otimes \mathcal{H}^{\otimes sk-1} \text{ is weakly positive}\}.$$
By definition, we can find a positive integer ν such that
$$\widehat{S}^{\nu}(f_*\omega_{V/W}^{\otimes k})\otimes \mathcal{H}^{\otimes rk\nu-\nu}\otimes \mathcal{H}^{\otimes \nu}$$
is generically generated by global sections. By Lemma 5.4, we have that $f_*\omega_{V/W}^{\otimes k}\otimes \mathcal{H}^{\otimes rk-r}$ is weakly positive. The choice of r allows this only if $(r-1)k-1<rk-r$, equivalently, $r\leq k$. Therefore, $f_*\omega_{V/W}^{\otimes k}\otimes \mathcal{H}^{\otimes k^2-k}$ is weakly positive.

Step 2. Let α be a positive integer. By Lemma 5.6 below, we can take a finite flat morphism $\tau: W\to W'$ from a smooth projective variety W' such that $\tau^*\mathcal{H}=\mathcal{H}'^{\otimes d}$ for $d=2\alpha(k^2-k)+1$. We put $V'=V\times_W W'$. Then we may assume that V' is a smooth projective variety by Lemma 5.6 below. Let $f': V'\to W'$ be the induced morphism.

$$\begin{tikzcd} V & V' \arrow[l] \arrow[d, "f'"] \\ W \arrow[from=u, "f"'] & W' \arrow[l, "\tau"] \end{tikzcd}$$

By applying the result obtained in Step 1 to $f' : V' \to W'$, we obtain that

$$f'_*\omega_{V'/W'}^{\otimes k} \otimes \mathcal{H}'^{\otimes k^2-k}$$

is weakly positive. Since $f'_*\omega_{V'/W'}^{\otimes k} = \tau^* f_*\omega_{V/W}^{\otimes k}$, we see that $\tau^* f_*\omega_{V/W}^{\otimes k} \otimes \mathcal{H}'^{\otimes k^2-k}$ is weakly positive. Let β be a large positive integer such that

$$\widehat{S}^{2\alpha\beta}(\tau^* f_*\omega_{V/W}^{\otimes k} \otimes \mathcal{H}'^{\otimes k^2-k}) \otimes \mathcal{H}'^{\otimes\beta} = \tau^*\widehat{S}^{2\alpha\beta}(f_*\omega_{V/W}^{\otimes k}) \otimes \tau^*\mathcal{H}^{\otimes\beta}$$

is generically generated by global sections. Let $\widehat{W}$ be a nonempty Zariski open set of W such that $\widehat{S}^{2\alpha\beta}(f_*\omega_{V/W}^{\otimes k})$ is locally free and that $\operatorname{codim}_W(W \setminus \widehat{W}) \geq 2$. By shrinking W, we may assume that $W = \widehat{W}$. Then we have a surjection

$$\tau_*\tau^*\widehat{S}^{2\alpha\beta}(f_*\omega_{V/W}^{\otimes k}) \otimes \mathcal{H}^{\otimes\beta} \to \widehat{S}^{2\alpha\beta}(f_*\omega_{V/W}^{\otimes k}) \otimes \mathcal{H}^{\otimes\beta}.$$

Therefore, we obtain a homomorphism

$$\tau_*\mathcal{O}_{W'} \otimes \mathcal{H}^{\otimes\beta} \to \widehat{S}^{2\alpha\beta}(f_*\omega_{V/W}^{\otimes k}) \otimes \mathcal{H}^{\otimes 2\beta}$$

which is surjective over a nonempty Zariski open set. Without loss of generality, we may assume that $\tau_*\mathcal{O}_{W'} \otimes \mathcal{H}^{\otimes\beta}$ is generated by global sections (see Remark 3.2). Thus, $\widehat{S}^{2\alpha\beta}(f_*\omega_{V/W}^{\otimes k}) \otimes \mathcal{H}^{\otimes 2\beta}$ is generated by global sections over a nonempty Zariski open set.

This means that $f_*\omega_{V/W}^{\otimes k}$ is weakly positive. □

The following covering construction is very important and useful. We have already used it in the proof of Theorem 5.5. The description of Kawamata's covering trick in [EV, 3.19. Lemma] is very useful for our purpose (see also [AbK, 5.3. Kawamata's covering] and [Vi7, Lemma 2.5]).

Lemma 5.6. *Let $f : V \to W$ be a projective surjective morphism between smooth quasi-projective varieties and let H be a Cartier divisor on W. Let d be an arbitrary positive integer. Then we can take a finite flat morphism $\tau : W' \to W$ from a smooth quasi-projective variety W' and a Cartier divisor H' on W' such that $\tau^*H \sim dH'$ and that $V' = V \times_W W'$ is a smooth quasi-projective variety with $\omega_{V'/W'} = \rho^*\omega_{V'/W'}$, where $\rho : V' \to V$.*

Proof. We take general very ample Cartier divisors D_1 and D_2 with the following properties.

(i) $H \sim D_1 - D_2$,
(ii) D_1, D_2, f^*D_1, and f^*D_2 are smooth,
(iii) D_1 and D_2 have no common components, and

(iv) $\mathrm{Supp}(D_1+D_2)$ and $\mathrm{Supp}(f^*D_1+f^*D_2)$ are simple normal crossing divisors.

We take a finite flat cover due to Kawamata with respect to W and D_1+D_2. Then we obtain $\tau : W' \to W$ and H' such that $\tau^*H \sim dH'$. By the construction of the above Kawamata cover $\tau : W' \to W$, we may assume that the ramification locus Σ of τ in W is a general simple normal crossing divisor. This means that f^*P is a smooth divisor for any irreducible component P of Σ and that $f^*\Sigma$ is a simple normal crossing divisor on V. In this situation, we can easily check that $V' = V \times_W W'$ is a smooth quasi-projective variety.

$$\begin{tikzcd} V' \arrow[r, "\rho"] \arrow[d, "f'"'] & V \arrow[d, "f"] \\ W' \arrow[r, "\tau"'] & W \end{tikzcd}$$

By construction, we can also easily check that $\omega_{V'/W'} = \rho^*\omega_{V/W}$ by the Hurwitz formula. $\square$

Remark 5.7. In the proof of Lemma 5.6, let S be any simple normal crossing divisor on V. Then we can choose the ramification locus Σ of τ such that $f^*P \not\subset S$ for any irreducible component P of Σ and that $f^*\Sigma \cup S$ is a simple normal crossing divisor on V. If we choose Σ as above, then we obtain that ρ^*S is a simple normal crossing divisor on V'.

Remark 5.8. As an interesting and useful generalization of Theorem 5.5, we have the twisted weak positivity theorem mainly due to Viehweg and Campana. For the details, see [F9] (see also [F13, Section 8]).

The following lemma is also an application of Lemma 5.3.

Lemma 5.9 ([Vi3, Lemma 5.4])**.** *Let $f : V \to W$ be a projective surjective morphism between smooth quasi-projective varieties. Let k be a positive integer and let k' be any multiple of k with $k' \geq 2$. Assume that we have an inclusion*

$$\mathcal{H} \hookrightarrow (f_*\omega_{V/W}^{\otimes k})^{**}$$

for some ample invertible sheaf $\mathcal{H}$ on W. Then there exists a finite surjective morphism $\tau : W' \to W$ from a smooth quasi-projective variety W' such that $V' = V \times_W W'$ is a smooth quasi-projective variety with the following properties:

(i) $\tau^*f_*\omega_{V/W}^{\otimes \nu} = f'_*\omega_{V'/W'}^{\otimes \nu}$ *for every positive integer ν, and*

(ii) *there exists an ample invertible sheaf $\mathcal{H}'$ on W' such that*

$$f'_*\omega_{V'/W'}^{\otimes k'} \otimes \mathcal{H}'^{\otimes -1}$$

is weakly positive.

$$\begin{tikzcd} V & V' \arrow[l] \arrow[d, "f'"] \\ W \arrow[u, leftarrow, "f"'] & W' \arrow[l, "\tau"] \end{tikzcd}$$

Proof. By the natural map

$$\mathcal{H}^{\otimes a} \to \widehat{S}^a(f_*\omega_{V/W}^{\otimes k}) \to \widehat{S}^1(f_*\omega_{V/W}^{\otimes ak}) = (f_*\omega_{V/W}^{\otimes ak})^{**},$$

we may assume that $k = k' > 1$. By taking blow-ups, if necessary, we may assume that there exist an invertible sheaf $\mathcal{N}$ on V and a simple normal crossing divisor $\sum_j \overline{E_j}$ on V, where $\overline{E_j}$ is smooth for every j and $\overline{E_i} \neq \overline{E_j}$ for $i \neq j$, such that

$$\begin{aligned} \mathcal{N} &= \mathrm{Im}\left(f^*f_*\omega_{V/W}^{\otimes k} \to \omega_{V/W}^{\otimes k}\right), \\ \mathcal{N} \otimes \mathcal{O}_V(\sum_j \overline{\mu}_j \overline{E_j}) &= \omega_{V/W}^{\otimes k}, \\ \mathcal{N} &= f^*\mathcal{H} \otimes \mathcal{O}_V(\sum_j \overline{\nu}_j \overline{E_j}), \end{aligned}$$

such that $\overline{\nu}_j \geq 0$ if $\overline{E_j}$ is not f-exceptional. We take a nonempty Zariski open set U' of W such that f is flat over U'. By shrinking U', we may assume that $E_j = \overline{E_j}|_{f^{-1}(U')}$ is dominant onto U' if $E_j \neq 0$. We put

$$\mu_j = \begin{cases} \overline{\mu}_j & \text{if } E_j \neq 0 \\ 0 & \text{if } E_j = 0 \end{cases} \quad \text{and} \quad \nu_j = \begin{cases} \overline{\nu}_j & \text{if } E_j \neq 0 \\ 0 & \text{if } E_j = 0. \end{cases}$$

We take a large integer b such that $b > \nu_j$ for all j. We take a general finite cover $\tau : W' \to W$ such that $V' = V \times_W W'$ is smooth, $\tau^*f_*\omega_{V/W}^{\otimes m} = f'_*\omega_{V'/W'}^{\otimes m}$ for every $m \geq 1$, and $\tau^*\mathcal{H} = \mathcal{A}^{\otimes bk+1}$ for some ample invertible sheaf $\mathcal{A}$ on W' by Lemma 5.6 (see also Remark 5.7). For simplicity, we may assume that $W = W'$ and that $\mathcal{H} = \mathcal{A}^{\otimes bk+1}$. By Theorem 5.5, $f_*\omega_{V/W}^{\otimes k}$ is weakly positive. Therefore, there exists some $\nu > 0$ such that

$$\widehat{S}^{\nu(b-1)}(f_*\omega_{V/W}^{\otimes k}) \otimes \mathcal{A}^{\otimes \nu}$$

is generically generated by global sections. We can take an effective f-exceptional divisor B on V such that $(f_*\omega_{V/W}^{\otimes \eta})^{**} = f_*(\omega_{V/W} \otimes$

$\mathcal{O}_V(B))^{\otimes \eta}$ for every $\eta \leq \nu(b-1)k$. We put $\mathcal{L} = \omega_{V/W} \otimes \mathcal{O}_V(B) \otimes f^*\mathcal{A}^{\otimes -1}$, $N = bk$, and

$$\mathcal{M} = \mathcal{L}^{\otimes N} \otimes \mathcal{O}_V(-\sum_j (b\mu_j + \nu_j)\overline{E_j}).$$

By construction, we may assume that there is an effective divisor F on V such that $\mathrm{Supp} F \subset V \setminus f^{-1}(U')$ and that

$$\mathcal{O}_V(F) = \omega_{V/W}^{\otimes k} \otimes \mathcal{O}_V(kB - \sum_j (\mu_j + \nu_j)\overline{E_j}) \otimes f^*\mathcal{H}^{\otimes -1}$$

by choosing B sufficiently large. Then we can check that

$$\mathcal{M} = \left(\omega_{V/W}^{\otimes k} \otimes \mathcal{O}_V(kB - \sum_j \mu_j \overline{E_j})\right)^{\otimes b-1} \otimes f^*\mathcal{A} \otimes \mathcal{O}_V(F).$$

The natural maps

$$\begin{aligned} f^*\widehat{S}^{\nu(b-1)}(f_*\omega_{V/W}^{\otimes k}) \longrightarrow & \left(\omega_{V/W}^{\otimes k} \otimes \mathcal{O}_V(kB - \sum_j \mu_j \overline{E_j})\right)^{\otimes \nu(b-1)} \\ & \longrightarrow \mathcal{M}^{\otimes \nu} \otimes f^*\mathcal{A}^{\otimes -\nu} \end{aligned}$$

are surjective on $f^{-1}(U')$. Thus the assumptions of Lemma 5.3 are satisfied, that is, $\mathcal{M}^{\otimes \nu}$ is generated by global sections over $f^{-1}(U)$ for some nonempty Zariski open set U of W. By the choice of b, we have

$$\left\lfloor \frac{(k-1)(b\mu_j + \nu_j)}{bk} \right\rfloor \leq \mu_j + \left\lfloor \frac{\nu_j}{b} \right\rfloor = \mu_j$$

for every j. This means that the sheaf $\omega_{V/W} \otimes \mathcal{L}^{(k-1)}$ contains $\mathcal{N} \otimes f^*\mathcal{A}^{\otimes -(k-1)}$ on $f^{-1}(U')$. We put $\mathcal{H}' = \mathcal{A}^{\otimes k-1}$. Then the inclusion

$$f_*(\omega_{V/W} \otimes \mathcal{L}^{(k-1)}) \to f_*(\omega_{V/W}^{\otimes k} \otimes \mathcal{O}_V(kB)) \otimes \mathcal{H}'^{\otimes -1}$$

is an isomorphism on U'. Thus, $f_*\omega_{V/W}^{\otimes k} \otimes \mathcal{H}'^{\otimes -1}$ is weakly positive by Lemma 5.3. □

As an application of Lemma 5.9, we have:

Proposition 5.10 ([Vi4, Proposition 3.4])**.** *Let $f: V \to W$ be a projective surjective morphism between smooth quasi-projective varieties. Let $\mathcal{H}$ be an ample invertible sheaf on W and let $\mathcal{M}$ be any invertible sheaf on W. Let k be a positive integer and let k' be any multiple of k with $k' \geq 2$. Assume that we have an inclusion $\mathcal{H} \hookrightarrow (f_*\omega_{V/W}^{\otimes k})^{**}$. Then*

$$\widehat{S}^{\gamma}(f_*\omega_{V/W}^{\otimes k'}) \otimes \mathcal{M}^{\otimes -1}$$

is weakly positive for every large positive integer γ. In particular, $f_*\omega_{V/W}^{\otimes k'}$ is big.

Proof. By Lemma 5.9, there exist a finite cover $\tau : W' \to W$ and an ample invertible sheaf $\mathcal{H}'$ on W' such that $\tau^*(f_*\omega_{V/W}^{\otimes k'}) \otimes \mathcal{H}'^{\otimes -1}$ is weakly positive. For every large positive integer γ, $\tau^*\mathcal{M}^{\otimes -1} \otimes \mathcal{H}'^{\otimes \gamma}$ has a nontrivial global section. Thus, $\widehat{S}^{\gamma}(\tau^* f_*\omega_{V/W}^{\otimes k'} \otimes \mathcal{H}'^{\otimes -1})$ is a subsheaf of $\tau^*(\widehat{S}^{\gamma}(f_*\omega_{V/W}^{\otimes k'}) \otimes \mathcal{M}^{\otimes -1})$. Note that the inclusion

$$\widehat{S}^{\gamma}(\tau^* f_*\omega_{V/W}^{\otimes k'} \otimes \mathcal{H}'^{\otimes -1}) \hookrightarrow \tau^*(\widehat{S}^{\gamma}(f_*\omega_{V/W}^{\otimes k'}) \otimes \mathcal{M}^{\otimes -1})$$

is an isomorphism at the generic point of W. This implies that $\widehat{S}^{\gamma}(f_*\omega_{V/W}^{\otimes k'}) \otimes \mathcal{M}^{\otimes -1}$ is weakly positive for every large positive integer γ by Lemma 3.6 (ii) and (v). $\square$

The following theorem is the main theorem of this section.

Theorem 5.11 ([Vi4, Theorem 3.5])**.** *Let $f : V \to W$ be a projective surjective morphism between smooth quasi-projective varieties. Assume that f is semistable in codimension one. We further assume that*

$$\kappa(W, \widehat{\det}(f_*\omega_{V/W}^{\otimes k})) = \dim W$$

for some positive integer k. Let $\mathcal{M}$ be any invertible sheaf on W and let k' be any multiple of k with $k' \geq 2$. Then we obtain that

$$\widehat{S}^{\gamma}(f_*\omega_{V/W}^{\otimes k'}) \otimes \mathcal{M}^{\otimes -1}$$

is weakly positive for every large and divisible positive integer γ. In particular, $f_*\omega_{V/W}^{\otimes k'}$ is big.

Proof. Let $\mathcal{H}$ be an ample invertible sheaf on W. By Kodaira's lemma, we can find $a > 0$ such that $\mathcal{H}$ is contained in $\widehat{\det}(f_*\omega_{V/W}^{\otimes k})^{\otimes a}$. Let U be a Zariski open set of W such that $\operatorname{codim}_W(W \setminus U) \geq 2$, f is semistable over U, and $f_*\omega_{V/W}^{\otimes k}$ is a locally free sheaf on U. We put $r = \operatorname{rank}(f_*\omega_{V/W}^{\otimes k})|_U$. Then we have an inclusion of $\det(f_*\omega_{V/W}^{\otimes k})|_U$ into $((f_*\omega_{V/W}^{\otimes k})|_U)^{\otimes r}$. Therefore, $\mathcal{H}$ can be seen as a subsheaf of $(f_*\omega_{V/W}^{\otimes k})^{\otimes s}$ for $s = ra$ on U. Let $f^{(s)} : V^{(s)} \to W$ be a desingularization of the s-fold fiber product $V \times_W V \times_W \cdots \times_W V$. Then

$$f_*^{(s)}\omega_{V^{(s)}/W}^{\otimes k} = (f_*\omega_{V/W}^{\otimes k})^{\otimes s}$$

holds on U (see Lemma 3.10 and Corollary 3.12). Thus, we have $\mathcal{H} \hookrightarrow (f_*^{(s)}\omega_{V^{(s)}/W}^{\otimes k})^{**}$. By Proposition 5.10, we obtain that

$$\widehat{S}^{\nu}(f_*^{(s)}\omega_{V^{(s)}/W}^{\otimes k'}) \otimes \mathcal{M}^{\otimes -1} = \widehat{S}^{\nu}(((f_*\omega_{V/W}^{\otimes k'})^{\otimes s})^{**}) \otimes \mathcal{M}^{\otimes -1}$$

is weakly positive for every large positive integer ν. Thus $\widehat{S}^{\nu s}(f_*\omega^{\otimes k'}_{V/W})\otimes \mathcal{M}^{\otimes -1}$ is also weakly positive for every large positive integer ν by Lemma 3.6 (ii). □

We close this section with an important remark on weakly semistable morphisms.

Remark 5.12. Theorem 5.11 holds under the assumption that

$$f : V \xrightarrow{\delta} V^\dagger \xrightarrow{f^\dagger} W$$

where δ is a resolution of singularities and $f^\dagger : V^\dagger \to W$ is weakly semistable. Since $f_*\omega^{\otimes k'}_{V/W} = f^\dagger_*\omega^{\otimes k'}_{V^\dagger/W}$, we may assume that $V = V^\dagger$ for the proof of Theorem 5.11. By induction on s, we see that V^s has only Gorenstein singularities by the flat base change theorem [Ve, Theorem 2] (see also [H1], [C], and so on).

$$\begin{tikzcd} V^s \arrow[d, "f^{s-1}"'] & V^s \arrow[l, "p"'] \arrow[d, "q"] \\ W & V \arrow[l, "f"] \end{tikzcd}$$

Since $f : V \to W$ is weakly semistable, we can easily see that V^s is normal and is local analytically isomorphic to a toric variety by induction on s. Anyway, V^s has only rational Gorenstein singularities and is flat over W. Therefore, $f^{(s)}_*\omega^{\otimes m}_{V^{(s)}/W} = f^s_*\omega^{\otimes m}_{V^s/W}$ is a reflexive sheaf for every positive integer m. By the flat base change theorem [Ve, Theorem 2] (see also [H1], [C], and so on), $\omega_{V^s/V} \simeq p^*\omega_{V^{s-1}/W}$. Therefore, we have

$$\begin{aligned} f^s_*\omega^{\otimes m}_{V^s/W} &\simeq f^{s-1}_* p_*(p^*\omega^{\otimes m}_{V^{s-1}/W} \otimes q^*\omega^{\otimes m}_{V/W}) \\ &\simeq f^{s-1}_*(\omega^{\otimes m}_{V^{s-1}/W} \otimes p_*q^*\omega^{\otimes m}_{V/W}) \\ &\simeq f^{s-1}_*(\omega^{\otimes m}_{V^{s-1}/W} \otimes (f^{s-1})^* f_*\omega^{\otimes m}_{V/W}) \\ &\simeq \Big(f_*\omega^{\otimes m}_{V/W} \otimes (f^{s-1}_*\omega^{\otimes m}_{V^{s-1}/W})\Big)^{**} \\ &\simeq \left(\bigotimes^s f_*\omega^{\otimes m}_{V/W}\right)^{**} \end{aligned}$$

by the flat base change theorem (see [H2, Chapter III, Proposition 9.3]) and the projection formula for every positive integer m and every positive integer s by induction on s. Therefore, the proof of Theorem 5.11 also works in this situation.

6. From Viehweg's conjecture to Iitaka's conjecture

This section is a slight reformulation of [Vi3, §7]. We prove that Viehweg's conjecture (see Conjecture 1.7) implies the generalized Iitaka conjecture (see Conjecture 1.6).

First, let us recall the definition of Viehweg's variation.

Definition 6.1 (Viehweg's variation)**.** Let $f : X \to Y$ be a surjective morphism between normal projective varieties. Let $K(\supset \mathbb{C})$ be an algebraically closed field contained in $\overline{\mathbb{C}(Y)}$ such that there is a smooth projective variety V defined over K and that $V \times_{\mathrm{Spec} K} \mathrm{Spec}\overline{\mathbb{C}(Y)}$ and $X \times_Y \mathrm{Spec}\overline{\mathbb{C}(Y)}$ are birational. The minimum of $\mathrm{trans.deg}_{\mathbb{C}} K$ for all such K is called the *variation* of f and is denoted by $\mathrm{Var}(f)$. We have $0 \leq \mathrm{Var}(f) \leq \dim Y$.

Next, we recall Viehweg's conjecture $Q_{n,m}$ (see Conjecture 1.7).

Conjecture 6.2 (Viehweg's conjecture $Q_{n,m}$)**.** *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers such that $\dim X = n$ and $\dim Y = m$. Assume that $\mathrm{Var}(f) = \dim Y$. Then $f_*\omega_{X/Y}^{\otimes k}$ is big for some positive integer k.*

Remark 6.3. Of course, we should assume that K_F is pseudo-effective in Conjecture 6.2, where F is the geometric generic fiber of $f : X \to Y$. We note that $f_*\omega_{X/Y}^{\otimes n} = 0$ for every positive integer n if K_F is not pseudo-effective.

We prepare Fujita's easy but important lemma (see [Ft, Proposition 1]).

Lemma 6.4 (Fujita's lemma)**.** *Let $f : X \to Y$ be a projective surjective morphism between normal projective varieties with connected fibers. Let $\mathcal{L}$ be an invertible sheaf on X and let $\mathcal{M}$ be an invertible sheaf on Y such that $\kappa(Y, \mathcal{M}) = \dim Y$ and $\kappa(X, \mathcal{L}^{\otimes a} \otimes f^*\mathcal{M}^{\otimes -b}) \geq 0$ for some positive integers a and b. then we have*

$$\kappa(X, \mathcal{L}) = \kappa(X_{\overline{\eta}}, \mathcal{L}|_{X_{\overline{\eta}}}) + \kappa(Y, \mathcal{M})$$

where $X_{\overline{\eta}}$ is the geometric generic fiber of $f : X \to Y$.

Proof. By Iitaka's easy addition formula (see Lemma 2.8), we have

$$\kappa(X, \mathcal{L}) \leq \dim Y + \kappa(X_{\overline{\eta}}, \mathcal{L}|_{X_{\overline{\eta}}}).$$

Therefore, it is sufficient to prove

$$\kappa(X, \mathcal{L}) \geq \kappa(Y, \mathcal{M}) + \kappa(X_{\overline{\eta}}, \mathcal{L}|_{X_{\overline{\eta}}}).$$

By Kodaira's lemma, we may assume that $\mathcal{M}$ is ample. We may further assume that $\mathcal{M}$ is very ample, the rational map $\Phi_{|\mathcal{L}|} : X \dashrightarrow V \subset$

$\mathbb{P}^{\dim|\mathcal{L}|}$ gives an Iitaka fibration, and $H^0(X, \mathcal{L} \otimes f^*\mathcal{M}^{\otimes -1}) \neq 0$ by replacing $\mathcal{L}$ and $\mathcal{M}$ with multiplies. An element $\alpha \neq 0$ of $H^0(X, \mathcal{L} \otimes f^*\mathcal{M}^{\otimes -1})$ defines an injection $H^0(Y, \mathcal{M}) \hookrightarrow H^0(X, \mathcal{L})$. Therefore, it gives a projection

$$\mathbb{P}^{\dim|\mathcal{L}|} \dashrightarrow \mathbb{P}^{\dim|\mathcal{M}|}.$$

Hence we obtain the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, dashed, "\Phi_{|\mathcal{L}|}"] \arrow[d, "f"'] & V \arrow[r, hook] \arrow[d, dashed, "\pi"] & \mathbb{P}^{\dim|\mathcal{L}|} \arrow[d, dashed] \\
Y \arrow[r, "\sim", "\Phi_{|\mathcal{M}|}"'] & W \arrow[r, hook] & \mathbb{P}^{\dim|\mathcal{M}|}
\end{tikzcd}$$

By taking suitable resolutions of X and V in the above diagram, we may assume that we have

$$\begin{tikzcd}
X \arrow[r, "\rho"] \arrow[d, "f"'] & \widetilde{V} \arrow[d, "\widetilde{\pi}"] \\
Y \arrow[r, equal] & Y
\end{tikzcd}$$

where $\widetilde{V}$ is a smooth projective variety which is birationally equivalent to V. We take a sufficiently general point y of Y and consider the mapping

$$\rho_y : X_y = f^{-1}(y) \to \widetilde{V}_y = \widetilde{\pi}^{-1}(y).$$

A sufficiently general fiber F of ρ_y is also a sufficiently general fiber of ρ. Therefore, we have $\kappa(F, \mathcal{L}|_F) = 0$. Note that ρ is an Iitaka fibration with respect to $\mathcal{L}$. Thus, we have

$$\begin{aligned}
\kappa(X_y, \mathcal{L}|_{X_y}) &\leq \kappa(F, \mathcal{L}|_F) + \dim \widetilde{V}_y \\
&= \dim \widetilde{V} - \dim Y \\
&= \kappa(X, \mathcal{L}) - \kappa(Y, \mathcal{M})
\end{aligned}$$

by Iitaka's easy addition formula (see Lemma 2.8). On the other hand, we have $\kappa(X_y, \mathcal{L}|_{X_y}) = \kappa(X_{\overline{\eta}}, \mathcal{L}|_{X_{\overline{\eta}}})$. Therefore, we obtain the desired inequality $\kappa(X, \mathcal{L}) \geq \kappa(Y, \mathcal{M}) + \kappa(X_{\overline{\eta}}, \mathcal{L}|_{X_{\overline{\eta}}})$. □

The big commutative diagram constructed in Lemma 6.5 plays important roles

Lemma 6.5. *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Then we have the following* *commutative diagram:*

$$\begin{tikzcd} X \arrow[d, "f"'] & V \arrow[l, "\alpha"'] \arrow[d, "g"] & V' \arrow[l, "\rho"'] \arrow[r, "\rho''"] \arrow[d, "g'"] & V'' \arrow[d, "g''"] \\ Y & W \arrow[l, "\beta"] & W' \arrow[l, "\tau"] \arrow[r, "\tau''"'] & W'' \end{tikzcd}$$

such that

(i) *V and W are smooth projective varieties.*
(ii) *α and β are birational.*
(iii) *all g-exceptional divisors are α-exceptional.*
(iv) *W'' is a smooth projective variety.*
(v) *V'' and W' are normal projective varieties.*
(vi) $\dim W'' = \mathrm{Var}(g'') = \mathrm{Var}(f)$.
(vii) *$\tau : W' \to W$ is a generically finite surjective morphism.*
(viii) *V' is a resolution of $W' \times_{W''} V''$ and is a resolution of the main component of $V \times_W W'$ at the same time.*
(ix) *$g'' : V'' \to W''$ and $\tau'' : W' \to W''$ have connected fibers and are weakly semistable.*

Proof. We divide the proof into several steps.

Step 1. By the flattening theorem (see, for example, [AbO, 3.3. The flattening lemma]), we can find a projective birational morphism $\beta : W \to Y$ from a smooth projective variety W such that $(W \times_Y X)_{\mathrm{main}} \to W$ induced by $\beta : W \to Y$ is flat, where $(W \times_Y X)_{\mathrm{main}}$ is the main component of $W \times_Y X$. Let $V \to (W \times_Y X)_{\mathrm{main}}$ be a projective birational morphism from a smooth projective variety V. Then we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[d, "f"'] & V \arrow[l, "\alpha"'] \arrow[d, "g"] \\ Y & W \arrow[l, "\beta"] \end{tikzcd}$$

satisfying (i), (ii), and (iii).

Step 2. Note that $\mathrm{Var}(f) = \mathrm{Var}(g)$ by definition. Therefore, we can construct the following commutative diagram:

$$\begin{tikzcd} V \arrow[d, "g"'] & V' \arrow[l, "\rho"'] \arrow[r, "\rho''"] \arrow[d, "g'"] & V'' \arrow[d, "g''"] \\ W & W' \arrow[l, "\tau"] \arrow[r, "\tau''"'] & W'' \end{tikzcd}$$

such that V', W', V'', and W'' are smooth projective varieties, g'' is a surjective morphism between smooth projective varieties with connected fibers, $\dim W'' = \mathrm{Var}(g'') = \mathrm{Var}(g) = \mathrm{Var}(f)$, $\tau : W' \to W$ is a generically finite surjective morphism, V' is a resolution of the main component of $V \times_W W'$ and is a resolution of the main component of $V'' \times_{W''} W'$ at the same time. Without loss of generality, we may assume that τ'' has connected fibers.

Step 3. By the weak semistable reduction theorem, we may assume that $g'' : V'' \to W''$ is weakly semistable by taking the base change by a generically finite surjective morphism $W^\dagger \to W''$ from a smooth projective variety $W^\dagger$. By applying the weak semistable reduction theorem to $\tau'' : W' \to W''$, we may further assume that $\tau'' : W' \to W''$ is also weakly semistable by the base change by a generically finite morphism $W^{\dagger\dagger} \to W''$ from a smooth projective variety $W^{\dagger\dagger}$ (see [AbK, Lemma 6.2]). Then we have a commutative diagram of V, V', V'', W, W' and W'' satisfying the properties (iv)–(ix).

Therefore, we have the desired big commutative diagram satisfying the properties (i)–(ix). $\square$

Lemma 6.6. *Let $\mathcal{L}$ be an invertible sheaf on Y. Then we have*

$$\kappa(X, \omega_{X/Y} \otimes f^*\mathcal{L}) \geq \kappa(V, \omega_{V/W} \otimes \mathcal{O}_V(B) \otimes \alpha^* f^*\mathcal{L})$$

for any effective g-exceptional divisor B on V.

Proof. We can write $K_V = \alpha^* K_X + E$ and $K_W = \beta^* K_Y + F$ such that E is an effective α-exceptional divisor and F is an effective β-exceptional divisor. Therefore,

$$K_{V/W} + B = K_V - g^*K_W + B = \alpha^* K_{X/Y} + E + B - g^*F \leq \alpha^* K_{X/Y} + E + B.$$

Note that $E + B$ is an effective α-exceptional divisor. Therefore, we obtain

$$\kappa(X, \omega_{X/Y} \otimes f^*\mathcal{L}) \geq \kappa(V, \omega_{V/W} \otimes \mathcal{O}_V(B) \otimes \alpha^* f^*\mathcal{L})$$

for any effective g-exceptional divisor B. $\square$

Lemma 6.7 essentially says that Viehweg's conjecture (see Conjecture 1.7) implies the generalized Iitaka conjecture (see Conjecture 1.6)

Lemma 6.7. *Assume that $\widehat{\det} g''_* \omega^{\otimes m}_{V''/W''}$ is a big invertible sheaf for some positive integer m. Then we obtain*

$$\kappa(X, \omega_{X/Y} \otimes f^*\mathcal{L}) \geq \kappa(X_{\overline{\eta}}) + \max\{\mathrm{Var}(f), \kappa(Y, \mathcal{L})\}$$

for every invertible sheaf $\mathcal{L}$ on Y with $\kappa(Y, \mathcal{L}) \geq 0$, where $X_{\overline{\eta}}$ is the geometric generic fiber of $f : X \to Y$.

Proof. We need several steps for the proof of Lemma 6.7.

Step 1. By the proof of Theorem 5.11 (see also Remark 5.12), we have that $g''_*\omega^{\otimes k}_{V''/W''}$ is big for some positive integer k. Therefore, there is a positive integer ν such that $\widehat{S}^\nu(g''_*\omega^{\otimes k}_{V''/W''})$ contains an ample Cartier divisor on W''. By the nonzero map

$$\widehat{S}^\nu(g''_*\omega^{\otimes k}_{V''/W''}) \to g''_*\omega^{\otimes \nu k}_{V''/W''},$$

we may assume that $g''_*\omega^{\otimes k}_{V''/W''}$ contains an ample Cartier divisor H on W'' by replacing νk with k.

Step 2. We consider the following commutative diagram:

$$\begin{tikzcd} \widetilde{V} \arrow[r, "\widetilde{\rho}"] \arrow[d, "\widetilde{g}"'] & V'' \arrow[d, "g''"] \\ W' \arrow[r, "\tau''"'] & W'' \end{tikzcd}$$

where $\widetilde{V} = W' \times_{W''} V''$. Then we obtain

$$(\tau'')^* g''_*\omega^{\otimes k}_{V''/W''} \simeq \widetilde{g}_*\omega^{\otimes k}_{\widetilde{V}/W'}$$

by the flat base change theorem [Ve, Theorem 2] (see also [H1], [C], and so on). Note that $\widetilde{V}$ has only rational Gorenstein singularities (see Lemma 2.14). This implies that $g'_*\omega^{\otimes k}_{V'/W'} \simeq \widetilde{g}_*\omega^{\otimes k}_{\widetilde{V}/W'}$. We obtain that $\widetilde{g}_*\omega^{\otimes k}_{\widetilde{V}/W'}(\simeq g'_*\omega^{\otimes k}_{V'/W'})$ contains $(\tau'')^*H$. So we have that $\omega^{\otimes k}_{\widetilde{V}/W'}$ contains $(\tau'' \circ \widetilde{g})^*H$. We note that $\widetilde{g}_*\omega^{\otimes k}_{\widetilde{V}/W'}$ is a reflexive sheaf on W'.

Step 3. In this step, we will check

$$\begin{aligned} \kappa(V', \omega_{V'/W'} \otimes \rho^*\alpha^*f^*\mathcal{L}) &\geq \kappa(V_{\overline{\eta}}) + \max\{\mathrm{Var}(g), \kappa(W, \beta^*\mathcal{L})\} \\ &= \kappa(X_{\overline{\eta}}) + \max\{\mathrm{Var}(f), \kappa(Y, \mathcal{L})\}, \end{aligned}$$

where $V_{\overline{\eta}}$ is the geometric generic fiber of g and $X_{\overline{\eta}}$ is the geometric generic fiber of f.

Since $\omega^{\otimes k}_{\widetilde{V}/W'}$ contains $(\tau'' \circ \widetilde{g})^*H$ and $\kappa(Y, \mathcal{L}) \geq 0$, we obtain

$$\kappa(\widetilde{V}, (\omega_{\widetilde{V}/W'} \otimes \widetilde{g}^*\tau^*\beta^*\mathcal{L})^{\otimes a} \otimes (\tau''^* \circ \widetilde{g})^*\mathcal{O}_{W''}(-bH)) \geq 0$$

for some positive integers a and b. Then, by Lemma 6.4, we obtain

$$\begin{aligned}\kappa(V', \omega_{V'/W'} \otimes \rho^*\alpha^*f^*\mathcal{L}) &= \kappa(V', \omega_{V'/W'} \otimes g'^*\tau^*\beta^*\mathcal{L})\\ &= \kappa(\widetilde{V}, \omega_{\widetilde{V}/W'} \otimes \widetilde{g}^*\tau^*\beta^*\mathcal{L})\\ &= \dim W'' + \kappa(\widetilde{V}_{\overline{w''}}, (\omega_{\widetilde{V}/W'} \otimes \widetilde{g}^*\tau^*\beta^*\mathcal{L})|_{\widetilde{V}_{\overline{w''}}})\\ &= \dim W'' + \kappa(V''_{\overline{w''}}, \omega_{V''_{\overline{w''}}}) + \kappa(W'_{\overline{w''}}, \tau^*\beta^*\mathcal{L}|_{W'_{\overline{w''}}})\\ &= \dim W'' + \kappa(V_{\overline{\eta}}) + \kappa(W'_{\overline{w''}}, \tau^*\beta^*\mathcal{L}|_{W'_{\overline{w''}}}).\end{aligned}$$

Note that $\widetilde{V}_{\overline{w''}} = W'_{\overline{w''}} \times V''_{\overline{w''}}$, where $\overline{w''}$ is the geometric generic point of W''. Since $\dim W'' = \mathrm{Var}(g)$ and

$$\kappa(W, \beta^*\mathcal{L}) = \kappa(W', \tau^*\beta^*\mathcal{L}) \leq \dim W'' + \kappa(W'_{\overline{w''}}, \tau^*\beta^*\mathcal{L}|_{W'_{\overline{w''}}})$$

by Lemma 2.8, we obtain

$$\begin{aligned}\kappa(V', \omega_{V'/W'} \otimes \rho^*\alpha^*f^*\mathcal{L}) &\geq \kappa(V_{\overline{\eta}}) + \max\{\mathrm{Var}(g), \kappa(W, \beta^*\mathcal{L})\}\\ &= \kappa(X_{\overline{\eta}}) + \max\{\mathrm{Var}(f), \kappa(Y, \mathcal{L})\}.\end{aligned}$$

Step 4. Let U be a Zariski open set of W such that g is flat over U and that $\mathrm{codim}_W(W \setminus U) \geq 2$. By restricting

$$\begin{tikzcd} V \arrow[d, "g"'] & V' \arrow[l, "\rho"'] \arrow[d, "g'"] \\ W & W' \arrow[l, "\tau"] \end{tikzcd}$$

to U, we obtain

$$\begin{tikzcd} V_U \arrow[d, "g"'] & V'_U \arrow[l, "\rho"'] \arrow[d, "g'"] \\ U & W'_U. \arrow[l, "\tau"] \end{tikzcd}$$

Without loss of generality, we may assume that W'_U is smooth and $\tau : W'_U \to U$ is flat by shrinking U. By the base change theorem (see Lemma 3.10), we obtain

$$g'_*\omega^{\otimes l}_{V'_U/W'_U} \hookrightarrow \tau^*(g_*\omega^{\otimes l}_{V_U/U}) \simeq g'_*(\rho^*\omega^{\otimes l}_{V_U/U})$$

for every positive integer l. Therefore, we have

$$g'_*(\omega_{V'_U/W'_U} \otimes g'^*\tau^*\beta^*\mathcal{L})^{\otimes l} \hookrightarrow g'_*(\rho^*(\omega_{V_U/U} \otimes g^*\beta^*\mathcal{L})^{\otimes l}).$$

Thus,

$$\begin{aligned}&H^0(V', (\omega_{V'/W'} \otimes g'^*\tau^*\beta^*\mathcal{L})^{\otimes l})\\ &\hookrightarrow H^0(V', \rho^*((\omega_{V/W} \otimes g^*\beta^*\mathcal{L})^{\otimes l} \otimes \mathcal{O}_V(D)) \otimes \mathcal{O}_{V'}(E))\\ &\simeq H^0(V, (\omega_{V/W} \otimes g^*\beta^*\mathcal{L})^{\otimes l} \otimes \mathcal{O}_V(D))\end{aligned}$$

for some effective Cartier divisor D on V such that $\mathrm{Supp} D \subset V \setminus V_U$ and some effective ρ-exceptional divisor E on V' such that $\mathrm{Supp} E \subset V' \setminus V'_U$. This implies that

$$\kappa(V', \omega_{V'/W'} \otimes \rho^*\alpha^*f^*\mathcal{L}) \leq \kappa(V, \omega_{V/W} \otimes \mathcal{O}_V(B) \otimes \alpha^*f^*\mathcal{L}).$$

for some effective Cartier divisor B on V such that $\mathrm{Supp} B \subset V \setminus V_U$. Note that B is g-exceptional. Therefore, B is α-exceptional.

Thus, by Lemma 6.6 and the inequalities obtained above, we obtain

$$\begin{aligned}\kappa(X, \omega_{X/Y} \otimes f^*\mathcal{L}) &\geq \kappa(V, \omega_{V/W} \otimes \mathcal{O}_V(B) \otimes \alpha^*f^*\mathcal{L})\\ &\geq \kappa(V', \omega_{V'/W'} \otimes \rho^*\alpha^*f^*\mathcal{L})\\ &\geq \kappa(X_{\overline{\eta}}) + \max\{\mathrm{Var}(f), \kappa(Y, \mathcal{L})\}.\end{aligned}$$

This is the desired inequality. □

We will apply Lemma 6.7 to algebraic fiber spaces whose geometric generic fiber is of general type and elliptic fibrations in Section 7 and Section 8 respectively.

Lemma 6.8. *Assume that* $\widehat{\det} g''_*\omega^{\otimes m}_{V''/W''}$ *is a big invertible sheaf for some positive integer* m*. Then we have*

$$\kappa(Y, \widehat{\det}(f_*\omega^{\otimes m}_{X/Y})) \geq \dim W'' = \mathrm{Var}(f).$$

Proof. Note that

$$\tau''^* g''_*\omega^{\otimes m}_{V''/W''} = \widetilde{g}_*\omega^{\otimes m}_{\widetilde{V}/W'} = g'_*\omega^{\otimes m}_{V'/W'}.$$

Therefore, we have

$$\begin{aligned}\kappa(W', \widehat{\det} g'_*\omega^{\otimes m}_{V'/W'}) &= \kappa(W', \widehat{\det}\widetilde{g}_*\omega^{\otimes m}_{V'/W'})\\ &= \kappa(W', \tau''^*\widehat{\det} g''_*\omega^{\otimes m}_{V''/W''}) = \dim W''.\end{aligned}$$

Let $U^\dagger$ be a Zariski open set of Y such that $\tau \circ \beta : W' \to Y$ is flat over $U^\dagger$ and that $\mathrm{codim}_Y(Y \setminus U^\dagger) \geq 2$. By restricting

$$\begin{array}{ccc} X & \xleftarrow{\alpha\circ\rho} & V' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g'} \\ Y & \xleftarrow[\beta\circ\tau]{} & W' \end{array}$$

to $U^\dagger$, we obtain

$$\begin{tikzcd} X_{U^\dagger} \arrow[d, "f"'] & V'_{U^\dagger} \arrow[l, "\alpha\circ\rho"'] \arrow[d, "g'"] \\ U^\dagger & W'_{U^\dagger}. \arrow[l, "\beta\circ\tau"] \end{tikzcd}$$

Without loss of generality, we may further assume that $W'_{U^\dagger}$ is smooth. By the base change theorem (see Lemma 3.10), we obtain a generically isomorphic inclusion

$$g'_*\omega^{\otimes m}_{V'_{U^\dagger}/W'_{U^\dagger}} \hookrightarrow (\beta\circ\tau)^*(f_*\omega^{\otimes m}_{X_{U^\dagger}/U^\dagger}).$$

This implies that there exists an inclusion of invertible sheaves:

$$\widehat{\det} g'_*\omega^{\otimes m}_{V'_{U^\dagger}/W'_{U^\dagger}} \hookrightarrow (\beta\circ\tau)^*\widehat{\det}(f_*\omega^{\otimes m}_{X_{U^\dagger}/U^\dagger}).$$

Therefore, we obtain an injection

$$\widehat{\det} g'_*\omega^{\otimes m}_{V'/W'} \hookrightarrow (\beta\circ\tau)^*\widehat{\det}(f_*\omega^{\otimes m}_{X/Y}) \otimes \mathcal{O}_{W'}(E^\dagger)$$

for some effective $(\beta\circ\tau)$-exceptional divisor on W'. Thus, we obtain

$$\begin{aligned} \kappa(Y, \widehat{\det}(f_*\omega^{\otimes m}_{X/Y})) &= \kappa(W', (\beta\circ\tau)^*\widehat{\det}(f_*\omega^{\otimes m}_{X/Y}) \otimes \mathcal{O}_{W'}(E^\dagger)) \\ &\geq \kappa(W', \widehat{\det} g'_*\omega^{\otimes m}_{V'/W'}) \\ &\geq \dim W'' = \mathrm{Var}(f). \end{aligned}$$

This is the desired inequality. $\square$

For some future references, we write the following lemma. The proof of Lemma 6.8 says:

Lemma 6.9. *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties and let $\tau : Y' \to Y$ be a generically finite surjective morphism from a smooth projective variety Y'. We take the following commutative diagram:*

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d, "f'"] \\ Y & Y' \arrow[l, "\tau"] \end{tikzcd}$$

where X' is a resolution of the main component of $X \times_Y Y'$. Let m be a positive integer. Then there exists an effective τ-exceptional divisor E on Y' such that

$$\widehat{\det} f'_*\omega^{\otimes m}_{X'/Y'} \hookrightarrow \tau^*(\widehat{\det} f_*\omega^{\otimes m}_{X/Y}) \otimes \mathcal{O}_{Y'}(E).$$

In particular, we have

$$\kappa(Y, \widehat{\det} f_*\omega_{X/Y}^{\otimes m}) \geq \kappa(Y', \widehat{\det} f'_*\omega_{X'/Y'}^{\otimes m}).$$

Remark 6.10. As in 3.15, by Lemma 6.9, we may assume that f is semistable in codimension one or

$$f : X \xrightarrow{\delta} X^\dagger \xrightarrow{f^\dagger} Y$$

such that $f^\dagger : X^\dagger \to Y$ is weakly semistable and that δ is a resolution of singularities when we want to prove $\kappa(Y, \widehat{\det}(f_*\omega_{X/Y}^{\otimes m})) = \dim Y$.

7. Fiber spaces whose general fibers are of general type

In this section, we discuss projective surjective morphisms between smooth projective varieties whose general fibers are of general type. The main purpose of this section is to prove:

Theorem 7.1. *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Assume that the geometric generic fiber $X_{\overline{\eta}}$ of $f : X \to Y$ is of general type. Then there exists a generically finite surjective morphism $\tau : Y' \to Y$ from a smooth projective variety Y' with the following property.*

Let X' be any resolution of the main component of $X \times_Y Y'$ sitting in the commutative diagram below:

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "\tau"'] & Y. \end{tikzcd}$$

Then $f'_*\omega_{X'/Y'}^{\otimes m}$ is a semipositive locally free sheaf for every nonnegative integer m. In particular, $\det f'_*\omega_{X'/Y'}^{\otimes m}$ is a nef invertible sheaf for every nonnegative integer m. We further assume that $\mathrm{Var}(f) = \dim Y$. Then $\det f'_*\omega_{X'/Y'}^{\otimes k}$ is a nef and big invertible sheaf for some large and divisible positive integer k.

Theorem 7.1 is slightly better than the well-known results by Kawamata, Kollár, Viehweg, and others (see [Kaw4], [Ko2], and [Vi6]).

The following remark is very important for various applications.

Remark 7.2. In Theorem 7.1, it is sufficient to assume that $\tau : Y' \to Y$ is a generically finite surjective morphism from a smooth projective variety Y' such that there exists a weakly semistable morphism $f^\dagger : X^\dagger \to Y'$ in the sense of Abramovich–Karu (see 2.13), where $X^\dagger \to X \times_Y Y'$ is a projective birational morphism and $f^\dagger : X^\dagger \to Y'$ is the induced morphism.

Remark 7.3. In Theorem 7.1, the bigness of $\det f'_*\omega^{\otimes k}_{X'/Y'}$ implies that $f'_*\omega^{\otimes k'}_{X'/Y'}$ is a big locally free sheaf, where k' is any multiple of k with $k' \geq 2$. For the details, see Theorem 5.11 and Remark 5.12 (see also Remark 7.2). Therefore, Theorem 1.8 follows from Theorem 7.1.

By the results explained in Section 6, we have the following result as an application of Theorem 7.1. Corollary 7.4 says that the generalized Iitaka conjecture (see Conjecture 1.6) holds for projective surjective morphisms between smooth projective varieties with connected fibers whose general fibers are of general type.

Corollary 7.4 (see [Kaw4], [Ko2], and [Vi6])**.** *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Assume that the geometric generic fiber $X_{\overline{\eta}}$ of $f : X \to Y$ is of general type. Then we have the following properties.*

(i) *There exists a positive integer k such that*
$$\kappa(Y, \widehat{\det}(f_*\omega^{\otimes k}_{X/Y})) \geq \mathrm{Var}(f).$$
(ii) *If $\kappa(Y, \mathcal{L}) \geq 0$, then we have*
$$\begin{aligned}\kappa(X, \omega_{X/Y} \otimes f^*\mathcal{L}) &\geq \kappa(X_{\overline{\eta}}) + \max\{\kappa(Y, \mathcal{L}), \mathrm{Var}(f)\}\\ &= \dim X - \dim Y + \max\{\kappa(Y, \mathcal{L}), \mathrm{Var}(f)\}.\end{aligned}$$
(iii) *We have*
$$\begin{aligned}\kappa(X, \omega_{X/Y}) &\geq \kappa(X_{\overline{\eta}}) + \mathrm{Var}(f)\\ &= \dim X - \dim Y + \mathrm{Var}(f).\end{aligned}$$
(iv) *If $\kappa(Y) \geq 0$, then we have*
$$\begin{aligned}\kappa(X) &\geq \kappa(X_{\overline{\eta}}) + \max\{\kappa(Y), \mathrm{Var}(f)\}\\ &= \dim X - \dim Y + \max\{\kappa(Y), \mathrm{Var}(f)\}.\end{aligned}$$

Proof. Note that (iii) and (iv) are important special cases of the statement (ii). In the big commutative diagram constructed in Lemma 6.5, we apply Theorem 7.1 to $g'' : V'' \to W''$ (see also Remark 7.2). Then we obtain that $\det g''_*\omega^{\otimes m}_{V''/W''}$ is nef and big for some positive integer m. Therefore, we obtain the desired inequality in (ii) by Lemma 6.7. We also obtain the desired inequality in (i) by Lemma 6.8. □

Before we start the proof of Theorem 7.1, we prepare several lemmas for the reader's convenience.

Lemma 7.5. *Let X be a normal variety with only canonical singularities. Then $\mathcal{O}_X(mK_X)$ is Cohen–Macaulay for every integer m.*

Proof. We note that X has only rational singularities when X is canonical. Let r be the smallest positive integer such that rK_X is Cartier. Since the problem is local, we may assume that $rK_X \sim 0$ by shrinking X. If $r = 1$, then $\mathcal{O}_X(mK_X) \simeq \mathcal{O}_X$ for every integer m. In this case, $\mathcal{O}_X(mK_X)$ is Cohen–Macaulay for every integer m since X has only rational singularities. From now on, we assume that $r \geq 2$. Let $\pi : \widetilde{X} \to X$ be the index one cover. Then we have

$$\pi_*\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) \simeq \bigoplus_{i=1}^{r} \mathcal{O}_X(iK_X).$$

Since $\widetilde{X}$ has only canonical singularities and $K_{\widetilde{X}}$ is Cartier, $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})$ is Cohen–Macaulay. Since π is finite, $\mathcal{O}_X(iK_X)$ is Cohen–Macaulay for $1 \leq i \leq r$. By $rK_X \sim 0$, we obtain that $\mathcal{O}_X(mK_X)$ is Cohen–Macaulay for every integer m. □

Let us recall the following well-known lemma, which is a special case of [N1, Corollary 3].

Lemma 7.6 (cf. [N1, Corollary 3])**.** *Let $g : V \to C$ be a projective surjective morphism from a normal quasi-projective variety V to a smooth quasi-projective curve C. Assume that V has only canonical singularities and that K_V is g-semi-ample. Then $R^i g_*\mathcal{O}_V(mK_V)$ is locally free for every i and every positive integer m.*

Proof. Let $h : V' \to V$ be a resolution of singularities such that $\mathrm{Exc}(h)$ is a simple normal crossing divisor on V'. We write

$$K_{V'} = h^*K_V + E,$$

where E is an effective h-exceptional $\mathbb{Q}$-divisor. Then we have

$$\lceil mh^*K_V + E\rceil - (K_{V'} + \{-(mh^*K_V + E)\}) = (m-1)h^*K_V.$$

We note that the right hand side is semi-ample over C. Therefore,

$$R^i(g \circ h)_*\mathcal{O}_{V'}(\lceil mh^*K_V + E\rceil)$$

is locally free for every i and every positive integer m (see, for example, [F6, Theorem 6.3 (i)]). On the other hand, we have

$$R^i h_*\mathcal{O}_{V'}(\lceil mh^*K_V + E\rceil) = 0$$

for every $i > 0$ by the relative Kawamata–Viehweg vanishing theorem, and

$$h_*\mathcal{O}_{V'}(\lceil mh^*K_V + E\rceil) \simeq \mathcal{O}_V(mK_V).$$

Therefore, we obtain that

$$R^i g_*\mathcal{O}_V(mK_V)$$

is locally free for every i and every positive integer m. $\square$

We will use the following easy criterion of semipositivity in the proof of Theorem 7.1.

Lemma 7.7. *Let $\mathcal{E}$ be a locally free sheaf of finite rank on a smooth projective variety V. Assume that there exists an invertible sheaf $\mathcal{M}$ such that $\mathcal{E}^{\otimes s} \otimes \mathcal{M}$ is generated by global sections for every positive integer s. Then $\mathcal{E}$ is semipositive.*

Proof. We put $\pi : W = \mathbb{P}_V(\mathcal{E}) \to V$ and $\mathcal{O}_W(1) = \mathcal{O}_{\mathbb{P}_V(\mathcal{E})}(1)$. Since $\mathcal{E}^{\otimes s} \otimes \mathcal{M}$ is generated by global sections, $\mathrm{Sym}^s\mathcal{E} \otimes \mathcal{M}$ is also generated by global sections for every positive integer s. This implies that $\mathcal{O}_W(s) \otimes \pi^*\mathcal{M}$ is generated by global sections for every positive integer s. Thus, we obtain that $\mathcal{O}_W(1)$ is nef, equivalently, $\mathcal{E}$ is semipositive. $\square$

Let us start the proof of Theorem 7.1.

Proof of Theorem 7.1. Let us divide the proof into several steps. First, let us prove the existence of $f' : X' \to Y'$ such that $f'_*\omega^{\otimes m}_{X'/Y'}$ is locally free.

Step 1 (Weak semistable reduction)**.** By [AbK, Theorem 0.3], there exist a generically finite morphism $\tau : Y' \to Y$ from a smooth projective variety Y' and $f^\dagger : X^\dagger \to Y'$ with the following properties.

(i) $X^\dagger$ is a normal projective Gorenstein (see [AbK, Lemma 6.1]) variety which is birationally equivalent to $X \times_Y Y'$.

(ii) $(U_{X^\dagger} \subset X^\dagger)$ and $(U_{Y'} \subset Y')$ are toroidal embeddings without self-intersection, with $U_{X^\dagger} = (f^\dagger)^{-1}(U_{Y'})$.

(iii) $f^\dagger : (U_{X^\dagger} \subset X^\dagger) \to (U_{Y'} \subset Y')$ is toroidal and equidimensional.

(iv) all the fibers of the morphism $f^\dagger$ are reduced.

Note that $f^\dagger : X^\dagger \to Y'$ is weakly semistable (see 2.13) and is called a weak semistable reduction of $f : X \to Y$. We also note that $X^\dagger$ has only rational singularities since $X^\dagger$ is toroidal. Therefore, $X^\dagger$ has only canonical Gorenstein singularities and is Cohen–Macaulay. Thus, we have

$$f^\dagger_*\mathcal{O}_{X^\dagger}(mK_{X^\dagger/Y'}) \simeq f'_*\omega^{\otimes m}_{X'/Y'}$$

for every positive integer m. Therefore, it is sufficient to prove that $f^\dagger_*\mathcal{O}_{X^\dagger}(mK_{X^\dagger/Y'})$ is locally free for every positive integer m. Note that $f^\dagger$ is flat since Y' is smooth, $X^\dagger$ is Cohen–Macaulay, and $f^\dagger$ is equidimensional (see [H2, Chapter III, Exercise 10.9] and [AlK, Chapter V, Proposition (3.5)]).

Step 2 (Relative canonical models)**.** By assumption, the geometric generic fiber of $f^\dagger : X^\dagger \to Y'$ is of general type. Therefore, $f^\dagger : X^\dagger \to Y'$ has the relative canonical model $\widetilde{f} : \widetilde{X} \to Y'$ by [BCHM]. Note that

$$f^\dagger_* \mathcal{O}_{X^\dagger}(mK_{X^\dagger/Y'}) \simeq \widetilde{f}_* \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}/Y'})$$

for every positive integer m. Therefore, it is sufficient to prove that $\widetilde{f}_* \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}/Y'})$ is locally free for every positive integer m.

Step 3 (Local freeness via the flat base change theorem)**.** We take an arbitrary point $P \in Y'$. We take general very ample Cartier divisors $H_1, H_2, \cdots, H_{n-1}$, where $n = \dim Y$, such that $C = H_1 \cap H_2 \cap \cdots \cap H_{n-1}$ is a smooth projective curve passing through P. By [AbK, Lemma 6.2], we see that $X^\dagger_C = X^\dagger \times_{Y'} C \to C$ is weakly semistable. In particular, $X^\dagger_C$ has only rational Gorenstein singularities (see [AbK, Lemma 6.1]). By adjunction, we see that $\widetilde{X}_C = \widetilde{X} \times_{Y'} C$ is normal and has only canonical singularities. More precisely, $(f^\dagger)^* H_1 = X^\dagger \times_{Y'} H_1 = X^\dagger_{H_1}$ has only rational Gorenstein singularities since $X^\dagger_{H_1} \to H_1$ is weakly semistable by [AbK, Lemma 6.1 and Lemma 6.2]. In particular, $(f^\dagger)^* H_1$ has only canonical singularities. Therefore, $(X^\dagger, (f^\dagger)^* H_1)$ is plt by the inversion of adjunction (see [KoM, Theorem 5.50]). So we have that $(\widetilde{X}, \widetilde{f}^* H_1)$ is plt by the negativity lemma (see, for example, [KoM, Proposition 3.51]). Thus, $\widetilde{X}_{H_1} = \widetilde{X} \times_{Y'} H_1 = \widetilde{f}^* H_1$ is normal (see [KoM, Proposition 5.51]). By adjunction and the negativity lemma again, we obtain that $\widetilde{X}_{H_1}$ has only canonical singularities. By repeating this process $(n-1)$-times, we obtain that $\widetilde{X}_C$ has only canonical singularities. Note that $\widetilde{X}_C \to C$ is equidimensional. Therefore, we see that $\widetilde{f} : \widetilde{X} \to Y'$ is equidimensional by the choice of C. Since $\widetilde{X}$ is Cohen–Macaulay and Y' is smooth, $\widetilde{f}$ is flat (see [H2, Chapter III, Exercise 10.9] and [AlK, Chapter V, Proposition (3.5)]). Moreover, $\mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}})$ is flat over Y' for every integer m since $\mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}})$ is Cohen–Macaulay (see Lemma 7.5) and $\widetilde{f}$ is equidimensional (see [AlK, Chapter V, Proposition (3.5)]). By applying Lemma 7.6 and the base change theorem (see [H2, Chapter III, Theorem 12.11]) to $\widetilde{X}_C \to C$, we obtain that

$$\dim H^0(\widetilde{X}_y, \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}/Y'})|_{\widetilde{X}_y})$$

is independent of $y \in Y'$ for every positive integer m. By the base change theorem (see [H2, Chapter III, Corollary 12.9]), we obtain that $f'_* \omega^{\otimes m}_{X'/Y'} \simeq \widetilde{f}_* \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}/Y'})$ is locally free for every positive integer m.

We complete the proof of the local freeness of $f'_*\omega^{\otimes m}_{X'/Y'}$. Next, we will prove that $f'_*\omega^{\otimes m}_{X'/Y'}$ is semipositive. Our proof depends on the effective freeness due to Popa–Schnell (see Theorem 4.1). We do not need the difficult semipositivity theorem in [F8].

Step 4 (Semipositivity)**.** By the proof of the local freeness of $f'_*\omega^{\otimes m}_{X'/Y'}$, we may assume that $f' : X' \to Y'$ is weakly semistable. For simplicity, we denote $f' : X' \to Y'$ by $f : X \to Y$ in this step. We take the s-fold fiber product

$$f^s : X^s = X \times_Y X \times_Y \cdots \times_Y X \to Y.$$

Then we see that X^s is normal and Gorenstein. Moreover, X^s has only rational singularities because X^s is local analytically isomorphic to a toric variety. Therefore, X^s has only canonical singularities (see Lemma 2.14 and Remark 5.12). By the flat base change theorem [Ve, Theorem 2] (see also [H1], [C], and so on), we have $\omega_{X^s/X} \simeq p^*\omega_{X^{s-1}/Y}$. Thus we have

$$\begin{aligned}\omega_{X^s/Y} &\simeq \omega_{X^s/X} \otimes q^*\omega_{X/Y}\\ &\simeq p^*\omega_{X^{s-1}/Y} \otimes q^*\omega_{X/Y}.\end{aligned}$$

We note the following commutative diagram.

$$\begin{tikzcd} X^{s-1} \arrow[d, "f^{s-1}"'] & X^s \arrow[l, "p"'] \arrow[d, "q"] \arrow[dl, "f^s"] \\ Y & X \arrow[l, "f"] \end{tikzcd}$$

Therefore, by the flat base change theorem (see [H2, Chapter III, Proposition 9.3]) and the projection formula, we obtain

$$\begin{aligned}f^s_*\omega^{\otimes m}_{X^s/Y} &\simeq f^{s-1}_* p_*(p^*\omega^{\otimes m}_{X^{s-1}/Y} \otimes q^*\omega^{\otimes m}_{X/Y})\\ &\simeq f^{s-1}_*(\omega^{\otimes m}_{X^{s-1}/Y} \otimes p_*q^*\omega^{\otimes m}_{X/Y})\\ &\simeq f^{s-1}_*(\omega^{\otimes m}_{X^{s-1}/Y} \otimes (f^{s-1})^* f_*\omega^{\otimes m}_{X/Y})\\ &\simeq f_*\omega^{\otimes m}_{X/Y} \otimes f^{s-1}_*\omega^{\otimes m}_{X^{s-1}/Y}\\ &\simeq \bigotimes^s f_*\omega^{\otimes m}_{X/Y}\end{aligned}$$

for every positive integer m and every positive integer s by induction on s. Note that $f_*\omega^{\otimes m}_{X/Y}$ is locally free for every positive integer m. By

Corollary 4.5, we see that

$$\begin{aligned} & f^s_*\omega^{\otimes m}_{X^s/Y} \otimes \omega_Y^{\otimes m} \otimes \mathcal{L}^{\otimes m(\dim Y+1)} \\ \simeq & \left(\bigotimes^s f_*\omega^{\otimes m}_{X/Y}\right) \otimes \omega_Y^{\otimes m} \otimes \mathcal{L}^{\otimes m(\dim Y+1)} \end{aligned}$$

is generated by global sections for every positive integer s, where $\mathcal{L}$ is an ample invertible sheaf on Y such that $|\mathcal{L}|$ is free. Therefore, by Lemma 7.7, we obtain that the locally free sheaf $f_*\omega^{\otimes m}_{X/Y}$ is semipositive for every positive integer m.

Finally, we will prove that $\det f'_*\omega^{\otimes k}_{X'/Y'}$ is big for some positive integer k under the assumption that $\mathrm{Var}(f) = \dim Y$. We closely follow the proof of [Vi7, Theorem 4.34] (see also [Ko3, 3.13. Lemma]).

Step 5 (Bigness)**.** In this step, we denote $\widetilde{f} : \widetilde{X} \to Y'$ by $f : X \to Y$ for simplicity. We take a positive integer l such that $lK_{X/Y}$ is f-very ample such that the multiplication map

$$\delta : S^\mu(f_*\mathcal{O}_X(lK_{X/Y})) \to f_*\mathcal{O}_X(\mu l K_{X/Y})$$

is surjective for every positive integer μ. We put $\mathcal{E} = f_*\mathcal{O}_X(lK_{X/Y})$. Then we obtain the following commutative diagram.

$$\begin{tikzcd} X \arrow[rr, hook, "\iota"] \arrow[dr, "f"'] & & \mathbb{P}(\mathcal{E}) \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

If $\mathcal{I}$ is the ideal sheaf of $\iota(X)$ on $\mathbb{P}(\mathcal{E})$, then we can find some positive integer μ such that

$$p^*p_*(\mathcal{I} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mu)) \to \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mu)$$

is surjective. We fix this positive integer μ throughout this step. We consider

$$\mathbb{P} = \mathbb{P}(\bigoplus^r \mathcal{E}^*) \xrightarrow{\pi} Y$$

for $r = \mathrm{rank}\mathcal{E}$. We have the universal basis map

$$s : \bigoplus \mathcal{O}_{\mathbb{P}}(-1) \to \pi^*\mathcal{E}.$$

The map s is injective. Let Δ be the zero divisor of $\det(s)$. We put $\mathcal{Q} = f_*\mathcal{O}_X(\mu l K_{X/Y})$ and consider the surjective map

$$\delta : S^\mu(\mathcal{E}) \to \mathcal{Q}.$$

Let $\mathcal{B} \subset \pi^*\mathcal{Q}$ be the image of the morphism

$$S^\mu(\bigoplus^r \mathcal{O}_{\mathbb{P}}(-1)) = S^\mu(\bigoplus^r \mathcal{O}_{\mathbb{P}}) \otimes \mathcal{O}_{\mathbb{P}}(-\mu) \xrightarrow{S^\mu(s)} S^\mu(\pi^*\mathcal{E}) \xrightarrow{\pi^*(\delta)} \pi^*\mathcal{Q}.$$

By taking blow-ups of $\mathbb{P}$ with centers in Δ, we can obtain a projective birational morphism $\tau : \mathbb{P}' \to \mathbb{P}$ such that $\mathcal{B}' = \tau^*\mathcal{B}/\text{torsion}$ is locally free. We put $\mathcal{O}_{\mathbb{P}'}(1) = \tau^*\mathcal{O}_{\mathbb{P}}(1)$ and $\pi' = \pi \circ \tau$. Then we obtain a surjective morphism

$$\theta : S^\mu(\bigoplus^r \mathcal{O}_{\mathbb{P}'}(-1)) \to \mathcal{B}'.$$

We have the Plücker embedding

$$\mathrm{Grass}(\mathrm{rank}(\mathcal{Q}), S^\mu(\mathbb{C}^r)) \hookrightarrow \mathbb{P}^M$$

and the surjection θ corresponds to the morphism

$$\rho' : \mathbb{P}' \to \mathrm{Grass}(\mathrm{rank}(\mathcal{Q}), S^\mu(\mathbb{C}^r)) \hookrightarrow \mathbb{P}^M$$

such that

$$\det(\mathcal{B}') \otimes \mathcal{O}_{\mathbb{P}'}(\gamma) \simeq \rho'^*\mathcal{O}_{\mathbb{P}^M}(1)$$

where $\gamma = \mu \cdot \mathrm{rank}\mathcal{Q}$. By assumption, we have $\mathrm{Var}(f) = \dim Y$. Note that the general fiber X_y of $f : X \to Y$ is a canonically polarized variety with only canonical singularities. Thus, the automorphism group of X_y is finite. Therefore, the morphism $\rho' : \mathbb{P}' \to \mathbb{P}^M$ is generically finite over its image. Thus $\rho'^*\mathcal{O}_{\mathbb{P}^M}(1)$ is nef and big on $\mathbb{P}'$. Let H be an ample Cartier divisor on Y. By Kodaira, we have

$$H^0(\mathbb{P}', \rho'^*\mathcal{O}_{\mathbb{P}^M}(\nu) \otimes \pi'^*\mathcal{O}_Y(-H)) \neq 0$$

for some large positive integer ν. Note that $\pi'^*\mathcal{Q}$ and its subsheaf $\mathcal{B}'$ coincide over a nonempty Zariski open set of $\mathbb{P}'$. Thus

$$\pi'^*(\mathcal{O}_Y(-H) \otimes \det(\mathcal{Q})^\nu) \otimes \mathcal{O}_{\mathbb{P}'}(\nu \cdot \gamma)$$

has a section. We put $\alpha = \nu \cdot \gamma$. Then we obtain a nontrivial map

$$\varphi : (\pi'_*\mathcal{O}_{\mathbb{P}'}(\alpha))^* = S^\alpha(\bigoplus^r \mathcal{E}) \to \mathcal{O}_Y(-H) \otimes \det(\mathcal{Q})^\nu.$$

By taking a birational modification $g : Y' \to Y$, we have

$$\mathcal{G} \otimes \mathcal{O}_{Y'}(F) = g^*\mathcal{O}_Y(-H) \otimes g^*(\det(\mathcal{Q})^\nu)$$

where F is an effective divisor on Y' and $\mathcal{G}$ is a quotient invertible sheaf of $g^*(S^\alpha(\bigoplus^r \mathcal{E}))$. Note that $\mathcal{G}$ is nef since $g^*(S^\alpha(\bigoplus^r \mathcal{E}))$ is semipositive.

We put $n=\dim Y=\dim Y'$. Then we obtain

$$\begin{aligned}(\det(\mathcal{Q})^{\nu})^n &= (g^*\det(\mathcal{Q})^{\nu})^n\\ &= (g^*\mathcal{O}_{Y'}(H)\otimes\mathcal{G}\otimes\mathcal{O}_{Y'}(F))\cdot(g^*\det(\mathcal{Q})^{\nu})^{n-1}\\ &\geq g^*\mathcal{O}_{Y'}(H)\cdot(g^*\det(\mathcal{Q})^{\nu})^{n-1}\\ &= g^*\mathcal{O}_{Y'}(H)\cdot(g^*\mathcal{O}_{Y'}(H)\otimes\mathcal{G}\otimes\mathcal{O}_{Y'}(F))\cdot(g^*\det(\mathcal{Q})^{\nu})^{n-2}\\ &\geq (g^*\mathcal{O}_{Y'}(H))^2\cdot(g^*\det(\mathcal{Q})^{\nu})^{n-2}\\ &\geq \cdots\\ &\geq (g^*\mathcal{O}_{Y'}(H))^n\\ &= H^n>0.\end{aligned}$$

This means that $\det f_*\mathcal{O}_X(\mu l K_{X/Y})$ is a nef and big invertible sheaf on Y.

Therefore, we obtain that $\det f'_*\omega^{\otimes k}_{X'/Y'}$ is a nef and big invertible sheaf on Y' for some positive integer k. □

We close this section with a remark on adjunction.

Remark 7.8. In general, $\widetilde{X}_y$ may be non-normal. However, we see that the canonical divisor $K_{\widetilde{X}_y}$ is well-defined, $\widetilde{X}_y$ has only semi-log-canonical singularities, and $\mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}/Y'})|_{\widetilde{X}_y}\simeq\mathcal{O}_{\widetilde{X}_y}(mK_{\widetilde{X}_y})$ for every positive integer m, by adjunction. For the details of semi-log-canonical singularities and pairs, see [F7].

8. Elliptic fibrations

Although the results in this section are more or less well known to the experts, we discuss elliptic fibrations for the reader's convenience. We will use Corollary 8.3 in the proof of Theorem 1.1 in Section 9. First, let us recall:

Theorem 8.1 ($\cdots$, Kawamata, Nakayama, $\cdots$)**.** *Let $f:V\to W$ be a surjective morphism between smooth projective varieties whose general fibers are elliptic curves. Assume that there exists a simple normal crossing divisor Σ on W such that f is smooth over $W_0=W\setminus\Sigma$. We further assume that all the local monodromies on $R^1f_{0*}\mathbb{C}_{V_0}$ around Σ are unipotent, where $f_0=f|_{V_0}:V_0=f^{-1}(W_0)\to W_0$. Then we have*

$$(f_*\omega_{V/W})^{\otimes 12}\simeq J^*\mathcal{O}_{\mathbb{P}^1}(1),$$

where $J:W\to\mathbb{P}^1$ is the natural extension of the period map $p:W_0\to\mathbb{C}\simeq\mathfrak{h}/SL(2,\mathbb{Z})$. Note that $\mathfrak{h}=\{z\in\mathbb{C};\mathrm{Im}(z)>0\}$.

Proof. We do not prove this theorem here. Note that this theorem is a special case of [Kaw3, Theorem 20]. For more detailed description of the period map $p_0 : W_0 \to \mathfrak{h}/SL(2,\mathbb{Z})$, see [N2, Corollary 3.2.1]. For a higher-dimensional generalization, see [F3, Theorem 2.11], where we discuss period maps of polarized variations of Hodge structure of weight one. Of course, this theorem is also a special case of [F3, Theorem 2.11]. □

8.2. Let $f : X \to Y$ be a projective surjective morphism between smooth projective varieties whose general fibers are elliptic curves. We can construct the following commutative diagram:

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "\rho"'] \arrow[d, "f'"] \\ Y & Y' \arrow[l, "\tau"] \end{tikzcd}$$

such that

(i) $\tau : Y' \to Y$ is a generically finite surjective morphism from a smooth projective variety Y'.
(ii) X' is a smooth projective variety which is a resolution of the main component of $X \times_Y Y'$.
(iii) There exists a simple normal crossing divisor Σ on Y' such that f' is smooth over $Y'_0 = Y' \setminus \Sigma$, $f'_0 = f'|_{X'_0} : X'_0 = f'^{-1}(Y'_0) \to Y'_0$ has a section, $f'_0 : X'_0 \to Y'_0$ is an elliptic curve with level 3-structure.
(iv) All the local monodromies on $R^1 f'_{0*}\mathbb{C}_{X'_0}$ around Σ are unipotent.

For the details, see, for example, [KatM, Theorem 2.1.2, Theorem 3.7.1, and so on]. Let $M_1^{(3)}$ be the fine moduli scheme of elliptic curves with level 3-structure (see, for example, [AbO, Theorem 13.1]). Note that $M_1^{(3)}$ is a finite cover of $\mathbb{C} = \mathfrak{h}/SL(2,\mathbb{Z})$. Let $\mathcal{C} \to M_1^{(3)}$ be the universal family. Then there exists a morphism $\alpha : Y'_0 \to M_1^{(3)}$ such that $X'_0 = \mathcal{C} \times_{M_1^{(3)}} Y'_0$. By Theorem 8.1, we have the period map $p : Y'_0 \to \mathbb{C} = \mathfrak{h}/SL(2,\mathbb{Z})$ and its extension $J : Y' \to \mathbb{P}^1$. We note the following commutative diagram.

$$
\begin{tikzcd}
 & M_1^{(3)} \arrow[d] \\
Y'_0 \arrow[ur, "\alpha"] \arrow[r, "p"] \arrow[d, hook] & \mathbb{C} \arrow[d, hook] \\
Y' \arrow[r, "J"'] & \mathbb{P}^1
\end{tikzcd}
$$

Therefore, we see

$$\mathrm{Var}(f) = \dim \overline{\alpha(Y'_0)} = \dim J(Y').$$

By Theorem 8.1,

$$(f'_*\omega_{X'/Y'})^{\otimes 12} \simeq J^*\mathcal{O}_{\mathbb{P}^1}(1).$$

This implies that

$$\kappa(Y', \det f'_*\omega_{X'/Y'}) = \kappa(Y', f'_*\omega_{X'/Y'}) = \kappa(Y', J^*\mathcal{O}_{\mathbb{P}^1}(1)) = \mathrm{Var}(f).$$

By [AbK, Theorem 0.3 and Lemma 6.3], we can take a generically finite morphism $\tau' : Y'' \to Y'$ from a smooth projective variety such that

(v) $\mathrm{Supp}\tau'^*\Sigma$ is a simple normal crossing divisor on Y''.
(vi) There exists a projective birational morphism $X^\dagger \to X' \times_{Y'} Y''$ such that the induced morphism $f^\dagger : X^\dagger \to Y''$ is weakly semistable.

Let $X'' \to X^\dagger$ be a birational morphism from a smooth projective variety X'' such that $f'' : X'' \to Y''$ is the induced morphism. In this case, we see that

$$\tau'^* f'_*\omega_{X'/Y'} = f''_*\omega_{X''/Y''}.$$

This is because $f'_*\omega_{X'/Y'}$ is characterized as the canonical extension of a suitable Hodge bundle and $\mathrm{Supp}\tau'^*\Sigma$ is a simple normal crossing divisor on Y''. Therefore, we have

$$\kappa(Y'', \det f''_*\omega_{X''/Y''}) = \kappa(Y'', f''_*\omega_{X''/Y''}) = \kappa(Y', f'_*\omega_{X'/Y'}) = \mathrm{Var}(f).$$

Moreover, by [F12, Theorem 1.6] (see also Step 4 in the proof of Theorem 7.1), we see that $f''_*\omega^{\otimes m}_{X''/Y''}$ is nef for every positive integer m.

By the above description of elliptic fibrations, we have:

Corollary 8.3. *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers whose general fibers are elliptic curves. Then we have the following properties.*

(i) *We have*
$$\kappa(Y, (f_*\omega_{X/Y})^{**}) \geq \mathrm{Var}(f).$$
Note that $(f_*\omega_{X/Y})^{**}$ *is an invertible sheaf on* Y*.*

(ii) *If* $\kappa(Y, \mathcal{L}) \geq 0$*, then we have*
$$\begin{aligned}\kappa(X, \omega_{X/Y} \otimes f^*\mathcal{L}) &\geq \kappa(X_{\overline{\eta}}) + \max\{\kappa(Y, \mathcal{L}), \mathrm{Var}(f)\}\\ &= \max\{\kappa(Y, \mathcal{L}), \mathrm{Var}(f)\}.\end{aligned}$$

(iii) *We have*
$$\begin{aligned}\kappa(X, \omega_{X/Y}) &\geq \kappa(X_{\overline{\eta}}) + \mathrm{Var}(f)\\ &= \mathrm{Var}(f).\end{aligned}$$

(iv) *If* $\kappa(Y) \geq 0$*, then we have*
$$\begin{aligned}\kappa(X) &\geq \kappa(X_{\overline{\eta}}) + \max\{\kappa(Y), \mathrm{Var}(f)\}\\ &= \max\{\kappa(Y), \mathrm{Var}(f)\}.\end{aligned}$$

Proof. The statements (iii) and (iv) are important special cases of (ii). In the big commutative diagram constructed in Lemma 6.5, we can choose a weakly semistable morphism $g'' : V'' \to W''$ such that we can apply the result in 8.2 to $g'' : V'' \to W''$, that is, $\kappa(W'', g''_*\omega_{V''/W''}) = \mathrm{Var}(f)$. Note that $g''_*\omega_{V''/W''}$ is an invertible sheaf. Therefore, we obtain the desired inequality in (ii) by Lemma 6.7. We also obtain the desired inequality in (i) by Lemma 6.8. □

9. $\overline{C}_{n,n-1}$

In this final section, we give a proof of the following theorem (see Theorem 1.1), which is the main theorem of [Kaw1]. This section is a revised version of the author's unpublished short note [F1] written in 2003 in Princeton.

Theorem 9.1 ([Kaw1, Theorem 1])**.** *Let* $f : V \to W$ *be a dominant morphism of algebraic varieties defined over the complex number field* $\mathbb{C}$*. We assume that the general fiber* $V_w = f^{-1}(w)$ *is an irreducible curve. Then we have the following inequality for logarithmic Kodaira dimensions*:
$$\overline{\kappa}(V) \geq \overline{\kappa}(W) + \overline{\kappa}(V_w).$$

It is easy to see that this statement is equivalent to Theorem 9.2 by the basic properties of the logarithmic Kodaira dimension.

Theorem 9.2 ($\overline{C}_{n,n-1}$)**.** *Let* $f : V \to W$ *be a surjective morphism between smooth projective varieties with connected fibers. Let* C *and* D *be simple normal crossing divisors on* V *and* W *respectively. We put*

$V_0 := V \setminus C$ and $W_0 := W \setminus D$. Assume that $f(V_0) \subset W_0$. Then the inequality

$$\overline{\kappa}(V_0) \geq \overline{\kappa}(W_0) + \overline{\kappa}(F_0)$$

holds, where F_0 is a general fiber of $f_0 = f|_{V_0} : V_0 \to W_0$.

Precisely speaking, we will prove the following theorem in this section.

Theorem 9.3 ($\overline{C}'_{n,n-1}$)**.** *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Let C and D be simple normal crossing divisors on X and Y respectively. We put $X_0 := X \setminus C$ and $Y_0 := Y \setminus D$. Assume that $f(X_0) \subset Y_0$. Then the inequality*

$$\kappa(X, K_X + C - f^*(K_Y + D)) \geq \overline{\kappa}(F_0)$$

holds, where F_0 is a general fiber of $f_0 = f|_{X_0} : X_0 \to Y_0$.

We note:

Proposition 9.4. *Theorem 9.3 implies Theorem 9.2.*

By this proposition, we see that Theorem 9.3 is sufficient for Theorem 9.1.

Proof of Proposition 9.4. Without loss of generality, we may assume that $\kappa(W, K_W + D) \geq 0$ and $\overline{\kappa}(F_0) \geq 0$ in Theorem 9.2. Therefore, we have

$$\kappa(V, K_V + C - f^*(K_W + D)) \geq \overline{\kappa}(F_0) \geq 0$$

by Theorem 9.3. We take a sufficiently large and divisible positive integer m such that

$$H^0(V, \mathcal{O}_V(m(K_V + C) - f^*(m(K_W + D)))) \neq 0,$$

and $\alpha = \Phi_{|m(K_V+C)|} : V \dashrightarrow \mathbb{P}^N$ and $\beta = \Phi_{|m(K_W+D)|} : Y \dashrightarrow \mathbb{P}^M$ are Iitaka fibrations of $K_V + C$ and $K_W + D$ respectively. Since

$$0 \neq a \in H^0(V, \mathcal{O}_V(m(K_V + C) - f^*(m(K_W + D))))$$

gives an injection

$$\iota : H^0(W, \mathcal{O}_W(m(K_W + D))) \hookrightarrow H^0(V, \mathcal{O}_V(m(K_V + C))),$$

we have $\kappa(V, K_V + C) \geq \kappa(W, K_W + D)$. Therefore, we obtain

$$\overline{\kappa}(V_0) \geq \overline{\kappa}(W_0) + \overline{\kappa}(F_0)$$

when $\overline{\kappa}(F_0) = 0$. This is the desired inequality when $\overline{\kappa}(F_0) = 0$.

From now on, we assume that $\overline{\kappa}(F_0) = 1$. We consider the following commutative diagram:

$$\begin{tikzcd}
V \arrow[r, dashed, "\alpha"] \arrow[d, "f"'] & V_m \arrow[r, hook] \arrow[d, dashed, "q"] & \mathbb{P}^N \arrow[d, dashed, "p"] \\
W \arrow[r, dashed, "\beta"'] & W_m \arrow[r, hook] & \mathbb{P}^M,
\end{tikzcd}$$

where V_m and W_m are the images of α and β respectively. Note that the projection $p : \mathbb{P}^N \dashrightarrow \mathbb{P}^M$ is induced by the inclusion ι. We assume that $\kappa(V, K_V + C) = \kappa(W, K_W + D)$. Then q is birational. By taking suitable birational modifications, we may assume that α and β are morphisms.

$$\begin{tikzcd}
V \arrow[r, "\alpha"] \arrow[d, "f"'] & V_m \arrow[d, dashed, "q"] \\
W \arrow[r, "\beta"'] & W_m
\end{tikzcd}$$

We take a sufficiently general point $P \in W_m$ and consider

$$\begin{tikzcd}
V & V' \arrow[l, hook'] \arrow[r] \arrow[d, "f'"'] & P \arrow[d, equal] \\
W \arrow[from=u, "f"'] & W' \arrow[l, hook'] \arrow[r] & P
\end{tikzcd}$$

where $V' = f^{-1}\beta^{-1}(P)$ and $W' = \beta^{-1}(P)$. We put $C' = C|_{V'}$ and $D' = D|_{W'}$. Then we have $\kappa(V', K_{V'} + C') = \kappa(W', K_{W'} + D') = 0$. By Theorem 9.3, we obtain

$$\begin{aligned}
0 = \kappa(V', K_{V'} + C') &\geq \kappa(V', K_{V'} + C' - f'^*(K_{W'} + D')) \\
&\geq \overline{\kappa}(F_0) = 1.
\end{aligned}$$

This is a contradiction. Therefore, we obtain

$$\kappa(V, K_V + C) \geq \kappa(W, K_W + D) + 1 = \kappa(W, K_W + D) + \overline{\kappa}(F_0).$$

This is the desired inequality when $\overline{\kappa}(F_0) = 1$. □

Before we start the proof of Theorem 9.3, let us recall the following trivial lemma. We will frequently use it in the proof of Theorem 9.3 without mentioning it.

Lemma 9.5. *Let X be a normal projective variety. Let D_1 and D_2 be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. Assume that $D_1 \geq D_2$. Then we have $\kappa(X, D_1) \geq \kappa(X, D_2)$.*

Proof of Theorem 9.3. We divide the proof into several steps.

Step 1. By Theorem 2.1 in [AbK] (see also [Kar, Chapter 2, Remark 4.5 and Section 9]), we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "p"'] \arrow[d, "f'"'] & U_{X'} \arrow[l, hook'] \arrow[d] \\ Y & Y' \arrow[l, "q"] & U_{Y'} \arrow[l, hook'] \end{tikzcd}$$

such that $p : X' \to X$ and $q : Y' \to Y$ are projective birational morphisms, X' has only quotient singularities, Y' is smooth, the inclusion on the right are toroidal embeddings without self-intersection, and such that

(i) $f' : (U_{X'} \subset X') \to (U_{Y'} \subset Y')$ is toroidal and equidimensional.
(ii) We put $C' := (p^*C)_{\mathrm{red}}$ and $D' := (q^*D)_{\mathrm{red}}$. Then $C' \subset X' \setminus U_{X'}$ and $D' \subset Y' \setminus U_{Y'}$.

Note that

$$\overline{\kappa}(X_0) = \kappa(X, K_X + C) = \kappa(X', K_{X'} + C')$$

and

$$\overline{\kappa}(Y_0) = \kappa(Y, K_Y + D) = \kappa(Y', K_{Y'} + D').$$

Since

$$\kappa(X, K_X + C - f^*(K_Y + D)) \geq \kappa(X', K_{X'} + C' - f'^*(K_{Y'} + D')),$$

we may replace $f : X \to Y$ with $f' : X' \to Y'$. From now on, we omit the superscript $'$ for simplicity of the notation. So, we may assume that $f : X \to Y$ is toroidal with the above extra assumptions.

Step 2. By taking a Kawamata cover $q : Y' \to Y$, we obtain the following commutative diagram:

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "p"'] \arrow[d, "f'"] \\ Y & Y' \arrow[l, "q"] \end{tikzcd}$$

such that $f' : X' \to Y'$ is weakly semistable, where X' is the normalization of $X \times_Y Y'$ (see [AbK, Section 5]). Note that X' is Gorenstein by [AbK, Lemma 6.1]. We put $G := X \setminus U_X$ and $H := Y \setminus U_Y$. Then we have

$$K_X + C - f^*(K_Y + D) \geq K_X + C_{\mathrm{hor}} + G_{\mathrm{ver}} - f^*(K_Y + H).$$

Therefore, we can check that

$$p^*(K_X + C - f^*(K_Y + D)) \geq K_{X'/Y'} + (p^*C)_{\mathrm{hor}}.$$

We note that $(p^*C)_{\mathrm{hor}} = p^*(C_{\mathrm{hor}})$. So, it is sufficient to prove that $\kappa(X', K_{X'/Y'} + (p^*C)_{\mathrm{hor}}) \geq \overline{\kappa}(F_0)$.

Step 3. Let F be a general fiber of $f : X \to Y$. We put $g := g(F)$: the genus of F.

Case $(g \geq 2)$**.** In this case,

$$\kappa(X', K_{X'/Y'} + (p^*C)_{\mathrm{hor}}) \geq \kappa(X', K_{X'/Y'}) \geq 1 = \overline{\kappa}(F_0)$$

by Corollary 7.4 (iii).

Case $(g = 1)$**.** By the description in 8.2 and Corollary 8.3, we have

$$\kappa(X', K_{X'/Y'}) \geq \mathrm{Var}(f') = \mathrm{Var}(f) \geq 0.$$

So, if C is vertical or $\mathrm{Var}(f) \geq 1$, then we obtain

$$\kappa(X', K_{X'/Y'} + (p^*C)_{\mathrm{hor}}) \geq \overline{\kappa}(F_0).$$

Therefore, we may assume that $\mathrm{Var}(f) = 0$ and C is not vertical. Since $\mathrm{Var}(f) = 0$, there is a finite surjective morphism $\tau : Y'' \to Y'$ from a normal projective variety Y'' such that $X'' = X' \times_{Y'} Y''$ is birationally equivalent to $Y'' \times E$, where E is an elliptic curve.

Lemma 9.6. *Let $\overline{\tau} : \overline{Y} \to Y'$ be a birational morphism from a smooth projective variety $\overline{Y}$ such that $\overline{\tau}^{-1}(Y' \setminus U_{Y'})$, where $(U_{Y'} \subset Y')$ is the toroidal structure of Y', is a simple normal crossing divisor on $\overline{Y}$. We have the following commutative diagram:*

$$\begin{tikzcd} X' \arrow[d, "f'"'] & \overline{X} \arrow[l, "\overline{\pi}"'] \arrow[d, "\overline{f}"] \\ Y' & \overline{Y} \arrow[l, "\overline{\tau}"] \end{tikzcd}$$

where $\overline{X} = X' \times_{Y'} \overline{Y}$. Then $\overline{f} : \overline{X} \to \overline{Y}$ is weakly semistable and

$$\kappa(X', K_{X'/Y'} + (p^*C)_{\mathrm{hor}}) \geq \kappa(\overline{X}, K_{\overline{X}/\overline{Y}} + (\overline{\pi}^*p^*C)_{\mathrm{hor}}).$$

Proof of Lemma 9.6. Note that $\overline{f} : \overline{X} \to \overline{Y}$ is weakly semistable by [AbK, Lemma 6.2]. We also note that

$$K_{\overline{Y}} = \overline{\tau}^* K_{Y'} + E$$

and

$$K_{\overline{X}} = \overline{\pi}^* K_{X'} + F,$$

where E is an effective $\overline{\tau}$-exceptional divisor on $\overline{Y}$ and F is an effective $\overline{\pi}$-exceptional divisor on $\overline{X}$. Therefore, we obtain

$$K_{\overline{X}/\overline{Y}} + (\overline{\pi}^*p^*C)_{\mathrm{hor}} \leq \overline{\pi}^* K_{X'/Y'} + F + \overline{\pi}^*(p^*C)_{\mathrm{hor}}.$$

This implies the desired inequality

$$\kappa(X', K_{X'/Y'} + (p^*C)_{\mathrm{hor}}) \geq \kappa(\overline{X}, K_{\overline{X}/\overline{Y}} + (\overline{\pi}^* p^* C)_{\mathrm{hor}})$$

holds. □

By modifying Y' birationally, we may assume that there exists a simple normal crossing divisor Σ on Y' such that $\tau : Y'' \to Y'$ is étale over $Y' \setminus \Sigma$ (see Lemma 9.6). By Lemma 2.15, we may further assume that Y'' is a smooth projective variety. Anyway, we obtain the following commutative diagram:

$$\begin{tikzcd} X' \arrow[d, "f'"'] & X'' \arrow[l, "\pi"'] \arrow[d, "f''"] \\ Y' & Y'' \arrow[l, "\tau"] \end{tikzcd}$$

where $\tau : Y'' \to Y'$ is a finite cover from a smooth projective variety Y'', $f'' : X'' := X' \times_{Y'} Y'' \to Y''$ is weakly semistable, and f'' is birationally equivalent to $Y'' \times E \to Y''$. Since

$$\pi^*(K_{X'/Y'} + (p^*C)_{\mathrm{hor}}) = K_{X''/Y''} + \pi^*((p^*C)_{\mathrm{hor}}),$$

it is sufficient to prove $\kappa(X'', K_{X''/Y''} + \pi^*((p^*C)_{\mathrm{hor}})) \geq 1$. Let $\alpha : \widetilde{X} \to Y'' \times E$ and $\beta : \widetilde{X} \to X''$ be a common resolution. Since X'' has only rational Gorenstein singularities, X'' has at worst canonical Gorenstein singularities. Thus, we obtain

$$\kappa(X'', K_{X''/Y''} + \pi^*((p^*C)_{\mathrm{hor}})) = \kappa(\widetilde{X}, K_{\widetilde{X}/Y''} + \beta^*\pi^*((p^*C)_{\mathrm{hor}})).$$

On the other hand,

$$K_{\widetilde{X}/Y''} = K_{\widetilde{X}/Y''\times E} + K_{Y''\times E/Y''} =: A$$

is an effective α-exceptional divisor such that $\mathrm{Supp}A = \mathrm{Exc}(\alpha)$. Let B be an irreducible component of $\beta^*\pi^*((p^*C)_{\mathrm{hor}})$ such that B is dominant onto Y''. Then

$$m(A + \beta^*\pi^*((p^*C)_{\mathrm{hor}})) \geq \alpha^*\alpha_*B,$$

for a sufficiently large integer m. Therefore, if is sufficient to prove $\kappa(Y'' \times E, \alpha_*B) \geq 1$. This holds true by Corollary 2.19. Thus, we finish the proof when $g = 1$.

Case $(g = 0)$**.** As in the above case, we can take a finite cover and obtain the following commutative diagram:

$$\begin{tikzcd} X' \arrow[d, "f'"'] & X'' \arrow[l, "\pi"'] \arrow[d, "f''"] \\ Y' & Y'' \arrow[l, "\tau"] \end{tikzcd}$$

where f'' is birationally equivalent to $Y'' \times \mathbb{P}^1 \to Y''$. We can further assume that all the horizontal components of $\pi^*((p^*C)_{\mathrm{hor}})$ are mapped onto Y'' birationally.

Lemma 9.7 (cf. [F3, Section 7]). *Let $f : V \to W$ be a surjective morphism between smooth projective varieties with connected fibers. Assume that f is birationally equivalent to $W \times \mathbb{P}^1 \to W$. Let $\{C_k\}$ be a set of distinct irreducible divisors such that $f : C_k \to W$ is birational for every k with $1 \leq k \leq 3$. Then*

$$\kappa(V, K_{V/W} + C_1 + C_2) \geq 0$$

and

$$\kappa(V, K_{V/W} + C_1 + C_2 + C_3) \geq 1.$$

Proof of Lemma 9.7. By modifying V and W birationally and replacing C_k with its strict transform, we may assume that there exists a simple normal crossing divisor Σ on W such that

$$\varphi_{ij} : V_0 := f^{-1}(W_0) \simeq W_0 \times \mathbb{P}^1$$

with $\varphi_{ij}(C_i|_{V_0}) = W_0 \times \{0\}$ and $\varphi_{ij}(C_j|_{V_0}) = W_0 \times \{\infty\}$ for $i \neq j$, where $W_0 := W \setminus \Sigma$. We may further assume that there exists $\psi_{ij} : V \to \mathbb{P}^1$ such that $\psi_{ij}|_{V_0} = p_2 \circ \varphi_{ij}$, where p_2 is the second projection $W_0 \times \mathbb{P}^1 \to \mathbb{P}^1$. We may also assume that $\bigcup_k C_k \cup \operatorname{Supp} f^*\Sigma$ is a simple normal crossing divisor on V. Then we obtain

$$\begin{aligned}
&\wedge {\psi_{ij}}^* \left(\frac{dz}{z}\right) \\
&\in \operatorname{Hom}_{\mathcal{O}_V}(f^*\mathcal{O}_W(K_W + \Sigma), \mathcal{O}_V(K_V + C_i + C_j + (f^*\Sigma)_{\mathrm{red}})) \\
&\simeq H^0(V, \mathcal{O}_V(K_{V/W} + C_i + C_j + (f^*\Sigma)_{\mathrm{red}} - f^*\Sigma)) \\
&\subset H^0(V, \mathcal{O}_V(K_{V/W} + C_i + C_j))
\end{aligned}$$

for $i \neq j$, where z denotes a suitable inhomogeneous coordinate of $\mathbb{P}^1$. Therefore, we have

$$\dim_{\mathbb{C}} H^0(V, \mathcal{O}_V(K_{V/W} + C_1 + C_2)) \geq 1$$

and

$$\dim_{\mathbb{C}} H^0(V, \mathcal{O}_V(K_{V/W} + C_1 + C_2 + C_3)) \geq 2.$$

Thus, we obtain the required result. □

Apply Lemma 9.7 to $\widetilde{X} \to Y''$, where $\beta : \widetilde{X} \to X''$ is a resolution of X''. Then we obtain

$$\kappa(\widetilde{X}, K_{\widetilde{X}/Y''} + \beta^*\pi^*((p^*C)_{\mathrm{hor}})) \geq \overline{\kappa}(F_0).$$

Thus, we complete the proof of Theorem 9.3. □

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