

TOPOLOGY OF POLYHEDRAL PRODUCTS AND THE GOLOD PROPERTY OF STANLEY-REISNER RINGS

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ABSTRACT. The polyhedral product is a space constructed from a simplicial complex and a collection of pairs of spaces, which is connected with the Stanley Reisner ring of the simplicial complex via cohomology. Generalizing the previous work [GT], [GW] and [IK], we show a decomposition of polyhedral products for a large class of simplicial complexes including the ones whose Alexander duals are shellable or sequentially Cohen-Macaulay. This implies the property, called Golod, of the corresponding Stanley-Reisner rings [HRW].

1. INTRODUCTION

In this paper, we study topological properties of spaces called polyhedral products and their reduction to algebraic and combinatorial properties of Stanley-Reisner rings.

Let us first introduce the main object to study. Let K be a simplicial complex on the vertex set $[m] = \{1, \dots, m\}$ and let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ be a collection of pairs of spaces. The *polyhedral product* $\mathcal{Z}_K(\underline{X}, \underline{A})$ is defined as

$$\mathcal{Z}_K(\underline{X}, \underline{A}) = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma \quad (\subset X_1 \times \cdots \times X_m)$$

for $(\underline{X}, \underline{A})^\sigma = Y_1 \times \cdots \times Y_m$ where $Y_i = X_i$ and A_i according as $i \in \sigma$ and $i \notin \sigma$. Since special polyhedral products first appeared in the work of Porter [P] in the 60's, they have been studied in a variety of contexts and directions. Notably, after the seminal work of Davis and Januszkiewicz [DJ] which introduces quasitoric manifolds, particular polyhedral products called the Davis-Januszkiewicz space and the moment-angle complex have been energetically investigated, where one of the most interesting points of them is to yield connections with combinatorial commutative algebra as below.

Let us next introduce an algebraic and combinatorial object connected with polyhedral products. Let \mathbb{k} be a commutative ring with unity. The *Stanley-Reisner ring* of K is defined as

$$\mathbb{k}[K] = \mathbb{k}[v_1, \dots, v_m] / (v_{i_1} \cdots v_{i_k} \mid \{v_{i_1} \cdots v_{i_k}\} \notin K),$$

where we conventionally put $|v_i| = 2$. The Stanley-Reisner ring is a central object in combinatorial commutative algebra and has been producing many results and applications in a

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wide area of mathematics. As is pointed out in [DJ, Theorem 4.8], it immediately follows from definition that there is an isomorphism

$$H^*(\mathcal{Z}_K(\underline{CP}^\infty, *); \mathbb{k}) \cong \mathbb{k}[K],$$

where the polyhedral product $\mathcal{Z}_K(\underline{CP}^\infty, *)$ is called the *Davis-Januszkiewicz space* for K . Then several algebras derived from $\mathbb{k}[K]$ are known to be realized as the cohomology of spaces related with the Davis-Januszkiewicz space for K . In particular, Buchstaber and Panov [BP, Theorem 7.6] showed

$$(1.1) \quad H^*(\mathcal{Z}_K(\underline{D}^2, \underline{S}^1); \mathbb{k}) \cong \text{Tor}_{\mathbb{k}[v_1, \dots, v_m]}(\mathbb{k}[K], \mathbb{k}),$$

where the polyhedral product $\mathcal{Z}_K(\underline{D}^2, \underline{S}^1)$ is called the *moment-angle complex* for K . With these connections, one might expect that some algebraic and combinatorial properties of Stanley-Reisner rings follow from stronger topological properties of polyhedral products. Indeed, there are some results confirming this. For example, Hochster's result [S, Theorem 4.8 in Chapter II] which computes the Poincaré series of $\text{Tor}_{\mathbb{k}[v_1, \dots, v_m]}(\mathbb{k}[K], \mathbb{k})$ in terms of the cohomology of induced subcomplexes (or full subcomplexes) of K is an immediate consequence of the decomposition of the suspension of the polyhedral product $\mathcal{Z}_K(C\underline{X}, \underline{X})$ due to Bahri, Bendersky, Cohen and Gitler [BBCG, Theorem 2.21], which is thought as a generalization of the standard decomposition $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$. In this paper, we consider the Golod property of Stanley-Reisner rings for the above expectation on polyhedral products.

Definition 1.1. A simplicial complex K on the vertex set $[m]$ is called *Golod* over \mathbb{k} if all products in $\text{Tor}_{\mathbb{k}[v_1, \dots, v_m]}(\mathbb{k}[K], \mathbb{k})$ vanish.

The Golod property was originally introduced by the equality involving Poincaré series of algebras derived from $\mathbb{k}[K]$, for which coefficient-wise inequality holds in general, and it is Golod [G] who proved that this equality is equivalent to that all products and (higher) Massey products in $\text{Tor}_{\mathbb{k}[v_1, \dots, v_m]}(\mathbb{k}[K], \mathbb{k})$ vanish. Recently, Berglund and Jöllenbeck [BJ, Theorem 3] showed that the condition of (higher) Massey products is unnecessary. So, we employ the above simple definition for the Golod property. The Golod property has been studied especially in connection with shellability of simplicial complexes. We here give the definition of shellable complexes due to Björner and Wachs [BW1].

Definition 1.2. A simplicial complex K is *shellable* if there is given an ordering of facets F_1, \dots, F_t satisfying that $F_k \cap (\bigcup_{i < k} F_i)$ is pure and $(\dim F_k - 1)$ -dimensional for $k > 1$.

There are two handy subclasses of shellable complexes introduced in [BW2]; shifted and vertex-decomposable complexes. As in [BW2, Theorem 11.3], we have implications:

$$\text{shifted} \quad \Rightarrow \quad \text{vertex-decomposable} \quad \Rightarrow \quad \text{shellable}$$

There is also a homological generalization of shellable complexes as follows. For $\sigma \in K$, let $\text{lk}_K(\sigma)$ denote the link of σ in K , i.e. $\text{lk}_K(\sigma) = \{\tau \in K \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in K\}$, and let $K^{(i)}$ be the subcomplex of K generated by facets of dimension $\geq i$.

Definition 1.3. A simplicial complex K is *sequentially Cohen-Macaulay* over \mathbb{k} if for any $\sigma \in K$ and $i \geq 0$,

$$\tilde{H}_k(\mathrm{lk}_K(\sigma)^{\langle i \rangle}; \mathbb{k}) = 0 \quad \text{whenever } k < i,$$

where σ can be \emptyset , that is, K itself satisfies this condition.

As in [BWW], we, of course, have an implication:

$$\text{shellable} \quad \Rightarrow \quad \text{sequentially Cohen-Macaulay over } \mathbb{Z}$$

The Golod property of the Alexander duals of the above complexes are proved as follows. Let K^\vee denote the Alexander dual of K .

Theorem 1.4 (Herzog, Reiner and Welker [HRW, Theorem 4 and 9]). *If \mathbb{k} is a field and K^\vee is sequentially Cohen-Macaulay over \mathbb{k} , then K is Golod over \mathbb{k} .*

We want to prove that this result is a consequence of much stronger topological property of the polyhedral product $\mathcal{Z}_K(C\underline{X}, \underline{X})$. There are some results confirming this. Grbić and Theriault [GT] showed that the moment-angle complex for a shifted complex has the homotopy type of a wedge of spheres. This is generalized by the authors [IK] to $\mathcal{Z}_K(C\underline{X}, \underline{X})$, which was conjectured in [BBCG, Conjecture 2.29]. We notice that the Alexander dual of a shifted complex is shifted, so these results guarantee Theorem 1.4 for shifted complexes. Recently, by a noble use of discrete Morse theory, Grujić and Welker [GW] showed if the Alexander dual of K is vertex-decomposable, $\mathcal{Z}_K(\underline{D}^k, \underline{S}^{k-1})$ for $k > 1$ has the homotopy type of a wedge of spheres, implying Theorem 1.4 in the vertex-decomposable case. We now state our main results. Let $|K|$ stand for the geometric realization of K and let K_I be the induced subcomplex (or the full subcomplex) on I for a subset $I \subset [m]$, i.e. $K_I = \{\sigma \in K \mid \sigma \subset I\}$. Let ΣK denote the suspension of K , that is, $\Sigma K = \{(\emptyset, \sigma), (\{1\}, \sigma), (\{2\}, \sigma) \mid \sigma \in K\}$, and we choose $(\{1\}, \emptyset)$ as the basepoint of $|\Sigma K|$, where $|\Sigma K|$ is equal to the unreduced suspension of $|K|$. For a collection of spaces $\{X_i\}_{i=1}^m$ and a subset $I \subset [m]$, we put $\widehat{X}^I = \bigwedge_{i \in I} X_i$.

Theorem 1.5. *Let K be a simplicial complex on the vertex set $[m]$ and let $\underline{X} = \{X_i\}_{i=1}^m$ be a collection of connected CW-complexes. If K^\vee is sequentially Cohen-Macaulay over \mathbb{Z} and each X_i is finite, there is a homotopy equivalence*

$$\mathcal{Z}_K(C\underline{X}, \underline{X}) \simeq \bigvee_{I \subset [m]} |\Sigma K_I| \wedge \widehat{X}^I.$$

Remark 1.6. As we will see in Theorem 4.4 below, if K^\vee is shellable, the finiteness assumption on X_i in Theorem 1.5 is unnecessary.

We will see that $|\Sigma K|$ has the homotopy type of a wedge of spheres if K^\vee is sequentially Cohen-Macaulay over \mathbb{Z} ((5.2) and Proposition 5.7). Then we obtain:

Corollary 1.7. *If K^\vee is sequentially Cohen-Macaulay over \mathbb{Z} , the moment-angle complex for K has the homotopy type of a wedge of spheres of dimension > 1 .*

Corollary 1.8. *If K^\vee is sequentially Cohen-Macaulay over \mathbb{Z} , K is Golod over any ring.*

Theorem 1.5 is actually deduced from the following p -local result.

Theorem 1.9. *Let K be a simplicial complex on the vertex set $[m]$ and let $\underline{X} = \{X_i\}_{i=1}^m$ be a collection of connected CW-complexes. If K^\vee is sequentially Cohen-Macaulay over \mathbb{Z}/p , there is a p -local homotopy equivalence*

$$\mathcal{Z}_K(C\underline{X}, \underline{X}) \simeq_{(p)} \bigvee_{I \subset [m]} |\Sigma K_I| \wedge \widehat{X}^I.$$

We will also see below that if K^\vee is sequentially Cohen-Macaulay over \mathbb{Z}/p , $\Sigma|\Sigma K|$ has the p -local homotopy type of a wedge of spheres of dimension > 1 . See (5.2). Then we have:

Corollary 1.10. *If K^\vee is sequentially Cohen-Macaulay over \mathbb{Z}/p , the moment-angle complex for K has the p -local homotopy type of a wedge of spheres of dimension > 1 .*

From this, we can recover (a slightly generalized version of) the result of Herzog, Reiner and Welker [HRW, Theorem 4 and 9] (Theorem 1.4 above).

Corollary 1.11. *If K^\vee is sequentially Cohen-Macaulay over \mathbb{Z}/p , K is Golod over any field of characteristic p .*

The paper is structured as follows. Section 2 reviews the construction of the decomposition of polyhedral products after a suspension given by Bahri, Bendersky, Cohen and Gitler [BCCG, Theorem 2.21] and elucidates its naturality which will be used later. Section 3 introduces new simplicial complexes called extractible complexes over \mathbb{k} by a recursive condition on deletions of vertices, which summarizes the inductive structure of the Alexander duals of shellable and sequentially Cohen-Macaulay complexes, and proves the decomposition of polyhedral products for extractible complexes. Section 4 shows the extractibility over \mathbb{Z} of simplicial complexes whose Alexander duals are shellable complexes by a mixture of combinatorial and homotopy theoretical arguments, which implies the decomposition of the corresponding polyhedral products. Section 5 deals with the extractibility over $\mathbb{Z}_{(p)}$ of the Alexander duals of sequentially Cohen-Macaulay complexes over \mathbb{Z}/p by a homologically generalized method of Section 4, which implies the p -local decomposition. Section 5 also deals with the integral decomposition of polyhedral products from the p -local ones.

Throughout the paper, we assume that spaces have basepoints and maps between spaces preserve basepoints. We also assume that every nonempty simplicial complex has the empty subset of the index set as its simplex.

2. REVIEW OF THE RESULT OF BAHRI, BENDERSKY, COHEN AND GITLER

In this section, we review the decomposition of polyhedral products after a suspension due to Bahri, Bendersky, Cohen and Gitler [BCCG, Theorem 2.21] and show its naturality which will be used below.

Let K be a simplicial complex on the index set $[m]$, possibly with ghost vertices, i.e. elements of $[m]$ which are not vertices of K . For the possibility of existence of ghost vertices, we use the terminology “index set” instead of “vertex set”. The definition of polyhedral products in the previous section also applies, without any change, to simplicial complexes with ghost vertices. For a subcomplex $L \subset K$ on the same index set $[m]$ and a collection of maps between pairs of spaces $\underline{f} = \{f_i : (X_i, A_i) \rightarrow (Y_i, B_i)\}_{i=1}^m$ with $(\underline{Y}, \underline{B}) = \{(Y_i, B_i)\}_{i=1}^m$, there are induced maps

$$\mathcal{Z}_L(\underline{X}, \underline{A}) \rightarrow \mathcal{Z}_K(\underline{X}, \underline{A}) \quad \text{and} \quad \underline{f} : \mathcal{Z}_K(\underline{X}, \underline{A}) \rightarrow \mathcal{Z}_K(\underline{Y}, \underline{B}).$$

For a subset $I \subset [m]$, let $(\underline{X}_I, \underline{A}_I)$ be a subcollection $\{(X_i, A_i)\}_{i \in I}$ of $(\underline{X}, \underline{A})$. By definition, the projection $\prod_{i=1}^m X_i \rightarrow \prod_{i \in I} X_i$ restricts to a map

$$\pi_I : \mathcal{Z}_K(\underline{X}, \underline{A}) \rightarrow \mathcal{Z}_{K_I}(\underline{X}_I, \underline{A}_I).$$

Replacing the direct product in the definition of polyhedral products with the smash product, we can also define the smash product analogue of $\mathcal{Z}_K(\underline{X}, \underline{A})$ which we denote by $\widehat{\mathcal{Z}}_K(\underline{X}, \underline{A})$. For a subcomplex $L \subset K$ on the same index set $[m]$ and a map $\underline{f} : (\underline{X}, \underline{A}) \rightarrow (\underline{Y}, \underline{B})$, there are also induced maps

$$\widehat{\mathcal{Z}}_L(\underline{X}, \underline{A}) \rightarrow \widehat{\mathcal{Z}}_K(\underline{X}, \underline{A}) \quad \text{and} \quad \underline{f} : \widehat{\mathcal{Z}}_K(\underline{X}, \underline{A}) \rightarrow \widehat{\mathcal{Z}}_K(\underline{Y}, \underline{B}).$$

For any subset $I \subset [m]$, the pinch map $\prod_{i \in I} X_i \rightarrow \bigwedge_{i \in I} X_i$ restricts to a map

$$\rho_I : \mathcal{Z}_{K_I}(\underline{X}_I, \underline{A}_I) \rightarrow \widehat{\mathcal{Z}}_{K_I}(\underline{X}_I, \underline{A}_I).$$

Let $\nabla : \Sigma X \rightarrow \Sigma X \vee \Sigma X$ be the suspension comultiplication and let ∇_n be the composite

$$\Sigma X \xrightarrow{\nabla} \Sigma X \vee \Sigma X \xrightarrow{1 \vee \nabla} \dots \xrightarrow{1 \vee \dots \vee 1 \vee \nabla} \underbrace{\Sigma X \vee \dots \vee \Sigma X}_n$$

for $n \geq 2$. Let $\emptyset = I_1 < \dots < I_{2^m} = [m]$ be the lexicographic order on the power set of $[m]$. We now define the map

$$\hat{\epsilon}_K = (\Sigma(\rho_{I_1} \circ \pi_{I_1}) \vee \dots \vee \Sigma(\rho_{I_{2^m}} \circ \pi_{I_{2^m}})) \circ \nabla_{2^m} : \Sigma \mathcal{Z}_K(\underline{X}, \underline{A}) \rightarrow \Sigma \bigvee_{I \subset [m]} \widehat{\mathcal{Z}}_{K_I}(\underline{X}_I, \underline{A}_I).$$

Generalizing the standard decomposition $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$, i.e. the composite

$$(2.1) \quad \Sigma(X \times Y) \xrightarrow{\nabla_3} \bigvee^3 \Sigma(X \times Y) \xrightarrow{\text{proj}} \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y),$$

Bahri, Bendersky, Cohen and Gitler [BBCG, Theorem 2.10] proved:

Theorem 2.1. *The map $\hat{\epsilon}_K$ is a homotopy equivalence if each (X_i, A_i) is a connected CW-pair.*

Remark 2.2. Notice that $\hat{\epsilon}_K$ is defined by using the lexicographic order on the power set of $[m]$. If we choose another order on the power set of $[m]$, we get another homotopy equivalence. However we easily see that these homotopy equivalences become homotopic after a suspension by the cocommutativity of a double suspension.

From now on, we fix a collection of spaces $\underline{X} = \{X_i\}_{i=1}^m$ and specialize polyhedral products and related spaces to the collection $(C\underline{X}, \underline{X})$. Then it is useful to put for $I \subset [m]$,

$$\mathcal{Z}_K^I = \mathcal{Z}_{K_I}(C\underline{X}_I, \underline{X}_I), \quad \widehat{\mathcal{Z}}_K^I = \widehat{\mathcal{Z}}_{K_I}(C\underline{X}_I, \underline{X}_I) \quad \text{and} \quad \mathcal{W}_K^I = \bigvee_{J \subset I} |\Sigma K_J| \wedge \widehat{X}^J.$$

Using (pointed) homotopy colimits, it is proved in [ZZ, Lemma 1.8] that there is a homotopy equivalence

$$\varpi_I : \widehat{\mathcal{Z}}_K^I \xrightarrow{\sim} |\Sigma K_I| \wedge \widehat{X}^I.$$

Then by putting

$$(2.2) \quad \bar{\epsilon}_K = \left(\left(\bigvee_{I \subset [m]} \Sigma \varpi_I \right) \circ \hat{\epsilon}_K \right)^{-1} : \Sigma \mathcal{W}_K^{[m]} \rightarrow \Sigma \mathcal{Z}_K^{[m]},$$

we obtain the result of Bahri, Bendersky, Cohen and Gitler [BBCG, Theorem 2.21] which is mentioned in the previous section.

Theorem 2.3. *Let K be a simplicial complex on the index set $[m]$, possibly with ghost vertices. If each X_i is a connected CW-complex, the map $\bar{\epsilon}_K$ is a homotopy equivalence.*

Let us consider a simple case that $\bar{\epsilon}_K$ desuspends. If $\mathcal{Z}_K(\underline{X}, \underline{A})$ is a co-H-space, the map ∇_{2^m} in the definition of $\hat{\epsilon}_K$ desuspends, i.e. there is a map $\mathcal{Z}_K(\underline{X}, \underline{A}) \rightarrow \bigvee^{2^m} \mathcal{Z}_K(\underline{X}, \underline{A})$, defined by using the comultiplication of $\mathcal{Z}_K(\underline{X}, \underline{A})$, whose suspension is homotopic to ∇_{2^m} .

Proposition 2.4. *If $\mathcal{Z}_K^{[m]}$ is a co-H-space, $\bar{\epsilon}_K$ desuspends.*

Let us consider the naturality of $\bar{\epsilon}_K$. We start with the map $\hat{\epsilon}_K$. By definition, $\hat{\epsilon}_K$ has the naturality such that for a subcomplex $L \subset K$ on the same index set $[m]$ and a map $\underline{f} : (\underline{X}, \underline{A}) \rightarrow (\underline{Y}, \underline{B})$, there are homotopy commutative squares

$$(2.3) \quad \begin{array}{ccc} \Sigma \mathcal{Z}_L(\underline{X}, \underline{A}) & \xrightarrow{\hat{\epsilon}_L} & \Sigma \bigvee_{I \subset [m]} \widehat{\mathcal{Z}}_{L_I}(\underline{X}_I, \underline{A}_I) & \text{and} & \Sigma \mathcal{Z}_L(\underline{X}, \underline{A}) & \xrightarrow{\hat{\epsilon}_L} & \Sigma \bigvee_{I \subset [m]} \widehat{\mathcal{Z}}_{L_I}(\underline{X}_I, \underline{A}_I) \\ \downarrow \text{incl} & & \downarrow \text{incl} & & \downarrow \Sigma \underline{f} & & \downarrow \Sigma \bigvee_{I \subset [m]} \underline{f}_I \\ \Sigma \mathcal{Z}_K(\underline{X}, \underline{A}) & \xrightarrow{\hat{\epsilon}_K} & \Sigma \bigvee_{I \subset [m]} \widehat{\mathcal{Z}}_{K_I}(\underline{X}_I, \underline{A}_I) & & \Sigma \mathcal{Z}_K(\underline{Y}, \underline{B}) & \xrightarrow{\hat{\epsilon}_K} & \Sigma \bigvee_{I \subset [m]} \widehat{\mathcal{Z}}_{K_I}(\underline{Y}_I, \underline{B}_I) \end{array}$$

where \underline{f}_I is a subcollection of \underline{f} corresponding to I . If v is a ghost vertex of K , by the same reason as Remark 2.2, the following square becomes homotopy commutative after a suspension.

$$(2.4) \quad \begin{array}{ccc} \Sigma \mathcal{Z}_K(\underline{X}, \underline{A}) & \xlongequal{\quad} & \Sigma(\mathcal{Z}_K(\underline{X}_{[m] \setminus v}, \underline{A}_{[m] \setminus v}) \times A_v) \\ \downarrow \hat{\epsilon}_K & & \downarrow \delta \\ \Sigma \mathcal{Z}_K(\underline{X}_{[m] \setminus v}, \underline{A}_{[m] \setminus v}) \vee \Sigma A_v & \vee & \Sigma(\mathcal{Z}_K(\underline{X}_{[m] \setminus v}, \underline{A}_{[m] \setminus v}) \wedge A_v) \\ \downarrow \hat{\epsilon}_K \vee 1 \vee (\hat{\epsilon}_K \wedge 1) & & \downarrow \hat{\epsilon}_K \vee 1 \vee (\hat{\epsilon}_K \wedge 1) \\ \Sigma \bigvee_{I \subset [m]} \widehat{\mathcal{Z}}_{K_I}(\underline{X}_I, \underline{A}_I) & \xlongequal{\quad} & \Sigma \widehat{\mathcal{Z}}_K(\underline{X}_{[m] \setminus v}, \underline{A}_{[m] \setminus v}) \vee \Sigma A_v \vee \Sigma(\widehat{\mathcal{Z}}_K(\underline{X}_{[m] \setminus v}, \underline{A}_{[m] \setminus v}) \wedge A_v), \end{array}$$

where $\hat{\delta}$ is the composite (2.1). We next consider the naturality of ϖ_I . By definition, ϖ_I has the naturality analogous to (2.3). Moreover, if $v \in [m]$ is a ghost vertex of K , there is a homotopy commutative diagram

$$(2.5) \quad \begin{array}{ccc} \widehat{\mathcal{Z}}_K^{[m]\setminus v} \wedge X_v & \xlongequal{\quad} & \widehat{\mathcal{Z}}_K^{[m]} \\ \downarrow \varpi_{[m]\setminus v} \wedge 1 & & \downarrow \varpi_{[m]} \\ (|\Sigma K| \wedge \widehat{X}^{[m]\setminus v}) \wedge X_v & \xlongequal{\quad} & |\Sigma K| \wedge \widehat{X}^{[m]}. \end{array}$$

We here record the naturality of $\bar{\epsilon}_K$ which will be used below. Let $X \rtimes Y = X \times Y / * \times Y$ and let $\delta : \Sigma X \rtimes Y \rightarrow \Sigma X \vee (\Sigma X \wedge Y)$ be a homotopy equivalence defined as the composite

$$(2.6) \quad \Sigma X \rtimes Y \xrightarrow{\nabla} (\Sigma X \rtimes Y) \vee (\Sigma X \rtimes Y) \xrightarrow{\text{proj}} \Sigma X \vee (\Sigma X \wedge Y),$$

where ∇ is the suspension comultiplication.

Proposition 2.5. *For a subcomplex $L \subset K$ on the same index set $[m]$ and a subset $I \subset [m]$, there are homotopy commutative diagrams*

$$\begin{array}{ccc} \Sigma \mathcal{W}_L^{[m]} \xrightarrow{\text{incl}} \Sigma \mathcal{W}_K^{[m]} & \text{and} & \Sigma \mathcal{W}_{K_I}^I \xrightarrow{\text{incl}} \Sigma \mathcal{W}_K^{[m]} \\ \downarrow \bar{\epsilon}_L & & \downarrow \bar{\epsilon}_{K_I} \\ \Sigma \mathcal{Z}_L^{[m]} \xrightarrow{\text{incl}} \Sigma \mathcal{Z}_K^{[m]} & & \Sigma \mathcal{Z}_{k_I}^I \xrightarrow{\text{incl}} \Sigma \mathcal{Z}_K^{[m]}. \end{array}$$

Moreover, if $v \in [m]$ is a ghost vertex of K , the following diagram becomes homotopy commutative after a suspension.

$$\begin{array}{ccc} \Sigma \mathcal{W}_K^{[m]} \xrightarrow{\text{proj}} \Sigma \mathcal{W}_K^{[m]\setminus v} \vee \Sigma(\mathcal{W}_K^{[m]\setminus v} \wedge X_v) \xrightarrow{\delta^{-1}} \Sigma \mathcal{W}_K^{[m]\setminus v} \rtimes X_v \\ \downarrow \bar{\epsilon}_K & & \downarrow \bar{\epsilon}_K \rtimes 1 \\ \Sigma \mathcal{Z}_K^{[m]} \xlongequal{\quad} \Sigma(\mathcal{Z}_K^{[m]\setminus v} \times X_v) \xrightarrow{\text{proj}} \Sigma \mathcal{Z}_K^{[m]\setminus v} \rtimes X_v \end{array}$$

Proof. The first two squares follow from the combination of (2.3) and its analogue for ϖ_K . Consider the following diagram.

$$\begin{array}{ccc} \Sigma \mathcal{W}_K^{[m]} \xrightarrow{\text{proj}} \Sigma \mathcal{W}_K^{[m]\setminus v} \vee \Sigma(\mathcal{W}_K^{[m]\setminus v} \wedge X_v) \xrightarrow{\delta^{-1}} \Sigma \mathcal{W}_K^{[m]\setminus v} \rtimes X_v \\ \downarrow \bigvee_{I \subset [m]} \varpi_I^{-1} & & \downarrow (\bigvee_{I \subset [m]\setminus v} \varpi_I^{-1}) \rtimes 1 \\ \Sigma \bigvee_{\substack{I \subset [m] \\ I \neq v}} \widehat{\mathcal{Z}}_K^I \xrightarrow{\text{proj}} \Sigma \bigvee_{\substack{I \subset [m] \\ I \neq v}} \widehat{\mathcal{Z}}_K^I \xrightarrow{\delta^{-1}} \Sigma \bigvee_{I \subset [m]\setminus v} \widehat{\mathcal{Z}}_K^I \rtimes X_v \\ \downarrow \hat{\epsilon}_K^{-1} & & \downarrow \hat{\epsilon}_K^{-1} \rtimes 1 \\ \Sigma \mathcal{Z}_K^{[m]} \xlongequal{\quad} \Sigma(\mathcal{Z}_K^{[m]\setminus v} \times X_v) \xrightarrow{\text{proj}} \Sigma \mathcal{Z}_K^{[m]\setminus v} \rtimes X_v \end{array}$$

The upper diagram is homotopy commutative by (2.5) and the lower diagram becomes homotopy commutative after a suspension by (2.4). Therefore by the definition of $\bar{\epsilon}_K$, we obtain the third commutativity. \square

We close this section by evaluating the connectivity of $\mathcal{Z}_K^{[m]}$.

Proposition 2.6. *If K has no ghost vertex and each X_i is path-connected, $\mathcal{Z}_K^{[m]}$ is simply connected.*

Proof. For a simplex $\sigma \in K$, we put

$$D(\sigma) = \mathcal{Z}_\Delta^{[m]} \cup (C\underline{X}, \underline{X})^\sigma,$$

where Δ is the discrete simplicial complex on the vertex set $[m]$, i.e. $\Delta = \{\emptyset, \{1\}, \dots, \{m\}\}$. As in [P] (cf. [IK]), $\mathcal{Z}_\Delta^{[m]}$ is simply connected, hence so is $D(\sigma)$ by the van Kampen theorem. By definition, we have $\mathcal{Z}_K^{[m]} = \bigcup_F D(F)$, where F ranges over all facets of K . We prove the proposition by induction on the number of facets of K . If K has only one facet, K is a simplex, implying that $\mathcal{Z}_K^{[m]}$ is contractible hence simply connected. If $K = \Delta$, $\mathcal{Z}_K^{[m]}$ is simply connected as above. Then we may assume there is a facet F with $\dim F \geq 1$, that is, $K \setminus F$ has no ghost vertex. By the induction hypothesis, $\mathcal{Z}_{K \setminus F}^{[m]}$ is simply connected. Thus since $\mathcal{Z}_{K \setminus F}^{[m]} \cap D(\sigma)$ is path-connected and $D(F)$ is simply connected, the result follows from the van Kampen theorem. \square

Corollary 2.7. *If $\mathcal{Z}_K^{[m]}$ is a co- H -space and each X_i is a connected CW-complex, there is a homotopy equivalence*

$$\mathcal{Z}_K^{[m]} \simeq \mathcal{W}_K^{[m]}.$$

Proof. By Proposition 2.3, the map of Proposition 2.4 induces an isomorphism in homology. Note that $\mathcal{W}_K^{[m]}$ is simply connected since each X_i is path-connected. Thus the proof is completed by Proposition 2.6 and the J.H.C. Whitehead theorem. \square

3. EXTRACTIBLE COMPLEXES

In this section, we introduce new simplicial complexes called extractible complexes by a recursive homotopical condition on deletions of vertices and prove the decomposition of polyhedral products for extractible complexes.

We first set notation for simplicial complexes. Let K be a simplicial complex on the index set $[m]$, possibly with ghost vertices. The link, the star and the deletion of a simplex $\sigma \in K$ is defined respectively as

$$\begin{aligned} \text{lk}_K(\sigma) &= \{\tau \in K \mid \sigma \cup \tau \in K, \sigma \cap \tau = \emptyset\}, \\ \text{st}_K(\sigma) &= \{\tau \in K \mid \sigma \cup \tau \in K\}, \\ \text{dl}_K(\sigma) &= \{\tau \in K \mid \sigma \not\subset \tau\}. \end{aligned}$$

The Alexander dual of K is defined as

$$K^\vee = \{\sigma \subset [m] \mid [m] \setminus \sigma \notin K\}.$$

Since Alexander duals depend on index sets, we must be careful for them.

We now introduce extractible complexes over a commutative ring \mathbb{k} .

Definition 3.1. A simplicial complex K with no ghost vertex is called *extractible* over \mathbb{k} if

- (1) $\text{dl}_K(v)$ is a simplex for some vertex v , or
- (2) $\text{dl}_K(v)$ is extractible over \mathbb{k} for any vertex v and there is a map $|\Sigma K| \rightarrow \bigvee_{v \in [m]} |\Sigma \text{dl}_K(v)|$ satisfying that the composite with the wedge of inclusions

$$|\Sigma K| \rightarrow \bigvee_{v \in [m]} |\Sigma \text{dl}_K(v)| \rightarrow |\Sigma K|$$

induces the identity map in homology with \mathbb{k} coefficient.

We prove a wedge decomposition of polyhedral products for extractible complexes.

Theorem 3.2. *If K is extractible over \mathbb{k} , there is a map*

$$\epsilon_K : \mathcal{W}_K^{[m]} \rightarrow \mathcal{Z}_K^{[m]}$$

inducing the same map as $\bar{\epsilon}_K$ of (2.2) in homology with \mathbb{k} coefficient.

Proof. Induct on m . If $m = 1$, both $\mathcal{W}_K^{[m]}$ and $\mathcal{Z}_K^{[m]}$ are contractible, hence the constant map is the desired ϵ_K . Suppose we have proved the case $m - 1$ and then consider the case m . Suppose $\text{dl}_K(v)$ is a simplex for some vertex v . Consider the pushout

$$(3.1) \quad \begin{array}{ccc} \mathcal{Z}_{\text{lk}_K(v)}^{[m]} & \longrightarrow & \mathcal{Z}_{\text{st}_K(v)}^{[m]} \\ \downarrow & & \downarrow \\ \mathcal{Z}_{\text{dl}_K(v)}^{[m]} & \longrightarrow & \mathcal{Z}_K^{[m]} \end{array}$$

induced from the corresponding pushout of simplicial complexes, where arrows are inclusions.

Note that

$$\mathcal{Z}_{\text{lk}_K(v)}^{[m]} = \mathcal{Z}_{\text{lk}_K(v)}^{[m] \setminus v} \times X_v, \quad \mathcal{Z}_{\text{st}_K(v)}^{[m]} = \mathcal{Z}_{\text{lk}_K(v)}^{[m] \setminus v} \times CX_v \quad \text{and} \quad \mathcal{Z}_{\text{dl}_K(v)}^{[m]} = \mathcal{Z}_{\text{dl}_K(v)}^{[m] \setminus v} \times X_v.$$

Include the pushout

$$\begin{array}{ccc} X_v & \longrightarrow & CX_v \\ \parallel & & \parallel \\ X_v & \longrightarrow & CX_v \end{array}$$

into (3.1) and take the cofiber of each corner. Then we get a pushout

$$(3.2) \quad \begin{array}{ccc} \mathcal{Z}_{\mathbb{k}_K(v)}^{[m]\setminus v} \rtimes X_v & \longrightarrow & \mathcal{Z}_{\mathbb{k}_K(v)}^{[m]\setminus v} \rtimes CX_v \\ \downarrow & & \downarrow \\ \mathcal{Z}_{\mathrm{dl}_K(v)}^{[m]\setminus v} \rtimes X_v & \longrightarrow & \mathcal{Z}_K^{[m]}/CX_v. \end{array}$$

Since $\mathrm{dl}_K(v)$ is a simplex by assumption, $\mathcal{Z}_{\mathrm{dl}_K(v)}^{[m]\setminus v} = \prod_{i \in [m] \setminus v} CX_v$ which is contractible, hence so is $\mathcal{Z}_{\mathrm{dl}_K(v)}^{[m]\setminus v} \rtimes X_v$. Then it follows from (3.2) that there is a homotopy equivalence $\mathcal{Z}_K^{[m]}/CX_v \simeq \Sigma(\mathcal{Z}_{\mathbb{k}_K(v)}^{[m]\setminus v} \wedge X_v)$, so $\mathcal{Z}_K^{[m]}$ has the homotopy type of a suspension. Thus by Corollary 2.7, we obtain the desired result.

Suppose next that $\mathrm{dl}_K(v)$ is extractible over \mathbb{k} for any vertex v and there is a map $s : |\Sigma K| \rightarrow \bigvee_{v \in [m]} |\Sigma \mathrm{dl}_K(v)|$ satisfying that the composite with the wedge of inclusions

$$|\Sigma K| \xrightarrow{s} \bigvee_{v \in [m]} |\Sigma \mathrm{dl}_K(v)| \rightarrow |\Sigma K|$$

induces the identity map in homology with \mathbb{k} coefficient. By the induction hypothesis, there is a map $\epsilon_{\mathrm{dl}_K(v)} : \mathcal{W}_{\mathrm{dl}_K(v)}^{[m]\setminus v} \rightarrow \mathcal{Z}_{\mathrm{dl}_K(v)}^{[m]\setminus v}$ with the desired property for any $v \in [m]$. Then by Proposition 2.5, the composite

$$\bigvee_{I \subsetneq [m]} |\Sigma K_I| \wedge \widehat{X}^I \xrightarrow{\mathrm{incl}} \bigvee_{v \in [m]} \mathcal{W}_{\mathrm{dl}_K(v)}^{[m]\setminus v} \xrightarrow{\bigvee_{v \in [m]} \epsilon_{\mathrm{dl}_K(v)}} \bigvee_{v \in [m]} \mathcal{Z}_{\mathrm{dl}_K(v)}^{[m]\setminus v} \rightarrow \mathcal{Z}_K^{[m]}$$

induces the same map as $\bar{\epsilon}_K$ in homology with \mathbb{k} coefficient on the wedge summand $\bigvee_{I \subsetneq [m]} |\Sigma K_I| \wedge \widehat{X}^I$ of $\mathcal{W}_K^{[m]}$, where the last arrow is the wedge of inclusions. We here notice that there are many choices for the first arrow but any choice will do. Now our remaining task is to construct a map $|\Sigma K| \wedge \widehat{X}^{[m]} \rightarrow \mathcal{Z}_K^{[m]}$ which induces the same map as the restriction of $\bar{\epsilon}_K$ in homology with \mathbb{k} coefficient. Define a map θ_v as the composite

$$\begin{aligned} |\Sigma \mathrm{dl}_K(v)| \wedge \widehat{X}^{[m]} &\xrightarrow{\mathrm{incl}} \mathcal{W}_{\mathrm{dl}_K(v)}^{[m]\setminus v} \vee (\mathcal{W}_{\mathrm{dl}_K(v)}^{[m]\setminus v} \wedge X_v) \xrightarrow{\delta^{-1}} \mathcal{W}_{\mathrm{dl}_K(v)}^{[m]\setminus v} \rtimes X_v \\ &\xrightarrow{\epsilon_{\mathrm{dl}_K(v)} \rtimes 1} \mathcal{Z}_{\mathrm{dl}_K(v)}^{[m]\setminus v} \rtimes X_v \xrightarrow{\mathrm{incl}} \mathcal{Z}_K^{[m]}/CX_m \xrightarrow{\simeq} \mathcal{Z}_K^{[m]}, \end{aligned}$$

where δ is as in (2.6) and the last arrow is the homotopy inverse of the projection $\mathcal{Z}_K^{[m]} \rightarrow \mathcal{Z}_K^{[m]}/CX_v$. By Proposition 2.5, we see that $\Sigma \theta_v$ is homotopic to

$$\Sigma |\Sigma \mathrm{dl}_K(v)| \wedge \widehat{X}^{[m]} \xrightarrow{\mathrm{incl}} \Sigma |\Sigma K| \wedge \widehat{X}^{[m]} \xrightarrow{\mathrm{incl}} \Sigma \mathcal{W}_K^{[m]} \xrightarrow{\bar{\epsilon}_K} \Sigma \mathcal{Z}_K^{[m]}.$$

Thus the composite

$$|\Sigma K| \wedge \widehat{X}^{[m]} \xrightarrow{s \wedge 1} \bigvee_{v \in [m]} |\Sigma \mathrm{dl}_K(v)| \wedge \widehat{X}^{[m]} \xrightarrow{\bigvee_{v \in [m]} \theta_v} \mathcal{Z}_K^{[m]}$$

is the desired map, and therefore the proof is completed. \square

Corollary 3.3. *If K is extractible over \mathbb{Z} (resp. $\mathbb{Z}_{(p)}$) and each X_i is a connected CW-complex, there is a homotopy equivalence*

$$\mathcal{Z}_K^{[m]} \simeq \mathcal{W}_K^{[m]} \quad (\text{resp. } \mathcal{Z}_K^{[m]} \simeq_{(p)} \mathcal{W}_K^{[m]}).$$

Proof. Combine Proposition 2.6, Theorem 3.2 and the J.H.C. Whitehead theorem. □

4. SHELLABLE COMPLEXES

In this section, we prove the extractibility over \mathbb{Z} of a simplicial complex whose Alexander dual is shellable, which implies the decomposition of the corresponding polyhedral product. The proof includes the core idea to show the extractibility in the case of sequentially Cohen-Macaulay complexes, so we set up this section.

We prepare two simple lemmas.

Lemma 4.1. *If a simplicial complex K is collapsible, $|K^\vee|$ is contractible.*

Proof. If $\sigma \in K$ is a free face of K and $\tau \in K$ satisfies $\sigma \subset \tau$ and $\dim \tau = \dim \sigma + 1$, then it is straightforward to see that $\tau^\vee = [m] \setminus \tau$ is a free face of $(K \setminus \{\sigma, \tau\})^\vee$ and $\sigma^\vee = [m] \setminus \sigma \in (K \setminus \{\sigma, \tau\})^\vee$ satisfies $\tau^\vee \subset \sigma^\vee$ and $\dim \sigma^\vee = \dim \tau^\vee + 1$. Then since $(K \setminus \{\sigma, \tau\})^\vee = K^\vee \cup \{\sigma^\vee, \tau^\vee\}$, if K is collapsible, $|K^\vee|$ has the homotopy type of a simplex which is contractible. □

Lemma 4.2. *Let K be a simplicial complex on the index set $[m]$ and choose the index set of $\text{lk}_K(v)$ to be $[m] \setminus v$. Then*

$$\text{lk}_K(v)^\vee = \text{dl}_{K^\vee}(v).$$

Proof. By definition, we have

$$\text{lk}_K(v)^\vee = \{\sigma \subset [m] \setminus v \mid (([m] \setminus v) \setminus \sigma) \cup v \notin K\} = \{\sigma \subset [m] \setminus v \mid [m] \setminus \sigma \notin K\} = \text{dl}_{K^\vee}(v).$$

□

Given a shelling F_1, \dots, F_t of K , F_k with $k > 1$ is called a spanning facet if the boundary of F_k is contained in $\bigcup_{i=1}^{k-1} F_i$. It is easy to see that if F_{i_1}, \dots, F_{i_k} are all spanning facets, $K \setminus \{F_{i_1}, \dots, F_{i_k}\}$ is collapsible.

Proposition 4.3. *If K has no ghost vertex and K^\vee is shellable, K is extractible over \mathbb{Z} .*

Proof. The proof is done by induction on m , where we put the index set of K to be $[m]$. The case $m = 1$ is trivial. Assuming the case $m - 1$, we prove the case m . If K is Δ^{m-1} or $\partial\Delta^{m-1}$, the first condition for extractible complexes is satisfied. Then we assume that K is neither Δ^{m-1} nor $\partial\Delta^{m-1}$, or equivalently, K^\vee has at least one vertex. By Lemma 4.2, we have $\text{dl}_K(v) = \text{lk}_{K^\vee}(v)^\vee$. In [BW2, Proposition 10.14], it is shown that $\text{lk}_{K^\vee}(v)$ is shellable, so $\text{dl}_K(v)$ is extractible over \mathbb{Z} by the induction hypothesis. Let F_1, \dots, F_t be a shelling of K^\vee and let

Γ_{K^\vee} be the set of all spanning facets of K^\vee for this shelling. Put $\Delta_{K^\vee} = K^\vee \setminus \bigcup_{F \in \Gamma_{K^\vee}} F$. Since each $F \in \Gamma_{K^\vee}$ is a minimal non-face of Δ_{K^\vee} , we have

$$(4.1) \quad (\Delta_{K^\vee})^\vee = K \cup \bigcup_{F \in \Gamma_{K^\vee}} F^\vee,$$

where $F^\vee = [m] \setminus F$. Since Δ_{K^\vee} is collapsible, $|(\Delta_{K^\vee})^\vee|$ is contractible by Lemma 4.1, and then

$$|\Sigma K| \simeq |(\Delta_{K^\vee})^\vee|/|K| = \bigvee_{F \in \Gamma_{K^\vee}} S^{|F^\vee|-1} = \bigvee_{F \in \Gamma_{K^\vee}} S^{m-|F|-1}.$$

Let F_{j_1}, \dots, F_{j_l} be all facets of K^\vee such that $v \in F_{j_i}$ and $j_1 < \dots < j_l$. Then $F_{j_1} \setminus v, \dots, F_{j_l} \setminus v$ is a shelling of $\text{lk}_{K^\vee}(v)$, and with this shelling, $\Gamma_{\text{lk}_{K^\vee}(v)} = \{F \setminus v \mid F \in \Gamma_{K^\vee}, v \in F\}$. Choose the index set of $\text{lk}_{K^\vee}(v)$ to be $[m] \setminus v$ and identify $\text{lk}_{K^\vee}(v)^\vee$ with $\text{dl}_K(v)$ by Lemma 4.2. Then by (4.1), we have the inclusion $\Delta_{\text{lk}_{K^\vee}(v)} \rightarrow \Delta_{K^\vee}$ whose Alexander dual $(\Delta_{\text{lk}_{K^\vee}(v)})^\vee \rightarrow (\Delta_{K^\vee})^\vee$ is the union of the inclusion $\iota : \text{dl}_K(v) \rightarrow K$ and the identity map $([m] \setminus v) \setminus (F \setminus v) \rightarrow [m] \setminus F$ for $F \in \Gamma_{K^\vee}$ with $v \in F$. Then it induces a map

$$(4.2) \quad |(\Delta_{\text{lk}_{K^\vee}(v)})^\vee|/|\text{dl}_K(v)| \rightarrow |(\Delta_{K^\vee})^\vee|/|K|$$

which is identified with the inclusion

$$(4.3) \quad \bigvee_{v \in F \in \Gamma_{K^\vee}} S^{m-|F|-1} \rightarrow \bigvee_{F \in \Gamma_{K^\vee}} S^{m-|F|-1}.$$

Consider the homotopy commutative diagram of homotopy cofiber sequences

$$\begin{array}{ccccccc} |\text{dl}_K(v)| & \longrightarrow & |(\Delta_{\text{lk}_{K^\vee}(v)})^\vee| & \longrightarrow & |(\Delta_{\text{lk}_{K^\vee}(v)})^\vee|/|\text{dl}_K(v)| & \longrightarrow & |\Sigma \text{dl}_K(v)| \\ \downarrow |\iota| & & \downarrow & & \downarrow & & \downarrow |\Sigma \iota| \\ |K| & \longrightarrow & |(\Delta_{K^\vee})^\vee| & \longrightarrow & |(\Delta_{K^\vee})^\vee|/|K| & \longrightarrow & |\Sigma K|. \end{array}$$

Since $|(\Delta_{\text{lk}_{K^\vee}(v)})^\vee|$ and $|(\Delta_{K^\vee})^\vee|$ are contractible, the right horizontal arrows are homotopy equivalences, so $|\Sigma \iota|$ is identified with the inclusion (4.2) hence with (4.3). Now it is easy to construct the desired map s and therefore the proof is completed. \square

By Corollary 3.3, we obtain:

Theorem 4.4. *Let K be a simplicial complex on the index set $[m]$ with no ghost vertex. If K^\vee is shellable and each X_i is a connected CW-complex, there is a homotopy equivalence*

$$\mathcal{Z}_K(C\underline{X}, \underline{X}) \simeq \bigvee_{I \subset [m]} |\Sigma K_I| \wedge \widehat{X}^I.$$

Remark 4.5. In the proof of Theorem 4.3, we have actually proved that $|\Sigma K_I|$ has the homotopy type of a wedge of spheres for any $I \subset [m]$. Then Corollary 1.7 in the shellable case follows from Theorem 4.4.

5. SEQUENTIALLY COHEN-MACAULAY COMPLEXES

In this section, we prove the extractibility over $\mathbb{Z}_{(p)}$ of a simplicial complex whose Alexander dual is sequentially Cohen-Macaulay over \mathbb{Z}/p by a homologically generalized technique in the proof of Proposition 4.3, which implies the p -local decomposition of polyhedral products. From this, we deduce the integral decomposition by using the result of McGibbon [M, Corollary 5.1] on the Mislin genus of a co-H-space.

By definition, if a simplicial complex K is sequentially Cohen-Macaulay over \mathbb{k} , so is $\text{lk}_K(v)$ for any vertex v . We record an immediate consequence from this together with Lemma 4.2.

Lemma 5.1. *Let K be a simplicial complex on the index set $[m]$ and choose the index set of $\text{dl}_K(v)$ to be $[m] \setminus v$. If K^\vee is sequentially Cohen-Macaulay over \mathbb{k} , so is $\text{dl}_K(v)^\vee$.*

The following simple lemma will be useful.

Lemma 5.2. *Let K be a simplicial complex with $\tilde{H}_i(K^{(i+1)}; \mathbb{k}) = 0$. Then any i -cycle of K over \mathbb{k} which is not a boundary involves a facet of dimension i .*

Proof. Let x be an i -cycle of K over \mathbb{k} . If x involves no facet, it is a cycle of $K^{(i+1)}$ over \mathbb{k} . Then since $\tilde{H}_i(K^{(i+1)}; \mathbb{k}) = 0$, x is a boundary, completing the proof. \square

We consider a connection between K and $\text{lk}_K(v)$ in homology. For a chain $x = \sum a_j \sigma_j$ of K ($a_j \in \mathbb{k}$, $\sigma_j \in K$) and a vertex v , let $x_v = \sum_{v \in \sigma_j} a_j \sigma_j$.

Proposition 5.3. *If a cycle x of K over \mathbb{k} involves a facet F with $v \in F$, ∂x_v is a cycle of $\text{lk}_K(v)$ over \mathbb{k} which involves a facet $F \setminus v$ of $\text{lk}_K(v)$ and is not a boundary.*

Proof. Consider the Mayer-Vietoris exact sequence

$$\cdots \rightarrow H_*(\text{lk}_K(v); \mathbb{k}) \rightarrow H_*(\text{dl}_K(v); \mathbb{k}) \bigoplus H_*(\text{st}_K(v); \mathbb{k}) \rightarrow H_*(K; \mathbb{k}) \xrightarrow{\partial_*} H_{*-1}(\text{lk}_K(v); \mathbb{k}) \rightarrow \cdots$$

By a straightforward calculation, $\partial_*[x]$ is represented by ∂x_v . Notice that if a cycle involves a facet, it is not a boundary. By definition, ∂x_v involves a facet $F \setminus v$, therefore it is not a boundary, completing the proof. \square

Since sequentially Cohen-Macaulay complexes are characterized by homology, not by facets, we do not have the notion of spanning facets as in the case of shellable complexes above, which play the central role in the proof of Proposition 4.3. We then generalize the notion of spanning facets in a homological setting. Facets F_1, \dots, F_r of a simplicial complex K are called *spanning facets* over \mathbb{k} if

- (1) $\Delta_K = K \setminus \{F_1, \dots, F_r\}$ is \mathbb{k} -acyclic, i.e. $\tilde{H}_*(\Delta_K; \mathbb{k}) = 0$.
- (2) The boundary of F_i is in Δ_K for all i .

Let us recall the (almost) localization of a suspension. For a space X , we choose one point from each path-component of X in such a way that the basepoint of X is involved in this choice.

Let V be the set of these points. Then the homotopy cofiber sequence $\Sigma V \rightarrow \Sigma X \rightarrow \Sigma(X/V)$ splits as

$$\Sigma X \simeq \Sigma V \vee \Sigma(X/V),$$

which is natural with respect to X . Using this splitting, the (almost) p -localization of ΣX is defined as

$$\Sigma X_{(p)} = \Sigma V \vee \Sigma(X/V)_{(p)}.$$

Proposition 5.4. *If K has no ghost vertex and K^\vee is sequentially Cohen-Macaulay over \mathbb{Z}/p , K is extractible over \mathbb{Z}/p .*

Proof. Choose a basis $x_i^1, \dots, x_i^{n_i}$ of $H_i(K^\vee; \mathbb{Z}/p)$. By Lemma 5.2, x_i^1 involves a facet F_i^1 and, by subtracting x_i^1 from $x_i^2, \dots, x_i^{n_i}$, we may assume that $x_i^2, \dots, x_i^{n_i}$ do not involve F_i^1 . Then by induction, we see that for $j = 1, \dots, n_i$, x_i^j involves a facet F_i^j which is not involved in x_i^k for $k \neq j$ and the coefficient of F_i^j in x_i^j is nontrivial.

Claim 5.5. $F_1^1, \dots, F_1^{n_1}, \dots, F_d^1, \dots, F_d^{n_d}$ are spanning facets of K^\vee over \mathbb{Z}/p , where $d = \dim K$.

Proof. The second condition for spanning facets is satisfied by the choice of F_j^i and x_j^i . Put Γ_{K^\vee} to be the set of all F_j^i and $\Delta_{K^\vee} = K^\vee \setminus \bigcup_{F \in \Gamma_{K^\vee}} F$. Then it remains to show that Δ_{K^\vee} is acyclic over \mathbb{Z}/p . Since each $F \in \Gamma_{K^\vee}$ satisfies the second condition for spanning facets, we have

$$|K^\vee|/|\Delta_{K^\vee}| = \bigvee_{F \in \Gamma_{K^\vee}} S^{|F|-1}$$

and by the choice of x_j^i , the pinch map $|K^\vee| \rightarrow |K^\vee|/|\Delta_{K^\vee}|$ sends x_j^i to a generator of $H_i(S^i; \mathbb{Z}/p)$ in homology. Then the Puppe exact sequence

$$\cdots \rightarrow H_*(|\Delta_{K^\vee}|; \mathbb{Z}/p) \rightarrow H_*(|K^\vee|; \mathbb{Z}/p) \rightarrow H_*(|K^\vee|/|\Delta_{K^\vee}|; \mathbb{Z}/p) \rightarrow \cdots$$

shows that Δ_{K^\vee} is acyclic over \mathbb{Z}/p . □

As in the proof of Proposition 4.3, we have

$$(\Delta_{K^\vee})^\vee = K \cup \bigcup_{F \in \Gamma_{K^\vee}} F^\vee, \quad \text{hence} \quad |(\Delta_{K^\vee})^\vee|/|K| = \bigvee_{F \in \Gamma_{K^\vee}} S^{m-|F|-1},$$

where $F^\vee = [m] \setminus F$. Consider the edges $F^\vee = [m] \setminus F$ of $(\Delta_{K^\vee})^\vee$ for $F \in \Gamma_{K^\vee}$ with $|F| = m - 2$ where facets of K^\vee are of dimension $\leq m - 2$ since K has no ghost vertex. If two such edges have end points in the same connected component of K , we connect these ends by an edge path in K having no self-intersection and take one vertex from this path, where the connecting path is a single point if two ends coincide. Let V_K be the set of all these vertices and remaining ends of such edges. Then there is a one-to-one correspondence between V_K and the connected components of K , and we have constructed a tree T_K in $|(\Delta_{K^\vee})^\vee|$ whose vertex set is V_K ,

where we put the edge path between two vertices in V_K as the edge of T_K . There is a homotopy commutative diagram of homotopy cofiber sequences

$$(5.1) \quad \begin{array}{ccccccc} |V_K| & \longrightarrow & T_K & \longrightarrow & T_K/|V_K| & \xrightarrow{\tilde{\alpha}} & |\Sigma V_K| \\ \downarrow |\iota| & & \downarrow & & \downarrow & & \downarrow |\Sigma \iota| \\ |K| & \longrightarrow & |(\Delta_{K^\vee})^\vee| & \longrightarrow & |(\Delta_{K^\vee})^\vee/|K| & \xrightarrow{\alpha} & |\Sigma K| \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ |K|/|V_K| & \longrightarrow & |(\Delta_{K^\vee})^\vee/T_K & \longrightarrow & |(\Delta_{K^\vee})^\vee/(|K| \cup T_K) & \xrightarrow{\bar{\alpha}} & \Sigma(|K|/|V_K|). \end{array}$$

Since T_K is contractible, $\tilde{\alpha}$ is a homotopy equivalence. Since $|(\Delta_{K^\vee})^\vee|$ is acyclic over \mathbb{Z}/p , α and then $\bar{\alpha}$ induces an isomorphism in the mod p homology. Then since both $|(\Delta_{K^\vee})^\vee/(|K| \cup T_K)$ and $\Sigma(|K|/|V_K|)$ are simply connected and finite complexes, it follows from [HMR, Theorem 1.14 in Chapter II] that there is a homotopy equivalence

$$(5.2) \quad |\Sigma K|_{(p)} \simeq \bigvee_{\substack{F \in \Gamma_{K^\vee} \\ |F|=m-2}} S^{m-|F|-1} \vee \bigvee_{\substack{F \in \Gamma_{K^\vee} \\ |F|<m-2}} S_{(p)}^{m-|F|-1}.$$

We induct on m to prove Proposition 5.4. The case $m = 1$ is trivial. Suppose the case $m - 1$ holds. By Lemma 5.1 and the induction hypothesis, $\text{dl}_K(v)$ is extractible over \mathbb{Z}/p for any vertex v . By Lemma 5.3, if $F_1, \dots, F_r \in \Gamma_{K^\vee}$ involve a vertex v , $\partial(x_1)_v, \dots, \partial(x_r)_v$ form a part of a basis of $H_*(\text{lk}_{K^\vee}(v); \mathbb{Z}/p)$, where x_i is a cycle corresponding to F_i . Then in the above way, we can choose spanning facets of $\text{lk}_{K^\vee}(v)$ over \mathbb{Z}/p which include $F \setminus v$ for all $F \in \Gamma_{K^\vee}$ with $v \in F$. Hence with these spanning facets over \mathbb{Z}/p , we have the homotopy commutative diagram (5.1) for $\text{lk}_{K^\vee}(v)^\vee = \text{dl}_K(v)$ which is compatible with that for K , so through the homotopy equivalence (5.2), the inclusion $|\Sigma \text{dl}_K(v)|_{(p)} \rightarrow |\Sigma K|_{(p)}$ is identified with the wedge of the identity map of $\bigvee_{v \in F \in \Gamma_{K^\vee}} S^{m-|F|-1}$ and the constant map on the remaining wedge summand. It is now easy to construct the desired map, completing the proof. \square

Proof of Theorem 1.9. Combine Corollary 3.3 and Proposition 5.4. \square

We want to integrate the p -local homotopy equivalence of Theorem 1.9 for each prime p to obtain Theorem 1.5. To this end, let us recall the result of McGibbon [M, Corollary 5.1] on the Mislin genus of a co-H-space.

Proposition 5.6 (McGibbon [M, Corollary 5.1]). *Let X, Y be simply connected finite complexes. If $X \simeq_{(p)} Y$ for any prime p and Y is a co-H-space, X is also a co-H-space.*

Proof of Theorem 1.5. By Theorem 1.9, $\mathcal{Z}_K^{[m]} \simeq_{(p)} \mathcal{W}_K^{[m]}$ for any prime p and $\mathcal{W}_K^{[m]}$ is a suspension. Since each X_i is a finite complex, so are $\mathcal{Z}_K^{[m]}$ and $\mathcal{W}_K^{[m]}$. Then by Proposition 5.6, $\mathcal{Z}_K^{[m]}$ is a co-H-space. Therefore Theorem 1.5 follows from Corollary 2.7. \square

In order to prove Corollary 1.7, we prepare the following simple lemma.

Lemma 5.7. *Let X be a connected finite type CW-complex. If $\Sigma X_{(p)}$ has the homotopy type of a wedge of p -local spheres for any prime p , ΣX itself has the homotopy type of a wedge of spheres.*

Proof. By assumption, $H_i(\Sigma X; \mathbb{Z})$ is a free abelian group of finite rank. Choose a basis $x_1^i, \dots, x_{n_i}^i$ of $H_i(X; \mathbb{Z})$. Using a p -local homotopy equivalence between ΣX and a wedge of spheres, we can easily construct a map ${}_p\theta_j^i : S^i \rightarrow \Sigma X_{(p)}$ satisfying $({}_p\theta_j^i)_*(u_i) = x_j^i$ in homology with $\mathbb{Z}_{(p)}$ coefficient for any i, j , where u_i is a generator of $H_i(S^i; \mathbb{Z}) \cong \mathbb{Z}$. Let $\{p_1, p_2, \dots\}$ be the set of all primes with $p_i \neq p$. It is well known that $\Sigma X_{(p)}$ is given as the homotopy colimit of the sequence

$$\Sigma X \xrightarrow{l_1} \Sigma X \xrightarrow{l_2} \Sigma X \xrightarrow{l_3} \Sigma X \xrightarrow{l_4} \dots$$

where $l_k = p_1 \cdots p_k$ and $\underline{q} : \Sigma X \rightarrow \Sigma X$ is the degree q map. By the compactness of S^i , ${}_p\theta_j^i$ factors through the finite step of the above sequence. Then there is a map ${}_p\bar{\theta}_j^i : S^i \rightarrow \Sigma X$ satisfying $({}_p\bar{\theta}_j^i)_*(u_i) = {}_p a_j^i x_j^i$ with $p \nmid {}_p a_j^i$ in the integral homology. Now we can choose primes q_1, \dots, q_n such that ${}_{q_1} a_j^i, \dots, {}_{q_n} a_j^i$ are relatively prime. There are integers d_1, \dots, d_n such that $d_1({}_{q_1} a_j^i) + \dots + d_n({}_{q_n} a_j^i) = 1$ hence the map

$$\lambda_j^i = \underline{d}_1 \circ {}_{q_1} \bar{\theta}_j^i + \dots + \underline{d}_n \circ {}_{q_n} \bar{\theta}_j^i$$

satisfies $(\lambda_j^i)_*(u_i) = x_j^i$ in the integral homology, where the sum is defined by using the suspension comultiplication of ΣX . Thus the map $\bigvee_{i \geq 1} \bigvee_{j=1}^{n_i} \lambda_j^i$ induces an isomorphism in the integral homology, and therefore the proof is completed by the J.H.C. Whitehead theorem. \square

Proof of Corollary 1.7. By Theorem 1.5, there is a homotopy equivalence

$$\mathcal{Z}_K(\underline{D}^2, \underline{S}^1) \simeq \bigvee_{\emptyset \neq I \subset [m]} |\Sigma K_I| \wedge S^{|I|},$$

where we exclude the case $I = \emptyset$ since the corresponding wedge summand is a single point. By Lemma 5.1 and (5.2), $\Sigma|\Sigma K_I|_{(p)}$ has the homotopy type of a wedge of p -local spheres of dimension > 1 for any prime p and $I \subset [m]$. Then it follows from Lemma 5.7 that $\Sigma|\Sigma K_I|$ has the homotopy type of a wedge of spheres of dimension > 1 for any $I \subset [m]$ hence the result. \square

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