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by

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Abstract. Babson and Kozlov [BK] studied $\mathtt{Hom}$-complexes of graphs with a focus on graph colorings. In this paper, we generalize $\mathtt{Hom}$-complexes to r-uniform hypergraphs (with multiplicities) and study them mainly in connection with hypergraph colorings. We reinterpret a result of Alon, Frankl and Lovász [AFL] by $\mathtt{Hom}$-complexes and show a hierarchy of known lower bounds for the chromatic numbers of r-uniform hypergraphs (with multiplicities) using $\mathtt{Hom}$-complexes.

1. Introduction

1.1. **$\mathtt{Hom}$-complexes of graphs.** Since Lovász solved the famous Kneser conjecture by relating the chromatic number of a given graph to connectivity of its neighborhood complex [Lo], it is a standard method to study combinatorial properties of graphs by relating them with topological properties of appropriately constructed polyhedral complexes from graphs. Then as is seen in [Jo], a plenty of complexes have been constructed from graphs. Among others, let us consider $\mathtt{Hom}$-complexes which were first introduced by Lovász and studied further by Babson and Kozlov [BK], [Ko1], [Ko2]. Compared to other complexes of graphs, the construction of $\mathtt{Hom}$-complex $\mathtt{Hom}(G,H)$ for graphs G,H is quite natural; it is a *space* of maps from G to H. Moreover, some complexes of graphs concerning colorings are realized by special $\mathtt{Hom}$-complexes [BK], [Ko1] by which one can easily understand related construction. For example, a result of Lovász [Lo] can be reproved easily by using $\mathtt{Hom}$-complexes as follows.

Let us start with a standard observation. Recall that an n-coloring of a graph G is a labelling of vertices of G by n colors in such a way that adjacent vertices have distinct colors. Then if K_n denotes the complete graph with n vertices, there is a one-to-one correspondence between n-colorings of G and homomorphisms of G into K_n. Suppose G admits an n-coloring. Then since the $\mathtt{Hom}$-complex $\mathtt{Hom}(G,H)$ is natural with respect to G,H, there is a map

$$\mathtt{Hom}(T,G)\to\mathtt{Hom}(T,K_n) \tag{1.1}$$

for any graph T. Specialize T to the complete graph K_2 with 2 vertices. Then a natural C_2-action on K_2 yields C_2-actions on both $\mathtt{Hom}(T,G)$ and $\mathtt{Hom}(T,K_n)$, and furthermore, the map (1.1) is a C_2-map for $T=K_2$, where C_k denotes the cyclic group of order k. One can easily see that the C_2-actions are free and can also easily count the dimension of $\mathtt{Hom}(K_2,K_n)$ as $n-2$ by definition. Then it follows from the Borsuk-Ulam theorem that

$$\operatorname{conn}\mathtt{Hom}(K_2,G)\le n-3,$$

where $\operatorname{conn} X$ denotes connectivity of a space X. Finally, since $\mathtt{Hom}(K_2, G)$ has the homotopy type of the neighborhood complex of G as in [BK], we obtain the result of Lovász [Lo]. The point of this proof is that we can get C_2-actions and a C_2-map quite naturally, which is often the most difficult part of the above mentioned topological method for graphs.

1.2. **Generalization to r-graphs.** Let us now generalize graphs to r-uniform hypergraphs. Recall that an r-uniform hypergraph (or an r-graph, for short) G consists of the vertex set and the edge set which is a collection of r elements subsets of the vertex set. Then 2-graphs are simple graphs, for instance. Homomorphisms of r-graphs are obviously defined. In [Ko2], Kozlov suggested a recipe to construct a *space* of a collection of maps between finite sets. Then one can define Hom-complexes for r-graphs as well. We would like to study colorings of r-graphs by using Hom-complexes as in the above case of graphs. Colorings of graphs are generalized to r-graphs as follows. An n-coloring of an r-graph is a labelling of vertices by n colors such that each edge contains more than 2 colors. Then for $r \geq 3$, colorings of r-graphs cannot be realized as homomorphisms. Then in order to study r-graph colorings by Hom-complexes, we must extend the category of r-graphs so that colorings become homomorphisms. If we extend the category of r-graphs to that of all hypergraphs, colorings become homomorphisms. However, this category is too big to control objects. So we need a much smaller extension of the category of r-graphs. For this purpose, we will consider r-graphs with multiplicities which were first introduced by Lange [La] in a different context. Then we will study colorings of r-graphs with multiplicities through Hom-complexes. More precisely, we will give a lower bound for the chromatic numbers of r-graphs with multiplicities using group actions on special Hom-complexes. Alon, Frankl and Lovász [AFL] defined certain simplicial complexes of r-graphs (without multiplicities) and gave a lower bound for the chromatic numbers by a rather tricky construction. We will show that these complexes are essentially the same as the above special Hom-complexes, and then we can interpret their construction in terms of Hom-complexes, which will make things clear. We will also consider $\mathtt{Hom}_+$-complexes of r-graphs with multiplicities (cf. [Ko2]) and show a hierarchy among lower bounds for the chromatic numbers.

1.3. **Organization.** The organization of the paper is as follows. In §2, we introduce r-graphs with multiplicities generalizing r-graphs by which we can study colorings of r-graphs as special homomorphisms. In §3, we recall a general construction of Hom-complexes of classes of maps between finite sets and then apply it to r-graphs with multiplicities. We show analogy of results of Babson and Kozlov [BK] for Hom-complexes of r-graphs with multiplicities and give a lower bound for the chromatic number by special Hom-complexes. In §4, we show that the box-edge complexes of Alon, Frankl and Lovász [AFL] are realized by the above special Hom-complexes, by which we see that the above lower bound is the same as the one given by Alon, Frankl and Lovász [AFL]. In §5, we consider $\mathtt{Hom}_+$-complexes of r-graphs with multiplicities and give another lower bound for the chromatic number. By comparing Hom-complexes and $\mathtt{Hom}_+$-complexes, we show a hierarchy among the above two lower bounds.

2. r-graphs with multiplicities

2.1. **r-graphs.** Let us explain in detail why we introduce r-graphs with multiplicities. Recall that an r-uniform hypergraph (r-graph, for short)G is a pair of a finite set $V(G)$ and a collection $E(G)$ of r elements subsets of $V(G)$. $V(G)$ and $E(G)$ are respectively called the vertex set and the edge set of G. For r-graphs G, H, a homomorphism $f : G \to H$ is a map $f : V(G) \to V(H)$ satisfying $f_*(E(G)) \subset E(H)$. Our objects are colorings of r-graphs. An n-coloring of an r-graph G is a map $c : V(G) \to [n]$ such that if $\{v_1, \dots, v_r\} \in E(G)$, $\{c(v_1), \dots, c(v_r)\} \subset [n]$ is not a singleton, where $[n] = \{1, 2, \dots, n\}$. Then one sees that colorings cannot be realized by homomorphisms in general as in the case of graphs. Then generalizing r-graphs, we introduce r-graphs with multiplicities among which colorings are homomorphisms.

2.2. **r-graphs with multiplicities.** Recall that the n^{th} symmetric product of a set V is defined as

$$\mathrm{SP}^n(V) = \underbrace{V \times \cdots \times V}_{n} / \Sigma_n,$$

where the action of the symmetric group Σ_n is given as $\sigma(v_1, \dots, v_n) = (v_{\sigma(1)}, \dots, v_{\sigma(n)})$ for $\sigma \in \Sigma_n$ and $v_1, \dots, v_n \in V$. We denote an element of $\mathrm{SP}^n(V)$ by $v_1 \cdots v_n$ for $v_1, \dots, v_n \in V$. In [La], Lange used multisets which are naturally identified with elements of symmetric products. We now define r-graphs with multiplicities and their homomorphisms and colorings.

Definition 2.1. (1) An r-graph with multiplicities G consists of a finite set $V(G)$ and a subset $E(G)$ of $\mathrm{SP}^r(V(G)) \setminus \Delta$, where $\Delta = \{v \cdots v \in \mathrm{SP}^r(V(G))\}$. $V(G)$ and $E(G)$ are called the vertex set and the edge set of G, respectively.

(2) Let G, H be r-graphs with multiplicities. A homomorphism $f : G \to H$ is a map $g : V(G) \to V(H)$ satisfying $f_*(E(G)) \subset E(H)$.

(3) An n-coloring of an r-graph with multiplicities G is a map $c : V(G) \to [n]$ such that if $v_1 \cdots v_r \in E(G)$, $\{c(v_1), \dots, c(v_r)\} \subset [n]$ is not a singleton.

(4) The chromatic number $\chi(G)$ of an r-graph with multiplicities G is the minimum integer n such that G admits an n-coloring.

Remark 2.2. If we allow r-graphs with multiplicities to have diagonal edges in Δ, some r-graphs do not admit any colorings. Since we will study colorings, we have omitted diagonal edges from r-graphs with multiplicities.

Since r elements subsets of a set V may be regarded as elements of $\mathrm{SP}^r(V) \setminus \Delta$, r-graphs with multiplicities and their homomorphisms and colorings include r-graphs and their homomorphisms and colorings.

Define an r-graph with multiplicities $\mathcal{K}_n^{(r)}$ as the maximum r-graph with multiplicities with n vertices. Namely,

$$V(\mathcal{K}_n^{(r)}) = [n] \quad \text{and} \quad E(\mathcal{K}_n^{(r)}) = \mathrm{SP}^r([n]) \setminus \Delta.$$

It is clear that there is the desired property as follows.

Proposition 2.3. *There is a one-to-one correspondence between n-colorings of an r-graph with multiplicities G and homomorphisms from G to $\mathcal{K}_n^{(r)}$.*

3. `Hom`-complexes of r-graphs with multiplicities

3.1. **General `Hom`-complexes.** Let us first recall a recipe of general `Hom`-complexes suggested by Kozlov [Ko2]. Let S, T be finite sets. Then a map $S \to T$ is identified with an element of T^S. Let Δ^T be the simplex whose vertex set is T. Since T^S is the vertex set of a direct product $\prod_S \Delta^T$, a map $S \to T$ is identified with a vertex of $\prod_S \Delta^T$. This simple observation leads us to the following definition of `Hom`-complexes which may be regarded as *spaces* of given maps between finite sets.

Definition 3.1. Let S, T be finite sets and $\mathcal{C}$ be a class of maps from S to T. The `Hom`-complex $\mathtt{Hom}^{\mathcal{C}}(S, T)$ is the maximum subcomplex of $\prod_S \Delta^T$ whose vertex set is $\mathcal{C}$.

Let S, T be finite sets and $\mathcal{C}$ be a class of maps from S to T. Given a map $f : T \to T'$ with T' finite and a class $\mathcal{D}$ of maps from S to T'. If $f_*(\mathcal{C}) \subset \mathcal{D}$, we can define a map of polyhedral complexes

$$f_* : \mathtt{Hom}^{\mathcal{C}}(S, T) \to \mathtt{Hom}^{\mathcal{D}}(S, T')$$

by sending $h \in \mathcal{C}$ to $f \circ h \in \mathcal{D}$. Dually, given a map $g : S' \to S$ and a class $\mathcal{E}$ of maps from S' to T satisfying $g^*(\mathcal{C}) \subset \mathcal{E}$, we can also define a map of polyhedral complexes

$$g^* : \mathtt{Hom}^{\mathcal{C}}(S, T) \to \mathtt{Hom}^{\mathcal{E}}(S', T)$$

by sending $h \in \mathcal{E}$ to $h \circ g \in \mathcal{C}$.

By definition of the above induced maps, we have the following functoriality.

Proposition 3.2. *Let S, T be finite sets and $\mathcal{C}$ be a class of maps from S to T.*

(1) *Let T_1, T_2 be finite sets and $\mathcal{D}_1, \mathcal{D}_2$ be classes of maps from S to T_1 and T_2, respectively. If maps $f_1 : T \to T_1$ and $f_2 : T_1 \to T_2$ satisfy $(f_1)_*(\mathcal{C}) \subset \mathcal{D}_1$ and $(f_2)_*(\mathcal{D}_1) \subset \mathcal{D}_2$, the induced maps on `Hom`-complexes satisfy*

$$(f_2 \circ f_1)_* = (f_2)_* \circ (f_1)_*.$$

(2) *Let S_1, S_2 be finite sets and $\mathcal{E}_1, \mathcal{E}_2$ be classes of maps from S_1 and S_2 to T, respectively. If maps $g_1 : S_1 \to S$ and $g_2 : S_2 \to S_1$ satisfy $(g_1)^*(\mathcal{C}) \subset \mathcal{E}_1$ and $(g_2)^*(\mathcal{E}_1) \subset \mathcal{E}_2$, the induced maps on `Hom`-complexes satisfy*

$$(g_2 \circ g_1)^* = (g_1)^* \circ (g_2)^*.$$

3.2. **`Hom`-complexes of r-graphs with multiplicities.** Let us return to r-graphs with multiplicities. Homomorphisms of r-graphs with multiplicities are maps between vertices satisfying certain properties. Since we are assuming vertex sets of r-graphs with multiplicities to be finite, we can apply the above general construction of `Hom`-complexes to r-graphs with multiplicities.

Definition 3.3. Let G, H be r-graphs with multiplicities and $\mathcal{C}$ be the set of homomorphisms from G to H. The `Hom`-complex $\mathtt{Hom}(G, H)$ is defined as $\mathtt{Hom}^{\mathcal{C}}(V(G), V(H))$.

By Proposition 3.2, we have the following.

Proposition 3.4. *Let* $\mathbf{Graph}^{(r)}$ *and* $\mathbf{Poly}$ *be the categories of r-graphs with multiplicities and polyhedral complexes, respectively. Then*

$$(\mathbf{Graph}^{(r)})^{\mathrm{op}} \times \mathbf{Graph}^{(r)} \to \mathbf{Poly}, \quad (G, H) \mapsto \mathtt{Hom}(G, H)$$

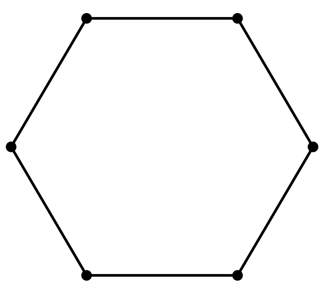
is a functor.

By Proposition 2.3, the $\mathtt{Hom}$-complex $\mathtt{Hom}(G, \mathcal{K}_n^{(r)})$ for an r-graph with multiplicities G is considered as a *space* of n-colorings of G. Then $\mathtt{Hom}(G, \mathcal{K}_n^{(r)})$ is especially important, and hence we here give some easy examples. Let $L_n^{(r)}$ denote the line r-graph with n vertices. Namely, $L_n^{(r)}$ is defined as

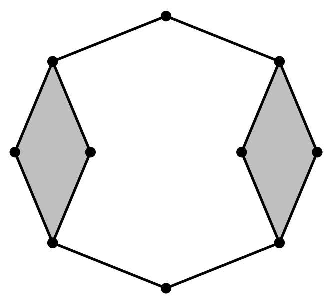
$$V(L_n^{(r)}) = [n] \quad \text{and} \quad E(L_n^{(r)}) = \{\{i, i+1, \ldots, i+r-1\} \mid i = 1, \ldots, n-r+1\}.$$

Then $\mathtt{Hom}(L_n^{(3)}, \mathcal{K}_2^{(3)})$ for $n = 3, 4, 5$ are given as follows.

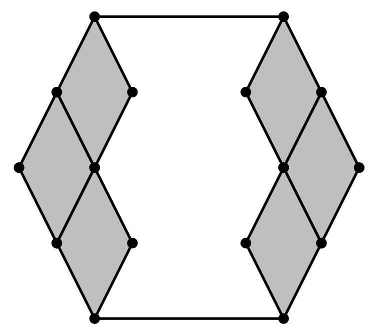
$\mathtt{Hom}(L_3^{(3)}, \mathcal{K}_2^{(3)})$



$\mathtt{Hom}(L_4^{(3)}, \mathcal{K}_2^{(3)})$



$\mathtt{Hom}(L_5^{(3)}, \mathcal{K}_2^{(3)})$



Note that $\mathtt{Hom}(L_n^{(3)}, \mathcal{K}_2^{(3)})$ for $n = 3, 4, 5$ have the same homotopy type. This will be justified below in a more general setting.

Let $C_n^{(r)}$ be the cyclic r-graph with n vertices. That is, $C_n^{(r)}$ is given as

$$V(C_n^{(r)}) = \mathbb{Z}/n \quad \text{and} \quad E(C_n^{(r)}) = \{\{i, i+1, \ldots, i+r-1\} \mid i \in \mathbb{Z}/n\}.$$

Let us next consider $\mathtt{Hom}(C_n^{(3)}, \mathcal{K}_2^{(3)})$. Since $C_3^{(3)} = L_3^{(3)}$, $\mathtt{Hom}(C_3^{(3)}, \mathcal{K}_2^{(3)})$ is a hexagon. One can easily see that $\mathtt{Hom}(C_4^{(3)}, \mathcal{K}_2^{(3)})$ consists of discrete six points and that $\mathtt{Hom}(C_5^{(3)}, \mathcal{K}_2^{(3)})$ is the outer polygon of $\mathtt{Hom}(L_5^{(3)}, \mathcal{K}_2^{(3)})$. Then their homotopy types are not the same.

3.3. **Lower bound for the chromatic number.** By functoriality of $\mathtt{Hom}(G, H)$, group actions on G and H induce those on $\mathtt{Hom}(G, H)$. We next consider these group actions for special G. Let $K_n^{(r)}$ be the maximum r-graph with n vertices. Namely,

$$V(K_n^{(r)}) = [n] \quad \text{and} \quad E(K_n^{(r)}) = \mathrm{SP}^r([n]) \setminus \Delta_n,$$

where $\Delta_n = \{v_1 \cdots v_r \in \mathrm{SP}^r([n]) \mid v_i \neq v_j \text{ for } i \neq j\}$. Notice that by a cyclic permutation of vertices, the cyclic group C_n acts on $K_n^{(r)}$.

Lemma 3.5. *If r is a prime, the induced C_r-action on $\mathtt{Hom}(K_r^{(r)}, G)$ is free.*

Proof. Any face of $\mathtt{Hom}(K_r^{(r)}, G)$ is of the form $\Delta^{S_1} \times \cdots \times \Delta^{S_r}$ such that each $(v_1, \ldots, v_r) \in S_1 \times \cdots \times S_r$ satisfies $v_1 \cdots v_r \in E(G)$. Then in particular, there is no diagonal element $(v, \ldots, v)$ in $S_1 \times \cdots \times S_r$. Let g be a non-trivial element of C_r. Then by renumbering if necessary, we have

$$g \cdot (v_1, \ldots, v_r) = (v_r, v_1, \ldots, v_{r-1})$$

for $(v_1, \ldots, v_r) \in S_1 \times \cdots \times S_r$ since r is a prime. Suppose $g(\Delta^{S_1} \times \cdots \times \Delta^{S_r}) = \Delta^{S_1} \times \cdots \times \Delta^{S_r}$. Then we have observed that elements of S_1 belong to all $S_1, \ldots, S_r$, a contradiction. □

Using the index of the above free group action [M], we give a lower bound for the chromatic numbers of r-graphs with multiplicities. We set some notation. Let Γ be a non-trivial finite group (with the discrete topology). Let $E_n\Gamma$ be the join of $n+1$ copies of Γ on which Γ acts diagonally. Then this Γ-action on $E_n\Gamma$ is free and $E_n\Gamma$ has the homotopy type of a wedge of n dimensional spheres. For a free Γ-complex X, the Γ-index of X is defined as

$$\mathrm{ind}_\Gamma X = \min\{n \mid \text{there is a } \Gamma\text{-map } X \to E_n\Gamma\}.$$

Let us list basic properties of $\mathrm{ind}_\Gamma X$.

Proposition 3.6. *Let Γ be a non-trivial finite group and let X, Y be free Γ-complexes.*

(1) *If there is a Γ-map from X to Y, we have*

$$\mathrm{ind}_\Gamma X \leq \mathrm{ind}_\Gamma Y.$$

(2) *The join $X * Y$ is a free Γ-space by the diagonal Γ-action for which it holds that*

$$\mathrm{ind}_\Gamma(X * Y) \leq \mathrm{ind}_\Gamma X + \mathrm{ind}_\Gamma Y + 1.$$

(3) *It holds that*

$$\mathrm{conn}\, X + 1 \leq \mathrm{ind}_\Gamma X \leq \dim\, X.$$

Proof. (1) follows from definition and (2) follows from the fact that $E_n\Gamma = E_m\Gamma * E_{n-m-1}\Gamma$. By the Borsuk-Ulam theorem due to Dold [D], we have $\mathrm{ind}_\Gamma E_n\Gamma = n$. Then (3) is shown by an easy obstruction argument. □

Put $B_n\Gamma = E_n\Gamma/\Gamma$, $B\Gamma = \bigcup_{n\geq 1} B_n\Gamma$ and $E\Gamma = \bigcup_{n\geq 1} E_n\Gamma$. Then the natural projection $E\Gamma \to B\Gamma$ is the well-known Milnor's universal principal Γ-bundle. Let $\varphi : X/\Gamma \to B\Gamma$ be the classifying map of a free Γ-complex X. Then it follows that $\mathrm{ind}_\Gamma X$ coincides with the minimum integer n such that φ factors through the inclusion $B_n\Gamma \to B\Gamma$, up to homotopy. By [Ja], we obtain that $\mathrm{ind}_\Gamma X$ is equal to the LS-category of the classifying map φ, implying that there are a lot of quantities estimating $\mathrm{ind}_\Gamma X$ other than connectivity and dimension.

We now give a lower bound for the chromatic number of r-graphs with multiplicities.

Theorem 3.7. *Let G be an r-graph with multiplicities. If r is a prime, there holds*

$$\chi(G) \geq \frac{\mathrm{ind}_{C_r}\mathtt{Hom}(K_r^{(r)}, G) + 1}{r-1} + 1.$$

Proof. By Lemma 3.5, $\mathtt{Hom}(K_r^{(r)}, H)$ is a free C_r-complex for any r-graph with multiplicities H. Suppose there is an n-coloring of G, or equivalently, a homomorphism $f : G \to \mathcal{K}_n^{(r)}$. By Proposition 3.4, the induced map $f_* : \mathtt{Hom}(K_r^{(r)}, G) \to \mathtt{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)})$ is a C_r-map, implying

$$\mathrm{ind}_{C_r}\mathtt{Hom}(K_r^{(r)}, G) \leq \dim \mathtt{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)})$$

by Proposition 3.6. We then count the dimension of $\mathtt{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)})$. Any face of $\mathtt{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)})$ is given as $\Delta^{S_1} \times \cdots \times \Delta^{S_r}$ such that $S_1, \ldots, S_r \subset [n]$ and $S_1 \cap \cdots \cap S_r = \emptyset$. The maximum of $|S_1| + \cdots + |S_r|$ is $nr - n$ and then the dimension of $\mathtt{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)})$ is $nr - n - r = (r-1)(n-1) - 1$, completing the proof. □

By Proposition 3.6, we obtain the following.

Corollary 3.8. *Let G be an r-graph with multiplicities. If r is a prime, we have*

$$\chi(G) \geq \frac{\mathrm{conn}\,\mathtt{Hom}(K_r^{(r)}, G) + 2}{r-1} + 1.$$

3.4. **Homotopy lemmas.** Let us recall three lemmas from [BK] and [Ko2] which will be used below. We first set some notation. Let P be a poset. We denote the order complex of P by $\Delta(P)$. That is, $\Delta(P)$ is a simplicial complex whose n-simplices are chains in P of length $n+1$. For $p \in P$, let

$$P_{\leq p} = \{q \in P \mid q \leq p\} \quad \text{and} \quad P_{\geq p} = \{q \in P \mid q \geq p\}.$$

We first state the famous Quillen fiber lemma.

Lemma 3.9 (cf. [Ko2])**.** *Let $\varphi : P \to Q$ be a poset map between finite posets. If $\Delta(\varphi^{-1}(Q_{\leq} q))$ is contractible for any $q \in Q$, then $\Delta(\varphi) : \Delta(P) \to \Delta(Q)$ is a homotopy equivalence.*

We next state a variant of the Quillen fiber lemma proved in [BK].

Lemma 3.10 ([BK])**.** *For a poset map $\varphi : P \to Q$ between finite posets, suppose the following conditions.*

(1) *$\Delta(\varphi^{-1}(q))$ is contractible for any $q \in Q$.*
(2) *For any $q \in Q$ and $p \in \varphi^{-1}(Q_{\geq q})$, the poset $\varphi^{-1}(q) \cap P_{\leq p}$ has the maximum.*

Then $\Delta(\varphi) : \Delta(P) \to \Delta(Q)$ is a homotopy equivalence.

Finally, we recall the generalized nerve lemma which is frequently used in combinatorial algebraic topology. Let $\mathcal{A}$ be a covering of a space X by non-empty subspaces $A_1, \ldots, A_n$. Then we associate to $\mathcal{A}$ a poset whose elements are non-empty intersections of $A_1, \ldots, A_n$ and the order is defined by inclusions. The nerve of $\mathcal{A}$ is by definition the order complex of this poset associated to $\mathcal{A}$.

Lemma 3.11 (cf. [Ko2])**.** *Let $\mathcal{A}$ be a covering of a polyhedral complex K by non-empty subcomplexes $A_1, \ldots, A_n$. Suppose that for any $i_1 < \cdots < i_t$, there exists k such that $A_{i_1} \cap \cdots \cap A_{i_t}$ is either empty or $(k-t+1)$-connected. Then K is k-connected if and only if so is the nerve of $\mathcal{A}$.*

3.5. **Homotopy type of** $\mathtt{Hom}(K_m^{(r)},\mathcal{K}_n^{(r)})$**.** As is mentioned above, for an r-graph with multiplicities G, the $\mathtt{Hom}$-complex $\mathtt{Hom}(G,\mathcal{K}_n^{(r)})$ is especially important. Then we determine the homotopy type of this $\mathtt{Hom}$-complex in the special case $G=K_m^{(r)}$.

For $\boldsymbol{t}=(t_1,\dots,t_n)$ with non-negative integers $t_1,\dots,t_n$, we define a polyhedral complex $\Delta^m(\boldsymbol{t})$ as the subcomplex of $\underbrace{\Delta^{[n]}\times\cdots\times\Delta^{[n]}}_{m}$ whose faces are $\Delta^{S_1}\times\cdots\times\Delta^{S_m}$ such that $|\{k\in[m]\mid i\in S_k\}|\le t_i$ for $i=1,\dots,n$. Note that if $\boldsymbol{t}'=(t_1',\dots,t_n')$ satisfies $t_k,t_k'\ge m$ for some k and $t_i=t_i'$ for $i\ne k$, then $\Delta^m(\boldsymbol{t})=\Delta^m(\boldsymbol{t}')$. As in the proof of Theorem 3.7, for $\boldsymbol{s}=(r-1,\dots,r-1)\in[m]^n$, we have

$$\Delta^m(\boldsymbol{s})=\mathtt{Hom}(K_m^{(r)},\mathcal{K}_n^{(r)}).$$

We determine the homotopy type of $\Delta^m(\boldsymbol{t})$ and, consequently, the homotopy type of $\mathtt{Hom}(K_m^{(r)},\mathcal{K}_n^{(r)})$. For a polyhedral complex K, let $\mathcal{F}(K)$ denote the face poset of K.

Theorem 3.12. *For $\boldsymbol{t}=(t_1,\dots,t_n)$ with $0\le t_i\le m$, $\Delta^m(\boldsymbol{t})$ has the homotopy type of a wedge of $(t_1+\cdots+t_n-m)$-dimensional spheres.*

Proof. Put $|\boldsymbol{t}|=t_1+\cdots+t_n$. As in the proof of Theorem 3.7, one can easily deduce that the dimension of $\Delta^m(\boldsymbol{t})$ is $|\boldsymbol{t}|-m$. Then we only have to show that $\Delta^m(\boldsymbol{t})$ is $(|\boldsymbol{t}|-m-1)$-connected.

For $F\subset[n]$, put $\boldsymbol{t}-F=(t_1',\dots,t_n')$ such that

$$t_i'=\begin{cases}\max\{t_i-1,0\} & i\in F\\ t_i & i\notin F.\end{cases}$$

Then if $F\subset F'\subset[n]$, we have $\Delta^\ell(\boldsymbol{t}-F)\supset\Delta^\ell(\boldsymbol{t}-F')$. We also have that if $|\boldsymbol{t}|-|F|-\ell<0$ for $F\subset[n]$, $\Delta^\ell(\boldsymbol{t}-F)=\emptyset$. We now define a functor

$$\rho:\mathcal{F}(\mathrm{sk}_{|\boldsymbol{t}|-m}\Delta^{[n]})^{\mathrm{op}}\to\mathbf{Poly}$$

by $\rho(F)=\Delta^{m-1}(\boldsymbol{t}-F)$ and inclusions $\Delta^{m-1}(\boldsymbol{t}-F)\supset\Delta^{m-1}(\boldsymbol{t}-F')$ for $F\subset F'\in\mathcal{F}(\mathrm{sk}_{|\boldsymbol{t}|-m}\Delta^{[n]})$, where sk_kK denotes the k-skeleton of a polyhedral complex K. By definition, $\Delta^m(\boldsymbol{t})$ is the union of $\Delta^F\times\Delta^{m-1}(\boldsymbol{t}-F)$ for all non-empty $F\in\mathcal{F}(\mathrm{sk}_{|\boldsymbol{t}|-m}\Delta^{[n]})$. Namely, we have

$$\Delta^m(\boldsymbol{t})=\operatorname{hocolim}\rho.$$

Since ρ maps every arrow to a cofibration, we get a homotopy equivalence

$$\operatorname{hocolim}\rho\xrightarrow{\simeq}\operatorname{colim}\rho.$$

See [Ko2]. Notice that $\operatorname{colim}\rho$ is covered by subcomplexes $\Delta^{m-1}(\boldsymbol{t}-\{i\})$ for $i\in[n]$ and that $\Delta^{m-1}(\boldsymbol{t}-F)\cap\Delta^{m-1}(\boldsymbol{t}-F')=\Delta^{m-1}(\boldsymbol{t}-F\cup F')$ for $F,F'\in\mathcal{F}(\mathrm{sk}_{|\boldsymbol{t}|-m}\Delta^{[n]})$.

If $|\boldsymbol{t}|=m$, $\Delta^m(\boldsymbol{t})$ is a discrete finite set. Apply Lemma 3.11 to the above covering of $\operatorname{colim}\rho$ inductively on $|\boldsymbol{t}|-m$. Thus we obtain the desired result. □

Corollary 3.13. $\mathtt{Hom}(K_m^{(r)},\mathcal{K}_n^{(r)})$ *has the homotopy type of a wedge of $((r-1)n-m)$-dimensional spheres.*

3.6. **Vertex deletion and $\mathtt{Hom}$-complexes.** In [BK], a relation between vertex deletion of G and the homotopy type of $\mathtt{Hom}(G, H)$ is considered when G, H are graphs. We prove analogy for r-graphs with multiplicities here by a quite similar way. In [BK], a condition for vertex deletion is given by a neighborhood of a vertex. As for graphs, a neighborhood of a vertex v is considered as both the set of vertices adjacent to v and the set of edges with the end v. As for r-graphs with multiplicities, these two sets cannot be identified for $r \geq 3$, and then we define two kinds of neighborhoods of vertices.

Let G be an r-graph with multiplicities. For a vertex v of G, we define $\mathtt{N}(v)$ as the set of $v_1 \cdots v_s \in \mathrm{SP}^s(V(G))$ for some $1 \leq s \leq r-1$ satisfying $\underbrace{v \cdots v}_{r-s} v_1 \cdots v_s \in E(G)$ and $v_1, \ldots, v_s \neq v$. For $v_1 \cdots v_s \in \mathrm{SP}^s(V(G))$ with $1 \leq s \leq r-1$, we also define $\check{\mathtt{N}}(v_1 \cdots v_s)$ as the set of vertices w of G satisfying $\underbrace{w \cdots w}_{r-s} v_1 \cdots v_s \in E(G)$.

For a vertex v of G, let $G \setminus v$ denote the maximum r-subgraph with multiplicities of G whose vertex set is $V(G) \setminus v$. We now state our result.

Theorem 3.14. *Let G, H be an r-graph with multiplicities. Suppose that there are vertices u, v of G satisfying $\mathtt{N}(u) \supset \mathtt{N}(v)$. Then the inclusion $i : G \setminus v \to G$ induces a homotopy equivalence*

$$i^* : \mathtt{Hom}(G, H) \xrightarrow{\simeq} \mathtt{Hom}(G \setminus v, H).$$

Proof. As is mentioned above, the proof is quite analogous to [BK, Proposition 5.1]. Note that any face of $\mathtt{Hom}(T, H)$ for an r-graph with multiplicities T is identified with a map $V(T) \to 2^{V(H)} \setminus \emptyset$. For $\eta \in \mathcal{F}(\mathtt{Hom}(G \setminus v, H))$, the fiber $\mathcal{F}(i^*)^{-1}(\eta)$ is the set of $\tau \in \mathcal{F}(\mathtt{Hom}(G, H))$ satisfying

$$\tau|_{V(G)\setminus v} = \eta.$$

Since

$$\begin{aligned} \bigcap_{v_1 \cdots v_s \in \mathtt{N}(v)} \bigcap_{(w_1, \ldots, w_s) \in \eta(v_1) \times \cdots \times \eta(v_s)} \check{\mathtt{N}}(w_1 \cdots w_s) &\supset \bigcap_{v_1 \cdots v_s \in \mathtt{N}(u)} \bigcap_{(w_1, \ldots, w_s) \in \eta(v_1) \times \cdots \times \eta(v_s)} \check{\mathtt{N}}(w_1 \cdots w_s) \\ &\supset \eta(u) \neq \emptyset, \end{aligned}$$

we can define $\nu \in \mathcal{F}(i^*)^{-1}(\eta)$ by

$$\nu(v) = \bigcap_{v_1 \cdots v_s \in \mathtt{N}(v)} \bigcap_{(w_1, \ldots, w_s) \in \eta(v_1) \times \cdots \times \eta(v_s)} \check{\mathtt{N}}(w_1 \cdots w_s)$$

and $\nu|_{V(G)\setminus v} = \eta$. By definition, ν is the maximum of $\mathcal{F}(i^*)^{-1}(\eta)$, and thus in particular, the order complex $\Delta(\mathcal{F}(i^*)^{-1}(\eta))$ is contractible.

Choose $\tau \in \mathcal{F}(\mathtt{Hom}(G, H))$ and $\eta \in \mathcal{F}(\mathtt{Hom}(G \setminus v, H))$ satisfying $\tau(w) \supset \eta(w)$ for $w \in V(G) \setminus v$. Observe that $\mathcal{F}(i^*)^{-1}(\eta) \cap \mathcal{F}(\mathtt{Hom}(G, H))_{\leq \tau}$ consists of $\sigma \in \mathcal{F}(\mathtt{Hom}(G, H))$ satisfying $\sigma(v) \subset \tau(v)$ and $\sigma|_{V(G)\setminus v} = \eta$. Then it has the maximum μ such that $\mu(v) = \tau(v)$ and $\mu|_{V(G)\setminus v} = \eta$. We have seen that Lemma 3.10 can be applied to $\mathcal{F}(i^*) : \mathcal{F}(\mathtt{Hom}(G, H)) \to \mathcal{F}(\mathtt{Hom}(G \setminus v, H))$, which completes the proof. $\square$

We generalize the above observation on the homotopy type of $\mathtt{Hom}(L_n^{(3)}, \mathcal{K}_2^{(3)})$.

Corollary 3.15. *Let $L_n^{(r)}$ be the line r-graph with n vertices as above, and let G be an r-graph with multiplicities. Then for $n \geq r$, we have*

$$\mathtt{Hom}(L_n^{(r)}, G) \simeq \mathtt{Hom}(L_r^{(r)}, G).$$

Proof. If $n > r$, we have $\mathtt{N}(n) \subset \mathtt{N}(n-r)$. Then by Theorem 3.14, it holds that $\mathtt{Hom}(L_n^{(r)}, G) \simeq \mathtt{Hom}(L_{n-1}^{(r)}, G)$. Thus the result follows by induction on n. □

4. Relation between box-edge complexes and Hom-complexes

4.1. **Box-edge complexes.** Let G be an r-graph (without multiplicities). In [AFL], Alon, Frankl and Lovász introduced a simplicial complex $\mathtt{B}_{\mathrm{edge}}(G)$ with a C_r-action which we call the box-edge complex of G, where we follow the name and the notation of [MZ]. By an ad-hoc and tricky construction concerning $\mathtt{B}_{\mathrm{edge}}(G)$, they gave a lower bound for the chromatic number of G. We will show that this construction is realized by special Hom-complexes of r-graphs with multiplicities, by which we can reprove and interpret a result of Alon, Frankl and Lovász [AFL] in a quite natural way.

Let $\pi : V^n \to \mathrm{SP}^n(V)$ denote the projection for a set V. Originally, the box-edge complexes were defined only for r-graphs (without multiplicities). However, their definition can be applied to r-graphs with multiplicities straightforwardly.

Definition 4.1. Let G be an r-graph with multiplicities. The box-edge complex of G is an abstract simplicial complex defined as

$$\mathtt{B}_{\mathrm{edge}}(G) = \{F \subset V(G)^r \mid \pi(F) \subset E(G)\} \tag{4.1}$$

on which the cyclic group C_r acts as the restriction of the permutation action on $V(G)^r$.

Notice here that as is shown in [AFL], if r is a prime, the C_r-action on $\mathtt{B}_{\mathrm{edge}}(G)$ is free.

4.2. **Result of Alon, Frankl and Lovász.** We prove that the box-edge complex $\mathtt{B}_{\mathrm{edge}}(G)$ is given by a special Hom-complex.

Theorem 4.2. *For an r-graph with multiplicities G, there is a C_r-map*

$$\mathtt{B}_{\mathrm{edge}}(G) \to \mathtt{Hom}(K_r^{(r)}, G)$$

which is a homotopy equivalence. In particular, if r is a prime, it is a C_r-homotopy equivalence.

Proof. The face poset of $\mathtt{Hom}(K_r^{(r)}, G)$ is given as

$$\mathcal{F}(\mathtt{Hom}(K_r^{(r)}, G)) = \{F_1 \times \cdots \times F_r \mid F_1, \ldots, F_r \subset V(G) \text{ and } \pi(F_1 \times \cdots \times F_r) \subset E(G)\}, \tag{4.2}$$

where the order is given by inclusions. Then as the face poset of $\mathtt{B}_{\mathrm{edge}}(G)$ is given in (4.1), we can define a map

$$\varphi : \mathcal{F}(\mathtt{B}_{\mathrm{edge}}(G)) \to \mathcal{F}(\mathtt{Hom}(K_r^{(r)}, G)), \quad F \mapsto \pi_1(F) \times \cdots \times \pi_r(F),$$

where $\pi_i : V(G)^r \to V(G)$ is the i^{th} projection. Then by definition, φ is a C_r-map and hence so is $\Delta(\varphi)$.

Take any $F_1 \times \cdots \times F_r \in \mathcal{F}(\mathtt{Hom}(K_r^{(r)}, G))$. Then the poset $\varphi^{-1}(\mathtt{Hom}(K_r^{(r)}, G)_{\le F_1 \times \cdots \times F_r})$ has the maximum $F_1 \times \cdots \times F_r$, implying that $\Delta(\varphi^{-1}(\mathtt{Hom}(K_r^{(r)}, G)_{\le F_1 \times \cdots \times F_r}))$ is contractible. Thus by Lemma 3.9, $\Delta(\varphi)$ is a homotopy equivalence. The desired map is the composite

$$\mathtt{B}_{\mathrm{edge}}(G) \xrightarrow{\cong} \Delta(\mathcal{F}(\mathtt{B}_{\mathrm{edge}}(G))) \xrightarrow{\Delta(\varphi)} \Delta(\mathcal{F}(\mathtt{Hom}(K_r^{(r)}, G))) \xrightarrow{\cong} \mathtt{Hom}(K_r^{(r)}, G),$$

where the first and the last arrows are the natural homeomorphisms between polyhedral complexes and their barycentric subdivision. Therefore we have established the first assertion. Suppose r is a prime. Then the C_r-action on $\mathtt{Hom}(K_r^{(r)}, G)$ is free by Lemma 3.5. Moreover, the C_r-action on $\mathtt{B}_{\mathrm{edge}}(G)$ is also free as is noted above. Thus the second assertion follows from the first one. $\square$

Remark 4.3. Recently, Thansri [T] showed that $\mathtt{B}_{\mathrm{edge}}(G)$ and $\mathtt{Hom}(K_r^{(r)}, G)$ has the same Σ_r-equivariant simple homotopy type for an r-graph (without multiplicities) G.

By Corollary 3.8, we obtain a result of Alon, Frankl and lovász [AFL].

Corollary 4.4. *Let G be an r-graph with multiplicities. If r is a prime, we have*

$$\chi(G) \ge \frac{\operatorname{conn} \mathtt{B}_{\mathrm{edge}}(G) + 2}{r-1} + 1.$$

Alon, Frankl and Lovász [AFL] proved Corollary 4.4 by constructing a map from $\mathtt{B}_{\mathrm{edge}}(G)$ into a Euclidean space with a certain C_r-action, which seems quite ad-hoc and tricky. Using $\mathtt{Hom}$-complexes, this construction will turn out to be the induced map between $\mathtt{Hom}$-complexes from a given coloring.

Let $\mathrm{M}_{r,n}(\mathbb{R})$ be the space of $r \times n$ real matrices. We let C_r act on $\mathrm{M}_{r,n}(\mathbb{R})$ as the cyclic permutation of rows. Let Y be a subspace of $\mathrm{M}_{r,n}(\mathbb{R})$ consisting of matrices (a_{ij}) satisfying

$$\sum_{i=1}^{r} a_{ik} = 0, \quad \sum_{j=1}^{n} a_{\ell j} = 0 \quad \text{and} \quad \sum_{i,j} a_{ij}^2 \neq 0$$

for $k = 1, \dots, n$ and $\ell = 1, \dots, r$. Then Y is also a C_r-subspace of $\mathrm{M}_{r,n}(\mathbb{R})$. Let G be an r-graph with multiplicities which admits an n-coloring, say c. Alon, Frankl and Lovász [AFL] defined a C_r-map

$$\bar{c} : \mathtt{B}_{\mathrm{edge}}(G) \to \mathrm{M}_{r,n}(\mathbb{R})$$

by sending a vertex $(v_1, \dots, v_r)$ of $\mathtt{B}_{\mathrm{edge}}(G)$ to a matrix $\sum_{i=1}^{r}(E_{i,c(i)} - E_{i,c(i)+1})$, where $E_{i,j}$ is the matrix whose (i,j) entry is 1 and other entries are 0 and $E_{i,n+1}$ means $E_{i,1}$. They showed that $\bar{c}$ has its image in Y and applied a special generalization of the Borsuk-Ulam theorem to obtain Corollary 4.4.

We now define a map $g : \mathtt{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)}) \to \mathrm{M}_{r,n}(\mathbb{R})$ by sending a vertex $(i_1, \dots, i_r) \in [n]^r$ to a matrix $\sum_{k=1}^{r}(E_{k,i_k} - E_{k,i_k+1})$. Then one can easily see that g is a C_r-map and has is image in Y. By definition, we have the following.

Proposition 4.5. *Let G be an r-graph with multiplicities which has an n-coloring c. Then there is a commutative diagram*

$$\begin{tikzcd}
\mathtt{Hom}(K_r^{(r)}, G) \arrow[r, "c_*"] \arrow[d, "\simeq"] & \mathtt{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)}) \arrow[d, "g"] \\
\mathtt{B}_{\mathrm{edge}}(G) \arrow[r, "\bar{c}"] & Y
\end{tikzcd}$$

where the left vertical arrow is as in Theorem 4.2.

We close this section by remarking that the complex of an r-graph (without multiplicities) introduced by Kříž [Kr] is the barycentric subdivision of $\mathtt{Hom}(K_r^{(r)}, G)$ and then essentially the same as the box-edge complex $\mathtt{B}_{\mathrm{edge}}(G)$.

5. $\mathtt{Hom}_+$-complexes and colorings

5.1. General $\mathtt{Hom}_+$-complexes.

In [Ko2], $\mathtt{Hom}_+$-complexes of graphs were introduced which are variants of $\mathtt{Hom}$-complexes. As in the case of $\mathtt{Hom}$-complexes, we can give a general recipe for $\mathtt{Hom}_+$-complexes of partial maps between finite sets and will apply it to r-graphs with multiplicities.

Let S, T be finite sets. A partial map from S to T is a map from a non-empty subset of S into T. Then a partial map from S to T is identified with an element of

$$(T \cup \{\emptyset\})^S \setminus (\emptyset, \ldots, \emptyset).$$

Let K, L be abstract simplicial complexes. Recall that the join $K * L$ is an abstract simplicial complex whose simplices are of the form (σ, τ) where $\sigma \in K$, $\tau \in L$ and either σ or τ is not empty. Then a partial map from S to T is identified with a vertex of the join $*_{s \in S} \Delta^T$. Analogously to $\mathtt{Hom}$-complexes, we are led to the following definition.

Definition 5.1. Let S, T be finite sets and $\mathcal{C}$ be a class of partial maps from S to T. The $\mathtt{Hom}_+$-complex $\mathtt{Hom}_+^{\mathcal{C}}(S, T)$ is defined as the maximum subcomplex of $*_{s \in S} \Delta^T$ whose vertex set is $\mathcal{C}$.

Analogously to $\mathtt{Hom}$-complexes, we can define induced maps between $\mathtt{Hom}_+$-complexes under certain conditions and see that these induced maps satisfy naturality corresponding to Proposition 3.2.

5.2. $\mathtt{Hom}_+$-complexes of r-graphs with multiplicities.

Let G, H be r-graphs with multiplicities. A partial homomorphism from G to H is a map from a subset V of $V(G)$ into $V(H)$ which is a homomorphism from the maximum r-subgraph with multiplicities of G whose vertex set is V into H. We now define $\mathtt{Hom}_+$-complexes of r-graphs with multiplicities.

Definition 5.2. Let G, H be r-graphs with multiplicities. The $\mathtt{Hom}_+$-complex $\mathtt{Hom}_+(G, H)$ is defined as $\mathtt{Hom}_+^{\mathcal{C}}(V(G), V(H))$ for the set $\mathcal{C}$ of all partial homomorphisms from G to H.

Similarly to Proposition 3.4, we have the following.

Proposition 5.3. *Let* $\mathbf{Graph}^{(r)}$ *and* $\mathbf{Poly}$ *be the categories of r-graphs with multiplicities and polyhedral complexes, respectively. Then*

$$(\mathbf{Graph}^{(r)})^{\mathrm{op}} \times \mathbf{Graph}^{(r)} \to \mathbf{Poly}, \quad (G, H) \mapsto \mathtt{Hom}_+(G, H)$$

is a functor.

Then as in the case of $\mathtt{Hom}$-complexes, we can construct group actions on $\mathtt{Hom}_+$-complexes by those on r-graphs with multiplicities. For instance, the natural C_r-action on $K_r^{(r)}$ induces a C_r-action on $\mathtt{Hom}_+(K_r^{(r)}, G)$ for an r-graph with multiplicities G. Analogously to Lemma 3.5, we can prove the following.

Lemma 5.4. *Let G be an r-graph with multiplicities. If r is a prime, the C_r-action on* $\mathtt{Hom}_+(K_r^{(r)}, G)$ *is free.*

Using this C_r-action, we obtain a lower bound for the chromatic numbers.

Theorem 5.5. *Let G be an r-graph with multiplicities. If r is a prime, it holds that*

$$\chi(G) \geq \frac{\mathrm{ind}_{C_r} \mathtt{Hom}_+(K_r^{(r)}, G) + 1}{r - 1}.$$

Proof. Note that the dimension of $\mathtt{Hom}_+(K_r^{(r)}, \mathcal{K}_n^{(r)})$ is $nr - n - 1 = (r-1)n - 1$. Then the result follow quite similarly to Theorem 3.7. □

Corollary 5.6. *Let G be an r-graph with multiplicities. If r is a prime, we have*

$$\chi(G) \geq \frac{\mathrm{conn}\, \mathtt{Hom}_+(K_r^{(r)}, G) + 2}{r - 1}.$$

In [La], Lange defined a complex $\mathtt{B}_0(G)$ for an r-graph with multiplicities and gave a lower bound for the chromatic number of G by using $\mathtt{B}_0(G)$. By definition, $\mathtt{B}_0(G)$ coincides with $\mathtt{Hom}_+(K_r^{(r)}, G)$ and a lower bound in Theorem 5.5 is the same as the one given by Lange.

As in §3, let us consider the homotopy type of $\mathtt{Hom}_+(K_m^{(r)}, \mathcal{K}_n^{(r)})$. In the case of $\mathtt{Hom}_+$-complexes, one can describe $\mathtt{Hom}_+(K_m^{(r)}, \mathcal{K}_n^{(r)})$ explicitly by Sarkaria's formula [S] as follows.

Theorem 5.7 (cf. [S])**.** *We have*

$$\mathtt{Hom}_+(K_m^{(r)}, \mathcal{K}_n^{(r)}) \cong *_n \mathrm{sk}_{r-2} \Delta^{[m]}.$$

In particular, $\mathtt{Hom}_+(K_m^{(r)}, \mathcal{K}_n^{(r)})$ *has the homotopy type of a wedge of* $\binom{m-1}{r-1}^n$ *copies of* $((r-1)n - 1)$*-dimensional spheres.*

5.3. **Hierarchy of lower bounds for the chromatic number.** Let G be an r-graph with multiplicities. We have obtained so far two kinds of lower bounds for the chromatic number of G, one is given by $\mathtt{Hom}(K_r^{(r)}, G)$ in Theorem 3.7 and the other is given by $\mathtt{Hom}_+(K_r^{(r)}, G)$ in Theorem 5.5. We have also seen that these lower bounds are related to formerly known ones [AFL], [La]. We describe $\mathtt{Hom}_+(K_r^{(r)}, G)$ by using $\mathtt{Hom}(K_r^{(r)}, G)$ and then get an inequality between the above lower bounds.

Theorem 5.8. *For an r-graph with multiplicities G, there is a C_r-map*

$$\mathtt{Hom}_+(K_r^{(r)}, G) \to \partial\Delta^{[r]} * \mathtt{Hom}(K_r^{(r)}, G)$$

which is a homotopy equivalence, where C_r acts diagonally on $\partial\Delta^{[r]} * \mathtt{Hom}(K_r^{(r)}, G)$.

Proof. Let P, Q be finite posets. Recall that the join $P * Q$ is a poset whose underlying set is $P \sqcup Q$ and order is defined as $x < y$ if either $x, y \in P$ with $x < y$, $x, y \in Q$ with $x < y$ or $x \in P, y \in Q$. Then it follows that

$$\Delta(P * Q) = \Delta(P) * \Delta(Q).$$

Note that the face poset of $\mathtt{Hom}_+(K_r^{(r)}, G)$ is the disjoint union of $\mathcal{F}(\mathtt{Hom}(K_r^{(r)}, G))$ in (4.2) and

$$\{F_1 \times \cdots \times F_r \mid F_1, \ldots, F_r \subset V(G),\ F_i = \emptyset \text{ for some } i \text{ and } \bigcup_{i=1}^{r} F_i \neq \emptyset\},$$

where the order is given by inclusions and $F_1 \times \cdots \times F_n$ with $F_{i_1}, \ldots, F_{i_k} \neq \emptyset$ and $F_j = \emptyset$ for $j \neq i_1, \ldots, i_k$ means $F_{i_1} \times \cdots \times F_{i_k}$. We then define a poset map

$$\varphi : \mathcal{F}(\mathtt{Hom}_+(K_r^{(r)}, G)) \to \mathcal{F}(\partial\Delta^{[r]}) * \mathcal{F}(\mathtt{Hom}(K_r^{(r)}, G))$$

as

$$\varphi(F_1 \times \cdots \times F_r) = \begin{cases} \{i_1, \ldots, i_k\} \in \mathcal{F}(\partial\Delta^{[r]}) & \bigcup_{i \neq i_1, \ldots, i_k} F_i = \emptyset \text{ and } F_{i_1}, \ldots, F_{i_k} \neq \emptyset \\ F_1 \times \cdots \times F_r \in \mathcal{F}(\mathtt{Hom}(K_r^{(r)}, G)) & F_1, \ldots, F_r \neq \emptyset. \end{cases}$$

By definition, φ is a C_r-map. For $F_1 \times \cdots \times F_r \in \mathcal{F}(\mathtt{Hom}(K_r^{(r)}, G)) \subset \mathcal{F}(\partial\Delta^{[r]}) * \mathcal{F}(\mathtt{Hom}(K_r^{(r)}, G))$, $\varphi^{-1}((\mathcal{F}(\partial\Delta^{[r]}) * \mathcal{F}(\mathtt{Hom}(K_r^{(r)}, G)))_{\le F_1 \times \cdots \times F_r})$ has the maximum $F_1 \times \cdots \times F_r$. For $\{i_1, \ldots, i_k\} \in \partial\Delta^{[r]} \subset \mathcal{F}(\partial\Delta^{[r]}) * \mathcal{F}(\mathtt{Hom}(K_r^{(r)}, G))$, $\varphi^{-1}((\mathcal{F}(\partial\Delta^{[r]}) * \mathcal{F}(\mathtt{Hom}(K_r^{(r)}, G)))_{\le \{i_1, \ldots, i_k\}})$ has the maximum

$$F_1 \times \cdots \times F_r, \quad F_{i_1} = \cdots = F_{i_k} = [n] \text{ and } \bigcup_{i \neq i_1, \ldots, i_k} F_i = \emptyset.$$

Then for any $x \in \mathcal{F}(\partial\Delta^{[r]}) * \mathcal{F}(\mathtt{Hom}(K_r^{(r)}, G))$, $\Delta((\mathcal{F}(\partial\Delta^{[r]}) * \mathcal{F}(\mathtt{Hom}(K_r^{(r)}, G)))_{\le x})$ is contractible, and it follows from Lemma 3.9 that $\Delta(\varphi)$ is a homotopy equivalence. Thus the composite

$$\begin{aligned} \mathtt{Hom}_+(K_r^{(r)}, G) \xrightarrow{\cong} \Delta(\mathcal{F}(\mathtt{Hom}_+(K_r^{(r)}, G))) &\xrightarrow{\Delta(\varphi)} \Delta(\mathcal{F}(\partial\Delta^{[r]}) * \mathcal{F}(\mathtt{Hom}(K_r^{(r)}, G))) \\ &= \Delta(\mathcal{F}(\partial\Delta^{[r]})) * \Delta(\mathcal{F}(\mathtt{Hom}(K_r^{(r)}, G))) \xrightarrow{\cong} \partial\Delta^{[r]} * \mathtt{Hom}(K_r^{(r)}, G) \end{aligned}$$

is the desired homotopy equivalence, where the first and the last arrows are the natural homeomorphisms. □

Corollary 5.9. *Let G be an r-graph with multiplicities. If r is a prime, there holds*

$$\begin{aligned} \chi(G) &\ge \frac{\mathrm{ind}_{C_r}\mathtt{Hom}(K_r^{(r)}, G) + 1}{r-1} + 1 \ge \frac{\mathrm{ind}_{C_r}\mathtt{Hom}_+(K_r^{(r)}, G) + 1}{r-1} \ge \frac{\mathrm{ind}_{C_r}\mathtt{Hom}(K_r^{(r)}, G) + 1}{r-1} \\ &\ge \frac{\mathrm{conn}\ \mathtt{Hom}(K_r^{(r)}, G) + 1}{r-1} = \frac{\mathrm{conn}\ \mathtt{Hom}_+(K_r^{(r)}, G) + 1}{r-1} - 1. \end{aligned}$$

Proof. The first inequality follows from Theorem 3.7 and the second from Proposition 3.6 and Theorem 5.8. As in the proof of Theorem 5.8, $\mathcal{F}(\mathtt{Hom}(K_r^{(r)}, G))$ is a subposet of $\mathcal{F}(\mathtt{Hom}_+(K_r^{(r)}, G))$ including the C_r-actions. Then there is a C_r-map $\mathtt{Hom}(K_r^{(r)}, G) \to \mathtt{Hom}_+(K_r^{(r)}, G)$, implying the third inequality by Proposition 3.6. The fourth inequality follows from Proposition 3.6 and the last equality from Theorem 5.8. □

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