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Equilibrium Measures for the Hénon Map at the First Bifurcation: Uniqueness and Geometric / Statistical Properties

by

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EQUILIBRIUM MEASURES FOR THE HÉNON MAP AT THE FIRST BIFURCATION: UNIQUENESS AND GEOMETRIC/STATISTICAL PROPERTIES

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Abstract. For strongly dissipative Hénon maps at the first bifurcation where the uniform hyperbolicity is destroyed by the formation of tangencies, we establish a thermodynamic formalism, i.e., prove the existence and uniqueness of an invariant probability measure which minimizes the free energy associated with a non continuous geometric potential $-t\log J^u$, where $t\in\mathbb{R}$ is in a certain large interval and J^u is the Jacobian in the unstable direction. We obtain geometric and statistical properties of these measures.

1. Introduction

In this paper we study the first bifurcation of the Hénon family

$$f_a\colon (x,y)\mapsto (1-ax^2+\sqrt{b}y, \pm\sqrt{b}x),\quad 0<b\ll 1. \tag{1}$$

There exists a parameter a^* near 2 such that the non-wandering set of f_a is a uniformly hyperbolic horseshoe for $a>a^*$, and (f_a) generically unfolds a quadratic tangency at $a=a^*$ [1, 2, 8]. We study the dynamics of f_{a^*} from the viewpoint of ergodic theory and thermodynamic formalism.

Write f for f_{a^*}. Let

$$K=\{z\in\mathbb{R}^2\colon\ \{f^nz\}_{n\in\mathbb{Z}} \text{ is bounded}\}.$$

This set is a compact set and it coincides with the transitive non-wandering set [7]. Let $\mathcal{M}(f)$ denote the space of all f-invariant Borel probability measures endowed with the topology of weak convergence. For a potential function $\varphi: K\to\mathbb{R}$ the associated free energy function $F_\varphi\colon\mathcal{M}(f)\to\mathbb{R}$ is given by

$$F_\varphi(\mu)=h(\mu)+\mu(\varphi),$$

where $h(\mu)$ denotes the entropy of μ and $\mu(\varphi)=\int\varphi d\mu$. An *equilibrium measure* associated to the potential φ is a measure $\mu_\varphi\in\mathcal{M}(f)$ which maximizes F_φ, i.e.

$$F_\varphi(\mu_\varphi)=\sup\{F_\varphi(\mu)\colon \mu\in\mathcal{M}(f)\}.$$

The main example of potential functions to which our theory applies is the family of potential functions

$$\varphi_t=-t\log J^u\quad t\in\mathbb{R},$$

where J^u denotes the Jacobian along the *unstable direction* that is defined as follows. At a point $z\in\mathbb{R}^2$, let $E^u(z)$ denote the one-dimensional subspace such that

$$\varlimsup_{n\to\infty}\frac{1}{n}\log\|Df^{-n}|E^u(z)\|<0. \tag{2}$$

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Since f^{-1} expands area, $E^u(z)$ is unique when it makes sense. We call E^u an *unstable direction*. Let $J^u(z) = \|Df|E^u(z)\|$ and $\varphi_t = -t \log J^u$.

This family of potentials was studied in [30] where it is proved [30, Proposition 4.1] that E^u is well-defined and is measurable on a Borel set with total probability, and it is continuous except at the fixed saddle Q near $(-1, 0)$. The particular equilibrium measure for the potential φ_t is called a *t-conformal measure*. We are concerned with the existence and uniqueness of t-conformal measures, and their geometric and statistical properties.

Define

$$P(t) := \sup\{F_{\varphi_t}(\mu) \colon \mu \in \mathcal{M}(f)\}.$$

The *pressure function* $t \mapsto P(t)$ is convex, and so is continuous. Let

$$t_0 := \inf\{t \in \mathbb{R} \colon P(s) > -(s/3)\log(4-\varepsilon) \text{ for any } s < t\}.$$

Considering the two fixed saddles (see FIGURE 1) we have $1 < t_0 < \infty$.

Theorem. [30, Theorem] *For any small $\varepsilon > 0$ there exists $b_0 > 0$ such that if $b < b_0$ and $t < t_0$, then there exists a t-conformal measure.*

The uniqueness of t-conformal measures does not follow from the argument in [30]. In addition, the range of positive t for which t-conformal measures exist is far from optimal. We show the existence and uniqueness of t-conformal measure with t in a certain interval containing much larger positive t.

Theorem A. *For any $\epsilon > 0$ there exists $b_1 \in (0, b_0)$ such that if $b < b_1$, then there exists a unique t-conformal measure for all $t \in (-1 + \epsilon, 1/\epsilon)$.*

Since entropies of invariant probability measures are written as linear combinations of the entropies of the ergodic components, and the same property holds for unstable Lyapunov exponents, all the t-conformal measures in the statement of Theorem A are ergodic. In addition, it follows from our construction and from transitivity that any t-conformal measure is supported on K, i.e. it gives positive weight to any open set intersecting K.

Our construction used in the proof of Theorem A leads to a version of Manning & McClusky's formula [16] (see [15, 34] for related results), which evaluates how substantial the set K is in terms of Hausdorff dimension. Given a C^1 one-dimensional submanifold γ of $\mathbb{R}^2$ and $p \in (0, 1]$, the Hausdorff p-measure of a set $A \subset \gamma$ is given by

$$m_p(A) = \lim_{\delta \to 0} \left(\inf \sum_{U \in \mathcal{U}} \ell(U)^p \right),$$

where ℓ denotes the length and the infimum is taken over all coverings $\mathcal{U}$ of A by open sets in γ with diameter $\leq \delta$. The Hausdorff dimension of A on γ, simply denoted by $\mathrm{HD}(A)$, is the unique number such that

$$\mathrm{HD}(A) = \sup\{p \colon m_p(A) = \infty\} = \inf\{p \colon m_p(A) = 0\}.$$

One has $P(0) > 0$, and Ruelle's inequality [23] gives $P(1) \leq 0$. Since f has no SRB measure [33], $P(1) < 0$ holds. Hence the equation $P(t) = 0$ has a unique solution in $(0, 1)$, which we denote by t^u.

Theorem B. *For any relatively open curve γ in the unstable manifold of the fixed saddle such that $\gamma \cap K \neq \emptyset$, one has $\mathrm{HD}(\gamma \cap K) = t^u$. In addition, $t^u \to 1$ as $b \to 0$.*

Let us here mention results of Leplaideur and Rios [14, 15] closely related to ours, in which a thermodynamical formalism for certain horseshoes with three branches and a single orbit of tangency was established. See [13] for a related result. However, their specific assumptions on the map, including the linearity and the balance between expansion/contraction rates, do not hold for the Hénon map f. Our argument is novel but exploits the well known line for the study of Hénon-like systems [3, 4, 18, 35].

A powerful approach in ergodic theory of dynamical systems is to "code" orbits of a system into symbolic sequences, by following their histories on a partition of the phase space. If this defines a nice shift system, then the construction of interesting invariant measures and the study of their properties can be carried out on the symbolic level. For uniformly hyperbolic systems, Markov partitions are used to code orbits with symbolic sequences with finite symbols. The existence and uniqueness of equilibrium measures for Hölder continuous potentials was established in [5, 24, 31].

However, at the first bifurcation the Hénon map f lacks such a nice partition. Indeed the natural partition of K into the "left" and the "right" of the point of tangency near the origin, constructed in [30] to prove the existence of equilibrium measures including t-conformal measures, only defines a semi-conjugacy between $f|K$ and the full shift on two symbols. In order to avoid the discontinuity of φ_t at Q, we must consider a (non-compact) subset of K which does not contain Q. We code the dynamics on this subset with a *countable* alphabet to establish the uniqueness (countable partitions were also constructed in [10, 15] albeit for other purposes/maps).

Our strategy for proving the uniqueness of the equilibrium measures is to construct an invariant measure as a candidate, and then show that it is indeed a unique measure which minimizes the free energy. The main step is to build an *inducing scheme* (S, τ). Here S is a countable collection of Borel subsets of K called *basic elements*. The union of all basic elements is denoted by X, and τ is the first return time to X, which is constant on each basic element. The inducing scheme allows us to represent the first return map to X as a countable (full) Markov shift. Under certain conditions on the potential function, which are proven to be satisfied by φ_t with $t \in (t_-, t_+)$, $t_- < 0 < t_+$ (see (24) for the definition) one can construct a Gibbs measure in the shift space following [17, 28]. This Gibbs measure is then used to obtain a unique invariant measure for the original system which minimizes the free energy among all measures which are *liftable* to the inducing scheme (i.e. those measures which can be obtained from symbolic shift invariant measures).

The set of non liftable measures is nonempty. For instance, it contains δ_Q. To show that the candidate measure is a unique equilibrium measure, we show that non liftable measures have smaller free energies. This can be undertaken by showing that our inducing scheme is efficient, in that any ergodic measure with not too small entropy gives positive weight to X, and hence is liftable. At this point it is worth noting that to study the Hénon maps it is usually necessary to exclude points from consideration for which "long stable leaves" cannot be constructed. Each basic element of the inducing scheme constructed here is a Cantor-like set, which makes the estimates more involved.

We now move on to geometric and statistical properties. In what follows, let μ_t denote the t-conformal measure in Theorem A. We first give a characterization of μ_{t^u} in terms of dimension. To give a precise statement let us recall general facts on nonuniformly hyperbolic

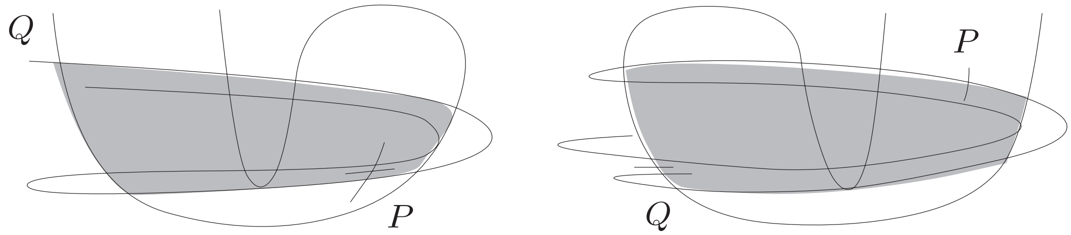

FIGURE 1. Manifold organization for $a = a^*$. There exist two hyperbolic fixed saddles P, Q near $(1/2, 0)$, $(-1, 0)$ correspondingly. In the orientation preserving case (left), $W^u(Q)$ meets $W^s(Q)$ tangentially. In the orientation reversing case (right), $W^u(P)$ meets $W^s(Q)$ tangentially. The shaded regions represent the region R (see Sect.3.1).

systems. For $x \in K$, let

$$W^u(x) = \left\{ y \in \mathbb{R}^2 \colon \varlimsup_{n\to\infty} \frac{1}{n} \log |f^{-n}x - f^{-n}y| < 0 \right\}, \tag{3}$$

which we call the *unstable manifold of* x. Let $\mathcal{M}^e(f)$ denote the set of ergodic elements of $\mathcal{M}(f)$. Since any $\mu \in \mathcal{M}^e(f)$ has exactly one positive Lyapunov exponent [7], there exists a set Γ of full μ-measure such that for any $x \in \Gamma$, $W^u(x)$ is an injectively immersed smooth submanifold of $\mathbb{R}^2$ [19, 25]. Let $\{\mu_x^u\}_{x\in\Gamma}$ denote the *canonical system of conditional measures of* μ *along unstable manifolds* [22]: μ_x^u is a probability measure supported on $W^u(x)$ such that $x \mapsto \mu_x^u(A)$ is measurable and $\mu(A) = \int \mu_x^u(A) d\mu(x)$ for any measurable set A. Let $\dim(\mu_x^u)$ denote the dimension of μ_x^u, namely

$$\dim(\mu_x^u) = \inf\{\mathrm{HD}(X) \colon X \subset W^u(x), \mu_x^u(X) = 1\}.$$

Then, $\dim(\mu_x^u)$ is constant μ-a.e. and this number is denoted by $\dim_H^u(\mu)$. We say $\mu \in \mathcal{M}^e(f)$ is a *measure of maximal unstable dimension* if

$$\dim_H^u(\mu) = \sup\{\dim_H^u(\nu) \colon \nu \in \mathcal{M}^e(f)\}.$$

Theorem C. μ_{t^u} *is the unique measure of maximal unstable dimension.*

Considering the tower associated to the inducing scheme allows us to apply the result of Young [36] to deduce several statistical properties of μ_t.

Theorem D. *The following holds for* (f, μ_t)*;*

(1) *for any* $\eta \in (0, 1]$ *there exists* $\tau \in (0, 1)$ *such that for any Hölder continuous* $\varphi \colon K \to \mathbb{R}$ *with Hölder exponent* η *and* $\psi \in L^\infty(\mu_t)$*, there exists a constant* $C(\varphi, \psi)$ *such that*

$$|\mu_t((\varphi \circ f^n)\psi) - \mu_t(\varphi)\mu_t(\psi)| \le C(\varphi, \psi)\tau^n \quad \textit{for every } n > 0;$$

(2) *for any Hölder continuous* $\phi \colon K \to \mathbb{R}$ *with* $\int \phi d\mu_t = 0$*, there exists* $\sigma \ge 0$ *such that*

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \phi \circ f^i \longrightarrow \mathcal{N}(0, \sigma) \quad \textit{in distribution},$$

where $\mathcal{N}(0, \sigma)$ *is the normal distribution with mean* 0 *and variance* σ^2*. In addition,* $\sigma > 0$ *if and only if* $\phi \ne \psi \circ g - \psi$ *for any* $\psi \in L^2(\mu_t)$*.*

The rest of this paper consists of four sections. In Sect.2 we recall the general thermodynamical formalism for maps admitting inducing schemes from [21]. In Sect.3 we construct an inducing scheme and show in Sect.4 that it is efficient in the above sense. In Sect.5 we define t_-, t^+ and then check all the conditions on φ_t, $t \in (t_-, t_+)$ necessary for implementing the theory in Sect.2. This yields an f-invariant measure μ_t which maximizes the free energy among all liftable measures. Using the results in Sect.4 we show then that μ_t is the unique measure which maximizes the free energy among *all* measures. This completes the proof of Theorem A. Other theorems are proven in Sect.6. The proofs of Proposition 3.1 and Lemma 3.3 require ingredients from [30] and are deferred to the Appendix.

2. Equilibrium measures for maps admitting inducing schemes

In this section we recall the construction of equilibrium measures for f associated to φ developed in [21]. The main idea is to use an inducing scheme to relate the induced system to a countable Markov shift, and construct a Gibbs measure in the shift space associated to the induced potential following [17, 28]. Gibbs measures with integrable inducing time are then used to construct an invariant equilibrium probability measures for the original map associated to the original potential function.

2.1. **Equilibrium states for countable Markov shifts.** Denote the set of all bi-infinite sequences over a countable alphabet S by $S^{\mathbb{Z}} := \{\underline{a} := (\dots, a_{-1}, a_0, a_1, \dots) \colon a_i \in S,\ i \in \mathbb{Z}\}$ and the (left) full shift by $\sigma : S^{\mathbb{Z}} \circlearrowleft$ i.e. $(\sigma(\underline{a}))_i = a_{i+1}$. The sets $[b_i, \dots, b_j] := \{\underline{a} \in S^{\mathbb{Z}} \colon a_k = b_k \text{ for all } i \leq k \leq j\}$ are called *cylinder sets*. Endow $S^{\mathbb{Z}}$ with the topology for which the cylinder sets form a base. The shift σ is continuous with respect to this topology. Denote by $\mathcal{M}(\sigma)$ the collection of σ-invariant Borel probability measures on $S^{\mathbb{Z}}$. Given a potential function $\Phi : S^{\mathbb{Z}} \to \mathbb{R}$, let

$$\mathcal{M}_\Phi(\sigma) := \{\nu \in \mathcal{M}(\sigma) \colon \nu(\Phi) > -\infty\}.$$

The n^{th}*variation* of Φ is defined by

$$V_n(\Phi) := \sup_{[b_{-n+1},\dots,b_{n-1}]} \sup_{\underline{a},\underline{a}' \in [b_{-n+1},\dots,b_{n-1}]} |\Phi(\underline{a}) - \Phi(\underline{a}')|.$$

The function Φ has *strongly summable variation* if

$$\sum_{n\geq 1} nV_n(\Phi) < \infty. \tag{4}$$

The *Gurevich pressure* of Φ is defined by

$$P_G(\Phi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n(\underline{a})=\underline{a}} \exp\left(\sum_{k=0}^{n-1} \Phi(\sigma^k(\underline{a}))\right) 1_{[b]}(\underline{a}),$$

where $b \in S$. Since it depends only on the positive side of the sequences, one can prove (as in [26, Theorem 1]) that $P_G(\Phi)$ exists and is independent of b whenever the variation

$$V_n^+(\Phi) := \sup_{[b_0,\dots,b_{n-1}]} \sup_{\underline{a},\underline{a}' \in [b_0,\dots,b_{n-1}]} |\Phi(\underline{a}) - \Phi(\underline{a}')|$$

over all *positive cylinders* is summable: $\sum_{n\geq 1} V_n^+(\Phi) < \infty$. Also $P_G(\Phi) > -\infty$ holds in this case.

We say $\nu_\Phi \in \mathcal{M}(\sigma)$ is a *Gibbs measure for* Φ if there exist constants $C_1 > 0$, $C_2 > 0$ such that for any cylinder set $[b_0, \ldots, b_{n-1}]$ and any $\underline{a} \in [b_0, \ldots, b_{n-1}]$ we have

$$C_1 \leq \frac{\nu_\Phi([b_0, \ldots, b_{n-1}])}{\exp\left(-nP_G(\Phi) + \sum_{k=0}^{n-1} \Phi(\sigma^k(\underline{a}))\right)} \leq C_2. \tag{5}$$

Note that this definition only involves *positive cylinders*.

We say $\nu_\Phi \in \mathcal{M}(\sigma)$ is an *equilibrium measure for* Φ if

$$h_{\nu_\Phi}(\sigma) + \nu_\Phi(\Phi) = \sup_{\nu \in \mathcal{M}_\Phi(\sigma)} \{h_\nu(\sigma) + \nu(\Phi)\}. \tag{6}$$

The thermodynamics of the full shift of countable type on the space of two-sided sequences $\sigma : S^{\mathbb{Z}} \circlearrowleft$ is described in the following theorem from [21].

Proposition 2.1. [21, Theorem 3.1] *Let* $\Phi : S^{\mathbb{Z}} \to \mathbb{R}$ *be a potential function with* $\sup \Phi < \infty$ *and strongly summable variation. The following statements hold:*

(a) *the variational principle holds for* Φ*:* $P_G(\Phi) = \sup_{\nu \in \mathcal{M}_\Phi(\sigma)} \{h_\nu(\sigma) + \nu(\Phi)\}$*;*
(b) *if* $P_G(\Phi) < \infty$ *there exists a unique Gibbs measure* ν_Φ *for* Φ*;*
(c) *if* $\nu_\Phi \in \mathcal{M}_\Phi(\sigma)$ *then it is a unique equilibrium measure for* Φ*.*

The main idea is to reduce the problem to the (left) full shift on the set of one-sided infinite sequences $S^{\mathbb{N}}$ by constructing a potential function cohomologous to the given potential Φ but which only depends on the positive coordinates of any point $\underline{a} \in S^{\mathbb{N}}$. The variational principle and the existence of a unique Gibbs and equilibrium measure for the one-sided shift and potential follows from [26, Theorem 3],[28, Theorem 1], [6, Theorem 1.1]. The statements of Proposition 2.1 follow by considering the natural extension of this one-sided Gibbs and equilibrium measure.

2.2. **Gibbs and equilibrium measures for the induced map.**

For the rest of this section we assume M is a compact metric space, and $f : M \circlearrowleft$ is a continuous map with finite topological entropy.

Definition 2.2. We say f admits an *inducing scheme* (S, τ) *of hyperbolic type*, if there exist a countable collection S of Borel sets in M called *basic elements* and an *inducing time function* $\tau : S \to \mathbb{N}$ such that the following holds for the *inducing domain* $X := \bigcup_{J \in S} J$ and the *induced map* $F : X \circlearrowleft$ defined by $F|J = f^{\tau(J)}|J$ for all $J \in S$. There exists a Borel set $X_0 \subset X$ such that:

- $\nu(X_0) = 0$ for any F-invariant probability measure ν;
- if $J_1, J_2 \in S$, $J_1 \neq J_2$ and $J_1 \cap J_2 \neq \emptyset$ then $J_1 \cap J_2 \subset X_0$;
- the coding map $h\colon S^{\mathbb{Z}} \to X$ given by

$$h(\underline{a}) := \bigcap_{n \in \mathbb{Z}} F^{-n}(J_{a_n}) \quad \text{where} \quad \underline{a} := (\ldots, a_{-1}, a_0, a_1, \ldots) \in S^{\mathbb{Z}} \tag{7}$$

  is well-defined, and is a measurable bijection between $S^{\mathbb{Z}} \setminus h^{-1}(X_0)$ and $X \setminus X_0$.

Remark 2.1. The induced map F is multi-valued on points of intersection between elements of S. Since no measure gives positive weight to the set of such points, this is not important for our purpose.

If f admits an inducing scheme (S,τ) of hyperbolic type, the *induced potential* $\overline{\varphi}\colon X \to \mathbb{R}$ associated to a given potential $\varphi\colon M \to \mathbb{R}$ is defined by

$$\overline{\varphi} := \sum_{i=0}^{\tau-1} \varphi \circ f^i$$

where the inducing time τ is viewed as a function on X in the obvious way. We say the induced potential $\overline{\varphi}$ has:

- (strongly) summable variations if $\Phi := \overline{\varphi} \circ h$ has (strongly) summable variations;
- finite Gurevich pressure if $P_G(\Phi) < \infty$.

Let $\mathcal{M}(F)$ denote the set of F-invariant Borel probability measures and let $\mathcal{M}_{\overline{\varphi}}(F) = \{\nu \in \mathcal{M}(F)\colon \nu(\overline{\varphi}) > -\infty\}$. An F-invariant probability measure $\nu_{\overline{\varphi}}$ is a *Gibbs measure for* $\overline{\varphi}$ if there exists an σ-invariant Gibbs measure ν_Φ for Φ such that $\nu_{\overline{\varphi}} = h_*\nu_\Phi$. We call $\nu_{\overline{\varphi}}$ an *equilibrium measure* for $\overline{\varphi}$ if $\nu_{\overline{\varphi}} \in \mathcal{M}_{\overline{\varphi}}(F)$ and

$$h_{\overline{\varphi}}(F) + \nu_{\overline{\varphi}}(\overline{\varphi}) = \sup\left\{\nu \in \mathcal{M}_{\overline{\varphi}}(F)\colon h_\nu(F) + \nu(\overline{\varphi})\right\}.$$

By definition, h_* preserves entropy, the Gibbs property and integrals of potentials. Hence the next statement is a direct consequence of Proposition 2.1.

Corollary 2.3. *Assume f admits an inducing scheme (S,τ) of hyperbolic type and let $\varphi : M \to \mathbb{R}$ be a potential with $\sup\overline{\varphi} < \infty$, strongly summable variations and finite Gurevich pressure. Then there exists a unique F-invariant Gibbs measure $\nu_{\overline{\varphi}}$ for $\overline{\varphi}$. If $\nu_{\overline{\varphi}} \in \mathcal{M}_{\overline{\varphi}}(F)$, then it is a unique equilibrium measure for $\overline{\varphi}$.*

2.3. **Candidate equilibrium measures for the original map.** We now use the Gibbs measure for the induced map F to construct an equilibrium measure for the original map f.

For $\nu \in \mathcal{M}(F)$ with $\nu(\tau) < \infty$, the measure given by

$$\mathcal{L}(\nu) := \frac{1}{\nu(\tau)} \sum_{k=0}^{\infty} (f^k)_*\nu|_{\{\tau \le k\}}$$

is an f-invariant Borel probability measure. Let

$$\mathcal{M}_L(f) := \{\mu \in \mathcal{M}(f)\colon \mu = \mathcal{L}(\nu) \text{ for some } \nu \in \mathcal{M}(F)\}.$$

Measures in $\mathcal{M}_L(f)$ are called *liftable* measures. Consider a potential $\varphi\colon M \to \mathbb{R}$, and let

$$P_L(\varphi) = \sup\{h_\mu(f) + \mu(\varphi)\colon \mu \in \mathcal{M}_L(f)\}. \tag{8}$$

We say $\mu \in \mathcal{M}_L(f)$ is a *candidate equilibrium measure for* φ if $F_\varphi(\mu) = P_L(\varphi)$. Candidate equilibrium measures are equilibrium measures in the classical sense when $P_L(\varphi) = P(\varphi)$.

Abramov's and Kac's formulæ [20, Theorem 2.3] relate the entropy of ν and the integral of a potential against ν to the entropy of $\mathcal{L}(\nu)$ and the integral of the induced potential against $\mathcal{L}(\nu)$. Note that the energy $F_\varphi(\mathcal{L}(\nu)) = \frac{1}{\nu(\tau)} F_{\overline{\varphi}}(\nu)$ and so it is not straightforward that an equilibrium measure for $\overline{\varphi}$ lifts to a candidate equilibrium measure for φ. However, this is the case for the equilibrium measure associated to the potential induced by $\varphi - P_L(\varphi)$ and the latter is cohomologous to φ. Observe that by [20, Theorem 4.2] $|P_L(\varphi)| < \infty$ whenever φ has summable variations and finite Gurevich pressure.

We say $\overline{\varphi}$ is *positive recurrent* if there exists $\varepsilon_0 > 0$ such that

$$P_G(\overline{\varphi - (P_L(\varphi) - \varepsilon)}) < \infty \text{ for all } 0 \le \varepsilon \le \varepsilon_0. \tag{9}$$

This condition implies positive recurrence condition in the sense of Sarig (e.g [28]). Indeed, [20, Theorem 4.4] and the continuity of $P_G(\overline{\varphi - (P_L(\varphi) - \varepsilon)})$ with respect to ε for positive recurrent potentials $\overline{\varphi}$ imply $P_G(\overline{\varphi - P_L(\varphi)}) = 0$. This implies the existence of some $N \in \mathbb{N}$ such that

$$\inf_{n \geq N} \left\{ \sum_{F^n x = x} \exp\left(\sum_{i=0}^{n-1} \overline{\varphi - P_L(\varphi)}(F^i x) \right) \right\} > 0$$

which is equivalent to the positive recurrence condition of Sarig (c.f. [28, Theorem 1]).

We obtain the following:

Proposition 2.4. [21, Theorem 4.7](Existence and uniqueness of candidate equilibrium measures) *Assume f admits an inducing scheme (S, τ) of hyperbolic type. Let $\varphi : M \to \mathbb{R}$ be such that $\sup \overline{\varphi} < \infty$ and $\overline{\varphi}$ has strongly summable variation, finite Gurevich pressure and is positive recurrent. Then there exists a Gibbs measure ν for $\overline{\varphi - P_L(\varphi)}$. If $\nu \in \mathcal{M}_{\overline{\varphi - P_L(\varphi)}}(F)$ then it is the unique equilibrium measure for $\overline{\varphi - P_L(\varphi)}$. If $\nu(\tau) < \infty$ then $\mathcal{L}(\nu)$ is the unique candidate equilibrium measure for φ.*

3. Construction of inducing scheme

In this section we construct an induced system (X, F) for the Hénon map f. After preliminary geometric considerations in Sect.3.1 we introduce a rectangle Θ and show that the first return map to it is uniformly hyperbolic with controlled distortion. In Sect.3.2 we construct two families Γ^u and Γ^s of C^1 curves in Θ and generate a *lattice* Λ. The first return map to Λ is denoted by F. In Sect.3.3 we show the uniform hyperbolicity of F, and that the set of points in Λ for which F is undefined has small Hausdorff dimension. We define the domain X of our induced system to be the subset of Λ on which F may be iterated indefinitely. In Sect.3.4 we show that the induced map $F \colon X \circlearrowleft$ is semi-conjugated to the countable Markov shift.

We deal with positive constants ε, ξ, N, the purpose of which is as follows:

- $\varepsilon \ll 1$ is the constant in the statements of Theorem and Theorem A. We shall construct an induced system (X, F) such that any ergodic measure with entropy $\geq 2\varepsilon$ gives positive weight to X (cf. Proposition 4.1);
- $\xi \gg 1$ determines the rate of approach of points in the lattice Λ to the point of tangency;
- N is a large integer and controls a lower bound of diameters of gaps of a Cantor set in the unstable manifold, constructed in Sect.3.

Any generic constant which only depends on the Chebyshev quadratic map (and hence is independent of ε, ξ, N, b) is simply denoted by C.

3.1. **Family of invariant manifolds.** Let us from now on assume that f preserves orientation, as the proofs for the orientation reversing case are identical. Recall that P, Q denote the fixed saddles near $(1/2, 0)$ and $(-1, 0)$ correspondingly. Let $W^u = W^u(Q)$. By a *rectangle* we mean any closed region bordered by two compact curves in W^u and two in the stable manifolds of P, Q. By an *unstable side* of a rectangle we mean any of the two boundary curves in W^u. A *stable side* is defined similarly.

Let R denote the largest possible rectangle determined by W^u and $W^s(Q)$, as indicated in Figure 1. One of its unstable sides contains the point of tangency near $(0, 0)$, which we denote

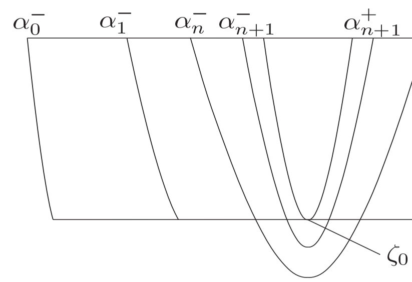

FIGURE 2. $(\alpha_n^\pm)$ accumulate on the parabola containing the point of tangency ζ_0.

by ζ. Let α_0^+ denote the stable side of R which contains $f\zeta$. Let α_0^- denote the other stable side of R.

Define a sequence $(\tilde\alpha_n)_{n\geq 0}$ of compact curves in $W^s(P)\cap R$ inductively as follows. First, let $\tilde\alpha_0$ be the component of $W^s(P)\cap R$ containing P. Given $\tilde\alpha_{n-1}$, define $\tilde\alpha_n$ to be one of the two components of $f^{-1}\tilde\alpha_{n-1}\cap R$ which is at the left of ζ. Observe that $\tilde\alpha_n\to\alpha_0^-$ as $n\to\infty$.

For each $n\geq 0$, $f^{-2}\tilde\alpha_n\cap R$ consists of four curves, two of them at the left of ζ and two at the right. Let α_{n+1}^- denote the one which is not $\tilde\alpha_{n+2}$ and is at the left of ζ. Among the two at the right of ζ, let α_{n+1}^+ denote the one which is at the left of the other. Then $\lim_{n\to\infty}\alpha_n^-$ (resp. $\lim_{n\to\infty}\alpha_n^+$) accumulates the component of $W^s(Q)\cap R$ containing ζ from the right (resp. left). Observe that $\tilde\alpha_1=\alpha_1^-$ and $\tilde\alpha_0=\alpha_1^+$. By definition, the curves obey the following diagram

$$\{\alpha_{n+1}^-,\alpha_{n+1}^+\}\xrightarrow{f^2}\tilde\alpha_n\xrightarrow{f}\tilde\alpha_{n-1}\xrightarrow{f}\tilde\alpha_{n-2}\xrightarrow{f}\cdots\xrightarrow{f}\tilde\alpha_1=\alpha_1^-\xrightarrow{f}\tilde\alpha_0=\alpha_1^+.$$

By a $C^2(b)$-*curve* we mean a closed curve such that the slopes of its tangent directions are $\leq\sqrt{b}$ and the curvature is everywhere $\leq\sqrt{b}$. For a $C^2(b)$-curve γ with endpoints in $\bigcup_{n\geq 1}\alpha_n^+\cup\alpha_n^-$ we define a *canonical partition*, by intersecting it with the countable family $(\alpha_n^\pm)$ of pieces of stable manifolds. This is feasible by the fact that each of these pieces intersect γ exactly one point (See [33, Remark 2.1]).

Let Θ denote the rectangle bordered by α_1^-, α_1^+ and the unstable sides of R. Any component γ of $\Theta\cap W^u$ is a $C^2(b)$-curve [30, Lemma 2.1]. For each $n>1$, let γ_n denote the element of the canonical partition of γ with endpoints in $\alpha_n^+,\alpha_{n-1}^+$. We also denote by γ_n the partition element with endpoints in $\alpha_{n-1}^-,\alpha_n^-$. Then, for each γ_n and every $1\leq i<n$, $f^i\gamma_n\cap\mathrm{int}\Theta=\emptyset$, and $f^n\gamma_n$ is a $C^2(b)$-curve in Θ with endpoints in α_1^-,α_1^+. Namely, n is the first return time of γ_n to Θ.

The next proposition, the proof of which is given in Appendix A1, states that the first return map to Θ is uniformly hyperbolic with controlled distortions. Let

$$\sigma_1=2-\varepsilon\quad\text{and}\quad\sigma_2=4+\varepsilon. \tag{10}$$

Proposition 3.1. *There exist $C>0$ and $N>0$ such that for any component γ of $\Theta\cap W^u$ and each γ_n, $n>N$ we have:*

(a) *for all $x\in\gamma_n$, $\sigma_1^n\leq\|D_xf^n|E^u\|\leq\sigma_2^n$;*

(b) *for all $x,y\in\gamma_n$, $\log\dfrac{\|D_xf^n|E^u\|}{\|D_yf^n|E^u\|}\leq C|f^nx-f^ny|$.*

3.2. Construction of families of curves.

Definition 3.2. Let Γ^u and Γ^s be two families of C^1 curves in Θ such that:

- curves in Γ^s are pairwise disjoint. At most countably many pairs of curves in Γ^u can intersect;
- every $\gamma^u \in \Gamma^u$ meets every $\gamma^s \in \Gamma^s$ in exactly one point;
- there is a minimum angle between γ^u and γ^s at the point of intersection;
- endpoints of curves in Γ^u (resp. Γ^s) are in the *stable* (resp. *unstable*) *sides of* Θ.

Call the set

$$\Lambda = \{\gamma^u \cap \gamma^s \colon \gamma^u \in \Gamma^u, \gamma^s \in \Gamma^s\}$$

a *lattice*.

We now construct two families Γ^u, Γ^s of C^1 curves in Θ which generate a lattice. Denote by $\tilde{\Gamma}^u$ the collection of $C^2(b)$-curves in W^u with endpoints in α_1^-, α_1^+ and let

$$\Gamma^u = \{\gamma^u \colon \gamma^u \text{ is the pointwise limit of a sequence in } \tilde{\Gamma}^u\}.$$

By the $C^2(b)$ property, the pointwise convergence is equivalent to the uniform convergence. Since two distinct curves in $\tilde{\Gamma}^u$ do not intersect each other, the uniform convergence is equivalent to the C^1 convergence. Hence, each curve in Γ^u is C^1 and the slopes of its tangent directions are $\leq \sqrt{b}$. Since every $\gamma^u \in \Gamma^u$ is the monotone limit of curves in $\tilde{\Gamma}^u$, there are at most countably many pairs of curves in Γ^u that intersect.

We construct Γ^s as follows. For each $n \geq N$, let Θ_n denote the rectangle bordered by α_n^-, α_n^+ and the unstable sides of Θ. Let $\widehat{\gamma}$ denote the lower unstable side of Θ. Let $\Omega_0 = \widehat{\gamma} \setminus \Theta_N$. We call $\widehat{\gamma} \cap \Theta_N$ a gap of order 0. Let $\hat{\mathcal{P}}_0$ denote the canonical partition of $\widehat{\gamma}$ and let $\mathcal{P}_0 = \hat{\mathcal{P}}_0 | \Omega_0$. For $n > 0$ define

$$\Omega_n = \{z \in \widehat{\gamma} \colon f^k z \notin \Theta_{\xi k+N} \text{ for every } 0 \leq k \leq n\}. \tag{11}$$

Any component of $\Omega_{n-1} \setminus \Omega_n$ is called a *gap of order* n. We set $\Omega_\infty = \bigcap_{n \geq 0} \Omega_n$. Observe that $\Omega_\infty \subset K$.

We call a *vertical* $C^2(b)$*-curve* a curve in Θ with endpoints in the unstable sides of Θ and of the form

$$\{(x(y), y) \colon |x'(y)| \leq C\sqrt{b}, |x''(y)| \leq C\sqrt{b}\}.$$

The next lemma is proven in Appendix A2.

Lemma 3.3. *For any $z \in \Omega_\infty$ there exists a vertical $C^2(b)$-curve $\gamma^s(z) \subset \Theta$ through z with the following properties:*

(a) *if $f^n \gamma^s(z_1) \cap \gamma^s(z_2) \neq \emptyset$ for $n \geq 0$, then $f^n \gamma^s(z_1) \subset \gamma^s(z_2)$;*
(b) *$|f^n x - f^n y| \leq (Cb)^{\frac{n}{2}}$ and $\|Df_x \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\| \leq 2 \cdot \|Df_y \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|$ for all x, $y \in \gamma^s(z)$ and $n \geq 0$;*
(c) *if $z_1, z_2 \in \Omega_\infty$ and $x_1 \in \gamma^s(z_1)$, $x_2 \in \gamma^s(z_2)$, then $\angle(u(x_1), u(x_2)) \leq C\sqrt{b}|x_1 - x_2|$, where $u(x_i)$ denotes any unit vector tangent to $\gamma^s(z_i)$ at x_i, $i = 1, 2$. In particular, if $z_1 \neq z_2$ then $\gamma^s(z_1) \cap \gamma^s(z_2) = \emptyset$.*

Define

$$\Gamma^s = \{\gamma^s(z) \colon z \in \Omega_\infty\},$$

where $\gamma^s(z)$ is the vertical $C^2(b)$-curve satisfying Lemma 3.3.

3.3. **First return map.** Consider the lattice Λ defined by Γ^u and Γ^s: $\Lambda = \{\gamma^u \cap \gamma^s \colon \gamma^u \in \Gamma^u, \gamma^s \in \Gamma^s\}$. We study the first return map to Λ. To this end we first study the transversal structure of Λ. Let

$$\mathcal{W}^s = \bigcup_{\gamma^s \in \Gamma^s} \gamma^s.$$

Since $\Omega_\infty \cap \Theta_{N+1} = \emptyset$ we have $\mathcal{W}^s \cap \Theta_{N+1} = \emptyset$.

For $z \in K$ let

$$\tau(z) = \inf\left(\{n > 0 \colon f^n z \in \Lambda\} \cup \{\infty\}\right),$$

which is the first return time to Λ.

Proposition 3.4. *There exists a collection $\mathcal{Q}$ of subsets of Ω_∞ such that:*

(a) $\bigcup_{\omega \in \mathcal{Q}} \omega = \{z \in \Omega_\infty \colon \tau(z) < \infty\}$;
(b) *τ is constant on each $\omega \in \mathcal{Q}$ (denote this value by $\tau(\omega)$);*
(c) *for each $\omega \in \mathcal{Q}$ there exists $\gamma \in \tilde{\Gamma}^u$ such that $f^{\tau(\omega)}\omega = \gamma \cap \mathcal{W}^s$.*

Proof. We construct $\mathcal{Q}$ by induction. Consider $\omega \in \mathcal{P}_0$ and let $1 < n(\omega) \leq N$ denote the smallest integer such that $f^{n(\omega)}\omega \subset \tilde{\Gamma}^u$. By construction, $n(\omega)$ is the first return time of ω to Θ. We let $f^{-n(\omega)}\left(f^{n(\omega)}\omega \cap \mathcal{W}^s\right) \in \mathcal{Q}$.

Lemma 3.5. *For each $\omega \in \mathcal{P}_0$, $f^{-n(\omega)}\left(f^{n(\omega)}\omega \cap \mathcal{W}^s\right) \subset \Omega_\infty$.*

Proof. By definition, $f^{-n(\omega)}\left(f^{n(\omega)}\omega \cap \mathcal{W}^s\right) \subset \Omega_{n(\omega)}$ and if $z \in f^{-n(\omega)}\left(f^{n(\omega)}\omega \cap \mathcal{W}^s\right)$, then $f^{n(\omega)}z \in \gamma^s(y)$ for some $y \in \Omega_\infty$. We now show $z \in \Omega_{n+n(\omega)}$ for every $n > 0$. Observe that $f^n y \notin \Theta_{\xi n+N}$ by (11). Then $f^{n+n(\omega)}z \notin \Theta_{\xi n+N}$, as otherwise $f^n\gamma^s(y)$ would intersect the stable side of $\Theta_{\xi n+N}$. However, because of contraction along γ^s and since the stable side of $\Theta_{\xi n+N}$ is contained in $W^s(P)$, this would imply that $f^n\gamma^s(y) \subset W^s(P)$ leading to a contradiction. Then $z \in \Omega_{n+n(\omega)}$ holds for all $n > 0$ and since the sets $\Omega_{n+n(\omega)}$ are nested this implies $z \in \Omega_\infty$. □

Definition 3.6. By a *gap of $\mathcal{W}^s$ of order n* we mean any rectangle bordered by a gap of Ω_∞ of order n, a segment in the upper unstable side of Θ, and two long stable leaves joining their endpoints.

For the next step of the induction, consider a gap G of $\mathcal{W}^s$ of order g and let $\gamma \subset \omega \in \mathcal{Q}$ be such that $f^{n(\omega)}\gamma$ stretches across G. Let $\omega' \subset \gamma$ be the preimage under $f^{n(\omega)+g}$ of an element of the canonical partition such that ω' contains points of Ω_∞. Then $f^{m+g+n(\omega)}\omega' \in \tilde{\Gamma}^u$, where $m = n(f^{n(\omega)+g}\omega')$. Let $\tilde{\omega} := f^{-m-g-n(\omega)}\left(f^{m+g+n(\omega)}\omega' \cap \mathcal{W}^s\right)$.

Lemma 3.7. $\tilde{\omega} \subset \Omega_\infty$.

Proof. Since $n(\omega)$ is the first return time of ω to Θ, $f^i\omega' \cap \Theta = \emptyset$ for $0 < i < n(\omega)$. We have $f^{n(\omega)}\omega' \subset G$, and so by definition of the gap G of order g, $f^j(f^{n(\omega)}\omega') \cap \Theta_{\xi j+N} = \emptyset$ for $0 \leq j \leq g-1$, and $f^{g+n(\omega)}\omega' \subset \Theta_{\xi g+N}$. Since $\Theta_{\xi(n(\omega)+j)+N} \subset \Theta_{\xi j+N}$, we in fact get $f^j(f^{n(\omega)}\omega') \cap \Theta_{\xi(j+n(\omega))+N} = \emptyset$, for $0 \leq j \leq g-1$. However, since $\omega' \cap \Omega_\infty \neq \emptyset$ then $f^{n(\omega)+g}\omega' \subset \Theta_{\xi g+N} \setminus \Theta_{\xi(g+n(\omega))+N}$, and thus $\omega' \subset \Omega_{g+n(\omega)}$.

Since m is the first return time of $f^{n(\omega)+g}\omega'$ to Θ, for every $g+n(\omega) < n < m+g+n(\omega)$ we have $f^{n+n(\omega)+g}\omega' \cap \Theta = \emptyset$, and so $\tilde{\omega} \subset \omega' \subset \Omega_{m+g+n(\omega)-1}$. Finally $f^{m+g+n(\omega)}\tilde{\omega} \subset \mathcal{W}^s$ and $\mathcal{W}^s \cap \Theta_{N+1} = \emptyset$ imply $\tilde{\omega} \subset \Omega_{m+g+n(\omega)}$. The argument of Lemma 3.5 shows $f^n(f^{m+g+n(\omega)}\tilde{\omega}) \cap \Theta_{\xi n+N} = \emptyset$ for every $n > 0$, and so $\tilde{\omega} \subset \Omega_{n+m+g+n(\omega)}$. □

This allows to complete the inductive construction of $\mathcal{Q}$. [30, Lemma 2.2] implies $K\cap\mathcal{W}^s\subset\Lambda$, and so $\tau(z)=\inf\left(\{n>0\colon f^nz\in\mathcal{W}^s\}\cup\{\infty\}\right)$.

Lemma 3.8. *Let G be a gap of order g. Then for $0\leq i\leq g$, $f^iG\cap\mathcal{W}^s=\emptyset$.*

Proof. Suppose there exists a point $x\in f^iG\cap\mathcal{W}^s\neq\emptyset$ for some $0\leq i\leq g$. Then $f^{g-i}x\notin\Theta_{\xi(g-i)+N}$. On the other hand, $f^{g-i}x\in f^gG\subset\Theta_{\xi g+N}\subset\Theta_{\xi(g-i)+N}$, a contradiction. □

The rest of the proof of Proposition 3.4 follows from the construction and Lemma 3.8. □

Definition 3.9. We say:

- $\Lambda'\subset\Lambda$ is a *u-sublattice* of Λ if there exists $\Gamma^{u\prime}\subset\Gamma^u$ such that $\Lambda'=\{\gamma^u\cap\gamma^s\colon\gamma^u\in\Gamma^{u\prime},\gamma^s\in\Gamma^s\}$. An *s-sublattice of* Λ is defined similarly;
- $Q\subset\mathbb{R}^2$ is the *rectangle spanned by* Λ' if $\Lambda'\subset Q$ and ∂Q is made up of two curves in $\Gamma^{u\prime}$ and two in Γ^s.

Define $\hat{S}$ to be the collection of s-sublattices of Λ whose defining s-families are of the form $\{\gamma^s(z)\colon z\in\omega\}$ for some $\omega\in\mathcal{Q}$. For $I\in\hat{S}$ let Q_I denote the rectangle spanned by I. From the construction it directly follows that τ is constant on each element of $\hat{S}$. We think of τ as a function on $\hat{S}$ in the obvious way.

Proposition 3.10. *The following statements hold:*

(P1) (Topological structure) *for any $I\in\hat{S}$, $f^{\tau(I)}I$ is a u-sublattice of Λ;*

(P2) (Backward contraction) *there exist $C>0$ and $\lambda>1$ such that for any $\gamma^u\in\Gamma^u$, $z\in\gamma^u$, any unit vector v at z and $n>0$, $\|D_{f^{-n}z}f^nv\|\geq C\lambda^n$. In particular, E^u makes sense on γ^u and coincides with its tangent directions;*

(P3) (Hyperbolicity) *for any $\gamma\in\Gamma^u$, $I\in\hat{S}$ and all $z\in\gamma\cap Q_I$,*

$$\sigma_1^{\tau(I)}\leq\|D_zf^{\tau(I)}|E^u\|\leq\sigma_2^{\tau(I)}.$$

Here, σ_1,σ_2 are the constants from (10)*;*

(P4) (Distortion control)

(a) *for any $\gamma\in\Gamma^u$ and $x,y\in\gamma\cap Q_I$,*

$$\log\frac{\|D_xf^{\tau(I)}|E^u\|}{\|D_yf^{\tau(I)}|E^u\|}\leq C|f^{\tau(I)}x-f^{\tau(I)}y|;$$

(b) *for any $\gamma\in\Gamma^s$ and all $x,y\in\gamma^s$, $n\geq1$,*

$$\|D_xf^n|E^u\|\leq2\cdot\|D_yf^n|E^u\|.$$

Proof. To show (P1) it suffices to show that for any $\gamma\in\Gamma^u$, $f^{\tau(I)}(\gamma\cap I)\in\Gamma^u$. This follows from the construction. (P2) follows from the backward contraction on the leaves in $\tilde{\Gamma}^u$ (see [30, Lemma 4.2]) and the fact that any leaf in Γ^u is a C^1-limit of leaves in $\tilde{\Gamma}^u$. Since $f^{\tau(J)}$ is a composition of first return maps to Θ, (P3) and (P4)(a) follow from the estimates in Proposition 3.1. (P4)(b) follows from Lemma 3.3(b). □

3.4. **Symbolic coding.** Let

$$B=\{z\in K\colon\tau(z)=\infty\}.$$

Define an induced map $F\colon\Lambda\setminus B\to\Lambda$ by $Fz=f^{\tau(z)}z$, which is the first return map to Λ. Observe that $\Lambda\cap B=\Lambda\setminus\bigcup_{I\in\hat{S}}I$. Let

$$E=B\cap\Lambda,$$

and define the corresponding inducing domain X by

$$X = \Lambda \setminus \bigcup_{n\geq 0} F^{-n}E.$$

This is the set of points in Λ for which F may be iterated indefinitely. The corresponding collection S of basic elements is defined by

$$S = \{I \cap X \colon I \in \hat{S}\}.$$

Observe that any basic element is an s-sublattice of Λ, and $X = \bigcup_{S\in J} J$. The next Markov property allows us to represent (the natural extension of) the induced system $F\colon X \circlearrowleft$ as a countable two-sided full shift.

Lemma 3.11. *For any $J \in S$, FJ is a u-sublattice of X.*

Proof. Any $J \in S$ has the form $J = I \setminus \bigcup_{n\geq 0} F^{-n}E$, $I \in \hat{S}$. Hence $FJ = FI \setminus \bigcup_{n\geq 0} F^{-n}E$, which is a u-sublattice of X. □

The next lemma ensures that the set of points for which the coding is not uniquely defined carries no invariant probability measure.

Lemma 3.12. *If $J_1, J_2 \in S$, $J_1 \neq J_2$ and $J_1 \cap J_2 \neq \emptyset$, then $J_1 \cap J_2 \subset W^u(P) \setminus \{P\}$.*

Proof. It is obvious that $\tau(J_1) \neq \tau(J_2)$. Without loss of generality we may assume $\tau(J_1) < \tau(J_2)$. By Lemma 3.11, $f^{\tau(J_1)}J_1$ is an u-sublattice of X, while $f^{\tau(J_1)}J_2$ is contained in some gap of $\mathcal{W}^s$, say G. Hence, the intersection is contained in the stable side of the rectangle G, which is contained in $W^s(P)$ by construction. Hence the desired inclusion holds. □

Define a coding map $h\colon S^{\mathbb{Z}} \to X$ by (7). Observe that $F \circ h = h \circ \sigma$, and no F-invariant probability measure gives positive weight to $X_0 := W^s(P) \setminus \{P\}$. Then Lemma 3.12 and the next lemma ensure that (S, τ) is an inducing scheme of hyperbolic type.

Lemma 3.13. *h defines a measurable bijection between $S^{\mathbb{Z}} \setminus h^{-1}X_0$ and $X \setminus X_0$.*

Proof. First we show that h is well-defined. Let $\underline{a} = (a_n)_n \in S^{\mathbb{Z}}$. For every $n > 0$, $h([a_0, \ldots, a_n])$ is an s-sublattice of Λ, strictly decreasing in n. By (P2) and Lemma 3.3 (b), the stable sides of the rectangles spanned by these sublattices converge, in the C^1 topology, to a curve whose tangent direction has large slope. On the other hand, for every $n > 0$, $h([a_{-n}, \ldots, a_{-1}])$ is a strictly decreasing u-sublattice of Λ. By (P1) the unstable sides of the rectangle spanned by these sublattices converge, in the C^1 topology, to a C^1 curve whose tangent direction has small slopes. Thus the intersection of the two sets $\bigcap_{n\geq 0} h([a_0, \ldots, a_n])$ and $\bigcap_{n>0} h([a_{-n}, \ldots, a_{-1}])$ are curves, intersecting each other exactly at one point. Hence, $h(\underline{a})$ is well-defined. From the uniform hyperbolicity of F and the fact that the cylinder sets form a base of the topology in $S^{\mathbb{Z}}$, h is continuous. It is obviously surjective, and from Lemma 3.12 it defines a bijection between $S^{\mathbb{Z}} \setminus h^{-1}X_0$ and $X \setminus X_0$. By the continuity of h and [30, Claim 3.3], it is a measurable bijection. □

3.5. **Hausdorff dimension of exceptional sets.** For $\gamma \in \tilde{\Gamma}^u$, let

$$\Omega_\infty(\gamma) = \{z \in \gamma \colon f^n z \notin \Theta_{\xi n+N} \text{ for every } n \geq 0\}.$$

In particular, $\Omega_\infty(\widehat{\gamma}) = \Omega_\infty$. Although $\Omega_\infty(\gamma)$ depends on N, we will not explicitly express this dependency in the notation (except in the proof of Lemma 4.3).

Lemma 3.14. *for each* $\gamma \in \tilde{\Gamma}^u$, $\mathrm{HD}\,(\Omega_\infty(\gamma) \cap B) \leq \varepsilon$.

Proof. Given $j > 1$ and a j-string $(k_1, \ldots, k_j)$ of positive integers, we define collections $\mathcal{Q}(k_1), \mathcal{Q}(k_1, k_2), \ldots, \mathcal{Q}(k_1, k_2, \ldots, k_j)$ of pairwise disjoint curves in γ inductively as follows. Let

$$\mathcal{Q}(k_1) = \{\omega_1 \subset \gamma \colon f^{k_1}\omega_1 \in \tilde{\Gamma}^u\}.$$

Given $\mathcal{Q}(k_1, \ldots, k_i)$, for each $\omega_i \in \mathcal{Q}(k_1, \ldots, k_i)$ let

$$\mathcal{Q}(\omega_i, k_{i+1}) = \{\omega_{i+1} \subset \omega_i \colon f^{k_1+\cdots+k_i+k_{i+1}}\omega_{i+1} \in \tilde{\Gamma}^u\},$$

and define

$$\mathcal{Q}(k_1, \ldots, k_{i+1}) = \bigcup_{\omega_i \in \mathcal{Q}(k_1,\ldots,k_i)} \mathcal{Q}(\omega_i, k_{i+1}).$$

For each sufficiently large integer n, let

$$\mathcal{Q}_n(k_1, \ldots, k_i) = \{\omega_i \in \mathcal{Q}(k_1, \ldots, k_i) \colon \sup\{\tau(z) \colon z \in \omega_i\} \geq n\}.$$

For each $\omega_i \in \mathcal{Q}_n(k_1, \ldots, k_i)$, let

$$\mathcal{Q}_n(\omega_i, k_{i+1}) = \{\omega_{i+1} \in \mathcal{Q}(\omega_i, k_{i+1}) \colon \sup\{\tau(z) \colon z \in \omega_{i+1}\} \geq n\}.$$

Let $\omega_0 = \gamma$ and $\mathcal{Q}_n(\omega_0, k_1) = \mathcal{Q}_n(k_1)$.

Sublemma 3.15. *If $\xi > 2/3$ and $N > 2(1+\xi)$, then for every $n > 6N$ and for any $z \in \gamma$ with $\tau(z) \geq n$ there exist an integer $1 \leq s \leq n/N$, and for each $i = 1, \ldots, s$ an integer $k_i \geq N$ and a curve $\omega_i \in \mathcal{Q}_n(k_1, \ldots, k_i)$ such that:*

(a) $k_1 + \cdots + k_s \geq \frac{n}{3\xi}$;
(b) $z \in \omega_s \subset \cdots \subset \omega_1$;
(c) $\ell(\omega_s) \leq C\sigma_1^{-(k_1+\cdots+k_s)}$;
(d) *for each* $i = 0, \ldots, s-1$, $\#\mathcal{Q}_n(\omega_i, k_{i+1}) < 2^{\frac{k_{i+1}}{\xi}}$.

Proof. Define a sequence $0 =: t_0 < t_1 < \cdots$ of return times to Θ inductively as follows: given t_i such that $f^{t_i}z$ is in the gap of $\mathcal{W}^s$ of order g_i, define

$$t_{i+1} = \min\{t \geq t_i + g_i + \xi g_i + N \colon f^t z \in \Theta\}.$$

Note that (t_i) are not the only return times of the orbit of z to Θ. Since $g_i \geq 0$ we have

$$t_{i+1} - t_i \geq N. \tag{12}$$

Define $s = \max\{i \colon t_i < n\} + 1$. (12) gives $s \leq n/N$. We also have

$$t_{s-1} + g_{s-1} \geq \frac{n}{3\xi}. \tag{13}$$

For otherwise $t_{s-1} + g_{s-1} < \frac{n}{3\xi}$, and so $\xi(t_{s-1} + g_{s-1}) + N < [n/2]$ and $\Theta_{[n/2]} \subsetneq \Theta_{\xi(t_{s-1}+g_{s-1})+N}$. The assumption $z \in \Omega_\infty(\gamma)$ gives $f^{t_{s-1}+g_{s-1}}z \notin \Theta_{\xi(t_{s-1}+g_{s-1})+N}$ (see (11)), and so $f^{t_{s-1}+g_{s-1}}z \notin \Theta_{[n/2]}$. On the other hand, since $f^{t_{s-1}}z$ is in a gap of order g_{s-1} we have $f^{t_{s-1}+g_{s-1}}z \in \Theta_{\xi g_{s-1}+N}$. Let r denote the first return time of $f^{t_{s-1}+g_{s-1}}z$ to Θ. Then $r \geq \xi g_{s-1} + N$, and so $t_s = t_{s-1} + g_{s-1} + r$. Since $r < [n/2]$ we have $t_s < \frac{n}{3\xi} + n/2 < n$, which is a contradiction.

For each $i = 0, \ldots, s-1$, define $k_{i+1} = t_{i+1} - t_i$. Since $k_1 + \cdots + k_s = t_s > t_{s-1} + g_{s-1}$, (a) follows from (13). For each $i = 1, \ldots, s$, let ω_i denote the curve in $\mathcal{Q}_n(k_1, \ldots, k_i)$ which contains z. Then (b) is straightforward. (c) follows from Proposition 3.1.

Claim 3.16. *For any $\omega_{i+1}\in\mathcal{Q}_n(\omega_i,k_{i+1})$, $f^{k_1+\cdots+k_i}\omega_{i+1}$ is contained in a gap of $\mathcal{W}^s$.*

Proof. Let $z\in\omega_{i+1}$ be such that $\tau(z)\geq n$, and assume that $f^{k_1+\cdots+k_i}\omega_{i+1}$ is not contained in the gap containing $f^{k_1+\cdots+k_i}z$. Then the interior of $f^{k_1+\cdots+k_i}\omega_{i+1}$ contains a boundary point of the gap. It follows that $f^{k_1+\cdots+k_i+k_{i+1}}\omega_{i+1}\notin\tilde{\Gamma}^u$, a contradiction. □

For (d), observe that for any gap G of $\mathcal{W}^s$ we have

$$\#\{\omega_{i+1}\in\mathcal{Q}_n(\omega_i,k_{i+1})\colon f^{k_1+\cdots+k_i}\omega_{i+1}\subset G\}=0 \text{ or } =2, \tag{14}$$

since gaps are not folded up to their order, and $f^jG\cap\Theta=\emptyset$ for $g_i<j<k_{i+1}-g_i$ by the definition of t_{i+1}.

Let g_0 denote the maximal order of the gap of $\mathcal{W}^s$ which contains $f^{k_1+\cdots+k_i}$-images of elements of $\mathcal{Q}_n(\omega_i,k_{i+1})$. If a gap G is of order g_0, then $f^{g_0}G\subset\Theta_{\xi g_0+N}$. Hence $g_0+\xi g_0+N\leq k_{i+1}$ holds. From (14) and the fact that the number of gaps of order g is $\leq 2^g$ we obtain $\#\mathcal{Q}_n(\omega_i,k_{i+1})\leq 2\sum_{i=1}^{g_0}2^i<2^{\frac{k_{i+1}-N}{1+\xi}+2}<2^{\frac{k_{i+1}}{\xi}}$. The last inequality holds provided $N>2(1+\xi)$. □

Returning to the proof of Lemma 3.14, we have

$$\{z\in\Omega_\infty(\gamma)\colon\tau(z)\geq n\}\subset\bigcup_{s=1}^{\left[\frac{n}{N}\right]}\bigcup_{l=\left[\frac{n}{3\xi}\right]}^{\infty}\bigcup_{k_1+\cdots+k_s=l}\bigcup_{\omega_s\in\mathcal{Q}_n(k_1,\ldots,k_s)}\omega_s.$$

By Sublemma 3.15(a)(c), the lengths of the curves ω_s in the union of the right-hand-side are exponentially small in n. We show that $\sum_{\text{all relevant }\omega_s}\ell(\omega_s)^\varepsilon$ is finite for all n.

Observe that

$$\sum_{\omega_{i+1}\in\mathcal{Q}_n(k_1,\ldots,k_{i+1})}\ell(\omega_{i+1})^\varepsilon=\sum_{\omega_i\in\mathcal{Q}_n(k_1,\ldots,k_i)}\ell(\omega_i)^\varepsilon\sum_{\omega_{i+1}\in\mathcal{Q}_n(\omega_i,k_{i+1})}\frac{\ell(\omega_{i+1})^\varepsilon}{\ell(\omega_i)^\varepsilon}. \tag{15}$$

On the second sum of the fractions, let $\omega_{i+1}\in\mathcal{Q}_n(\omega_i,k_{i+1})$. Since $\ell(f^{k_1+\cdots+k_{i+1}}\omega_{i+1})<2$ and $\|D_xf^{k_{i+1}}|E^u\|\geq\sigma_1^{k_{i+1}}$ for all $x\in f^{k_1+\cdots+k_i}\omega_{i+1}$, we have $\ell(f^{k_1+\cdots+k_i}\omega_{i+1})\leq C\sigma_1^{-k_{i+1}}$. From this and the bounded distortion in Proposition 3.1,

$$\frac{\ell(\omega_{i+1})}{\ell(\omega_i)}\leq C\cdot\frac{\ell(f^{k_1+\cdots+k_i}\omega_{i+1})}{\ell(f^{k_1+\cdots+k_i}\omega_i)}\leq C\sigma_1^{-k_{i+1}}. \tag{16}$$

Using (16) and Sublemma 3.15(d),

$$\sum_{\omega_{i+1}\in\mathcal{Q}_n(\omega_i,k_{i+1})}\frac{\ell(\omega_{i+1})^\varepsilon}{\ell(\omega_i)^\varepsilon}\leq\#\mathcal{Q}_n(\omega_i,k_{i+1})C^\varepsilon\sigma_1^{-\varepsilon k_{i+1}}\leq C^\varepsilon\sigma_1^{-\frac{\varepsilon}{2}k_{i+1}}.$$

Plugging this into the right-hand-side of (15) we get

$$\sum_{\omega_i\in\mathcal{Q}_n(k_1,\ldots,k_{i+1})}\ell(\omega_{i+1})^\varepsilon\leq C^\varepsilon\sigma_1^{-\varepsilon k_{i+1}}\sum_{\omega_i\in\mathcal{Q}_n(k_1,\ldots,k_i)}\ell(\omega_i)^\varepsilon. \tag{17}$$

The same arguments as above applied to any $\omega_1\in\mathcal{Q}_n(k_1)$ yield

$$\sum_{\omega_1\in\mathcal{Q}_n(k_1)}\ell(\omega_1)^\varepsilon\leq C^\varepsilon\#\mathcal{Q}_n(k_1)\sigma_1^{-\varepsilon k_1}\leq C^\varepsilon\sigma_1^{-\frac{\varepsilon}{2}k_1}. \tag{18}$$

Using (17) inductively and (18) yields

$$\sum_{\omega_s\in\mathcal{Q}_n(k_1,\ldots,k_s)} \ell(\omega_s)^\varepsilon \le C^{\varepsilon s}\sigma_1^{-\frac{\varepsilon}{2}(k_1+\cdots+k_s)}.$$

Using this and Stirling's formula for factorials we have

$$\begin{aligned}\sum_{l=\left[\frac{n}{3\xi}\right]}^{\infty}\sum_{k_1+\cdots+k_s=l}\sum_{\omega_s\in\mathcal{Q}_n(k_1,\ldots,k_s)} \ell(\omega_s)^\varepsilon &\le C^{\varepsilon s}\sum_{l=\left[\frac{n}{3\xi}\right]}^{\infty}\sigma_1^{-\varepsilon l}\#\left\{(k_1,\ldots,k_s)\colon \sum_{i=1}^{s}k_i=l\right\}\\ &\le C^{\varepsilon s}\sum_{l=\left[\frac{n}{3\xi}\right]}^{\infty}\sigma_1^{-\frac{\varepsilon}{2}l}\beta^l,\end{aligned}$$

where $\beta\to 1$ as $N\to\infty$. Hence

$$\sum_{s=1}^{\left[\frac{n}{N}\right]}\sum_{l=\left[\frac{n}{3\xi}\right]}^{\infty}\sum_{k_1+\cdots+k_s=l}\sum_{\omega_s\in\mathcal{Q}_n(k_1,\ldots,k_s)} \ell(\omega_s)^\varepsilon \le \frac{C^{\frac{\varepsilon n}{N}}}{C^\varepsilon-1}\sum_{l=\left[\frac{n}{3\xi}\right]}^{\infty}\sigma_1^{-\frac{\varepsilon}{2}l}\beta^l.$$

Since N is chosen after ε and ξ, one can choose N large enough so that the expression on the right-hand-side decays exponentially with n. Consequently the Hausdorff ε-measure of $\Omega_\infty(\gamma)\cap B$ is zero. □

3.6. **Small growth rate of the number of basic elements.** Let

$$S(n) := \#\{J\in S\colon \tau(J)=n\}.$$

Proposition 3.17. *For any $\epsilon>0$ there exist ξ and N large such that*

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log S(n)\le\epsilon.$$

Proof. For each $J\in S$ with $\tau(J)=n$, let ω_J denote the unstable side of the rectangle Q_J spanned by J which is contained in $\widehat{\gamma}$. Observe that there exists $1\le s\le n/N$ and a s-string $(k_1,\ldots,k_s)$ of positive integers such that $k_1+\cdots+k_s=n$ and $\omega_J\in\mathcal{Q}_n(k_1,\ldots,k_s)$. For two distinct $J_1,J_2\in S$ with $\tau(J_1)=\tau(J_2)=n$, one has $\omega_{J_1}\cap\omega_{J_2}=\emptyset$. Therefore,

$$S(n)\le\sum_{s=1}^{n/N}\sum_{k_1+\cdots+k_s=n}\#\mathcal{Q}_n(k_1,\ldots,k_s).$$

Sublemma 3.15(d) implies $\#\mathcal{Q}_n(k_1,\ldots,k_s)\le 2^{\frac{k_1+\cdots+k_s}{\xi}}\le 2^{\frac{n}{\xi}}$. Substituting this into the right-hand-side of the previous inequality we obtain $S(n)\le\frac{n}{N}\beta^n 2^{\frac{n}{\xi}}$, and thus $\overline{\lim_{n\to\infty}} n^{-1}\log S(n)\le \log\beta+(1/\xi)\log 2$, which can be made arbitrarily small by choosing large ξ, N. □

4. Efficiency of the inducing scheme

The purpose of this section is to prove the next

Proposition 4.1. *For any $\mu\in\mathcal{M}^e(f)$ with $h(\mu)\ge 2\varepsilon$, $\mu(\Lambda)>0$.*

It follows that any ergodic measure with not too small entropy is liftable to the tower associated with the induced system (X,F) constructed in Sect.3.

Corollary 4.2. *Any $\mu \in \mathcal{M}^e(f)$ with $h(\mu) \geq 2\varepsilon$ is liftable.*

Proof. Proposition 4.1 gives $\mu(\Lambda) > 0$. Since F is a first return map to Λ, the Poincaré recurrence gives $\mu(X) > 0$. Since F is the first return map to X as well, Kac's formula [29, Theorem 1.6] gives $\int_X \tau d\mu = 1$, and so τ is μ-integrable. By [37], μ is liftable. □

The rest of this section is devoted to the proof of Proposition 4.1. In Sect.4.1 we improve Lemma 3.14 and give a better control of the dimension of the set of points which do not return to Λ. In Sect.4.2 we recall some general results on invariant manifolds of nonuniformly hyperbolic systems which we then use to complete the proof of the proposition.

4.1. **Dimension of the set of points not returning to Λ.** We show that the set B is small in terms of Hausdorff dimension.

Lemma 4.3. *For any $\gamma \in \tilde{\Gamma}^u$ we have $\mathrm{HD}(\gamma \cap B) \leq \varepsilon$.*

Lemma 4.4. *For any relatively open curve γ in W^u intersecting K there exist a countable set $A \subset \gamma \cap W^s(Q)$, a countable collection $\{\gamma_n\}_n$ of curves in γ and a sequence $\{a_n\}$ of positive integers such that:*

(a) $(\gamma \cap K) \setminus A \subset \bigcup_n \gamma_n$;
(b) $f^{a_n}\gamma_n \in \tilde{\Gamma}^u$.

Proof. A successive use of Proposition 3.1 implies that all but countably many points in $\gamma \cap K$ have arbitrarily small neighborhoods in W^u which are mapped by some positive iterates to curves in $\tilde{\Gamma}^u$. □

The countable stability of Hausdorff dimension additionally yields:

Corollary 4.5. *For any relatively open curve $\gamma \subset W^u$, $\mathrm{HD}(\gamma \cap B) \leq \varepsilon$.*

Proof of Lemma 4.3. Choose $\xi = \xi(\varepsilon) \gg 1$ so that

$$\eta := 2\sigma_1^{-\varepsilon\xi} < 1. \tag{19}$$

We call $l > 0$ a *close return time of* $z \in \gamma$ if $l = \min\{i > 0 \colon f^i z \in \Theta_{\xi i+N}\}$. Let $l_1, l_2, \ldots$ be defined inductively as follows: l_1 is the first close return time of z; given $l_1, \ldots, l_{k-1}$, let l_k be close return time of $f^{l_1+\cdots+l_{k-1}}z$. Obviously $l_k \geq \xi l_{k-1} + N$ and $l_1 \geq 1$, and so

$$l_k \geq \xi^{k-1}. \tag{20}$$

If $l_1, \ldots, l_k$ are defined in this way, we say *z has k close returns* and denote by Ξ_k the set of $z \in \gamma$ which have k close returns. Let $\Xi_\infty = \bigcap_{k\geq 1} \Xi_k$.

Sublemma 4.6. $\mathrm{HD}(\Xi_\infty) \leq \varepsilon$.

Proof. Let $\mathcal{U}_k$ denote the collection of components of Ξ_k. Then for each $u_k \in \mathcal{U}_k$ there exist a sequence $l_1 < \cdots < l_k$ of positive integers and a nested sequence $u_1 \supset \cdots \supset u_k$ of curves such that for each $i = 1, \ldots, k$, $f^{l_1+\cdots+l_i}u_i$ is a $C^2(b)$-curve stretching across $\Theta_{\xi l_i+N}$. For $u_{k-1} \in \mathcal{U}_{k-1}$ and $l_k > 0$ let

$$\mathcal{R}(u_{k-1}, l_k) = \{u_k \in \mathcal{U}_k \colon l_k \text{ is the close return time of points in } f^{l_1+\cdots+l_{k-1}}u_k\}.$$

By definition,

$$\Xi_k = \bigcup_{u_{k-1}\in\mathcal{U}_{k-1}} \bigcup_{l_k} \bigcup_{u_k\in\mathcal{R}(u_{k-1},l_k)} u_k,$$

where the second union runs over all possible l_k. For each $u_k \in \mathcal{R}(u_{k-1}, l_k)$, let $\widehat{u}_k$ denote the curve in u_{k-1} containing u_k such that $f^{l_1+\cdots+l_k}\widehat{u}_k \in \tilde{\Gamma}^u$. Since $f^{l_1+\cdots+l_k}|\widehat{u}_k$ is a composition of first return maps to Θ, the distortion is uniformly bounded by Proposition 3.1. Hence

$$\frac{\ell(u_k)}{\ell(u_{k-1})} \leq \frac{\ell(u_k)}{\ell(\widehat{u}_k)} \leq \frac{\ell(f^{l_1+\cdots+l_k}u_k)}{\ell(f^{l_1+\cdots+l_k}\widehat{u}_k)} \leq C\sigma_1^{-\xi l_k}.$$

Using $\#\mathcal{R}(u_{k-1}, l_k) \leq 2^{l_k}$, (20) and then (19) we get

$$\sum_{l_k} \sum_{u_k \in \mathcal{R}(u_{k-1}, l_k)} \frac{\ell(u_k)^\varepsilon}{\ell(u_{k-1})^\varepsilon} \leq C^\varepsilon \sum_{l_k \geq \xi^{k-1}} 2^{l_k}\sigma_1^{-\varepsilon\xi l_k} \leq C^\varepsilon \eta^{\xi^{k-1}}.$$

Hence

$$\sum_{u_k \in \mathcal{U}_k} \ell(u_k)^\varepsilon = \sum_{u_{k-1} \in \mathcal{U}_{k-1}} \ell(u_{k-1})^\varepsilon \left(\sum_{l_k} \sum_{u_k \in \mathcal{R}(u_{k-1}, l_k)} \frac{\ell(u_k)^\varepsilon}{\ell(u_{k-1})^\varepsilon} \right) \leq C^\varepsilon \eta^{\xi^{k-1}} \sum_{u_{k-1} \in \mathcal{U}_{k-1}} \ell(u_{k-1})^\varepsilon.$$

Using this recursively for k we get

$$\sum_{u_k \in \mathcal{U}_k} \ell(u_k)^\varepsilon \leq C^{\varepsilon(k-1)} \eta^{\sum_{i=1}^{k-1} \xi^{i-1}} \sum_{u_1 \in \mathcal{U}_1} \ell(u_1)^\varepsilon.$$

The right-hand-side goes to 0 as $k \to \infty$, and thus the Hausdorff ε-measure of Ξ_∞ is 0. $\square$

Returning to the proof of Lemma 4.3, observe that $\Xi_n \setminus \Xi_{n+1}$ is decomposed into a countable collection of preimages of sets of the form $\Omega_\infty^M(\gamma)$, $M \geq N$, $\gamma \in \tilde{\Gamma}^u$ (see the definition before Lemma 3.14). Lemma 3.14 yields $\mathrm{HD}\,((\Xi_n \setminus \Xi_{n+1}) \cap B) \leq \varepsilon$, and also $\mathrm{HD}\,((\gamma \setminus \Xi_1) \cap B) \leq \varepsilon$. These two estimates and the one in Sublemma 4.6 yields the desired one. $\square$

4.2. **Positive measure of the set of points returning to Λ.** In order to complete the proof of Proposition 4.1 we need to recall a few general results on stable and unstable manifolds of nonuniformly hyperbolic systems from [19, 25] which hold for our system since any $\mu \in \mathcal{M}^e(f)$ has one positive and one negative Lyapunov exponent, by [7].

For any $\mu \in \mathcal{M}^e(f)$ there exist Borel subsets $\Gamma_1 \subset \Gamma_2 \subset \cdots \subset K$ such that $\mathrm{supp}(\mu) = \Gamma_\infty := \bigcup \Gamma_n$ and sequences of positive numbers $\delta_n \gg \epsilon_n$, possibly $\to 0$ as $n \to \infty$, such that, for $x \in \Gamma_n$:

(N1) the *unstable manifold* $W^u(x)$ of x (see (3)) is an injectively immersed C^2 submanifold with $T_xW^u(x) = E^u(x)$. An analogous statement holds for the *stable manifold* $W^s(x)$.

Let $B^u_\delta(x)$ (resp. $B^s_\delta(x)$) denote the ball of radius δ centered at the origin of $T_x\mathbb{R}^2$ in $E^u(x)$ (resp. $E^s(x)$) and $B_\delta(x) := B^u_\delta(x) \times B^s_\delta(x)$. Let $\Gamma_n(x) := \{y \in \Gamma_n \colon |x - y| < \epsilon_n\}$ and for $y \in \Gamma_n(x)$, let $W^u_x(y)$ denote the connected component of $\exp_x^{-1}(W^u(y) \cap \exp_x(B_{\delta_n}(x)))$ that contains $\exp_x^{-1} y$.

(N2) For all $y \in \Gamma_n(x)$, $W^u_x(y)$ is the graph of a function $\varphi \colon B^u_{\delta_n}(x) \to B^s_{\delta_n}(x)$ with $\|D\varphi\| \leq \frac{1}{100}$, for a conveniently chosen metric. An analogous statement holds for $W^s_x(y)$.

(N3) For $z \in \bigcup_{y \in \Gamma_n(x)} W^s_x(y)$, let $\mathcal{F}^s(z)$ denote the element of $\{W^s_x(y)\}_{y \in \Gamma_n(x)}$ which contains z. Then $z \mapsto T_z\mathcal{F}^s(z)$ is Lipschitz continuous.

(N4) The holonomy map $\pi \colon \Sigma_1 \cap \bigcup_{y \in \Gamma_n(x)} W^s_x(y) \to \Sigma_2$ defined by $\pi(y) = W^s_x(y) \cap \Sigma_2$ for any graph Σ_i $(i = 1, 2)$ of a C^1 function $\psi_i \colon B^u_{\delta_n}(x) \to B^s_{\delta_n}(x)$ with $\|D\psi_i\| \leq \frac{1}{99}$ is bi-Lipschitz continuous. In particular, it preserves Hausdorff dimension.

Remark 4.1. Since $\dim E^u = 1$ the constant α in the *bunching condition* [11, (19.1.1)] can be taken to be 1. Then (N3) follows from a slight modification of the proof of [11, Theorem 19.1.6]. (N4) follows from (N3) and the fact that $\dim E^s = 1$.

Let $x \in \Gamma_\infty$. For each $n > 0$ consider a countable covering $\{\Gamma_n(z_i)\}_i$ of $\Gamma_n \cap W^u(x)$ such that $\bigcup_i W^u_{\text{loc}}(z_i) = \Gamma_n \cap W^u(x)$, where $W^u_{\text{loc}}(z_i) := \exp_{z_i} W^u_{z_i}(z_i)$. Let $B_i = W^u_{\text{loc}}(z_i) \cap B$.

Lemma 4.7. $\text{HD}(B_i) \leq \varepsilon$.

Proof. By Katok's closing lemma [11, Theorem S.4.13], there exists a periodic saddle $p_i \in \Gamma_n(z_i)$ such that $W^u_{z_i}(p_i)$ is the graph of a function $\varphi\colon B^u_{\delta_n}(z_i) \to B^s_{\delta_n}(z_i)$ with $\|D\varphi\| \leq \frac{1}{100}$. Since $W^s(p_i)$ and W^u have transverse intersections, the Inclination Lemma implies the existence of a connected component of $\exp^{-1}_{z_i}(W^u) \cap B_{\delta_n}(z_i)$ that is the graph of a function $\psi\colon B^u_{\delta_n}(z_i) \to B^s_{\delta_n}(z_i)$ with $\|D\psi\| \leq \frac{1}{99}$. Let π be the holonomy map between $W^u_{\text{loc}}(z_i)$ and $\exp_{z_i}(\text{graph}(\psi))$.

Claim 4.8. *$\pi(x) \in B$ if and only if $x \in B$.*

Proof. If $x \notin B$ then there exist $k \geq 0$ and $\gamma^s \in \Gamma^s$ such that $f^k x \in \gamma^s$. We have $f^k W^s_{\text{loc}}(x) \subset W^s_{\text{loc}}(f^k x)$ and $\gamma^s \subset W^s(f^k x)$. We have $W^s_{\text{loc}}(f^k x) \subset \gamma^s$, for otherwise $W^s_{\text{loc}}(f^k x)$ contains points that escape to infinity. Since both x and $\pi(x)$ belong to $W^s_{\text{loc}}(x)$ then $f^k(\pi(x)) \in \Gamma^s$ so $\pi(x) \notin B$. The same reasoning yields the converse. □

By Claim 4.8, $\pi(B_i) \subset B$ and Lemma 4.5 gives $\text{HD}(\pi(B_i)) \leq \varepsilon$. (N4) yields $\text{HD}(B_i) \leq \varepsilon$. □

To complete the proof of Proposition 4.1, observe that since $\Gamma_n \cap W^u(x) \cap B \subset \bigcup_i B_i$, Lemma 4.7 yields $\text{HD}(\Gamma_n \cap W^u(x) \cap B) \leq \varepsilon$ for every $n > 0$, and thus $\text{HD}(\Gamma_\infty \cap W^u(x) \cap B) \leq \varepsilon$. Let $\{\mu_x\}_{x \in \Gamma_\infty}$ denote the canonical system of conditional measures of μ along unstable manifolds. The dimension formula [12] gives $\dim(\mu_x) = \dim_H(\mu) = \frac{h(\mu)}{\int \log J^u d\mu} > \varepsilon$, and thus $\mu_x(W^u(x) \cap \Gamma_\infty \cap B) < 1$ and $\mu_x((W^u(x))^c \cup \Gamma^c_\infty \cup B^c) > 0$. Since $\mu_x((W^u(x))^c) = 0 = \mu_x(\Gamma^c_\infty)$ we have $\mu(B^c) = \int_{x \in \Gamma_\infty} \mu_x(B^c) d\mu(x) > 0$. The f-invariance of μ yields $\mu(\Lambda) > 0$. □

5. Proofs of the theorems

In this last section we prove the theorems. Prior to Theorem A we prove Theorem B in Sect.5.1. In Sect.5.2 we show that the induced potential $\overline{\varphi_t}\colon X \to \mathbb{R}$ has strongly summabe variations and finite Gurevich pressure. In Sect.5.3 we define two numbers $t_- < 0 < t_+$ and show that $\overline{\varphi_t}$ is positive recurrent for any $t \in (t_-, t_+)$. From Proposition 2.4 it follows that for any $t \in (t_-, t_+)$ there exists a unique measure which minimizes the free energy among measures which are liftable to the inducing scheme. In Sect.5.4 we complete the proof of Theorem A by showing that this candidate measure is indeed a t-conformal measure. In Sect.6 we prove Theorem C and Theorem D.

5.1. **Unstable Hausdorff dimension of K.** In this subsection we prove Theorem B. To this end we need a couple of lemmas.

Lemma 5.1. $t^u \geq \frac{\log 2}{\log 5}$.

Proof. Consider the line through the points $(0, \log 2)$ and $(t^u, 0)$ which are on the pressure curve $\{(t, P(t))\colon t \in \mathbb{R}\}$. The point $(-1, (1/t^u)\log 2 + \log 2)$ lies on this line. Since the pressure curve is concave up, we have $(1/t^u)\log 2 + \log 2 \leq P(-1)$. Since $\|Df\| \leq \log 5$ we have $P(-1) \leq \log 2 + \log 5$, and thus the desired inequality holds. □

For $\mu \in \mathcal{M}(f)$, let

$$\lambda^u(\mu) = \int \log J^u d\mu.$$

A proof of the next lemma is given in Appendix A3.

Lemma 5.2. $\inf\{\lambda^u(\mu)\colon \mu \in \mathcal{M}^e(f)\} \geq \log(2-\varepsilon)$.

Proof of Theorem B. By Lemma 3.3, the tangent directions of the curves in Γ^s vary in a Lipschitz continuous way. Then the holonomy map between two curves in $\tilde{\Gamma}^u$ along γ^s-curves is Lipschitz continuous, and thus the Hausdorff dimension of $\gamma \cap X$ is independent of the choice of $\gamma \in \tilde{\Gamma}^u$. This number is denoted by $d^u(X)$.

Lemma 5.3. $d^u(X) = t^u$.

Proof. Fix $J_0 \in S$. Consider the covering $\mathcal{U}_n$ of $\widehat{\gamma} \cap J_0$ by n-cylinders. Using the bounded distortion of the inducing scheme, for some $C > 0$ we have

$$\sum_{U \in \mathcal{U}_n} \ell(U)^t \leq C^t \sum_{\substack{x \in \widehat{\gamma} \cap J_0 \\ F^n x \in \gamma^s(x)}} \exp\left(-t \sum_{i=0}^{n-1} \log \|DF|E^u(F^i x)\|\right).$$

The expression of the right-hand-side has the growth rate $P_G(\overline{\varphi_t})$ as n increases. Choose $\epsilon > 0$ so that $P(t) < 0$ holds for all $t \in I(\epsilon) := (t^u, t^u + \epsilon)$. By Lemma 5.8, $\overline{\varphi_t}$ has finite Gurevich pressure for all $t \in I(\epsilon)$. It is strongly summable by Proposition 5.7, and hence, there exists a unique F-invariant Gibbs measure $\nu_{\overline{\varphi_t}}$ for $\overline{\varphi_t}$. We also have $\nu_{\overline{\varphi_t}}(\tau) < \infty$. The variational principle and Abramov's and Kac's formulæ [20, Theorem 2.3] yield $P_G(\overline{\varphi_t}) < 0$. Hence the Hausdorff t-measure of $\widehat{\gamma} \cap J_0$ is 0. Since $t \in I(\epsilon)$ is arbitrary, $d^u(X) = \mathrm{HD}(\widehat{\gamma} \cap J_0) = \mathrm{HD}(\widehat{\gamma} \cap X) \leq t^u$.

To show the reverse inequality, pick an ergodic t^u-conformal measure, which was proved to exist in [30, Theorem] and denote it by μ_{t^u}. The dimension formula gives $h(\mu_{t^u}) = \dim_H(\mu_{t^u})\lambda^u(\mu_{t^u})$. Using the equation $F_{\varphi_{t^u}}(\mu_{t^u}) = 0$, $\varepsilon \ll 1$ and Lemma 5.1 we have $\dim_H(\mu_{t^u}) = t^u > 4\varepsilon$. From this and Lemma 5.2 we have $h(\mu_{t^u}) \geq 2\varepsilon$. By Proposition 4.1, μ_{t^u} is liftable. Let $\{\nu_x\}_x$ denote the canonical system of conditional measures of μ_{t^u} along unstable manifolds. Since μ_{t^u} gives full weight to the set $Y := \bigcup_{n\geq 0} f^n X$, $\nu_x(W^u(x) \cap Y) = 1$ holds for μ_{t^u}-a.e. x. (P3) gives $\gamma^u(x) \subset W^u(x)$, and thus $W^u(x) \cap Y = \bigcup_{n\geq 0} f^n(\gamma^u(x) \cap X)$. Since $\dim_H(\nu) = \dim(\nu_x) = t^u$ we have $\mathrm{HD}(\gamma^u(x) \cap X) \geq t^u$, and therefore $d^u(X) \geq t^u$. This completes the proof of Lemma 5.3. □

Take any relatively open curve $\gamma \subset W^u$ intersecting K. We show $\mathrm{HD}(\gamma \cap K) = d^u(X)$. The first statement of Theorem B follows from this and Lemma 5.3.

By Lemma 4.4, there exist $n \geq 0$ and a curve $\omega \subset \gamma$ such that $f^n \omega \in \tilde{\Gamma}^u$. Hence we have $d^u(X) \leq \mathrm{HD}(\omega \cap f^{-n} X) \leq \mathrm{HD}(\gamma \cap K)$. To show the reverse inequality, we use Lemma 4.4 to take a countable collection (γ_n) of curves in γ, and a sequence (a_n) of positive integers so that $f^{a_n}\gamma_n \in \tilde{\Gamma}^u$ holds. The set $(f^{a_n}\gamma_n \cap K) \setminus X$ is decomposed into a countable collection of sets which are sent by some positive iterates to sets of the form $\gamma \cap (\Lambda \setminus X)$, $\gamma \in \tilde{\Gamma}^u$. Lemma 4.3 implies $\mathrm{HD}((f^{a_n}\gamma_n \cap K) \setminus X) \leq \varepsilon$, and therefore $\mathrm{HD}(\gamma_n \cap K) = \mathrm{HD}(f^{a_n}\gamma_n \cap K) \leq \max\{\varepsilon, d^u(X)\} = d^u(X)$. The last inequality follows from Lemma 5.3. This yields $\mathrm{HD}(\gamma \cap K) \leq d^u(X)$.

To complete the proof of Theorem B it is left to show $t^u \to 1$ as $b \to 0$. Let S_n denote the open domain bounded by $\alpha_n^{\pm}$ and the unstable sides of R.

Lemma 5.4. *For any $n > 0$ and ε there exists $b' > 0$ such that if $b < b'$ then:*

(a) *if $i \geq 1$, $z \in \widehat{\gamma} \setminus S_n$ and $fz, \ldots, f^{i-1}z \notin S_n$, then $\|Df^i|E^u(z)\| \leq 2 + \varepsilon$.*
(b) *If $\gamma \subset \widehat{\gamma} \setminus S_n$ is a $C^2(b)$-curve, then $f\gamma$ is $C^2(b)$.*

Proof. Immediate from the form of our map (1). $\square$

Let γ_0 denote the $C^2(b)$-curve in $\widehat{\gamma}$ with endpoints in $\alpha_1^{\pm}$. Obviously $\widehat{\gamma} \cap K \supset \gamma_0 \setminus \bigcup_{i=0}^{\infty} f^{-i} S_n =: \bigcap_{k=0}^{\infty} E_k$, where $E_0 = \gamma_0$ and $E_k = E_{k-1} \setminus f^{-k+1} S_n$ for $k \geq 1$. Observe that $\bigcap_{k=0}^{\infty} E_k$ is a Cantor set in γ_0. For each component of F of E_{k-1}, either $F \setminus E_k$ is contained in the interior of F, or else F is a component of E_k. The next lemma indicates that the latter case rarely occurs.

Lemma 5.5. *Let $F^{(0)} \supset F^{(1)} \supset \cdots$ be a nested sequence of closed curves in γ_0 such that $F^{(i)}$ is a component of E_i $(i = 0, 1, \ldots)$. For every $k \geq 1$,*

$$\{i \in [1, k] \colon F^{(i-1)} = F^{(i)}\} \leq \frac{k}{n} + 1.$$

Proof. It is not hard to see that if $F^{(i-1)} = F^{(i)}$, then $F^{(j-1)} \neq F^{(j)}$ holds for $j = i+1, \ldots, i+n$. $\square$

Lemma 5.6. *For every $k \geq 0$ and any component F of E_k, $\ell(F) \geq (2+\varepsilon)^{-k-1}$.*

Proof. By Lemma 5.4(b), $f^k F$ is a $C^2(b)$-curve. We first treat the case where the endpoints of $f^k F$ are in $f(\alpha_n^- \cup \alpha_n^+)$ and α_p^-, $p \geq 1$. Then, $\ell(f^k F) > 1$ and so Lemma 5.4(a) yields $\ell(F) \geq (2+\varepsilon)^{-k}$.

Next we treat the case where the endpoints of $f^k F$ are in $f(\alpha_n^- \cup \alpha_n^+)$ and α_1^+. Then the endpoints of the $C^2(b)$-curve $f^{k+1}F$ are in $f^2(\alpha_n^- \cup \alpha_n^+) \subset \tilde{\alpha}_{n-1}$ and α_1^+, and so $\ell(f^{k+1}F) > 1$. Hence Lemma 5.4(a) yields $\ell(F) \geq (2+\varepsilon)^{-k-1}$. $\square$

Let μ be the natural mass distribution on $\bigcap_{k=0}^{\infty} E_k$, so that each of the components of E_k carry a mass $\leq 2^{-k(1-\frac{1}{n})}$ by Lemma 5.5. Let U be a small curve in γ_0 and $k > 0$ the large integer such that

$$(2+\varepsilon)^{-k-2} < \ell(U) \leq (2+\varepsilon)^{-k-1}.$$

By Lemma 5.6, U can intersect at most two of the components of E_k, and so

$$\mu(U) \leq 2^{-k(1-\frac{1}{n})+1} = (2+\varepsilon)^{\frac{-(k(1-\frac{1}{n})-1)\log 2}{\log(2+\varepsilon)}} \leq \ell(U)^{\frac{(k(1-\frac{1}{n})-1)\log 2}{(k+2)\log(2+\varepsilon)}} \leq \ell(U)^{\frac{\log 2}{\log(2+2\varepsilon)}},$$

where the last inequality holds for a given $\varepsilon > 0$ provided n, k are sufficiently large. The Mass Distribution Principle [9, p.60] yields $\mathrm{HD}(\bigcap_{k=0}^{\infty} E_k) \geq \frac{\log 2}{\log(2+\varepsilon)}$. The right-hand-side can be made arbitrarily close to 1 by choosing sufficiently small $\varepsilon > 0$ and then choosing sufficiently small b. Since $\widehat{\gamma} \cap K \supset \bigcap_{k=0}^{\infty} E_k$ and $t^u = \mathrm{HD}(\widehat{\gamma} \cap K)$ from the first statement of Theorem B, we obtain $t^u \to 1$ as $b \to 0$. $\square$

5.2. **Strong summability and finite Gurewich pressure.** Observe that for $z \in X$ we have $\sum_{i=0}^{\tau(J)-1} J^u(f^i z) = \|DF|E^u(z)\|$. We now prove that the potential function $\overline{\varphi_t}(z) := -t\log\|DF|E^u(z)\|$ has strongly summable variations (i.e. the potential $t\Phi = \overline{\varphi_t}\circ h$ has strongly summable variations).

Lemma 5.7. *There exists $C>0$ such that for every $n>0$, $V_n(\Phi) \le Cb\sigma_1^{-n}$. In particular, $t\Phi$ has strongly summable variations for any $t \in \mathbb{R}$.*

Proof. Take $\underline{a}, \underline{a}' \in S^{\mathbb{Z}}$ such that $a_i = a_i'$ for every $-n+1 \le i \le n-1$. Let $x_i = F^i(h(\underline{a}))$, $x_i' = F^i(h(\underline{a}'))$. Let y denote the point of intersection between $\gamma^u(x_{-n})$ and $\gamma^s(x_{-n}')$. We have

$$|\Phi(\underline{a}) - \Phi(\underline{a}')| = \left|\sum_{i=0}^{\tau(x_0)-1} \log\frac{J^u(f^i x_0)}{J^u(f^i x_0')}\right| \le \left|\sum_{i=0}^{\tau(x_0)-1} \log\frac{J^u(f^i x_0)}{J^u(f^i(F^n y))}\right| + \left|\sum_{i=0}^{\tau(x_0)-1} \log\frac{J^u(f^i (F^n y))}{J^u(f^i x_0')}\right|.$$

By the F-invariance of the γ^s-curves, $F^i(F^n y) \in \gamma^s(x_i')$ for $0 \le i \le n-1$. Hence x_i and $F^i(F^n y)$ belong to the same basic element for $0 \le i \le n-1$. By the F-invariance of the γ^u-curves, x_i and $F^i(F^n y)$ belong to the same γ^u-curves for $0 \le i \le n-1$. Proposition 3.10(P3) implies $|x_1 - F(F^n y)| \le \sigma_1^{-n}$. Using this and Proposition 3.10(P4)(a) we have

$$\left|\sum_{i=0}^{\tau(x_0)-1} \log\frac{J^u(f^i x_0)}{J^u(f^i(F^n y))}\right| \le C|f^{\tau(x_0)}x_0 - F(F^n y)| \le C\sigma_1^{-n}. \tag{21}$$

To estimate the second summand, for $z \in \Gamma^u$ let $e^u(z)$ denote the unit vector with a positive first component which spans $E^u(z)$. From the bounded distortion in (P4)(b) and the proof of Lemma 3.3 in Appendix A2 we have $\|Df^j(z)e^u(z)\| \ge (1/2)\kappa^j$ for every $j \ge 1$, where $\kappa = 5^{-(1+\xi)N}$. Then the angle estimate in [35, Claim 5.3] yields

$$\angle(Df^i(F^n y)e^u(F^n y), Df^i(x_0')e^u(x_0')) \le (Cb)^{\frac{i+n}{2}}.$$

From the contraction along the γ^s-curves we have

$$\|Df(f^i(F^n y)) - Df(f^i x_0')\| \le C|f^i(F^n y) - f^i x_0'| \le (Cb)^{\frac{i}{2}}|F^n y - x_0'| \le (Cb)^{\frac{i+n}{2}}.$$

Hence

$$\begin{aligned}
\left|\log\frac{J^u(f^i(F^n y))}{J^u(f^i x_0')}\right| &\le Cb^{-1}\left\|\frac{Df^{i+1}(F^n y)e^u(F^n y)}{\|Df^i(F^n y)e^u(F^n y)\|} - \frac{Df^{i+1}(x_0')e^u(x_0')}{\|Df^i(x_0')e^u(x_0')\|}\right\| \\
&\le Cb^{-1}\left(\|Df(f^i(F^n y)) - Df(f^i x_0')\| + C\angle(Df^i(F^n y)e^u(F^n y), Df^i(x_0')e^u(x_0'))\right) \\
&\le (Cb)^{\frac{i+n}{2}-1}.
\end{aligned}$$

The first inequality follows from the fact that $|\log(1+\psi)| \le |\psi|$ for $\psi \ge 0$ and $J^u \ge b/5$. The second one follows from the triangle inequality. Then

$$\sum_{i=0}^{\tau(x_0)-1} \left|\log\frac{J^u(f^i(F^n y))}{J^u(f^i x_0)}\right| \le \sum_{i=0}^{\tau(x_0)-1} (Cb)^{\frac{i+n}{2}-1} \le (Cb)^{\frac{n}{2}-1}. \tag{22}$$

(21) (22) yield the desired inequality. $\square$

We show the finiteness of the Gurewich pressure of the induced potential of a *"shifted"* potential. For $t, c \in \mathbb{R}$ define

$$T_{t,c} = \sum_{J \in S} e^{c\tau(J)} \ell(J)^t$$

and

$$c_0(t) = \begin{cases} t \log \sigma_2 - \overline{\lim\limits_{n\to\infty}}(1/n)\log S(n) & \text{if } t < 0; \\ t \log \sigma_1 - \overline{\lim\limits_{n\to\infty}}(1/n)\log S(n) & \text{if } t \geq 0. \end{cases}$$

By Proposition 3.10(P3)(P4), for some $C > 0$ we have

$$T_{t,c} \leq \begin{cases} C \sum_{n \geq N} S(n) e^{cn} \sigma_2^{-tn} \text{ if } t < 0; \\ C \sum_{n \geq N} e^{cn} \sigma_1^{-tn} \text{ if } t \geq 0. \end{cases} \tag{23}$$

Lemma 5.8. *If $c < c_0(t)$, then $T_{t,c} < \infty$ and $P_G\left(\overline{\varphi_t + c}\right) < \infty$.*

Proof. In the case $t \geq 0$, using the second alternative of (23) we have

$$T_{t,c} \leq C \sum_{n \geq N} \exp\left(n \left(c - t\log\sigma_1 + \frac{1}{n}\log S(n)\right)\right) < \infty.$$

The case $t < 0$ can be handled similarly.

As for the Gurewich pressure, fix $J_0 \in S$. Observe that $\overline{\varphi_t + c} = -t \log \|Df^\tau|E^u\| + c\tau$ and so,

$$\begin{aligned} P_G\left(\overline{\varphi_t + c}\right) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{\substack{x \in \widehat{\gamma} \cap J_0 \\ F^n x \in \gamma^s(x)}} \exp\left(\sum_{i=0}^{n-1} \overline{(\varphi_t + c)}(F^i x)\right) \\ &\leq \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{J \in S} \sup_{x \in J} \exp \overline{(\varphi_t + c)}(x)\right)^n \leq \lim_{n\to\infty} \frac{1}{n} \log(C \cdot T_{t,c})^n = \log T_{t,c} < \infty, \end{aligned}$$

where $C > 0$ is a uniform constant. $\square$

5.3. **Positive recurrence.** We now define

$$t_+ = \frac{t^u \lambda^u(\mu_{t^u})}{\lambda^u(\mu_{t^u}) - \log(2-\varepsilon) + \sqrt{\varepsilon}} \quad \text{and} \quad t_- = \frac{t^u \lambda^u(\mu_{t^u})}{\lambda^u(\mu_{t^u}) - \log(4+\varepsilon) - \sqrt{\varepsilon}}. \tag{24}$$

Lemma 5.9. *$\lambda^u(\mu_{t^u}) \to \log 2$ as $b \to 0$.*

Proof. The topological entropy of f is $\log 2$. The relation $F_{\varphi_t^u}(\mu_{t^u}) = 0$ and the variational principle give $\lambda^u(\mu_{t^u}) \leq \log 2/t^u$. On the other hand, Lemma 5.2 gives $\lambda^u(\mu_{t^u}) \geq \log(2-\varepsilon)$. Since $t^u \to 1$ as $b \to 0$ as in Theorem B and $\varepsilon > 0$ can be made arbitrarily small by choosing small b, we get the claim. $\square$

Lemma 5.9 implies that the definition of $t_\pm$ make sense. It also implies that for any given $\epsilon > 0$ one can choose ε and $b_1 \in (0, b_0)$ so that if $b < b_1$ then $(-1+\epsilon, 1/\epsilon) \subset (t_-, t_+)$.

Proposition 5.10. *If $t \in (t_-, t_+)$, then $\overline{\varphi_t}$ is positive recurrent.*

Proof. Let $\mathcal{M}_L(f)$ denote the set of liftable measures to the inducing scheme constructed in Sect. 3. Let

$$P_t := \sup\{F_{\varphi_t}(\mu)\colon \mu \in \mathcal{M}_L(f)\}.$$

In view of Lemma 5.8 it suffices to show that one can choose $\eta_0 > 0$ so that $T_{t,-(P_t-\eta)}$ is finite for all $0 \le \eta \le \eta_0$. To show this we first estimate P_t from below. In the proof of Theorem B we have shown that μ_{t^u} is liftable. Hence

$$P_t \ge F_{\varphi_t}(\mu_{t^u}) = h(\mu_{t^u}) - t\lambda^u(\mu_{t^u}) = (t^u - t)\lambda^u(\mu_{t^u}). \tag{25}$$

To show the finiteness of $T_{t,-(P_t-\eta)}$ we consider the following three cases.

Case I: $0 < t^u \le t < t_+$. Using (25) and the fact that $\sigma_1 = 2 - \varepsilon$ in (10) we have

$$-P_t - t\log\sigma_1 + \frac{1}{n}\log S(n) \le (t - t^u)\lambda^u(\mu_{t^u}) - t\log(2-\varepsilon) + \frac{1}{n}\log S(n).$$

By the definition of t_+ in (24) and Proposition 3.17, the number of the right-hand-side is strictly negative for all large n. Therefore for sufficiently small $\eta \ge 0$,

$$T_{t,-(P_t-\eta)} \le C\sum_{n>0}\exp\left(n\left(-P_t + \eta - t\log\sigma_1 + \frac{1}{n}\log S(n)\right)\right) < \infty.$$

Case II: $0 \le t < t^u$. Jensen's inequality applied to the convex function $x \to x^t$ yields

$$\sum_{\tau(J)=n}\ell(J)^t \le S(n)^{1-t}\left(\sum_{\tau(J)=n}\ell(J)\right)^t.$$

Using this and the upper bound of $S(n)$ in Proposition 3.17 we have

$$\begin{aligned} e^{-(P_t-\eta)n}\sum_{\tau(J)=n}\ell(J)^t &\le \exp\left(\left(\eta + (t-t^u)\lambda(\mu_{t^u}) - \frac{t}{2}\log\sigma_1\right)n\right) \\ &\le \exp\left(\left(\eta - t^u\lambda(\mu_{t^u}) + \frac{2t}{3}\log 2\right)n\right). \end{aligned}$$

Since $t^u \to 1$ and $\lambda(\mu_{t^u}) \to \log 2$ as $b \to 1$, the exponent is strictly negative for sufficiently small $\eta \ge 0$. Therefore $T_{t,-(P_t-\eta)} < \infty$ holds.

Case III: $t_- < t \le 0$. Using (8) and the fact that $\sigma_2 = 4 + \varepsilon$ in (10) we have

$$-P_t - t\log\sigma_2 + \frac{1}{n}\log S(n) \le (t - t^u)\lambda^u(\mu_{t^u}) - t\log(4+\varepsilon) + \frac{1}{n}\log S(n).$$

By the definition of t_- in (24) and Proposition 3.17, the number of the right-hand-side is strictly negative for all large n. Therefore for sufficiently small $\eta \ge 0$,

$$T_{t,-(P_t-\eta)} \le C\sum_{n>0}\exp\left(n\left(-(P_t-\eta) - t\log\sigma_2 + \frac{1}{n}\log S(n)\right)\right) < \infty.$$

This completes the proof of Proposition 5.10. □

Corollary 5.11. *For any $t \in (t_-, t_+)$ there exists a unique equilibrium measure for φ_t among all liftable measures.*

Proof. Choose $c < c_0(t)$ so that $-c \gg 1$. Then $\varphi_t + c$ has finite Gurevich pressure, and is strongly summable by Proposition 5.7. Observe that $P_L(\varphi_t + c) = P_L(\varphi_t) + c$ and so $\overline{\varphi_t + c - P_L(\varphi_t + c)} = \overline{\varphi_t - P_L(\varphi_t)}$. Since φ_t is positive recurrent by Lemma 5.10, so is $\varphi_t + c$. By Proposition 2.4, there exists a Gibbs measure $\nu_{\overline{\varphi_t+c}}$. For any $J \in S$ and for all $x \in J$,

$$\nu_{\overline{\varphi_t+c}}(J) \leq C \exp\left(-P_G(\overline{\varphi_t + c}) + \overline{\varphi_t + c}(x)\right) \leq Ce^{-P_G(\overline{\varphi_t+c})}e^{c\tau(J)} \max\left(\sigma_1^{-t\tau(J)}, \sigma_2^{-t\tau(J)}\right),$$

and therefore

$$\sum_{\substack{J \in S\\ \tau(J)=n}} \tau(J)\nu_{\overline{\varphi_t+c}}(J) \leq CS(n)e^{-P_G(\overline{\varphi_t+c})}e^{cn} \max\left(\sigma_1^{-tn}, \sigma_2^{-tn}\right). \tag{26}$$

The right-hand-side has a negative growth rate as n increases. Hence $\nu_{\overline{\varphi_t+c}}(\tau) < \infty$ holds. By Proposition 2.4, there exists a unique equilibrium measure for $\varphi_t + c$ among all liftable measures. Since $\varphi_t + c$ is cohomologous to φ_t, they yield the same equilibrium measures. □

5.4. **Uniqueness of t-conformal measures.** We finish the proof of Theorem A. We start with preliminary estimates of $t_\pm$. Define

$$\lambda^u_{\sup} := \sup\{\lambda^u(\mu)\colon \mu \in \mathcal{M}^e(f)\} \quad\text{and}\quad \lambda^u_{\inf} := \inf\{\lambda^u(\mu)\colon \mu \in \mathcal{M}^e(f)\}.$$

Lemma 5.12. *We have*

$$t_+ < \frac{t^u\lambda^u(\mu_{t^u}) - 2\varepsilon}{\lambda^u(\mu_{t^u}) - \lambda^u_{\inf}} \quad \textit{and} \quad t_- > \frac{t^u\lambda^u(\mu_{t^u}) - 2\varepsilon}{\lambda^u(\mu_{t^u}) - \lambda^u_{\sup}}.$$

Proof. A direct computation gives

$$\frac{t^u\lambda^u(\mu_{t^u}) - 2\varepsilon}{\lambda^u(\mu_{t^u}) - \lambda^u_{\inf}} - t_+ = \frac{t^u\lambda^u(\mu_{t^u})(\lambda^u_{\inf} - \log(2-\varepsilon) + \sqrt{\varepsilon}) - 2\varepsilon(\lambda^u(\mu_{t^u}) - \log(2-\varepsilon) + \sqrt{\varepsilon})}{(\lambda^u(\mu_{t^u}) - \lambda^u_{\inf})(\lambda^u(\mu_{t^u}) - \log(2-\varepsilon) + \sqrt{\varepsilon})}.$$

The denominator of the fraction of the right-hand-side is positive. Since $t^u \to 1$ and $\lambda^u(\mu_{t^u}) \to \log 2$ as $b \to 0$, the first term of the numerator is $\geq (1/2)\sqrt{\varepsilon}$. Hence the numerator is positive. Hence the first inequality holds. A proof of the second one is analogous. □

Proof of Theorem A. Let $t \in (t_-, t_+)$. In view of Corollary 5.11 we need to consider measures which do not give positive weight to X. Since

$$\sup\{F_{\varphi_t}(\mu)\colon \mu \in \mathcal{M}(f),\ \mu(X) = 0\} = \sup\{F_{\varphi_t}(\mu)\colon \mu \in \mathcal{M}^e(f),\ \mu(X) = 0\},$$

we may restrict ourselves to ergodic measures. It suffices to show

$$\sup\{F_{\varphi_t}(\mu)\colon \mu \in \mathcal{M}^e(f),\ \mu(X) = 0\} < P_t. \tag{27}$$

We argue by contradiction assuming (27) is false. Then, for any $\delta > 0$ there exists $\mu \in \mathcal{M}^e(f)$ such that $\mu(X) = 0$ and $h(\mu) - t\lambda^u(\mu) \geq P_t - \delta$. (8) yields

$$h(\mu) \geq t\left(\lambda^u(\mu) - \lambda^u(\mu_{t^u})\right) + t^u\lambda^u(\mu_{t^u}) - \delta.$$

For the rest of the proof we deal with two cases separately.

Case I: $t \geq 0$. We have $h(\mu) \geq t\left(\lambda^u_{\inf} - \lambda^u(\mu_{t^u})\right) + t^u\lambda^u(\mu_{t^u}) - \delta$. Since $\delta > 0$ is arbitrary we get

$$h(\mu) \geq t\left(\lambda^u_{\inf} - \lambda^u(\mu_{t^u})\right) + t^u\lambda^u(\mu_{t^u}). \tag{28}$$

(28) and the first inequality in Lemma 5.12 yield $h(\mu) > 2\varepsilon$. Proposition 4.1 gives $\mu(X) > 0$, which is a contradiction.

Case II: $t<0$. We have $h(\mu)\geq t\left(\lambda^u_{\sup}-\lambda^u(\mu_{t^u})\right)+t^u\lambda^u(\mu_{t^u})-\delta$. Since $\delta>0$ is arbitrary we get

$$h(\mu)\geq t\left(\lambda^u_{\sup}-\lambda^u(\mu_{t^u})\right)+t^u\lambda^u(\mu_{t^u}). \tag{29}$$

(29) and the second inequality in Lemma 5.12 yield $h(\mu)>2\varepsilon$. Proposition 4.1 gives $\mu(X)>0$, which is a contradiction. □

6. Proofs of Theorem C and Theorem D.

We prove Theorem C and Theorem D.

Proof of Theorem C. Let $\mu\in\mathcal{M}^e(f)$. If $\mu(X)>0$, then $\mu(\bigcup_{n\geq 0}f^nX)=1$ holds. Arguing similarly to the last paragraph in the proof of Theorem B we obtain $\dim_H(\mu)\leq d(X)=t^u$. If $\mu(X)=0$, then Proposition 4.1 gives $h(\mu)<2\varepsilon$, and so $\dim_H(\mu)<t^u$. Since $\dim_H(\mu_{t^u})=t^u$, μ_{t^u} is a measure of maximal unstable dimension.

As for the uniqueness, let μ be a measure of maximal unstable dimension. Then $\dim_H(\mu)=t^u$, and so $h(\mu)-t^u\lambda^u(\mu)=0$, namely μ is a t^u-conformal measure. The uniqueness in Theorem A yields $\mu=\mu_{t^u}$. □

Proof of Theorem D. Let ν_Φ denote the Gibbs measure on $S^{\mathbb{Z}}$ for the strongly summable potential $\Phi:=\overline{\varphi+c}\circ h$, where c is the one in Corollary 5.11. Let ν_Φ^+ denote the canonical projection of ν_Φ on $S^{\mathbb{N}}$, which is a σ^+-invariant Gibbs measure on $S^{\mathbb{N}}$.

For any cylinder $[a]\subset S^{\mathbb{N}}$ the measurable bijection $\sigma^+|[a]\colon[a]\to S^{\mathbb{N}}$ is non-singular. [26, Theorem 8, Lemma 10] gives $\nu_\Phi^+=h\nu$, where h is a strongly summable function bounded away from zero and infinity, and ν is a conformal measure in the sense that $\nu\ll\nu\circ\sigma^+$ and

$$\frac{d\nu}{d\nu\circ\sigma^+}=e^{\Phi-P_G(\Phi)}.$$

Here, define $\nu\circ\sigma^+$ by $(\nu\circ\sigma^+)(A)=\sum_{a\in S}\nu(\sigma^+([a]\cap A))$. Then $(\nu_\Phi^+\circ\sigma^+)|[a]\ll\nu_\Phi^+|[a]$ and

$$\frac{d(\nu_\Phi^+\circ\sigma^+)|[a]}{d\nu_\Phi^+|[a]}=\frac{d(\nu_\Phi^+\circ\sigma^+)|[a]}{d(\nu\circ\sigma^+)|[a]}\frac{d(\nu\circ\sigma^+)|[a]}{d\nu|[a]}\frac{d\nu|[a]}{d\nu_\Phi^+|[a]}=\frac{h\circ\sigma^+}{e^{\Phi-P_G(\Phi)}h}.$$

Thus there exist $C>0$ and $0<r<1$ such that for all $\underline{b},\underline{b}'\in[a]$,

$$\begin{aligned}\log\frac{d(\nu_\Phi^+\circ\sigma^+)|[a]}{d\nu_\Phi^+|[a]}(\underline{b})-\log\frac{d(\nu_\Phi^+\circ\sigma^+)|[a]}{d\nu_\Phi^+|[a]}(\underline{b}')&=\log\frac{h\circ\sigma^+(\underline{b})}{h\circ\sigma^+(\underline{b}')}-\log\frac{h(\underline{b})}{h(\underline{b}')}-(\Phi(\underline{b})-\Phi(\underline{b}'))\\&\leq Cr^{\min\{i>0\colon b_i\neq b_i'\}}.\end{aligned}$$

By the results in [36], the statistical properties can be deduced from this estimate which shows the Hölder continuity of the Jacobian, and the exponential tail estimate in (26). □

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Appendix: computational proofs

We refer the reader to [30, Sect.2] for relevant definitions and results used in this appendix.

A1. Proof of Proposition 3.1. Let ζ denote the critical point [30, Sect.2.2] on γ, and let $p(z)$ denote the corresponding *bound period* for $z \in \gamma_n$ [30, Sect.2.3]. [30, Proposition 2.6(e)] gives $\|Df^{p(z)}|E^u(z)\| \geq (4-2\varepsilon)^{\frac{p(z)}{2}}$ and $\mathrm{slope}(Df^{p(z)}|E^u(z)) \leq \sqrt{b}$. Since $p(z) < n$, the derivative estimate in [30, Lemma 2.3(a)] gives $\|Df^{n-p(z)}|E^u(f^{p(z)}z)\| \geq \sigma_1^{n-p(z)}$. Hence $\|Df^n|E^u(z)\| \geq (4-2\varepsilon)^{\frac{p(z)}{2}}\sigma_1^{n-p(z)} \geq \sigma_1^n$ and (a) holds.

For $z \in \gamma_n$, Let $e^u(z)$ denote the unit vector with positive first component which spans $E^u(z)$. Consider the *stable foliation* $\mathcal{F}^s$ [30, Sect.2.2], and let $\mathcal{F}^s(fz)$ denote the leaf through fz. Let $e^s(fz)$ denote the unit vector with positive second component which spans $T_{fz}\mathcal{F}^s(fz)$. Split $Df(z)e^u(z) = A(z)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + B(z)e^s(fz)$. [32, Lemma 2.2] gives

$$|A(z)| \approx |\zeta - z| \quad \text{and} \quad |B(z)| \leq C\sqrt{b}. \tag{30}$$

Let $p = \max\{p(z)\colon z \in \gamma_n\}$. Split $\|Df^p(x)e^u(x) - Df^p(y)e^u(y)\| \leq I_1 + I_2 + I_3 + I_4$, where

$$\begin{aligned}
I_1 &= |A(x) - A(y)| \cdot \|Df^{p-1}(fx)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|,\\
I_2 &= |B(x) - B(y)| \cdot \|Df^{p-1}(fx)e^s(fx)\|,\\
I_3 &= |B(y)| \cdot \|Df^{p-1}(fx)e^s(fx) - Df^{p-1}(fy)e^s(fy)\|,\\
I_4 &= |A(y)| \cdot \|Df^{p-1}(fx)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) - Df^{p-1}(fy)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|.
\end{aligned}$$

Estimates of I_1, I_2. Let $e^s(z) = \left(\begin{smallmatrix}e_1(z)\\e_2(z)\end{smallmatrix}\right)$, and let $\pi_1(z)$ denote the first component of z. Write

$$S(z) = \begin{pmatrix}1 & e_1(z)\\ 0 & e_2(z)\end{pmatrix}^{-1} = \begin{pmatrix}1+\epsilon_1 & \epsilon_2\\ \epsilon_3 & 1+\epsilon_4\end{pmatrix} \quad \text{and} \quad Df(z) = \begin{pmatrix}-2a^*\pi_1(z)+\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{pmatrix}.$$

Let $R(z)$ denote the rotation matrix by $\theta(z) := \angle(e^u(z), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))$. Then $A(z)$, $B(z)$ are equal to the $(1,1)$, $(2,1)$ entries of the matrix $S(z)\cdot Df(z)\cdot R(z)^{-1}$ correspondingly. A direct computation shows that A, B are linear combinations of α_i, ϵ_i $(1 \leq i \leq 4)$, $\cos\theta$, $\sin\theta$, all of which are Lipschitz continuous on γ_n, from (1), property (F3) of $\mathcal{F}^s$ (see [30, Sect.2.2]) and the $C^2(b)$-property of γ_n. Hence A, B are Lipschitz continuous on γ_n as well, which implies

$$I_1 \leq C|x-y|\cdot\|w_p(\zeta)\| \quad \text{and} \quad I_2 \leq (Cb)^{p-1}|x-y|. \tag{31}$$

Estimate of I_3. We start with an elementary geometric reasoning. Let v_1, v_2 be nonzero vectors in $\mathbb{R}^2$ such that $\|v_1\| \leq \|v_2\|$, $\theta \ll 1$ (See FIGURE 3). We have

$$\begin{aligned}
\|v_2 - v_1\| &< |\|v_2\| - \|v_1\|\cos\theta| + \|v_1\|\sin\theta\\
&= \cos\theta|\|v_2\| - \|v_1\|| + (1-\cos\theta)\|v_2\| + \|v_1\|\sin\theta\\
&\leq |\|v_2\| - \|v_1\|| + 2\theta\|v_2\|.
\end{aligned}$$

We use this to estimate I_3. Without loss of generality we may assume $\|Df^{p-1}(y)e^s(y)\| \geq \|Df^{p-1}(x)e^s(x)\|$. The angle between the two vectors involved in I_3 is small. The fact that $|B(y)| \leq C$ and the above reasoning show

$$I_3 \leq C\|Df^{p-1}(y)e^s(y)\|\left(\left|\frac{\|Df^{p-1}(x)e^s(x)\|}{\|Df^{p-1}(y)e^s(y)\|} - 1\right| + 3\|e^s(f^px) - e^s(f^py)\|\right). \tag{32}$$

We have $\|Df^{p-1}(y)e^s(y)\| \leq Cb$, and the first term in the parenthesis is $\leq C|f^px - f^py|$.

To estimate the second term in the parenthesis of (32) we argue as follows. The invariance of the stable foliation $\mathcal{F}^s$ gives

$$\log \frac{\|Df^{p-1}(x)e^s(x)\|}{\|Df^{p-1}(y)e^s(y)\|} \le \sum_{i=1}^{p-1} \log \frac{\|Df(f^ix)e^s(f^ix)\|}{\|Df(f^iy)e^s(f^iy)\|}. \tag{33}$$

Let $e^{s\perp}(z)$ denote any unit vector orthogonal to $e^s(z)$ and let $\theta(z) = \angle(Df(z)e^s(z), Df(z)e^{s\perp}(z))$. Then $e^{s\perp}$ and θ are Lipschitz continuous, $\|Dfe^{s\perp}\| > 2$ and $\theta \approx \pi/2$. Hence $\log\|Dfe^{s\perp}\|$ and $\sin\theta$ are Lipschitz continuous, with a Lipschitz constant independent of b. Since[1]

$$\|Df(f^ix)e^s(f^ix)\| \cdot \|Df(f^ix)e^{s\perp}(f^ix)\| \sin\theta(f^ix) = |\det Df(f^ix)| = b,$$

for $1 \le i < p$ we have

$$\log \frac{\|Df(f^ix)e^s(f^ix)\|}{\|Df(f^iy)fe^s(f^iy)\|} = \log \frac{\|Df(f^iy)(e^s)^\perp(f^iy)\|}{\|Df(f^ix)(e^s)^\perp(f^ix)\|} + \log\frac{\sin\theta(f^iy)}{\sin\theta(f^ix)} \le C|f^ix - f^iy|.$$

Lemma 6.1. $\sum_{i=1}^{p-1} |f^ix - f^iy| \le C|f^px - f^py|$.

Proof. Let $\pi_1, \pi_2 \colon \mathbb{R}^2 \to \mathbb{R}$ denote the projections to the first and the second coordinate. For $1 \le i < p$ we have $|\pi_1(f^ix) - \pi_1(f^iy)| \le \sigma_1^{-(i-p)}|\pi_1(f^px) - \pi_1(f^py)|$, and

$$\begin{aligned} |\pi_2(f^ix) - \pi_2(f^iy)| &\le (Cb)^{\frac{i-1}{2}}|\pi_2(fx) - \pi_2(fy)| \\ &\le (Cb)^{\frac{i}{2}}|\pi_1(x) - \pi_1(y)| \le (Cb)^{\frac{i}{2}}|\pi_1(f^px) - \pi_1(f^py)|, \end{aligned}$$

where the second inequality follows from integrating $B(z)$ in (30) along the path in γ_n from x to y. Summing these two inequalities over all $1 \le i < p$ yields the desired one. □

Lemma 6.1 implies that the right-hand-side of (33) is bounded by a constant $C > 0$ independent of b. Since there exists $\rho = \rho(C) > 0$ such that $e^\psi \le 1 + \rho\psi$ for $0 \le \psi \le C$, we have

$$\frac{\|Df^{p-1}(x)e^s(x)\|}{\|Df^{p-1}(y)e^s(y)\|} - 1 \le \rho\sum_{i=1}^{p-1} \log \frac{\|Df(f^ix)e^s(f^ix)\|}{\|Df(f^iy)e^s(f^iy)\|} \le \rho C\sum_{i=1}^{p-1}|f^ix - f^iy| \le C|f^px - f^py|. \tag{34}$$

Plugging (34) into (32) we obtain

$$I_3 \le C|f^px - f^py|. \tag{35}$$

Estimate of I_4. In the same way as in the proof of (32) we have

$$I_4 \le |A(y)| \cdot \|w_p(\zeta)\| \left(\left| \frac{\|Df^{p-1}(fx)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|}{\|Df^{p-1}(fy)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|} - 1 \right| + 2\angle(Df^{p-1}(x)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), Df^{p-1}(y)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)) \right).$$

From the distortion estimate in the proof of [30, Lemma 2.7] and Lemma 6.1, the first term in the parenthesis is $\le C|f^px - f^py|$. To estimate the second term in the parenthesis, take a point r so that the leaf $\mathcal{F}^s(fy)$ intersects the horizontal through fx at fr. By the angle estimate in [35, Claim 5.3],

$$\angle(Df^{p-1}(y)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), Df^{p-1}(r)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)) \le (Cb)^{p-1}|fy - fr| \le (Cb)^{p-1}|x - y| \le (Cb)^{p-1}|f^px - f^py|.$$

[1]Here we use the fact that the Jacobian of the Hénon map is constant equal to b. Essentially the same argument remains to hold for Hénon-like maps for which there exists $C > 0$ independent of b such that $\|D\log|\det Df|\| \le C$ (c.f. [18]). Therefore our main theorems hold for Hénon-like maps satisfying this assumption.

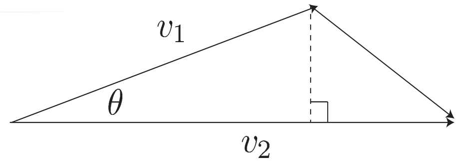

FIGURE 3. $\|v_1\| \le \|v_2\|$, $\theta \ll 1$

By the $C^2(b)$-property and the definition of r,

$$\angle(Df^{p-1}(x)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), Df^{p-1}(r)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)) \le \sqrt{b}|f^p x - f^p r| \le C\sqrt{b}|f^p x - f^p y|.$$

Hence we obtain

$$\angle(Df^{p-1}(x)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), Df^{p-1}(y)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)) \le C\sqrt{b}|f^p x - f^p y|.$$

Additionally (30) yields

$$|A(y)| \le C|\zeta - y| \le C(d(\zeta, \gamma_n) + \ell(\gamma_n))$$

where $d(\cdot,\cdot)$ denotes the minimal distance apart. Finally, from Claim 6.2 below we get

$$I_4 \le Cd(\zeta, \gamma_n)\|w_p(\zeta)\| \cdot |f^p x - f^p y|. \tag{36}$$

Claim 6.2. $\ell(\gamma_n) \le Cd(\zeta, \gamma_n)$.

Proof. Let M be a large integer such that $M \ll N$. Consider the leaf of the stable foliation $\mathcal{F}^s$ through $f\zeta$ which is of the form $\mathcal{F}^s(f\zeta) = \{(x(y), y)\colon |y| \le \sqrt{b}\}$. For $k > M$ define

$$U_k := \left\{(x, y)\colon D_k \le |x - x(y)| < D_{k-M}(\zeta), |y| \le \sqrt{b}\right\}$$

where $D_k := C\left[\sum_{i=1}^k \frac{\|\omega_i\|^2}{\|\omega_{i+1}\|}\right]^{-1}$ for some constant $C > 0$ and $w_i := w_i(\zeta)$. Let $k_0 := \max\{k > M\colon U_k \cap f\gamma_n \ne \emptyset\} - 1$. By [30, Lemma 2.5(a)], there exist constants $0 < C_1 < C_2 < 1/2$ such that

$$C_1 D_{k_0-M} \le D_{k_0} \le C_2 D_{k_0-M}. \tag{37}$$

We prove

$$f\gamma_n \subset U_{k_0} \cup U_{k_0+1}. \tag{38}$$

(37) (38) imply $\ell(\gamma_n) \le C\sqrt{D_{k_0-M}} \le C\sqrt{D_{k_0}} \le Cd(\gamma_n, \zeta)$, and thus Claim 6.2 holds.

It is left to prove (38). If the inclusion were false, then one could choose a curve $\delta \subset f\gamma_n \cap U_{k_0}$ with endpoints in the two vertical boundaries of U_{k_0}. Let x denote the endpoint of $f^2\gamma_n$ in $\tilde{\alpha}_{n-1}$. The bounded distortion and the second inequality in [30, Lemma 2.5(b)] give

$$d(\alpha_0^-, f^{k_0-M}x) \le 2D_{k_0}\|w_{k_0-M+1}\| \le 2 \cdot 3^{-M} D_{k_0}\|w_{k_0}\| \le 3^{-M},$$

and

$$\ell(f^{k_0-M}\delta) \ge C(D_{k_0-M} - D_{k_0})\|w_{k_0-M}\| \ge C(1 - C_2)D_{k_0-M}\|w_{k_0-M}\| \ge C.$$

From these two estimates and choosing large M if necessary we have that the interior of $f^{k-10}\delta$ intersects some $\tilde{\alpha}_i$. This yields a contradiction. $\square$

Overall estimates. Gluing (31) (35) (36) together,

$$\|Df^p(x)e^u(x) - Df^p(y)e^u(y)\| \leq C\|w_p(\zeta)\| \cdot |x-y| + Cd(\zeta,\gamma_n)\|w_p(\zeta)\| \cdot |f^px - f^py|.$$

Dividing both sides by $\|Df^p(z)e^u(z)\| \approx |\zeta - z| \cdot \|w_p(\zeta)\| \approx d(\zeta,\gamma_n)\cdot\|w_p(\zeta)\|$ which follows from the proof of [30, Proposition 2.6], we obtain for all $x, y \in \gamma_n$,

$$\log\frac{\|Df^p|E^u(x)\|}{\|Df^p|E^u(y)\|} \leq \frac{C|x-y|}{d(\zeta,\gamma_n)} + C|f^px - f^py|. \tag{39}$$

Since $f^p\gamma_n$ is $C^2(b)$ and $p < n$, we have $|f^px - f^py| \leq |f^nx - f^ny|$. (39) and Claim 6.3 below yield

$$\log\frac{\|Df^p|E^u(x)\|}{\|Df^p|E^u(y)\|} + \log\frac{\|Df^{n-p}|E^u(f^px)\|}{\|Df^{n-p}|E^u(f^py)\|} \leq C|f^nx - f^ny|,$$

which proves (b).

Claim 6.3. $\frac{|x-y|}{d(\zeta,\gamma_n)} \leq C|f^nx - f^ny|$.

Proof. The first estimate of (30) implies

$$\frac{|x-y|}{d(\zeta,\gamma_n)} \leq \frac{C|\pi_1(fx) - \pi_1(fy)|}{d(\zeta,\gamma_n)^2}. \tag{40}$$

By the bounded distortion outside of Θ, there exists $\theta \in f\gamma_n$ such that

$$|\pi_1(fx) - \pi_1(fy)| \cdot \|Df^{n-1}(\theta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\| \leq C|f^nx - f^ny|. \tag{41}$$

The bounded distortion outside of Θ and the quadratic behavior near ζ as in (30) imply

$$\ell(\gamma_n)d(\zeta,\gamma_n)\|Df^{n-1}(\theta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\| \geq C\ell(f^n\gamma_n).$$

Hence there exists $C > 0$ such that

$$d(\zeta,\gamma_n)^2\|Df^{n-1}(\theta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\| \geq C\ell(\gamma_n)d(\zeta,\gamma_n)\|Df^{n-1}(\theta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\| \geq C\ell(f^n\gamma_n) > C. \tag{42}$$

For the first inequality we have used Claim 6.2. The last inequality is because $f^n\gamma_n$ is a $C^2(b)$-curve with endpoints in $\alpha_1^{\pm}$. (40) (41) (42) yield

$$\frac{|x-y|}{d(\zeta,\gamma_n)} \leq \frac{C|\pi_1(fx) - \pi_1(fy)|}{d(\zeta,\gamma_n)^2} \leq \frac{C|f^nx - f^ny|}{d(\zeta,\gamma_n)^2\|Df^{n-1}(\theta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|} \leq C|f^nx - f^ny|. \qquad \square$$

A2. Proof of Lemma 3.3. Let $\kappa = 5^{-(1+\xi)N}$. For all $z \in \Omega_\infty$ we show $\|Df^n|E^u(z)\| \geq \kappa^n$ for every $n \geq 1$. Then (a) (b) follow from the results of [18].

With the terminology in [30, Sect.2.5] we introduce the bound/free structure on the orbit of z, using Θ_N as a critical neighborhood. If f^nz is free, then the orbit $z, \ldots, f^nz$ is decomposed into alternative bound and free segments. Applying the expansion estimates in [30, Lemma 2.3, Proposition 2.8(e)] alternatively we have $\|Df^n|E^u(z)\| \geq \kappa^n$. If f^nz is bound, then there exists an integer $0 < m < n$ such that $f^mz \in \Theta_N$ and $m < n < m+p$, where p is the bound period of f^mz. Since $f^{m+p}z$ is free and $\|Df\| < 5$ we have $\|Df^n|E^u(z)\| \geq 5^{-(m+p-n)}\|Df^{m+p}|E^u(z)\| > 5^{-p}$, and since $z \in \Omega_\infty$ we have $p \leq \xi m + N$. If $m \leq N$, then $p \leq (1+\xi)N$ and so $\|Df^n|E^u(z)\| \geq 5^{-(1+\xi)N} \geq \kappa^n$. If $m > N$, then $p \leq (1+\xi)m$ and so $\|Df^n|E^u(z)\| \geq 5^{-(1+\xi)m} \geq \kappa^n$.

Choose a large integer $M \gg n$ such that f^Mz_2 is free. Take $x_1 \in f^n\gamma^s(z_1)$, $x_2 \in \gamma^s(z_2)$ which are connected by a horizontal segment of length $b^{\frac{M}{3}}$. By construction, $\|Df^M|E^u(x_2)\| \geq \sigma_1^M$. By the bounded distortion, the f^M-iterate of the segment is $C^2(b)$ and $|f^Mx_1 - f^Mx_2| \geq$

$C\sigma_1^M|x_1-x_2| \geq C\sigma_1^M b^{\frac{M}{3}}$. If $q \in f^n\gamma^s(z_1)\cap\gamma^s(z_2)$, then $|f^Mx_1-f^Mx_2| \leq |f^Mx_1-f^Mq| + |f^Mq-f^Mx_2| \leq 2(Cb)^{\frac{M}{2}}$. These two estimates are incompatible. Hence (c) holds. □

A3. Proof of Lemma 5.2. Let $\mu \in \mathcal{M}^e(f)$. Take a point $\xi \in R$ such that $\lim_{n\to\infty}(1/n)\log\|Df^n_\xi|E^u\| = \lambda^u(\mu)$ and ξ is free. The orbit $\xi, f\xi, \dots$ is decomposed into alternative bound and free segments. Applying the expansion estimates in [30, Lemma 2.3, Proposition 2.8] alternatively we have $\|Df^n|E^u(\xi)\| \geq (2-\varepsilon)^n$ if $f^n\xi$ is free. This implies $\lambda^u(\mu) \geq \log(2-\varepsilon)$. □

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