Kyoto University

Kyoto-Math 2011-13

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by

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October 2011

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DECOMPOSITIONS OF POLYHEDRAL PRODUCTS

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Abstract. In [BBCG], Bahri, Bendersky, Cohen and Gitler posed a conjecture on decompositions of certain polyhedral products, supported by their own result on suspensions of polyhedral products and a result of Grbić and Theriault [GT]. We resolve this conjecture affirmatively.

1. Introduction

In this paper, we resolve a conjecture of Bahri, Bendersky, Cohen and Gitler [BBCG] on decompositions of polyhedral products which generalizes a result of Grbić and Theriault [GT]. Throughout the paper, we work in the category of compactly generated weak Hausdorff spaces with nondegenerate base points and base point preserving maps.

Let us first recall the definition of polyhedral products. Let K be a simplicial complex on the index set $[n] = \{1, \ldots, n\}$, and let $(\underline{X}, \underline{A})$ be a collection of pairs of spaces $\{(X_i, A_i)\}_{i=1}^n$. The *polyhedral product* or the *generalized moment-angle complex* of $(\underline{X}, \underline{A})$ with respect to K, denoted by $\mathcal{Z}_K(\underline{X}, \underline{A})$, is defined as the union of all $\prod_{i\in\sigma} X_i \times \prod_{i\notin\sigma} A_i$ for $\sigma \in K$. Polyhedral products first appeared in a work of Porter [P] which defines higher order Whitehead products. Since then, polyhedral products have been studied in connection with topology, algebra and combinatorics, and so there are diverse results [B, BBCG, BP, DJ, DO, DS, FT, GT, N]. Among these results, we are particularly interested in decompositions of polyhedral products. Bahri, Bendersky, Cohen and Gitler [BBCG] gave decompositions of polyhedral products after a suspension. Let us illustrate a special example of their result. We set some notation. Let $|K|$ denote the geometric realization of a simplicial complex K. A subcomplex L of K is called induced whenever all vertices of a simplex σ of K belong to L, σ is a simplex of L. We denote the induced subcomplex of K on the vertex set $I \subset V(K)$ by K_I, where $V(K)$ is the vertex set of K. For a collection of spaces $\{X_i\}_{i=1}^n$ and $\emptyset \neq J \subset [n]$, let $\widehat{X}^J$ denote the smash product $\bigwedge_{j\in J} X_j$. For spaces X, Y, let $X * Y$ be the join of X and Y.

Theorem 1.1 (Bahri, Bendersky, Cohen and Gitler [BBCG])**.** *Let K be a simplicial complex on the index set $[n]$ and let $(\underline{X}, \underline{A})$ be a collection of NDR pairs $\{(X_i, A_i)\}_{i=1}^n$. If each X_i is contractible, there is a homotopy equivalence*

$$\Sigma\mathcal{Z}_K(\underline{X}, \underline{A}) \simeq \Sigma \bigvee_{\emptyset\neq I\subset V(K)} |K_I| * \widehat{A}^I.$$

There is also a result due to Grbić and Theriault [GT] on decompositions of certain polyhedral products without a suspension. To state their result, we define special simplicial complexes. A simplicial complex K is called *shifted* if there is an order on $V(K)$ whenever $v \in \sigma \in K$ and $v < w \in V(K)$, $(\sigma - v) \cup w$ belongs to K.

Theorem 1.2 (Grbić and Theriault [GT]). *Let K be a shifted complex on the index set $[n]$ and let $(\underline{D}^2, \underline{S}^1)$ be a collection of n-copies of (D^2, S^1). Then $\mathcal{Z}_K(\underline{D}^2, \underline{S}^1)$ has the homotopy type of a wedge of spheres.*

Supported by Theorem 1.1 and 1.2, Bahri, Bendersky, Cohen and Gitler [BBCG] posed the following conjecture. For a collection of spaces $\underline{X} = \{X_i\}_{i=1}^n$, put $(C\underline{X}, \underline{X}) = \{(CX_i, X_i)\}_{i=1}^n$.

Conjecture 1.3 (Bahri, Bendersky, Cohen and Gitler [BBCG]). Let K be a shifted complex on the index set $[n]$ and let $\underline{X}$ be a collection of spaces $\{X_i\}_{i=1}^n$ such that each X_i is path-connected. Then

$$\mathcal{Z}_K(C\underline{X}, \underline{X}) \simeq \bigvee_{\emptyset \neq I \subset [n]} |K_I| * \widehat{X}^I$$

The proof of Theorem 1.2 in [GT] heavily relies on the fact that S^1 has a classifying space, and then we must make a different approach to Conjecture 1.3. Our approach is simple and straightforward compared to that of Grbić and Theriault [GT], and we resolve Conjecture 1.3 together with a certain naturality with respect to K (Theorem 5.2). As its corollary, we obtain:

Theorem 1.4. *Let K be a shifted complex on the index set $[n]$ and let $\underline{X}$ be a collection of spaces $\{X_i\}_{i=1}^n$. Then there is a homotopy equivalence*

$$\mathcal{Z}_K(C\underline{X}, \underline{X}) \simeq \bigvee_{\emptyset \neq I \subset [n]} |K_I| * \widehat{X}^I.$$

Corollary 1.5. *Let K and $\underline{X}$ be as in Theorem 1.4. If ΣX_i has the homotopy type of a wedge of spheres for each i, so does $\mathcal{Z}_K(C\underline{X}, \underline{X})$.*

Proof. By pinching out the star of the maximum vertex, any shifted complex becomes a wedge of spheres. See Lemma 3.5 below. Then since each induced subcomplex of a shifted complex is also shifted, the proof is completed by Theorem 1.4. □

2. Lemmas on pushouts

The purpose of this section is to collect some lemmas on pushouts of topological spaces. Let us first recall two basic properties of pushouts of topological spaces, where we omit the proofs. We will use these two properties implicitly in what follows.

Lemma 2.1. *Suppose there is a commutative diagram*

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "g"] \arrow[dd, "f_1"'] & & B \arrow[dr] \arrow[dd, "f_2"] & \\
& C \arrow[rr] \arrow[dd, "f_3"] & & D \arrow[dd, "f_4"] \\
E \arrow[rr] \arrow[dr, "h"] & & F \arrow[dr] & \\
& G \arrow[rr] & & H
\end{tikzcd}$$

in which the top and the bottom faces are pushouts and g, h are cofibrations. If f_1, f_2, f_3 are homotopy equivalences, so is f_4.

Lemma 2.2. *If a commutative diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

is a pushout, so is

$$\begin{tikzcd}
A \times E \arrow[r] \arrow[d] & B \times E \arrow[d] \\
C \times E \arrow[r] & D \times E.
\end{tikzcd}$$

We consider a special case that pushouts preserve cofiber sequences.

Lemma 2.3. *Suppose there is a commutative diagram*

$$\begin{tikzcd}
A_1 \arrow[rr] \arrow[dr, "f"] \arrow[dd] & & B_1 \arrow[rr] \arrow[dr, "g"] \arrow[dd] & & C_1 \arrow[dr] \arrow[dd] & \\
& A_2 \arrow[rr] \arrow[dd] & & B_2 \arrow[rr] \arrow[dd] & & C_2 \arrow[dd] \\
A_3 \arrow[rr] \arrow[dr] & & B_3 \arrow[rr] \arrow[dr] & & C_3 \arrow[dr] & \\
& A_4 \arrow[rr] & & B_4 \arrow[rr] & & C_4
\end{tikzcd}$$

in which all three side faces are pushouts and f, g are cofibrations. If $A_i \to B_i \to C_i$ is a cofiber sequence for $i = 1, 2, 3$, so is $A_4 \to B_4 \to C_4$.

Proof. Since colimits commute with colimits, $C_4 = B_4/A_4$ and the map $B_4 \to C_4$ is the projection. Then we show $A_4 \to B_4$ is a cofibration. By a result of Lillig [L], the map $Q \to B_2$ is a cofibration, where Q is a pushout of $A_2 \leftarrow A_1 \to B_1$. Then it follows from [H, Lemma 7.2.15] that $A_4 \to B_4$ is also a cofibration. $\square$

For spaces X, Y, we put $X \ltimes Y = X \times Y / X \times *$.

Lemma 2.4. *Define Q as a pushout*

$$\begin{tikzcd} A \times (B \vee C) \arrow[r, "i \times 1"] \arrow[d, "1 \times (1 \vee *)"] & CA \times (B \vee C) \arrow[d] \\ A \times (B \vee D) \arrow[r] & Q, \end{tikzcd}$$

where $i : A \to CA$ is the inclusion. Then there is a homotopy equivalence $Q \xrightarrow{\simeq} B \vee (A \ltimes D) \vee (\Sigma A \wedge C)$ which is natural with respect to A, B, C, D.

Proof. Apply Lemma 2.3 to a commutative diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d] & A \arrow[r, "i"] \arrow[d] & CA \arrow[d] \\ A \times (B \vee D) & A \times (B \vee C) \arrow[l, "1 \times (1 \vee *)"'] \arrow[r, "i \times 1"] & CA \times (B \vee C), \end{tikzcd}$$

where the vertical arrows are the inclusions into the first factors. Then we get a cofiber sequence $CA \to Q \xrightarrow{q} \overline{Q}$, where $\overline{Q}$ is defined as a pushout

$$\begin{tikzcd} A \ltimes (B \vee C) \arrow[r, "i \ltimes 1"] \arrow[d, "1 \ltimes (1 \vee *)"] & CA \ltimes (B \vee C) \arrow[d] \\ A \ltimes (B \vee D) \arrow[r] & \overline{Q}. \end{tikzcd}$$

Obviously, $\overline{Q} = (CA \ltimes B) \vee (A \ltimes D) \vee (\Sigma A \wedge C)$ and then the composite of $q : Q \to \overline{Q}$ and the projection $\overline{Q} \to B \vee (A \ltimes D) \vee (\Sigma A \wedge C)$ is the desired homotopy equivalence. $\square$

3. Topology of shifted complexes

In this section, we study the topology of shifted complexes. Let K be a shifted complex. If $V(K)$ is a subset of $[n]$, we assume that K is shifted by the order of $[n]$. A shifted subcomplex of K means a subcomplex of K which is shifted by the order of $V(K)$. We give two important examples of shifted subcomplexes. For a simplicial complex K and its vertex v, let $\mathrm{rest}_K(v)$, $\mathrm{star}_K(v)$ and $\mathrm{link}_K(v)$ be the induced subcomplex $K_{V(K)-v}$, the star and the link of v in K, respectively.

Example 3.1. Let K be a shifted complex. Any induced subcomplex of K is a shifted subcomplex of K. In particular, $\mathrm{rest}_K(v)$ is a shifted subcomplex of K.

Example 3.2. Let K be a shifted complex. If v is the maximum vertex, $\mathrm{star}_K(v)$ is a shifted subcomplex of K. Then $\mathrm{link}_K(v)$ is also a shifted subcomplex of K since it is an induced subcomplex of $\mathrm{star}_K(v)$.

Let us first consider the connected components of shifted complexes.

Proposition 3.3. *Let K be a shifted complex on the index set $[n_K, n] = \{n_K, n_K + 1, \dots, n\}$ and let K_0 be the connected component of K containing the maximum vertex n. Then $V(K_0) = V(\mathrm{star}_K(n)) = [n_{K_0}, n]$ for some $n_{K_0} \in [n_K, n]$ and $K - K_0$ is discrete.*

Proof. Choose any vertex v of K_0. Then v is adjacent to some vertex w of K_0. If $v < w$, v is also adjacent to n, equivalently, $v \in V(\mathrm{star}_K(n))$. If $v > w$, w is adjacent to n, implying $v \in V(\mathrm{star}_K(n))$. Then we obtain $V(K_0) \subset V(\mathrm{star}_K(n))$. The converse implication is obvious and thus the first assertion is proved. We next prove the second assertion. If there are adjacent vertices v, w of $K - K_0$ with $v < w$, one sees that $v \in V(\mathrm{star}_K(n))$ as above, a contradiction. Then there is no edge in $K - K_0$, that is, $K - K_0$ is discrete. $\square$

We next consider the quotient of a shifted complex by the star of the maximum vertex. We set some notation. Let K be a shifted complex with the maximum vertex v. Let $\mathfrak{m}(K)$ be the set of all maximal simplices of K which do not contain v. We put

$$\begin{aligned}
\mathfrak{m}_0(K) &= \mathfrak{m}(\mathrm{rest}_K(v)) \cap \mathfrak{m}(\mathrm{link}_K(v)),\\
\mathfrak{m}_1(K) &= \mathfrak{m}(\mathrm{rest}_K(v)) - \mathfrak{m}_0(K),\\
\mathfrak{m}_2(K) &= \mathfrak{m}(\mathrm{link}_K(v)) - \mathfrak{m}_0(K).
\end{aligned}$$

Proposition 3.4. *For a connected shifted complex K, the dimension of any simplex in $\mathfrak{m}_1(K)$ is positive.*

Proof. By Proposition 3.3, we have $V(\mathrm{link}_K(v)) = V(\mathrm{rest}_K(v))$, implying if $\sigma \in \mathfrak{m}(\mathrm{rest}_K(v))$ and $\dim \sigma = 0$, σ belongs to $\mathfrak{m}(\mathrm{link}_K(v))$, where v is the maximum vertex of K. $\square$

Let K be a shifted complex with the maximum vertex v. Put

$$\overline{K} = |K|/|\mathrm{star}_K(v)|.$$

For a shifted subcomplex L of K satisfying $v \in L$, let $\iota_{K,L} : \overline{L} \to \overline{K}$ be the induced map from the inclusion of L into K.

Proposition 3.5. *Let K be a shifted complex. Then*

$$\overline{K} = \bigvee_{\sigma \in \mathfrak{m}(K)} S^{\dim \sigma}$$

and $\iota_{K,L}$ is a wedge of the identity map of $S^{\dim \sigma}$ for $\sigma \in \mathfrak{m}(L) \cap \mathfrak{m}(K)$ and the constant map on other spheres, where L is a shifted subcomplex of K having the common maximum vertex.

Proof. Any simplex σ of K satisfies $\partial\sigma \in \mathrm{star}_K(v)$, where v is the maximum vertex of K. Then the first assertion follows. The second assertion is obvious. $\square$

Proposition 3.6. *Let K be a shifted complex with the maximum vertex v. There is an identification*

$$\overline{K} = (\bigvee_{\sigma\in\mathfrak{m}_1(K)} S^{\dim\sigma}) \vee (\bigvee_{\tau\in\mathfrak{m}_2(K)} S^{\dim\tau+1}).$$

Through this identification, $\iota_{K,L}$ is a wedge of the restriction of $\iota_{\mathrm{rest}_K(v),\mathrm{rest}_L(v)}$ and $\Sigma\iota_{\mathrm{link}_K(v),\mathrm{link}_L(v)}$, where L is a shifted subcomplex of K with $v \in L$.

Proof. Let w be the second greatest vertex of K. Notice that $\mathfrak{m}_1(K) = \{\sigma \in \mathfrak{m}(K) \mid w \notin \sigma\}$ and $\mathfrak{m}_2(K) = \{\tau \in K \mid w \notin \tau \text{ and } \tau \cup w \in \mathfrak{m}(K)\}$. Then the first assertion follows. The second assertion is clear. □

4. The space $\mathcal{W}_K^n$

The purpose of this section is to introduce the space $\mathcal{W}_K^n$ for a certain shifted complex K and to show that it is built up inductively by two kinds of pushouts. Hereafter, we fix a collection of spaces $\underline{X} = \{X_i\}_{i=1}^n$. We also fix K, L, M to be a shifted complex on the index set $[n_K, n]$, a shifted subcomplex of K on $[n_L, n]$ and a shifted subcomplex of L on $[n_M, n]$, respectively.

4.1. **Definition of $\mathcal{W}_K^n$.** For a shifted complex P on the index set $V(P) \subset [m]$, we define

$$\mathcal{W}_P^m = (\bigvee_{\emptyset\ne I\subset V(P)} \Sigma\overline{P_I} \wedge \widehat{X}^I) \times \prod_{j\in[m]-V(P)} X_j.$$

Then if $n_K = 1$, $\mathcal{W}_K^n$ has the homotopy type of the right hand side of the homotopy equivalence in Theorem 1.4 since $A * B \simeq \Sigma A \wedge B$ and $|K_I| \simeq \overline{K_I}$. We also define $\mathcal{W}_\emptyset^m = \prod_{j=1}^m X_j$.

We next define a map $\lambda_{K,L} : \mathcal{W}_L^n \to \mathcal{W}_K^n$. Let δ_i be the composite of maps

$$\Sigma A \rtimes X_i \xrightarrow{\nabla} (\Sigma A \rtimes X_i) \vee (\Sigma A \rtimes X_i) \to \Sigma A \vee (\Sigma A \wedge X_i),$$

where ∇ is the suspension comultiplication and the second arrow is a wedge of the projections. Then δ_i is a homotopy equivalence. We now define $\lambda_{K,L}$ as follows. We first apply the projection $\mathcal{W}_L^n \to ((\bigvee_{\emptyset\ne I\subset[n_L,n]} \Sigma\overline{L_I} \wedge \widehat{X}^I) \rtimes \prod_{j=n_K}^{n_L-1} X_j) \times \prod_{j<n_K} X_j$ and then $\delta_{n_L-1}, \delta_{n_L-2}, \dots, \delta_{n_K}$ in turn to get a map

$$\mathcal{W}_L^n \to (\bigvee_{\substack{\emptyset\ne I\subset[n_L,n]\\ J\subset[n_K,n_L-1]}} \Sigma\overline{L_I} \wedge \widehat{X}^{I\cup J}) \times \prod_{j<n_K} X_j.$$

We next apply the product of a wedge of $\Sigma\iota_{K_{I\cup J},L_I} \wedge 1 : \Sigma\overline{L_I} \wedge \widehat{X}^{I\cup J} \to \Sigma\overline{K_{I\cup J}} \wedge \widehat{X}^{I\cup J}$ and the identity map of $\prod_{j<n_K} X_j$ to the target of the above map. Then we obtain a map $\lambda_{K,L} : \mathcal{W}_L^n \to \mathcal{W}_K^n$. We also define $\lambda_{K,\emptyset} : \mathcal{W}_\emptyset^n \to \mathcal{W}_K^n$ as the composite

$$\mathcal{W}_\emptyset^n \xrightarrow{\mathrm{proj}} \prod_{j<n_K} X_j \xrightarrow{\mathrm{incl}} \mathcal{W}_K^n.$$

By definition, we have

$$\lambda_{K,L} \circ \lambda_{L,M} = \lambda_{K,M},$$

where M may be $\emptyset$. We will often abbreviate $\lambda_{K,L}$ by λ when K, L are clear in the context.

4.2. **Pushouts in $\mathcal{W}^n_K$.** Let us consider two kinds of pushouts by which $\mathcal{W}^n_K$ is constructed inductively. Define $\mathcal{W}^{n,c}_K$ as a pushout

$$\begin{CD} X_n \times \mathcal{W}^{n-1}_{\mathrm{link}_K(n)} @>{i\times 1}>> CX_n \times \mathcal{W}^{n-1}_{\mathrm{link}_K(n)} \\ @V{1\times\lambda}VV @VVV \\ X_n \times \mathcal{W}^{n-1}_{\mathrm{rest}_K(n)} @>>> \mathcal{W}^{n,c}_K \end{CD}$$

and a map $\lambda^c_{K,L} : \mathcal{W}^{n,c}_L \to \mathcal{W}^{n,c}_K$ as the induced map from a commutative diagram

$$\begin{CD} X_n \times \mathcal{W}^{n-1}_{\mathrm{rest}_L(n)} @<{1\times\lambda}<< X_n \times \mathcal{W}^{n-1}_{\mathrm{link}_L(n)} @>{i\times 1}>> CX_n \times \mathcal{W}^{n-1}_{\mathrm{link}_L(n)} \\ @V{1\times\lambda}VV @V{1\times\lambda}VV @V{1\times\lambda}VV \\ X_n \times \mathcal{W}^{n-1}_{\mathrm{rest}_K(n)} @<{1\times\lambda}<< X_n \times \mathcal{W}^{n-1}_{\mathrm{link}_K(n)} @>{i\times 1}>> CX_n \times \mathcal{W}^{n-1}_{\mathrm{link}_K(n)}. \end{CD}$$

Proposition 4.1. *Suppose K is connected. Then there is a homotopy equivalence $h^c_K : \mathcal{W}^{n,c}_K \xrightarrow{\simeq} \mathcal{W}^n_K$ satisfying*

$$h^c_K \circ \lambda^c_{K,L} = \lambda_{K,L} \circ h^c_L$$

if L is also connected.

Proof. Put

$$W_K(i) = \bigvee_{\emptyset\neq I\subset[n_K,n-1]} \Big(\bigvee_{\sigma\in\mathfrak{m}_i(K_{I\cup n})} \Sigma S^{\dim\sigma} \wedge \widehat{X}^I \Big)$$

for $i = 0, 1, 2$. Since $\mathrm{rest}_{K_{I\cup n}}(n) = \mathrm{rest}_K(n)_I$ and $\mathrm{link}_{K_{I\cup n}}(n) = \mathrm{link}_K(n)_I$ for $\emptyset \neq I \subset [n_K, n-1]$, it follows from Proposition 3.5 that $\mathcal{W}^{n-1}_{\mathrm{rest}_K(n)} = (W_K(0) \vee W_K(1)) \times \prod_{j<n_K} X_j$, $\mathcal{W}^{n-1}_{\mathrm{link}_K(n)} = (W_K(0) \vee W_K(2)) \times \prod_{j<n_K} X_j$ and $\lambda_{\mathrm{rest}_K(n),\mathrm{link}_K(n)}$ is the product of the identity map of $\prod_{j<n_K} X_j$ and a wedge of the identity map of $W_K(0)$ and the constant map. Put

$$W^c_K = (W_K(0) \vee (W_K(1) \rtimes X_n) \vee (\Sigma W_K(2) \wedge X_n)) \times \prod_{j<n_K} X_j$$

and $r^c_{K,L} : W^c_L \to W^c_K$ to be the induced map from $\lambda_{\mathrm{rest}_K(n),\mathrm{rest}_L(n)}$ and $\lambda_{\mathrm{link}_K(n),\mathrm{link}_K(n)}$. Then it follows from Proposition 3.5 that we can apply Lemma 2.4 to $\mathcal{W}^{n,c}_K$ and obtain a homotopy equivalence

$$g^c_K : \mathcal{W}^{n,c}_K \xrightarrow{\simeq} W^c_K$$

satisfying $g^c_K \circ \lambda^c_{K,L} = r^c_{K,L} \circ g^c_L$.

For $\emptyset \neq I \subset [n_K, n-1]$, $\mathrm{rest}_{K_{I\cup n}}(n) = K_I$. Then by Proposition 3.5, $W_K(0) \vee W_K(1) = \bigvee_{\emptyset\neq I\subset[n_K,n-1]} \Sigma\overline{K_I}\wedge\widehat{X}^I$. On the other hand, $(W_K(1)\wedge X_n)\vee(\Sigma W_K(2)\wedge X_n) = \bigvee_{\emptyset\neq I\subset[n_K,n-1]} \Sigma\overline{K_{I\cup n}}\wedge \widehat{X}^{I\cup n}$ by Proposition 3.6. Then it follows that

$$\mathcal{W}^n_K = (W_K(0) \vee W_K(1) \vee (W_K(1) \wedge X_n) \vee (\Sigma W_K(2) \wedge X_n)) \times \prod_{j<n_K} X_j.$$

Moreover, through this identification, one sees that $\lambda_{K,L}$ is induced from $\lambda_{\mathrm{rest}_K(n),\mathrm{rest}_L(n)}$ and $\lambda_{\mathrm{link}_K(n),\mathrm{link}_L(n)}$ by Proposition 3.5 and 3.6. Since K is connected, so is $K_{I\cup n}$ also by Proposition

3.3. Then by Proposition 3.4, $\dim\sigma > 0$ for $\sigma \in \mathfrak{m}_1(K_{I\cup n})$ and hence we can define a map $\delta_n : X_n \ltimes W_K(1) \xrightarrow{\simeq} W_K(1) \vee (X_n \wedge W_K(1))$ by using the suspension parameter of $S^{\dim\sigma}$ for $\sigma \in \mathfrak{m}_1(K_{I\cup n})$. We now define a homotopy equivalence h_K^c as the composite

$$\begin{aligned}\mathcal{W}_K^{n,c} &\xrightarrow{g_K^c} W_K^c\\ &= (W_K(0) \vee (W_K(1) \rtimes X_n) \vee (\Sigma W_K(2) \wedge X_n)) \times \prod_{j<n_K} X_j\\ &\xrightarrow{(1\vee\delta_n\vee 1)\times 1} (W_K(0) \vee W_K(1) \vee (W_K(1) \wedge X_n) \vee (\Sigma W_K(2) \wedge X_n)) \times \prod_{j<n_K} X_j\\ &= \mathcal{W}_K^n.\end{aligned}$$

Since the suspension parameters used for $r_{K,L}^c$ and the above δ_n are distinct, it holds that $((1 \vee \delta_n \vee 1) \times 1) \circ r_{K,L}^c = \lambda_{K,L} \circ ((1 \vee \delta_n \vee 1) \times 1)$. Then one has

$$\begin{aligned}h_K^c \circ \lambda_{K,L}^c &= ((1 \vee \delta_n \vee 1) \times 1) \circ g_K^c \circ \lambda_{K,L}^c\\ &= ((1 \vee \delta_n \vee 1) \times 1) \circ r_{K,L}^c \circ g_L^c\\ &= \lambda_{K,L} \circ ((1 \vee \delta_n \vee 1) \times 1) \circ g_L^c\\ &= \lambda_{K,L} \circ h_L^c,\end{aligned}$$

completing the proof. □

Define $\mathcal{W}_K^{n,d}$ as a pushout

$$\begin{tikzcd} \mathcal{W}_\emptyset^n \arrow[r, "\mathrm{incl}"] \arrow[d, "\lambda"] & CX_{n_K-1} \times \prod_{j\ne n_K-1} X_j \arrow[d] \\ \mathcal{W}_K^n \arrow[r] & \mathcal{W}_K^{n,d}. \end{tikzcd}$$

When $n_L = n_K$, we also define a map $\lambda_{K,L}^d : \mathcal{W}_L^{n,d} \to \mathcal{W}_K^{n,d}$ as the induced map from a commutative diagram

$$\begin{tikzcd} \mathcal{W}_L^n \arrow[d, "\lambda"] & \mathcal{W}_\emptyset^n \arrow[l, "\lambda"'] \arrow[r, "\mathrm{incl}"] \arrow[d, equal] & CX_{n_L-1} \times \prod_{j\ne n_L-1} X_j \arrow[d, equal] \\ \mathcal{W}_K^n & \mathcal{W}_\emptyset^n \arrow[l, "\lambda"'] \arrow[r, "\mathrm{incl}"] & CX_{n_K-1} \times \prod_{j\ne n_K-1} X_j. \end{tikzcd}$$

Proposition 4.2. *There is a homotopy equivalence $h_K^d : \mathcal{W}_K^{n,d} \xrightarrow{\simeq} \mathcal{W}_{K\cup(n_K-1)}^n$ such that when $n_L = n_K$,*

$$h_K^d \circ \lambda_{K,L}^d = \lambda_{K\cup(n_K-1),L\cup(n_L-1)} \circ h_L^d.$$

Proof. Let $\widehat{\delta}_i$ be the composite

$$\Sigma(A \times X_i) \xrightarrow{(1\vee\nabla)\circ\nabla} \bigvee^3 \Sigma(A \times X_i) \to \Sigma A \vee \Sigma X_i \vee (\Sigma A \wedge X_i),$$

where the second arrow is a wedge of the projections. Then $\widehat{\delta}_i$ is a homotopy equivalence. By applying $\widehat{\delta}_m, \widehat{\delta}_{m-1}, \dots, \widehat{\delta}_{i+2}$ in turn to $\Sigma X_i \wedge \prod_{j=i+1}^m X_j$, we obtain a homotopy equivalence

$$\delta^m(i) : \Sigma X_i \wedge \prod_{j=i+1}^m X_j \to \bigvee_{\emptyset \neq I \subset [i+1,m]} \Sigma \widehat{X}^{I \cup i} = \bigvee_{\emptyset \neq I \subset [i+1,m]} \Sigma\{I_0, i\} \wedge \widehat{X}^{I \cup i},$$

where I_0 is the maximum of I.

Put

$$W_K^d = ((\bigvee_{\emptyset \neq I \subset [n_K,n]} \Sigma \overline{K_I} \wedge \widehat{X}^I) \rtimes X_{n_K-1} \vee (\Sigma X_{n_K-1} \wedge \prod_{j \geq n_K} X_j)) \times \prod_{j<n_K-1} X_j$$

and $r_{K,L}^d : W_L^d \to W_K^d$ to be the induced map from $\lambda_{K,L}$, where $n_L = n_K$. Then by Lemma 2.4 and Proposition 3.5, there is a homotopy equivalence

$$g_K^d : \mathcal{W}_K^{n,d} \xrightarrow{\simeq} W_K^d$$

satisfying $g_K^d \circ \lambda_{K,L}^d = r_{K,L}^d \circ g_L^d$. Define h_K^d as the composite of g_K^d and a homotopy equivalence

$$\begin{aligned}
W_K^d &= ((\bigvee_{\emptyset \neq I \subset [n_K,n]} \Sigma \overline{K_I} \wedge \widehat{X}^I) \rtimes X_{n_K-1} \vee (\Sigma X_{n_K-1} \wedge \prod_{j \geq n_K} X_j)) \times \prod_{j<n_K-1} X_j \\
&\xrightarrow{(\delta_{n_K-1} \vee \delta^n(n_K-1)) \times 1} (\bigvee_{\emptyset \neq I \subset [n_K,n]} (\Sigma \overline{K_I} \wedge \widehat{X}^I) \vee (\Sigma(\overline{K_I} \vee \{I_0, n_K-1\}) \wedge \widehat{X}^{I \cup (n_K-1)})) \times \prod_{j<n_K-1} X_j \\
&= (\bigvee_{\emptyset \neq I \subset [n_K,n]} (\Sigma \overline{K_I} \wedge \widehat{X}^I) \vee (\Sigma \overline{K_{I \cup (n_K-1)}} \wedge \widehat{X}^{I \cup (n_K-1)}) \times \prod_{j<n_K-1} X_j \\
&= \mathcal{W}_{K \cup (n_K-1)}^n,
\end{aligned}$$

where the base point of $\{I_0, n_K-1\}$ is I_0. Then since $g_K^d \circ \lambda_{K,L}^d = r_{K,L}^d \circ g_L^d$ and $((\delta_{n_K-1} \vee \delta^n(n_K-1)) \times 1) \circ r_{K,L}^d = \lambda_{K \cup (n_K-1), L \cup (n_L-1)} \circ ((\delta_{n_L-1} \vee \delta^n(n_L-1)) \times 1)$, one can see quite similarly to h_K^c that

$$h_K^d \circ \lambda_{K,L}^d = \lambda_{K \cup (n_K-1), L \cup (n_L-1)} \circ h_L^d.$$

Thus the proof is completed. □

Let us show further properties of the homotopy equivalence h_K^d.

Proposition 4.3. *Define a map $\mu_{K,L} : \mathcal{W}_L^n \to \mathcal{W}_K^{n,d}$ as the induced map from the commutative diagram*

$$\begin{tikzcd}
\mathcal{W}_L^n \arrow[d, "\lambda"] & \mathcal{W}_\emptyset^n \arrow[l, "\lambda"'] \arrow[r, equal] \arrow[d, equal] & \mathcal{W}_\emptyset^n \arrow[d, "\mathrm{incl}"] \\
\mathcal{W}_K^n & \mathcal{W}_\emptyset^n \arrow[l, "\lambda"'] \arrow[r, "\mathrm{incl}"] & CX_{n_K-1} \times \prod_{j \neq n_K-1} X_j.
\end{tikzcd}$$

Then $h_K^d \circ \mu_{K,L} = \lambda_{K \cup (n_K-1), L}$.

Proof. By the definition of h_K^d, the statement is true for $\mu_{K,K}$. Since $\mu_{K,K} \circ \lambda_{K,L} = \mu_{K,L}$, the proof is completed. □

We put

$$S^m(i,k) = (\bigvee_{\substack{\emptyset \neq I \subset [i+1,m] \\ J \subset [k,i-1]}} \Sigma\{I_0, i\} \wedge \widehat{X}^{I \cup J \cup i}) \times \prod_{j<k} X_j.$$

Let us define a map $\delta^m(i,k) : CX_i \times \prod_{j \in [m]-i} X_j \to S^m(i,k)$ for $1 \leq k \leq i < m$. We first apply the projection $CX_i \times \prod_{j \notin [m]-i} X_j \to (\Sigma X_i \wedge \prod_{j=i+1}^m X_j) \times \prod_{j<i} X_j$ and then a homotopy equivalence $\delta^m(i)$ to get a map

$$CX_i \times \prod_{j \in [m]-i} X_j \to S^m(i,i).$$

We next apply the projection $S^m(i,i) \to ((\bigvee_{\emptyset \neq I \subset [i+1,m]} \Sigma\{I_0, i\} \wedge \widehat{X}^{I \cup i}) \rtimes \prod_{j=k}^{i-1} X_j) \times \prod_{j<k} X_j$ and then $\delta_{i-1}, \delta_{i-2}, \dots, \delta_k$ in turn to obtain $\delta^m(i,k)$.

Let $\iota_K(i) : S^n(i, n_K) \to \mathcal{W}_K^n$ be the product of the identity map of $\prod_{j<n_K} X_j$ and a wedge of $\Sigma\iota_{K_{I \cup J \cup i}, \{I_0, i\}} \wedge 1 : \Sigma\{I_0, i\} \wedge \widehat{X}^{I \cup J \cup i} \to \Sigma\overline{K_{I \cup J \cup i}} \wedge \widehat{X}^{I \cup J \cup i}$ for $\emptyset \neq I \subset [i+1, n]$, $J \subset [n_K, i-1]$. We now define

$$\theta_K(i) = \iota_K(i) \circ \delta^n(i, n_K)$$

for $n_K \leq i < n$ and $\theta_K(n)$ as the composite

$$CX_i \times \prod_{j \neq i} X_j \xrightarrow{\text{proj}} \prod_{j<n_K} X_j \xrightarrow{\text{incl}} \mathcal{W}_K^n.$$

Let us make a list of properties of the map $\theta_K(i)$. Let $\nabla_i : CX_i \to CX_i$ be the composite

$$CX_i \to CX_i \vee \Sigma X_i \xrightarrow{\text{proj}} CX_i,$$

where the first arrow is the co-action map.

Proposition 4.4. *The map $\theta_K(i)$ has the following properties.*

(1) *$\theta_K(i)$ restricts to $\lambda_{K,\emptyset}$ for $n_K \leq i \leq n$.*
(2) *$\theta_K(i) = \lambda_{K,L} \circ \theta_L(i)$ for $n_L \leq i \leq n$.*
(3) *If K is connected, the composite*

$$X_n \times CX_i \times \prod_{j \neq i,n} X_j \xrightarrow{1 \times \nabla_i \times 1} X_n \times CX_i \times \prod_{j \neq i,n} X_j \xrightarrow{1 \times \theta_{\mathrm{rest}_K(n)}(i)} X_n \times \mathcal{W}_{\mathrm{rest}_K(n)}^{n-1} \xrightarrow{\overline{\lambda}} \mathcal{W}_K^n$$

is equal to $\theta_K(i)$ for $n_K \leq i < n$, where $\overline{\lambda}$ is the composite of the canonical map $X_n \times \mathcal{W}_{\mathrm{rest}_K(n)}^{n-1} \to \mathcal{W}_K^{n,c}$ and h_K^c.
(4) *The composite*

$$CX_{n_K-1} \times \prod_{j \neq n_K-1} X_j \to \mathcal{W}_K^{n,d} \xrightarrow{h_K^d} \mathcal{W}_{K \cup (n_K-1)}^n$$

is equal to $\theta_{K \cup (n_K-1)}(n_K - 1)$, where the first arrow is the canonical map.

Proof. 1, 2 and 4 follow immediately from the definitions of $\theta_K(i)$ and the homotopy equivalence h_K^d. We prove 3. By the definition of $\bar{\lambda}$, it is true for $\theta_K(n)$ if $n_K = n-1$. Let $n_K \le i < n-1$. Notice that by Proposition 3.3, $\iota_K(i)$ maps $S^n(i, n_K)$ into $(\bigvee_I \Sigma\overline{K_I} \wedge \widehat{X}^I) \times \prod_{j<n_K} X_j \subset \mathcal{W}_K^n$, where I ranges over all nonempty subsets of $[n_K, n]$ such that K_I is discrete. It follows from Proposition 3.4 that there is a map $S^{n-1}(i, n_K) \to W_K(0) \times \prod_{j<n_k} X_j$ satisfying a commutative diagram

$$\begin{tikzcd}
S^n(i,n_K) \arrow[r, "\mathrm{proj}"] \arrow[d, "\iota_K(i)"] & S^{n-1}(i,n_K) \arrow[r, equal] \arrow[d] & S^{n-1}(i,n_K) \arrow[d, "\iota_{\mathrm{rest}_K(n)}(i)"] \\
\mathcal{W}_K^n & W_K(0) \times \prod_{j<n_k} X_j \arrow[l, "\mathrm{incl}"'] \arrow[r, "\mathrm{incl}"] & \mathcal{W}^{n-1}_{\mathrm{rest}_K(n)}.
\end{tikzcd}$$

Then since the restriction of $\bar{\lambda}$ to $X_n \times (W_K(0) \times \prod_{j<n_K} X_j) \subset X_n \times \mathcal{W}^{n-1}_{\mathrm{rest}_K(n)}$ is the composite $X_n \times W_K(0) \times \prod_{j<n_k} X_j \xrightarrow{\mathrm{proj}} W_K(0) \times \prod_{j<n_k} X_j \xrightarrow{\mathrm{incl}} \mathcal{W}_K^n$ as in the proof of Proposition 4.1, we get a commutative diagram

$$\begin{tikzcd}
X_n \times S^{n-1}(i,n_K) \arrow[rr, "1\times\iota_{\mathrm{rest}_K(n)}(i)"] \arrow[d, "\mathrm{proj}"] & & X_n \times \mathcal{W}^{n-1}_{\mathrm{rest}_K(n)} \arrow[d, "\bar{\lambda}"] \\
S^{n-1}(i,n_K) \arrow[r, "\mathrm{incl}"] & S^n(i,n_K) \arrow[r, "\iota_K(i)"] & \mathcal{W}_K^n.
\end{tikzcd}$$

On the other hand, there is a commutative diagram

$$\begin{tikzcd}
X_n \times CX_i \times \prod_{j\ne i,n} X_j \arrow[rr, "1\times\delta^{n-1}(i,n_K)"] & & X_n \times S^{n-1}(i,n_K) \arrow[d, "\mathrm{proj}"] \\
X_n \times CX_i \times \prod_{j\ne i,n} X_j \arrow[u, "1\times\nabla_i\times 1"] \arrow[r, "\delta^n(i,n_K)"] & S^n(i,n_K) \arrow[r, "\mathrm{proj}"] & S^{n-1}(i,n_K).
\end{tikzcd}$$

Then since $\iota_K(i)$ is equal to the composite $S^n(i,n_K) \xrightarrow{\mathrm{proj}} S^{n-1}(i,n_K) \xrightarrow{\mathrm{incl}} S^n(i,n_K) \xrightarrow{\iota_K(i)} \mathcal{W}_K^n$ as above, we obtain the desired result. $\square$

Proposition 4.5. *Suppose $n_K < n_L$ and define a map $\nu_{K,L} : \mathcal{W}_L^{n,d} \to \mathcal{W}_K^n$ as the induced map from a commutative diagram*

$$\begin{tikzcd}
\mathcal{W}_L^n \arrow[d, "\lambda"] & \mathcal{W}_\emptyset^n \arrow[l, "\lambda"'] \arrow[r, "\mathrm{incl}"] \arrow[d, "\lambda"] & CX_{n_L-1} \times \prod_{j\ne n_L-1} X_j \arrow[d, "\theta_K(n_L-1)"] \\
\mathcal{W}_K^n \arrow[r, equal] & \mathcal{W}_K^n \arrow[r, equal] & \mathcal{W}_K^n.
\end{tikzcd}$$

Then $\nu_{K,L} = \lambda_{K,L\cup(n_L-1)} \circ h_L^d$.

Proof. By the definition of h_L^d, $\nu_{L,L} = h_L^d$. Since $\theta_K(n_L-1) = \lambda_{K,L\cup(n_L-1)} \circ \theta_L(n_L-1)$ by Proposition 4.4, it holds that $\nu_{K,L} = \lambda_{K,L\cup(n_L-1)} \circ \nu_{L,L}$, completing the proof. $\square$

5. Decomposition of $\mathcal{Z}_K^n$

In this section, we introduce the space $\mathcal{Z}_K^n$ and prove its decomposition. As an immediate corollary, we obtain Theorem 1.4.

5.1. **The space $\mathcal{Z}_K^n$.** For a simplicial complex P on the index set $V(P) \subset [m]$, we define

$$\mathcal{Z}_P^m = \bigcup_{\sigma \in P} (\prod_{j \in \sigma} CX_j \times \prod_{j \notin [m]-\sigma} X_j).$$

Then if $n_K = 1$, $\mathcal{Z}_K^n = \mathcal{Z}_K(C\underline{X}, \underline{X})$. When Q is a subcomplex of P, we denote the inclusion $\mathcal{Z}_Q^m \to \mathcal{Z}_P^m$ by $\rho_{P,Q}$. We also put $\mathcal{Z}_\emptyset^m = \prod_{j=1}^m X_j$ and $\rho_{P,\emptyset}$ to be the inclusion $\mathcal{Z}_\emptyset^m \to \mathcal{Z}_P^m$. Notice that $\rho_{P,Q}$ is a cofibration, where Q may be $\emptyset$. Likewise $\lambda_{K,L}$, we will often abbreviate $\rho_{K,L}$ by ρ when K, L are clear.

Proposition 5.1. *There are pushouts*

$$\begin{array}{ccc} X_n \times \mathcal{Z}^{n-1}_{\mathrm{link}_K(n)} & \xrightarrow{i \times 1} & CX_n \times \mathcal{Z}^{n-1}_{\mathrm{link}_K(n)} \\ \downarrow{\scriptstyle 1\times\rho} & & \downarrow \\ X_n \times \mathcal{Z}^{n-1}_{\mathrm{rest}_K(n)} & \longrightarrow & \mathcal{Z}_K^n \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathcal{Z}_\emptyset^n & \xrightarrow{\mathrm{incl}} & CX_{n_K-1} \times \prod_{j \neq n_K-1} X_j \\ \downarrow{\scriptstyle \rho} & & \downarrow \\ \mathcal{Z}_K^n & \longrightarrow & \mathcal{Z}^n_{K \cup (n_K-1)}, \end{array}$$

where $i : X_n \to CX_n$ is the inclusion.

Proof. There is a pushout of simplicial complexes

$$\begin{array}{ccc} \mathrm{link}_K(n) & \longrightarrow & \mathrm{star}_K(n) \\ \downarrow & & \downarrow \\ \mathrm{rest}_K(n) & \longrightarrow & K \end{array}$$

resulting a pushout of topological spaces

$$\begin{array}{ccc} \mathcal{Z}^n_{\mathrm{link}_K(n)} & \xrightarrow{\rho} & \mathcal{Z}^n_{\mathrm{star}_K(n)} \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\ \mathcal{Z}^n_{\mathrm{rest}_K(n)} & \xrightarrow{\rho} & \mathcal{Z}^n_K. \end{array}$$

We have $\mathcal{Z}_P^n = X_n \times \mathcal{Z}_P^{n-1}$ for $P = \mathrm{link}_K(n), \mathrm{rest}_K(n)$ and $\mathcal{Z}^n_{\mathrm{star}_K(n)} = CX_n \times \mathcal{Z}^{n-1}_{\mathrm{link}_K(n)}$. Through this identification, we obtain the first pushout. The second pushout is obtained quite similarly. □

5.2. **Main theorem.** We now state the main result.

Theorem 5.2. *There is a homotopy equivalence $\epsilon_K : \mathcal{Z}_K^n \xrightarrow{\simeq} \mathcal{W}_K^n$ such that $\epsilon_K \circ \rho_{K,L} = \lambda_{K,L} \circ \epsilon_L$, where L may be $\emptyset$.*

Proof. First of all, we put $\epsilon_\emptyset : \mathcal{Z}^n_\emptyset \to \mathcal{W}^n_\emptyset$ to be the identity map of $\prod_{j=1}^n X_j$.

By induction on n, we construct ϵ_K satisfying $\epsilon_K \circ \rho_{K,L} = \lambda_{K,L} \circ \epsilon_L$ and $\epsilon_K \circ \rho_{K,i} = \theta_K(i)$. We will abbreviate ϵ_K by ϵ when K is clear in the context. For $n = 1$, K must be 1. We put $\epsilon_K : \mathcal{Z}^1_K = CX_1 \to * = \mathcal{W}^1_K$ to be the constant map and then it follows that $\epsilon_K \circ \rho_{K,\emptyset} = \lambda_{K,\emptyset} \circ \epsilon_\emptyset$ and $\epsilon_K \circ \rho_{K,1} = \theta_K(1)$.

Assume that the theorem holds less than n. Let K_0, L_0 be the connected components of K, L containing the vertex n, respectively. Then by Proposition 3.3, $V(K_0) = [n_{K_0}, n]$ and $V(L_0) = [n_{L_0}, n]$ for some $n_{K_0} \in [n_K, n]$ and $n_{L_0} \in [n_L, n]$. Put $K(m) = K_0, K_{[m,n]}, K$ according as $n_{K_0} \le m \le n, n_K \le m < n_{K_0}, 1 \le m < n_K$. We construct $\epsilon_{K(m)}$ by induction on m. By the hypothesis of induction on n, there is a commutative diagram

$$\begin{tikzcd}
X_n \times \mathcal{Z}^{n-1}_{\mathrm{rest}_{K_0}(n)} \arrow[d, "1\times\epsilon"] & X_n \times \mathcal{Z}^{n-1}_{\mathrm{link}_{K_0}(n)} \arrow[l, "1\times\rho"'] \arrow[r, "i\times 1"] \arrow[d, "1\times\epsilon"] & CX_n \times \mathcal{Z}^{n-1}_{\mathrm{link}_{K_0}(n)} \arrow[d, "1\times\epsilon"] \\
X_n \times \mathcal{W}^{n-1}_{\mathrm{rest}_{K_0}(n)} & X_n \times \mathcal{W}^{n-1}_{\mathrm{link}_{K_0}(n)} \arrow[l, "1\times\lambda"'] \arrow[r, "i\times 1"] & CX_n \times \mathcal{W}^{n-1}_{\mathrm{link}_{K_0}(n)}.
\end{tikzcd}$$

By Proposition 5.1, the pushout of the top row is $\mathcal{Z}^n_{K_0}$. Then we define a homotopy equivalence $\bar{\epsilon}_{K_0} : \mathcal{Z}^n_{K_0} \xrightarrow{\simeq} \mathcal{W}^{n,c}_{K_0}$ as the induced map from the above diagram. By the induction hypothesis, there is also a commutative diagram

$$\begin{tikzcd}
X_n \times \mathcal{Z}^{n-1}_{\mathrm{rest}_{L_0}(n)} \arrow[dr, "1\times\epsilon"] \arrow[dd, "1\times\rho"] & & X_n \times \mathcal{Z}^{n-1}_{\mathrm{link}_{L_0}(n)} \arrow[ll, "1\times\rho"'] \arrow[rr, "i\times 1"] \arrow[dr, "1\times\epsilon"] \arrow[dd, "1\times\rho"] & & CX_n \times \mathcal{Z}^{n-1}_{\mathrm{link}_{L_0}(n)} \arrow[dr, "1\times\epsilon"] \arrow[dd, "1\times\rho"] & \\
& X_n \times \mathcal{W}^{n-1}_{\mathrm{rest}_{L_0}(n)} \arrow[dd, "1\times\lambda"] & & X_n \times \mathcal{W}^{n-1}_{\mathrm{link}_{L_0}(n)} \arrow[ll, "1\times\lambda"'] \arrow[rr, "i\times 1"] \arrow[dd, "1\times\lambda"] & & CX_n \times \mathcal{W}^{n-1}_{\mathrm{link}_{L_0}(n)} \arrow[dd, "1\times\lambda"] \\
X_n \times \mathcal{Z}^{n-1}_{\mathrm{rest}_{K_0}(n)} \arrow[dr, "1\times\epsilon"] & & X_n \times \mathcal{Z}^{n-1}_{\mathrm{link}_{K_0}(n)} \arrow[ll, "1\times\rho"'] \arrow[rr, "i\times 1"] \arrow[dr, "1\times\epsilon"] & & CX_n \times \mathcal{Z}^{n-1}_{\mathrm{link}_{K_0}(n)} \arrow[dr, "1\times\epsilon"] & \\
& X_n \times \mathcal{W}^{n-1}_{\mathrm{rest}_{K_0}(n)} & & X_n \times \mathcal{W}^{n-1}_{\mathrm{link}_{K_0}(n)} \arrow[ll, "1\times\lambda"'] \arrow[rr, "i\times 1"] & & CX_n \times \mathcal{W}^{n-1}_{\mathrm{link}_{K_0}(n)}.
\end{tikzcd}$$

Then it holds that $\bar{\epsilon}_{K_0} \circ \rho_{K_0,L_0} = \lambda^c_{K_0,L_0} \circ \bar{\epsilon}_{L_0}$. Let $\nabla_K : \mathcal{Z}^n_K \to \mathcal{Z}^n_K$ be the restriction of $\prod_{j=1}^n \nabla_j : \prod_{j=1}^n CX_j \to \prod_{j=1}^n CX_j$. Then $\nabla_K \circ \rho_{K,L} = \rho_{K,L} \circ \nabla_L$. By Proposition 4.1, if we put $\epsilon_{K_0} = h^c_{K_0} \circ \bar{\epsilon}_{K_0} \circ \nabla_{K_0}$, it holds that

$$\epsilon_{K_0} \circ \rho_{K_0,L_0} = \lambda_{K_0,L_0} \circ \epsilon_{L_0},$$

where L may be $\emptyset$. Moreover, by the induction hypothesis and Proposition 4.4, we have

$$\begin{aligned}\epsilon_{K_0}\circ\rho_{K_0,i} &= h^c_{K_0}\circ\bar{\epsilon}_{K_0}\circ\nabla_{K_0}\circ\rho_{K_0,i}\\ &= h^c_{K_0}\circ\bar{\epsilon}_{K_0}\circ\rho_{K_0,i}\circ\nabla_i\\ &= \overline{\lambda}\circ(1\times(\epsilon_{\mathrm{rest}_{K_0}(n)}\circ\rho_{\mathrm{rest}_{K_0}(n),i}))\circ\nabla_i\\ &= \overline{\lambda}\circ(1\times\theta_{\mathrm{rest}_{K_0}(n)}(i))\circ\nabla_i\\ &= \theta_{K_0}(i)\end{aligned}$$

for $n_{K_0}\le i<n$, where $\overline{\lambda}$ is the composite of the canonical map $X_n\times\mathcal{W}^{n-1}_{\mathrm{rest}_{K_0}(n)}\to\mathcal{W}^{n,c}_{K_0}$ and $h^c_{K_0}$ as above. We also have

$$\epsilon_{K_0}\circ\rho_{K_0,n}=\widehat{\lambda}\circ(1\times\lambda_{\mathrm{link}_{K_0}(n),\emptyset})\circ\nabla_n,$$

where $\widehat{\lambda}$ is the composite of the canonical map $CX_n\times\mathcal{W}^{n-1}_{\mathrm{link}_{K_0}(n)}\to\mathcal{W}^{n,c}_{K_0}$ and $h^c_{K_0}$. By the definition of $h^c_{K_0}$, the right hand side is the composite $CX_n\times\prod_{j<n}X_j\xrightarrow{\mathrm{proj}}\prod_{j<n_{K_0}}X_j\xrightarrow{\mathrm{incl}}\mathcal{W}^n_{K_0}$, implying $\epsilon_{K_0}\circ\rho_{K_0,}=\theta_{K_0}(n)$. Summarizing, we have constructed a homotopy equivalence $\epsilon_{K_0}=\epsilon_{K(n)}:\mathcal{Z}^n_{K(n)}\xrightarrow{\simeq}\mathcal{W}^n_{K(n)}$ satisfying

$$\epsilon_{K(n)}\circ\rho_{K(n),L(n)}=\lambda_{K(n),L(n)}\circ\epsilon_{L(n)}\quad\text{and}\quad\epsilon_{K(n)}\circ\rho_{K(n),i}=\theta_{K(n)}(i)$$

for $n_{K_0}\le i\le n$.

Put

$$(P_K(m),Q_K(m),\overline{P}_K(m),\overline{Q}_K(m))=\begin{cases}(\mathcal{Z}^n_{K(m+1)},\mathcal{Z}^n_{K(m+1)},\mathcal{W}^n_{K(m+1)},\mathcal{W}^n_{K(m+1)}) & n_{K_0}\le m<n\\ (\mathcal{Z}^n_\emptyset,CX_m\times\prod_{j\ne m}X_j,\mathcal{W}^n_\emptyset,CX_m\times\prod_{j\ne m}X_j) & n_K\le m<n_{K_0}\\ (\mathcal{Z}^n_\emptyset,\mathcal{Z}^n_\emptyset,\mathcal{W}^n_\emptyset,\mathcal{W}^n_\emptyset) & 1\le m<n_K.\end{cases}$$

We also put $P_\emptyset(m)=Q_\emptyset(m)=\overline{P}_\emptyset(m)=\overline{Q}_\emptyset(m)=\prod_{j=1}^n X_j$. Then by the hypothesis of induction on m, there is a commutative diagram

$$\begin{tikzcd}\mathcal{Z}^n_{K(m+1)} \arrow[d, "\epsilon"] & P_K(m) \arrow[l, "\rho"'] \arrow[r, "\mathrm{incl}"] \arrow[d, "\epsilon"] & Q_K(m) \arrow[d, "\hat{\epsilon}"]\\ \mathcal{W}^n_{K(m+1)} & \overline{P}_K(m) \arrow[l, "\lambda"'] \arrow[r, "\mathrm{incl}"] & \overline{Q}_K(m),\end{tikzcd}$$

where $\hat{\epsilon}:Q_K(m)\to\overline{Q}_K(m)$ is defined as $\epsilon_{K(m+1)}$, the identity map of $CX_n\times\prod_{j\ne n}X_j$ and $\epsilon_\emptyset$ according as $n_{K_0}\le m<n, n_K\le m<n_{K_0}$ and $1\le m<n_K$. We define a homotopy equivalence $\epsilon_{K(m)}$ as the induced map from the above commutative diagram for $n_{K_0}\le m<n$ and $1\le m<n_K$. Then $\epsilon_{K(m)}=\epsilon_{K_0},\epsilon_K$ according as $n_{K_0}\le m<n, 1\le m<n_K$. By Proposition 4.2 and 5.1, we also define $\epsilon_{K(m)}$ as the composite of the induced map $\mathcal{Z}^n_{K(m)}\xrightarrow{\simeq}\mathcal{W}^{n,d}_{K(m)}$ from the above commutative diagram and $h^d_{K(m)}$ for $n_K\le m<n_{K_0}$. Then by the induction hypothesis and

Proposition 4.4,

$$\begin{aligned}
\epsilon_{K(m)} \circ \rho_{K(m),i} &= \epsilon_{K(m)} \circ \rho_{K(m),K(m+1)} \circ \rho_{K(m+1),i} \\
&= \lambda_{K(m),K(m+1)} \circ \epsilon_{K(m+1)} \circ \rho_{K(m+1),i} \\
&= \lambda_{K(m),K(m+1)} \circ \theta_{K(m+1)}(i) \\
&= \theta_{K(m)}(i)
\end{aligned}$$

for $\min\{n_{K_0}, m+1\} \leq i \leq n$. By Proposition 4.4, one also has $\epsilon_{K(m)} \circ \rho_{K(m),m} = \theta_{K(m)}(m)$ if $m < n_{K_0}$.

Define $\phi : Q_L(m) \to Q_K(m)$ as $\rho_{K(m+1),L(m+1)}, \theta_{K(m+1)}(m)$ and the inclusion according as $(Q_L(m), Q_K(m)) = (\mathcal{Z}^n_{L(m+1)}, \mathcal{Z}^n_{K(m+1)}), (CX_m \times \prod_{j \neq m} X_j, \mathcal{Z}^n_{K(m+1)})$ and otherwise. We also define $\overline{\phi} : \overline{Q}_L(m) \to \overline{Q}_K(m)$ similarly using $\lambda_{K(m+1),L(m+1)}$ and $\theta_{K(m+1)}(m)$. By the hypothesis of induction on m, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{Z}^n_{L(m+1)} \arrow[dr, "\epsilon"] \arrow[dd, "\rho"] & & P_L(m) \arrow[ll, "\rho"'] \arrow[rr, "\mathrm{incl}"] \arrow[dr, "\epsilon"] \arrow[dd, "\rho"] & & Q_L(m) \arrow[dr, "\hat{\epsilon}"] \arrow[dd, "\phi"] & \\
& \mathcal{W}^n_{L(m+1)} \arrow[dd, "\lambda"] & & \overline{P}_L(m) \arrow[ll, "\lambda"'] \arrow[rr, "\mathrm{incl}"] \arrow[dd, "\lambda"] & & \overline{Q}_L(m) \arrow[dd, "\overline{\phi}"] \\
\mathcal{Z}^n_{K(m+1)} \arrow[dr, "\epsilon"] & & P_K(m) \arrow[ll, "\rho"'] \arrow[rr, "\mathrm{incl}"] \arrow[dr, "\epsilon"] & & Q_K(m) \arrow[dr, "\hat{\epsilon}"] & \\
& \mathcal{W}^n_{K(m+1)} & & \overline{P}_K(m) \arrow[ll, "\lambda"'] \arrow[rr, "\mathrm{incl}"] & & \overline{Q}_K(m).
\end{tikzcd}$$

Then it follows from Proposition 4.2, 4.3 and 4.5 that

$$\epsilon_{K(m)} \circ \rho_{K(m),L(m)} = \lambda_{K(m),L(m)} \circ \epsilon_{L(m)}.$$

Therefore we complete the proof. $\square$

Proof of Theorem 1.4. As is noted above, if $n_K = 1$, $\mathcal{Z}^n_K = \mathcal{Z}_K(C\underline{X}, \underline{X})$ and $\mathcal{W}^n_K \simeq \bigvee_{\emptyset \neq I \subset [n]} |K_I| * \widehat{X}^I$. Then the result follows from Theorem 5.2. $\square$

References

[B] I. Baskakov, *Cohomology of K-powers of spaces and the combinatorics of simplicial divisions*, Russian Math. Surveys **57** (2002), no. 5, 989-990.

[BBCG] A. Bahri, M. Bendersky, F. R. Cohen, and S. Gitler, *The polyhedral product functor: a method of decomposition for moment-angle complexes, arrangements and related spaces*, Advances in Math. **225** (2010), 1634-1668.

[BP] V.M. Buchstaber and T.E. Panov, *Torus actions and their applications in topology and combinatorics*, University Lecture Series **24**, American Mathematical Society, Providence, RI, 2002.

[DJ] M.W. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Mathematical Journal **62** (1991), 417-452.

[DO] M.W. Davis and B. Okun, *Cohomology computations for Artin groups, Bestvina-Brady groups, and graph products*, preprint, 2010.

[DS] G. Denham and A. Suciu, *Moment-angle complexes, monomial ideals and Massey products*, Pure Appl. Math. Q. **3** (2007), no. 1, 25-60.

[FT] Y. Félix and D. Tanré, *Rational homotopy of the polyhedral product functor*, Proc. Amer. Math. Soc. **137** (2009), no. 3, 891-898.

[GT] J. Grbić and S. Theriault, *The homotopy type of the complement of a coordinate subspace arrangement*, Topology **46** (2007), 357-396.

[H] P.S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs **99**, American Mathematical Society, Providence, RI, 2003.

[L] J. Lillig, *A union theorem for cofibrations*, Arch. Math. **24** (1973), 410-415.

[N] D. Notbohm, *Colorings of simplicial complexes and vector bundles over Davis-Januszkiewicz spaces*, Math. Z. **266** (2010), no. 2, 399-405.

[P] T. Porter, *Higher-order Whitehead products*, Topology **3** (1965), 123-135.

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